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The R-matrix of the $\mathcal{U}_q\left(d_4^{(3)}\right)$ algebra and $g_2^{(1)}$ affine Toda field theory

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ABSTRACT

The R-matrix of the $\mathcal{U}_q\left(d_4^{(3)}\right)$ algebra is constructed in the 8-dimensional fundamental representation. Using this result, an exact S-matrix is conjectured for the imaginary coupled $g_2^{(1)}$ affine Toda field theory, the structure of which is found to be very similar to the previously investigated S-matrix of $d_4^{(3)}$ Toda theory. It is shown that this S-matrix is consistent with the results for the case of real coupling using the breather-particle correspondence. For q a root of unity it is argued that the theory can be restricted to yield $\Phi(11|12)$ perturbations of WA_2 minimal models.

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1 Introduction

Imaginary coupled affine Toda field theories have attracted a lot of interest recently (see [1] and references therein). These theories can be thought of as natural generalizations of sine-Gordon theory with solitonic excitations in their spectra. While in general these models are nonunitary as quantum field theories, their RSOS restrictions correspond to perturbations of W-symmetric rational conformal field theories (RCFTs), among them to unitary ones [2, 3].

In the case of theories associated to simply-laced affine Lie algebras, the semi-classical mass ratios are stable under quantum corrections [4] and the S-matrices can be obtained using the fact that the theories are invariant under a quantum affine symmetry algebra of nonlocal charges. In the nonsimply-laced case, while the mass ratios are not stable under quantum corrections [5], it is again thought that the S-matrix can be obtained using the representation theory of the nonlocal symmetry algebra. We remark that the existing computations of the mass renormalization [5] in the nonsimply-laced case do not agree with each other and also that the instability of the classical solutions [6] casts a big question mark over the validity of the results in [4, 5]. However, it is still plausible that the mass ratios of the nonsimply-laced theories would be changed by quantum corrections similarly to the real coupling theories [7], but it is unclear how to make a consistent semiclassical quantisation in the imaginary coupling case.

A nice review of the concept of applying the quantum symmetry algebra to construct exact S-matrices can be found in [8]. In several cases the exact S-matrices have been computed: for $a_n^{(1)}$ affine Toda theory [9], for the $d_n^{(2)}$ [10] and the $b_n^{(1)}$ [11] case. Recently, an exact S-matrix was conjectured for the $d_4^{(3)}$ theory as well [12].

In this paper we will treat the imaginary coupled $g_2^{(1)}$ affine Toda field theory. This theory is known to have a $\mathcal{U}_q\left(d_4^{(3)}\right)$ symmetry algebra [13, 14]. The fundamental excitations are solitons which are assigned to an 8-dimensional irreducible representation of $\mathcal{U}_q\left(d_4^{(3)}\right)$. However, the construction of the S-matrix has been hindered by a multiplicity problem in the tensor product of two copies of this representation, which prevented the application of the usual tensor product graph (TPG) method [15]. In this paper this difficulty is circumvented by explicitely constructing the invariant tensors and then computing the invariant amplitudes using computer algebra.

The layout of the paper is the following: in Section 2 we briefly review the known facts about the quantum symmetry. The derivation of the R-matrix is described in Section 3. The S-matrix is constructed in Section 4 and the scattering amplitudes associated to the first breather bound state of the fundamental solitons are analyzed. It is shown that they can be brought into corrspondence with the S-matrix of the second particle in the real coupling theory. We also discuss the issues related to the gradation of the quantum affine symmetry algebra and the link of the theory to $\Phi(11|12)$ perturbations of WA_2 minimal models. In Section 5 we draw our conclusions.

2 The quantum affine symmetry

Let us take an affine Lie algebra $\hat{g}$ and define the affine Toda field theory with the Lagrangian

$$S = \int d^2x \frac{1}{2} \partial_\mu \vec{\Phi} \partial_\mu \vec{\Phi} + \frac{\lambda}{2\pi} \int d^2x \sum_{\vec{\alpha}_j} \exp\left(i\beta \frac{2}{(\vec{\alpha}_j, \vec{\alpha}_j)} \vec{\alpha}_j \cdot \vec{\Phi} \right) , \tag{2.1}$$

where the vectors $\vec{\alpha}_j$, $j = 0 \ldots r$ are the simple roots of $\hat{g}$ (meaning the simple roots of g plus the extending or affine root with label 0). The normalization of the roots is given by taking $(\vec{\alpha}_j, \vec{\alpha}_j) = 2$ for the long roots.

In the usual nomenclature, (2.1) is referred to as the $\hat{g}^\vee$ affine Toda action, where $\hat{g}^\vee$ denotes the affine Lie algebra dual to $\hat{g}$, whose roots $\vec{\gamma}_j$ are the coroots of $\hat{g}$

$$\vec{\gamma}_j = \frac{2\vec{\alpha}_j}{(\vec{\alpha}_j, \vec{\alpha}_j)} . \tag{2.2}$$

It is immediately apparent that any simply-laced affine Lie algebra is self-dual, while for nonsimply-laced ones the dual is obtained by reversing the arrows in the Dynkin diagram.

The theory (2.1) is known to be integrable. Besides the infinite number of commuting charges, however, there is a nonlocal non-abelian symmetry algebra, which is given by the quantum symmetry algebra $\mathcal{U}_q(\hat{g})$ [13, 14]. The parameter q is related to the coupling constant by

$$q = \exp\left(\frac{4\pi^2 i}{\beta^2} \right) . \tag{2.3}$$

To fix our conventions of the quantum affine algebra, let us briefly summarize the defining relations. We define first the algebra $\widetilde{\mathcal{U}_q(\hat{g})}$, which is generated by elements $\{h_i,\ e_i,\ f_i,\ i = 0 \ldots r\}$, satisfying the following commutation relations:

$$\begin{aligned} &[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j, \\ &[e_i, f_j] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}},\ q_i = q^{(\alpha_i, \alpha_i)/2} , \end{aligned} \tag{2.4}$$

together with the quantum Serre relations

$$\begin{aligned} &\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix}_{q_i} e_i^k e_j e_i^{1-a_{ij}-k} = 0 , \\ &\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix}_{q_i} f_i^k f_j f_i^{1-a_{ij}-k} = 0 , \\ &\qquad\qquad i \neq j , \end{aligned} \tag{2.5}$$

where

$$\begin{pmatrix} m \\ k \end{pmatrix}_{q_i} = \frac{[m]_q!}{[k]_q! [m-k]_q!} ,\ [m]_q! = \prod_{1 \leq i \leq m} [i]_q ,\ [i]_q = \frac{q^i - q^{-i}}{q - q^{-1}} \tag{2.6}$$

are the usual quantum binomial coefficients and

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{2.7}$$

is the Cartan matrix of $d_4^{(3)}$. The coproduct is given by

$$\begin{aligned}
\Delta(e_i) &= q_i^{-h_i/2} \otimes e_i + e_i \otimes q_i^{h_i/2} \ , \\
\Delta(f_i) &= q_i^{-h_i/2} \otimes f_i + f_i \otimes q_i^{h_i/2} \ , \\
\Delta(h_i) &= 1 \otimes h_i + h_i \otimes 1 \ .
\end{aligned} \tag{2.8}$$

The conserved charges possess a definite Lorentz spin. Denoting the infinitesimal two-dimensional Lorentz generator by D we have

$$[D, e_i] = s_i e_i, \quad [D, f_i] = -s_i f_i, \quad [D, H_i] = 0, \quad i = 0, \ldots, r. \tag{2.9}$$

where s_i is the Lorentz spin of e_i. Adjoining the operator D to the algebra $\widetilde{\mathcal{U}_q(\hat{g})}$ results in the full algebra $\mathcal{U}_q(\hat{g})$.

Denoting the Lorentz spin of an operator A by spin(A), spin : $\mathcal{U}_q(\hat{g}) \to \mathrm{R}$ is a gradation of $\mathcal{U}_q(\hat{g})$, which is uniquely fixed by giving $s_0, \ldots, s_r$. The change between the gradations can be performed with similarity transformations by exponentials of the Cartan elements h_i.

For details of how the quantum affine symmetry acts on the multiparticle states and how this action can be used to constrain the two-particle S-matrix we refer the reader to [8].

3 The universal R-matrix of $\mathcal{U}_q\left(d_4^{(3)}\right)$ in the fundamental representation

3.1 The fundamental representation of $\mathcal{U}_q\left(d_4^{(3)}\right)$

The quantum symmetry of the $g_2^{(1)}$ theory is given by $\mathcal{U}_q\left(d_4^{(3)}\right)$. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{3.1}$$

and the simple roots are given by

$$\begin{aligned}
&\alpha_0 = (-3/\sqrt{2}, -3\sqrt{3/2}) \ , \\
&\alpha_1 = (\sqrt{2}, 0) \ , \ \alpha_2 = (-1/\sqrt{2}, 1/\sqrt{6}) \ .
\end{aligned} \tag{3.2}$$

The length of the roots was chosen in such a way that the short roots have length $\sqrt{2}$, which is different from the usual normalization, but in this way the normalization of the coupling β will be more convenient.

There are two important subalgebras of $\mathcal{U}_q\left(d_4^{(3)}\right)$: the generators $\{h_i, e_i, f_i, i = 1, 2\}$ form a subalgebra $\mathcal{A}_1$ isomorphic to $\mathcal{U}_q(a_2)$, while the algebra $\mathcal{A}_0$ generated by $\{h_i, e_i, f_i, i = 0, 1\}$ is isomorphic to $\mathcal{U}_{q^3}(g_2)$.

We will assume that the fundamental solitons transform in the 8-dimensional representation of the algebra, which is also the fundamental representation of $\mathcal{A}_1$. In this space, the algebra $\mathcal{U}_q\left(\widetilde{d_4^{(3)}}\right)$ is represented by the following matrices:

$$
h_0 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} , \; h_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} ,
$$

$$
h_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} , \; e_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ,
$$

$$
e_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{[3]_q[1/2]_q}{2[3/2]_q}} & \sqrt{\frac{[3/2]_q}{2[1/2]_q}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{[3]_q[1/2]_q}{2[3/2]_q}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{[3]_q[1/2]_q}{2[3/2]_q}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ,
$$

$$
e_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{\frac{[3]_q[1/2]_q}{2[3/2]_q}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{[3/2]_q}{2[1/2]_q}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{\frac{[3]_q[1/2]_q}{2[3/2]_q}} & \sqrt{\frac{[3/2]_q}{2[1/2]_q}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ,
$$

$$
f_i = e_i^{tr} \; , \; i = 0, 1, 2 \; , \tag{3.3}
$$

with the notation

$$
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \; , \tag{3.4}
$$

and ^{tr} denotes usual matrix transposition.

In the following we will use the a_2 homogeneous gradation, in which case all the rapidity dependence is shifted to the generators with index 0, i.e.

$$\pi_{a_2}(h_i) = h_i \ , \ \pi_{a_2}(e_i) = x^{\delta_{i0}} e_i \ , \ \pi_{a_2}(h_i) = x^{-\delta_{i0}} f_i \ , \tag{3.5}$$

where x is the spectral parameter (essentially the exponential of the rapidity; the precise correspondence will be given later). In this gradation, the generators $\{e_i, \ f_i, \ h_i, \ i = 1, 2\}$ are independent of the spectral parameter x and so they still form a $\mathcal{U}_q(a_2)$ algebra.

We will solve the intertvining equation for the operator

$$\widehat{\mathcal{R}}(x, q) = P_{12} \mathcal{R}(x, q) \ , \tag{3.6}$$

where $\mathcal{R}(x, q)$ denotes the universal R-matrix in the tensor product of two fundamental representations and x denotes the ratio x_1/x_2 of the spectral parameters in the first and second space, respectively.

The equations for the generators $X \in \mathcal{A}_1$ look like

$$[\widehat{\mathcal{R}}(x, q), \Delta(X)] = 0 \ . \tag{3.7}$$

This means that $\widehat{\mathcal{R}}(x, q)$ is a $\mathcal{U}_q(a_2)$ invariant operator in the space $\mathbf{8} \otimes \mathbf{8}$. This space decomposes into irreducible representations under $\mathcal{U}_q(g_2)$ in the following way

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27} \ , \tag{3.8}$$

where we denoted irreducible representations of $\mathcal{U}_q(a_2)$ by their dimensions. The fact that $\mathbf{8}$ occurs twice in the decomposition is the reason for which the usual tensor product graph method [15] does not apply.

3.2 Finding the invariants

Since the grading is put on the long root, the R-matrix must commute with the generators of the subalgebra $\mathcal{U}_q(a_2)$ in the representation $\mathbf{8} \otimes \mathbf{8}$:

$$[\widehat{\mathcal{R}}(x, q), \Delta(X)] = 0, \ \ X = h_1, h_2, e_1, e_2, f_1, f_2 \ . \tag{3.9}$$

The invariants in the tensor product space are given by the projectors to the irreducible components and, due to the fact that $\mathbf{8}$ occurs twice, two intertwiners from one to the other and vice versa. To compute these invariants, one first finds the highest weight vectors in $\mathbf{8} \otimes \mathbf{8}$, i.e. the common kernel of $E_i = \Delta(e_i), \ \ i = 1, 2$. Let us denote the basis vectors of the representation $\mathbf{8}$ (as given in the previous subsection) by $v_i, \ \ i = 1...8$. Then the highest weight vectors are:

$$|\mathbf{27} \ ; \ 2, 2\rangle = v_1 \otimes v_1$$
$$|\mathbf{10} \ ; \ 3, 0\rangle = q^{1/2} v_1 \otimes v_2 - \frac{1}{q^{1/2}} v_1 \otimes v_2$$

$$
\begin{aligned}
&|\overline{\mathbf{10}}\ ;\ 3,0\rangle = -\frac{1}{q} v_1 \otimes v_6 + v_6 \otimes v_1 \\
&|\mathbf{8_1}\ ;\ 1,1\rangle = q^{3/2} v_1 \otimes v_4 - \frac{1}{q^{3/2}} v_4 \otimes v_1 + \sqrt{\frac{q^2-q+1}{2q}} v_2 \otimes v_6 \\
&-\sqrt{\frac{q^2-q+1}{2q}} v_6 \otimes v_2 \\
&|\mathbf{8_2}\ ;\ 1,1\rangle = q^{3/2} v_1 \otimes v_5 + \frac{1}{q^{3/2}} v_5 \otimes v_1 - \sqrt{\frac{q^2+q+1}{2q}} v_2 \otimes v_6 \\
&-\sqrt{\frac{q^2+q+1}{2q}} v_6 \otimes v_2 \\
&|\mathbf{1}\ ;\ 0,0\rangle = q^2 v_1 \otimes v_8 + \frac{1}{q^2} v_8 \otimes v_1 - q v_2 \otimes v_7 - \frac{1}{q} v_7 \otimes v_1 - \frac{1}{q} v_3 \otimes v_6 + \\
&q v_6 \otimes v_1 + v_4 \otimes v_4 + v_5 \otimes v_5\ ,
\end{aligned} \tag{3.10}
$$

where $|\mathbf{R}\ ;\ a, b\rangle$ denotes a vector in the representation $\mathbf{R}$ with Dynkin labels $a,\ b$. These vectors are orthogonal to each other but unnormalized.

Using the formulae for the highest vectors, one then proceeds to build up the other vectors of the representation. The structure of the representations is depicted in Figure 1, showing the way the different vectors are obtained by the action of the step operators $F_i = \Delta(f_i),\ i = 1, 2$. The only point where one has to be careful that the two $(0, 0)$ vectors in $\mathbf{8}$ are not orthogonal, so an orthogonalization must be performed in this subspace. Using this information, it is possible to construct the basis of all irreducible components except $\mathbf{27}$ and to compute the invariant projectors $\mathcal{P}_{\mathbf{R}}$ on the irreducible subspaces $\mathbf{R} = \mathbf{1},\ \mathbf{8_1},\ \mathbf{8_2},\ \mathbf{10},\ \overline{\mathbf{10}}$. The projector $\mathcal{P}_{\mathbf{27}}$ on $\mathbf{27}$ was computed by subtracting the sum of the other projectors from identity.

The two intertwiners $\mathcal{I}_{12}$ and $\mathcal{I}_{21}$ between $\mathbf{8_1}$ and $\mathbf{8_2}$ can be computed in the following way:

$$
\mathcal{I}_{ij} = \sum_{(n_1,n_2)} |\mathbf{8_i}\ ;\ n_1, n_2\rangle \otimes \langle \mathbf{8_j}\ ;\ n_1, n_2|\ , \tag{3.11}
$$

where it is understood that the sum runs over an orthonormalized basis composed of basis vectors with Dynkin labels $n_1,\ n_2$ (the degeneracy indices in the $(0, 0)$ subspace are not shown explicitely, but should be understood). It is also necessary to note that in all scalar products the variable q must be treated as a formal one and should not be conjugated when computing the conjugate vector. This is equivalent to the requirement that the generators of the quantum affine algebra obey the hermiticity condition

$$
h_i^\dagger = h_i,\ e_i^\dagger = f_i\ . \tag{3.12}
$$

3.3 The fundamental R-matrix

Now let us introduce the spectral parameter. The universal R-matrix in the a_2 homogeneous gradation satisfies the following equations in addition to (3.9):

$$
[\widehat{\mathcal{R}}(x, q), \Delta(h_0)] = 0,
$$

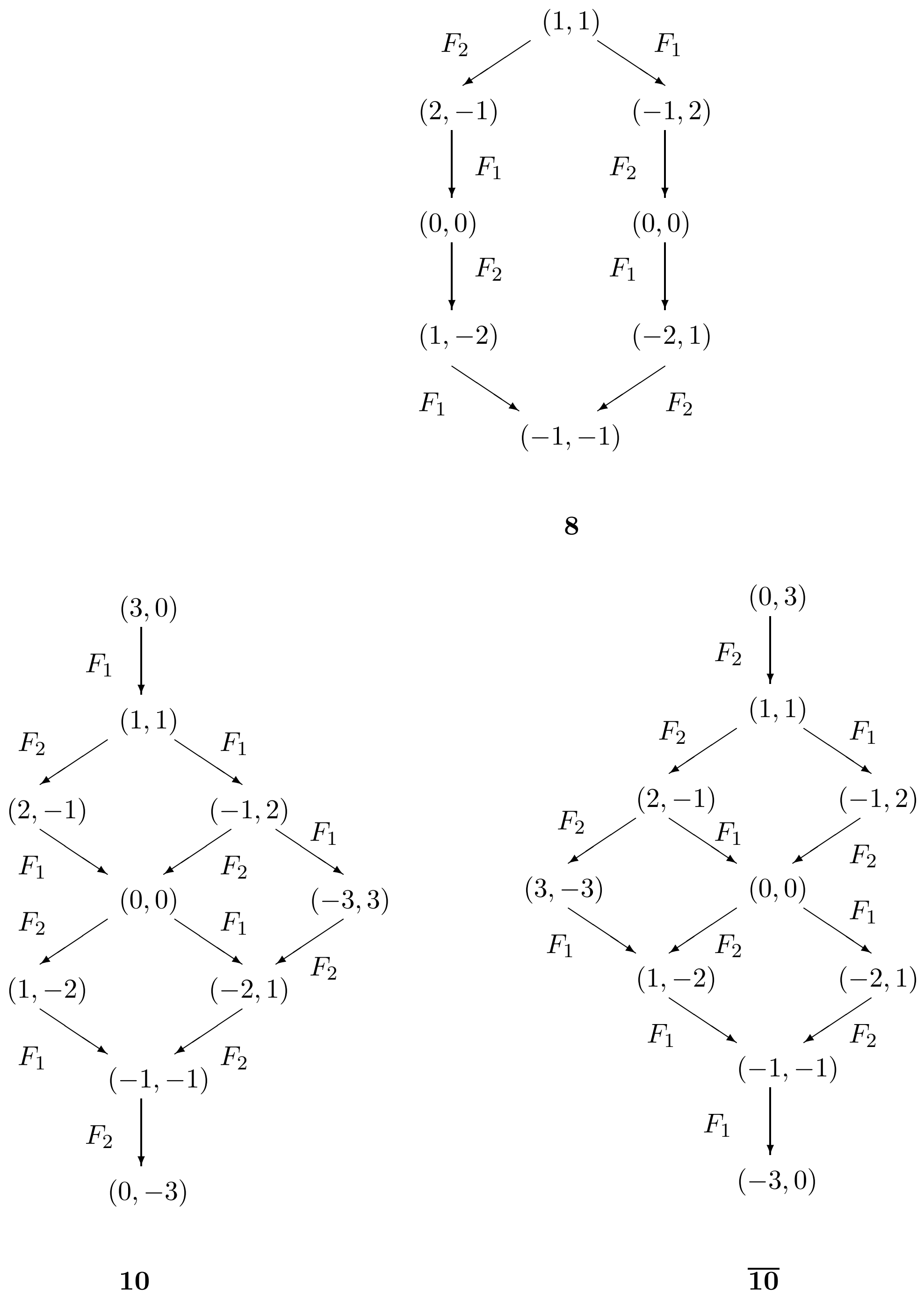

Figure 1: The structure of the representations **8**, **10** and $\overline{\mathbf{10}}$.

$$\widehat{\mathcal{R}}(x,q)(q^{-3h_0/2}\otimes e_0 + xe_0\otimes q^{3h_0/2}) = (xq^{-3h_0/2}\otimes e_0 + e_0\otimes q^{3h_0/2})\widehat{\mathcal{R}}(x,q) \ ,$$
$$\widehat{\mathcal{R}}(x,q)(q^{-3h_0/2}\otimes f_0 + \frac{1}{x}f_0\otimes q^{3h_0/2}) = (\frac{1}{x}q^{-3h_0/2}\otimes f_0 + f_0\otimes q^{3h_0/2})\widehat{\mathcal{R}}(x,q) \ . \tag{3.13}$$

We write the solution in the following form

$$\widehat{\mathcal{R}}(x,q) = \sum_{\mathbf{R}=\mathbf{1},\mathbf{10},\overline{\mathbf{10}},\mathbf{27}} A^{\mathbf{R}}(x,q)\mathcal{P}_{\mathbf{R}} + A^{\mathbf{8}}_{11}(x,q)\mathcal{P}_{\mathbf{8_1}} + A^{\mathbf{8}}_{22}(x,q)\mathcal{P}_{\mathbf{8_2}} + A^{\mathbf{8}}_{12}(x,q)\mathcal{I}_{12} + A^{\mathbf{8}}_{21}(x,q)\mathcal{I}_{21} \ . \tag{3.14}$$

Due to the relation $h_0 = -(2h_1+h_2)/3$, the first of the equations (3.13) is satisfied automatically. The third one follows from the second one if $\widehat{\mathcal{R}}(x,q)$ is a symmetric matrix (which turns out to be the case). The remaining equation can be solved using the computer algebra program MAPLE. The result is

$$\begin{aligned}
A^{\mathbf{27}}(x,q) &= 1 \ , \\
A^{\mathbf{10}}(x,q) = A^{\overline{\mathbf{10}}}(x,q) &= \frac{1-xq^2}{x-q^2} \ , \\
A^{\mathbf{1}}(x,q) &= \frac{1-xq^2}{x-q^2}\frac{1-xq^6}{x-q^6} \ , \\
A^{\mathbf{8}}_{11}(x,q) &= \frac{2q^5(1-x^3)-(q^6-1)((q^4+q^3-q^2-q+1)x^2+(q^4-q^3-q^2+q+1)x)}{2(x-q^2)(x^2+xq^4+q^8)} \ , \\
A^{\mathbf{8}}_{22}(x,q) &= \frac{2q^5(x^3-1)-(q^6-1)((q^4-q^3-q^2+q+1)x^2+(q^4+q^3-q^2-q+1)x)}{2(x-q^2)(x^2+xq^4+q^8)} \ , \\
A^{\mathbf{8}}_{12}(x,q) = A^{\mathbf{1}}_{21}(x,q) &= \frac{x(x-1)(q^6-1)\sqrt{q^8+q^6+q^4+q^2+1}}{2(x-q^2)(x^2+xq^4+q^8)} \ .
\end{aligned} \tag{3.15}$$

The coefficient of $\mathcal{P}_{\mathbf{27}}$ was normalized to one, using the fact that the R-matrix is only determined up to an arbitrary scalar factor. The above solution for $\widehat{\mathcal{R}}(x,q)$ satisfies

$$\widehat{\mathcal{R}}(x,q)\widehat{\mathcal{R}}(1/x,q) = \mathrm{I} \ , \tag{3.16}$$

$$\widehat{\mathcal{R}}(q^6/x,q)\frac{(x-1)(x-q^4)(x^2+xq^2+q^4)q^4}{(x-q^6)(x-q^2)(x^2+xq^4+q^8)} = (C\otimes\mathrm{I})(P_{12}\widehat{\mathcal{R}}(x,q))^{t_1}(C\otimes\mathrm{I})P_{12} \ , \tag{3.17}$$

$$C = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/q & 0 \\
0 & 0 & 0 & 0 & 0 & -q & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1/q & 0 & 0 & 0 & 0 & 0 \\
0 & -q & 0 & 0 & 0 & 0 & 0 & 0 \\
1/q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \ . \tag{3.18}$$

(3.16) is related to the unitarity of the scattering amplitudes, while (3.17) describes how the R-matrix transforms under crossing symmetry, with (3.18) as the charge conjugation matrix.

The R-matrix has the following pole structure:

- $x = q^6$: the R-matrix degenerates to a one-dimensional projector in the $\mathbf{1}$ channel. This should correspond to breathers occuring as bound states of fundamental solitons.

- $x = \omega q^4$ and $x = \omega^{-1} q^4$: the R-matrix degenerates to an eight-dimensional projector in a combination of the $\mathbf{8_1}$ and $\mathbf{8_2}$ channel. This corresponds to higher solitons in the $\mathbf{8}$ representation and also includes the fundamental soliton occurring as a bound state of itself.

- $x = q^2$: the R-matrix degenrates to a projector onto the reducible $\mathbf{1} + \mathbf{8} + \mathbf{10} + \overline{\mathbf{10}}$ representation, where the $\mathbf{8}$ is another combination of $\mathbf{8_1}$ and $\mathbf{8_2}$. This corresponds to a new solitonic multiplet.

The above result can be understood in the following way. The bootstrap structure corresponds to the $\mathcal{U}_q(a_2)$ subalgebra embedded in $\mathcal{U}_q\left(d_4^{(3)}\right)$. But the algebra $d_4^{(3)}$ is obtained by twisting $d_4^{(1)}$ using its Z_3 symmetry. The corresponding embedding of a_2 in $d_4^{(1)}$ is given by the nodes 4 and 0 in the Dynkin diagram. This embedding can be identified with a singular embedding of a_2 into d_4 [16].

Under this embedding, the branching rules are the following: the three 8-dimensional fundamental representations of d_4 (corresponding to the nodes 1, 2 and 3) become the 8-dimensional adjoint representation of a_2. The fourth fundamental representation of d_4 (node 4), which is just the 28-dimensional adjoint representation of $so(8)$, decomposes as $\mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}$. After the Z_3 twisting, the three 8-dimensional representations collapse to the 8-dimensional

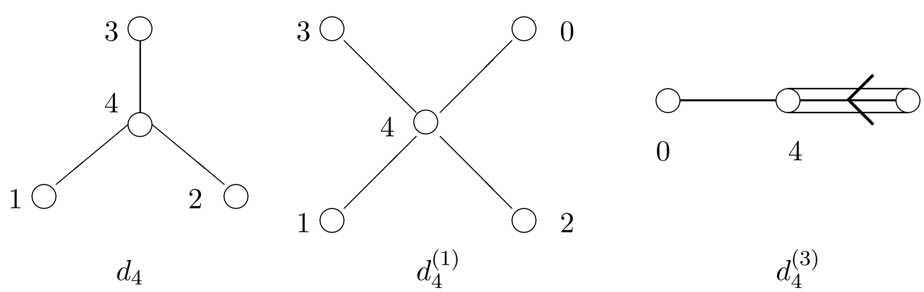

Figure 2: Dynkin diagrams of the algebras d_4, $d_4^{(1)}$ and $d_4^{(3)}$

fundamental representation of $d_4^{(3)}$ corresponding to the fundamental soliton. The coefficient of the pole at $x = q^2$ just projects onto the adjoint+singlet representation, where by adjoint we mean the reducible representation $\mathbf{8} + \mathbf{10} + \overline{\mathbf{10}}$, as justified by the above discussion. This is strikingly similar to the pattern which emerges in the case of the R-matrix of the $\mathcal{U}_q\left(g_2^{(1)}\right)$ algebra [12]. Note that the adjoint representation at the quantum level is extended to include a singlet particle, which is again similar to the case of $\mathcal{U}_q\left(g_2^{(1)}\right)$.

The result (3.15) shows that apart from the subspace containing $\mathbf{8_1}$ and $\mathbf{8_2}$ the R-matrix can be obtained by using the tensor product graph (TPG) method [15], which works well in the case when there are no multiplicities in the tensor product of the representations. This is not the case here, but in the $\mathbf{1} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{27}$ subspace the TPG works as it indeed should.

One can also compute the limit of the universal R-matrix as $x \to 0$ or ∞:

$$\mathcal{R}(0,q) = \mathcal{R}(\infty,q)^{-1} = \mathcal{P}_{\mathbf{27}} - q^2\mathcal{P}_{\mathbf{10}} - q^2\mathcal{P}_{\overline{\mathbf{10}}} - q^5\mathcal{P}_{\mathbf{8_1}} + q^5\mathcal{P}_{\mathbf{8_2}} + q^8\mathcal{P}_{\mathbf{1}} \ , \tag{3.19}$$

which coincides with the result in [17], where a solution to the Yang-Baxter equation was computed in the $\mathbf{8} \otimes \mathbf{8}$ representation of $\mathcal{U}_q(a_2)$. Note that in this limit the mixing terms in the $\mathbf{8_1} + \mathbf{8_2}$ subspace disappear.

4 The $\mathcal{U}_q\left(d_4^{(3)}\right)$ invariant S-matrix

4.1 The scalar factor and the pole structure

The S-matrix in the a_2 homogenous gradation takes the form

$$S(\theta) = \widehat{\mathcal{R}}(x,q) S_0(\theta). \tag{4.1}$$

$S_0(\theta)$ is a scalar function of the rapidity θ to be determined and we have the following relation between x, q and the physical parameters θ, β:

$$q = \exp\left(\frac{4\pi^2 i}{\beta^2}\right) \ , \ x = \exp\left(\left(\frac{4\pi^2}{\beta^2}h - h^\vee\right)\theta\right) \ , \tag{4.2}$$

where $h = 6$ and $h^\vee = 4$ denote the Coxeter and dual Coxeter number of $d_4^{(3)}$, respectively. In principle, the relation for x may contain a phase factor [11], but the above assignment turns out to be consistent with the particle-breather correspondence using the bootstrap. We introduce a new parametrization of the coupling constant with

$$\xi = \frac{\pi\beta^2}{12\pi - 2\beta^2} \ , \tag{4.3}$$

so that x and q take the form

$$x = \exp\left(\frac{2\pi}{\xi}\theta\right) \ , \ q = \exp\left(\frac{i\pi}{3}\left(\frac{\pi}{\xi} + 2\right)\right) \ . \tag{4.4}$$

In order to make the S-matrix crossing symmetric and unitary, the function $S_0(\theta)$ must satisfy

$$\begin{aligned} S_0(i\pi - \theta) = \ & \frac{\sinh\frac{\pi}{\xi}\theta \sinh\frac{\pi}{\xi}\left(\theta - i\frac{\pi}{3}\right) \sinh\frac{\pi}{\xi}\left(\theta - i\frac{\pi}{3} - i\frac{\xi}{3}\right)}{\sinh\frac{\pi}{\xi}\left(\theta - i\pi\right) \sinh\frac{\pi}{\xi}\left(\theta - i\frac{2\pi}{3}\right) \sinh\frac{\pi}{\xi}\left(\theta - i\frac{2\pi}{3} + i\frac{\xi}{3}\right)} \times \\ & \frac{\sinh\frac{\pi}{\xi}\left(\theta - i\frac{2\pi}{3} - i\frac{\xi}{3}\right)}{\sinh\frac{\pi}{\xi}\left(\theta - i\frac{\pi}{3} + i\frac{\xi}{3}\right)} S_0(\theta) \ . \end{aligned} \tag{4.5}$$

This equation has many solutions, among which we choose the so-called minimal one with the minimal number of poles in the physical strip $0 \le \theta < i\pi$. This solution is characterized by the property that it only has poles in locations where the S-matrix degenerates to some projector onto a proper subspace of the two-particle space and is given by the following formula:

$$S_0(\theta) = \prod_{k=0}^{\infty} \frac{(1)\left(\frac{2\pi}{\xi}\right)\left(\frac{\pi}{3\xi}\right)\left(1+\frac{5\pi}{3\xi}\right)\left(\frac{1}{3}+\frac{2\pi}{3\xi}\right)\left(\frac{2}{3}+\frac{4\pi}{3\xi}\right)}{\left(1+\frac{\pi}{\xi}\right)\left(\frac{\pi}{\xi}\right)\left(\frac{4\pi}{3\xi}\right)\left(1+\frac{2\pi}{3\xi}\right)\left(\frac{2}{3}+\frac{\pi}{3\xi}\right)\left(\frac{1}{3}+\frac{5\pi}{3\xi}\right)} \times \frac{\left(\frac{1}{3}+\frac{\pi}{3\xi}\right)\left(\frac{2}{3}+\frac{5\pi}{3\xi}\right)}{\left(\frac{2}{3}+\frac{2\pi}{3\xi}\right)\left(\frac{1}{3}+\frac{4\pi}{3\xi}\right)}, \tag{4.6}$$

where

$$(x) = \frac{\Gamma\left(x+\frac{2k\pi}{\xi}+\frac{i\theta}{\xi}\right)}{\Gamma\left(x+\frac{2k\pi}{\xi}-\frac{i\theta}{\xi}\right)}. \tag{4.7}$$

The pole structure of the S-matrix can be computed using the singularity structure of the R-matrix and the formula for $S_0(\theta)$. It can be seen that the poles come in crossing symmetric pairs and that the S-matrix degenerates to a projector at the direct channel pole and to the complementary projector at the crossed pole. Here we list the direct channel poles only:

- $\theta = i\pi - m\xi$, $m \geq 1$: singlet bound states of the fundamental soliton, so-called breathers. We denote the $m = 1$ case by B_1 and the rest for $m > 1$ with $B_1^{(m-1)}$, considered to be excited states of B_1, which is the ground state of the bound system of two solitons.
- $\theta = \frac{2i\pi}{3} - m\xi$, $m \geq 0$: for $m = 0$ this is the pole for the fundamental soliton K_1 to occur as a bound state of two K_1 particles. The higher cases are denoted by K_m and can be thought of as excited states of the fundamental soliton.
- $\theta = \frac{2i\pi}{3} - \left(m - \frac{2}{3}\right)\xi$, $m \geq 1$: another series of solitons in the fundamental representation, denoted by L_m.
- $\theta = \frac{i\pi}{3} - \left(m - \frac{2}{3}\right)\xi$, $m \geq 1$: a series of particles of the adjoint+singlet type (see discussion in subsection 3.3), which we denote by A_m.

Of course, the bootstrap structure turns out to be much more complicated since new poles can arise from amplitudes involving higher particles. We do not enter the question of closing the bootstrap here. The number of higher states strongly depends on the value of ξ, since only the poles falling into the physical strip can be candidates for new particles. In what follows we give an analysis of the amplitudes involving K_1 and B_1, which is useful for a comparison to the real coupling case via the breather-particle correspondence.

4.2 Breather-soliton and breather-breather scattering amplitudes

To get the S-matrix corresponding to the breather B_1, first one needs an equation for the bootstrap of the tensor part of the fundamental soliton S-matrix. This formula is the following:

$$(\mathcal{P}_{\mathbf{1}})_{12}\,\mathcal{R}_{13}(xq^2,q)\mathcal{R}_{23}(x/q^2,q)\,(\mathcal{P}_{\mathbf{1}})_{12} =$$

$$\frac{(xq^3-1)(x-q)(x^2q^2+xq+1)}{(x-q^3)(xq-1)(x^2+qx+q^2)}\,(\mathcal{P}_{\mathbf{1}})_{12}\otimes \mathrm{I}\ , \tag{4.8}$$

with $\mathcal{P}_{\mathbf{1}}$ denoting the projector onto the singlet in the tensor product $\mathbf{8}\otimes\mathbf{8}$ as before and the indices $1,2,3$ labelling the three one-particle spaces. Using formula (4.6) for the scalar factor $S_0(\theta)$, one finds the following results:

$$\begin{aligned} S_{K_1B_1} &= \left\langle \frac{\pi}{2}+\frac{\xi}{2}\right\rangle_{K_1} \left\langle \frac{5\pi}{6}-\frac{\xi}{2}\right\rangle_{K_2} \left\langle \frac{\pi}{6}+\frac{\xi}{6}\right\rangle \left\langle -\frac{\pi}{6}+\frac{\xi}{6}\right\rangle_{CDD_1} \\ S_{B_1B_1} &= \langle \xi\rangle_{B_1^{(1)}} \left\langle \frac{2\pi}{3}\right\rangle_{B_1} \left\langle -\frac{\pi}{3}+\xi\right\rangle \times \\ &\left\langle \frac{\pi}{3}-\frac{\xi}{3}\right\rangle \left\langle \frac{2\pi}{3}-\frac{\xi}{3}\right\rangle \left\langle -\frac{\pi}{3}-\frac{2\xi}{3}\right\rangle_{CDD2} \left\langle -\frac{\pi}{3}+\frac{2\xi}{3}\right\rangle\ . \end{aligned} \tag{4.9}$$

where we used the notation

$$\langle x\rangle = \frac{\sinh\theta + i\sin x}{\sinh\theta - i\sin x}\ . \tag{4.10}$$

The labels at the bottom of the blocks denote the bound states to which the corresponding poles belong. Blocks with no labels mean that there should be either a new particle associated to the pole, or there is some kind of multiparticle scattering process responsible for the singularity via a Coleman-Thun mechanism [18]. The poles labelled with CDD_1 and CDD_2 are always outside the physical strip whenever B_1 exists (i.e. for $\xi<\pi$).

The S-matrix of the real coupled $g_2^{(1)}$ Toda theory was computed using the hypothesis of 'floating masses' and strong–weak coupling duality [19, 20, 21]. Using the floating Coxeter number H, the S-matrix of the second particle reads [1]

$$S_{22} = \left\langle \frac{2\pi}{H}\right\rangle \left\langle \frac{2\pi}{3}\right\rangle \left\langle \frac{2\pi}{H}+\frac{\pi}{3}\right\rangle \left\langle \frac{\pi}{3}-\frac{4\pi}{H}\right\rangle \left\langle \frac{\pi}{3}-\frac{6\pi}{H}\right\rangle \left\langle \frac{4\pi}{H}-\pi\right\rangle \left\langle \frac{6\pi}{H}-\pi\right\rangle\ , \tag{4.11}$$

where

$$H = 12\frac{6\pi+\beta'^2}{12\pi+\beta'^2}\ , \tag{4.12}$$

and β' is the (real) coupling constant. Substituting $\beta=i\beta'$ we obtain the relation

$$\xi = \pi\left(\frac{6}{H}-1\right)\ , \tag{4.13}$$

and one can easily check that $S_{B_1B_1}$ and S_{22} become identical.

Now we face a puzzle: where is the first particle of the real coupling theory? To understand this let us recall the case of the $d_4^{(3)}$ theory with symmetry algebra $\mathcal{U}_q\left(g_2^{(1)}\right)$ [12]. There we have found two types of solitons: one in the 7-dimensional fundamental representation of g_2 and another one in the adjoint representation (extended by a singlet). The first particle of the real coupling theory was shown to correspond to the breather B_1 originating from the fundamental soliton, while the second one corresponds to a particle AB_1 which was conjectured to be a singlet bound state (i.e. a breather) of the soliton in the adjoint representation.

Using this as an analogy, we expect that in the case of $g_2^{(1)}$ Toda field theory the first particle has to come from the other type of solitons, the one in the $\mathbf{1}+\mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}$ representation.

Unfortunately, due to the reducibility of this representation the tensor product graph method is not applicable and the large number of dimensions prevents us from using a brute force approach with computer algebra. Therefore we can regard the statement on the origin of the 'missing' particle as a conjecture that can possibly be verified in the future with some other approach which makes the computation feasible.

4.3 Remarks on $\Phi(11|12)$ perturbations of WA_2 minimal models

The a_2 homogeneous gradation which was used so far is not the physical gradation of the affine Toda field theory. The physical gradation is the so-called spin gradation, in which the rapidity dependence of the charges generating the quantum affine symmetry algebra matches the spin of the nonlocal currents from which these charges can be derived [13, 14]. This gradation can be obtained by performing the following map on the generators:

$$a \to \alpha(x) a \alpha(x)^{-1}, \ \alpha(x) = x^{(h_1+h_2)/6} \ . \tag{4.14}$$

In this gradation, the evaluation representation becomes:

$$\begin{aligned}
\pi_{spin}(h_i) &= h_i \ , \ i = 0,1,2 \ , \\
\pi_{spin}(e_i) &= x^{1/6} e_i \ , \ i = 1,2 \ , \\
\pi_{spin}(f_i) &= x^{-1/6} f_i \ , \ i = 1,2 \ , \\
\pi_{spin}(e_0) &= x^{-1/2} e_0 \ , \\
\pi_{spin}(f_0) &= x^{1/2} f_0 \ .
\end{aligned} \tag{4.15}$$

From (4.15) one can read off the spin of the conserved charges, using the rapidity dependence of x (4.2). Note that the spins are proportional to the length of the corresponding root. The R-matrix in the spin gradation can be obtained via

$$P_{12}\widehat{\mathcal{R}}(x,q)_{spin} = \alpha(x_1) \otimes \alpha(x_2) P_{12} \widehat{\mathcal{R}}(x,q)_{a_2} \alpha(x_1)^{-1} \otimes \alpha(x_2)^{-1} \ , \ x = x_1/x_2 \ . \tag{4.16}$$

The charge conjugation matrix in this gradation is

$$C_{spin} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.17}$$

However, the a_2 gradation does play an important role which we now describe. It is well-known that imaginary coupled affine Toda field theory is nonunitary and that therefore a restriction is needed to get a sensible physical theory. This is called the RSOS restriction and leads to integrable perturbations of minimal models of W algebras [2, 3].

The action (2.1) can be rewritten as the action of a conformal A_2 Toda theory with a perturbation term:

$$
\begin{aligned}
&S = S_{A_2} + S_{pert} \ , \\
&S_{A_2} = \int d^2x \frac{1}{2}\partial_\mu \vec{\Phi} \partial_\mu \vec{\Phi} + \frac{\lambda}{2\pi}\int d^2x \sum_{j=1}^{2} \exp\left(i\beta \frac{2}{(\vec{\alpha}_j, \vec{\alpha}_j)} \vec{\alpha}_j \cdot \vec{\Phi} \right) \ , \\
&S_{pert} = \frac{\lambda}{2\pi}\int d^2x \exp\left(i\beta \frac{2}{(\vec{\alpha}_0, \vec{\alpha}_0)} \vec{\alpha}_0 \cdot \vec{\Phi} \right) \ . \qquad (4.18)
\end{aligned}
$$

S_{A_2} describes a WA_2-invariant conformal field theory with the central charge [22]

$$
c = 2\left(1 - 12\left(\frac{\beta}{\sqrt{4\pi}} - \frac{\sqrt{4\pi}}{\beta}\right)^2\right) \ . \qquad (4.19)
$$

When

$$
\frac{\beta}{\sqrt{4\pi}} = \sqrt{\frac{p}{p'}} \ , \ p, \ p' \text{ coprime integers} \qquad (4.20)
$$

this is just the central charge of the (p, p') minimal model of the WA_2 algebra, which we denote with $WA_2(p, p')$. The field content of the minimal model can be described by giving the spectrum of the primary fields. These are labelled by four integers $n_1, \ n_2, \ m_1, \ m_2$ and denoted $\Phi(n_1 n_2 | m_1 m_2)$. To each of these fields one can associate a vector

$$
\vec{\beta}(n_1 n_2 | m_1 m_2) = \sum_{i=1}^{2} (\alpha_-(1 - n_i) + \alpha_+(1 - m_i)\, \omega_i \ , \qquad (4.21)
$$

where ω_i are the fundamental weights of A_2 and we define

$$
\alpha_+^2 = \sqrt{\frac{p}{p'}} \ , \ \alpha_- = -\frac{1}{\alpha_+} \ , \ \alpha = \alpha_+ + \alpha_- \ . \qquad (4.22)
$$

The conformal weight of the field $\Phi(n_1 n_2 | m_1 m_2)$ is [23]

$$
h(n_1 n_2 | m_1 m_2) = \frac{1}{2}\vec{\beta}^2 - \alpha \vec{\rho} \vec{\beta} \ , \qquad (4.23)
$$

where

$$
\vec{\rho} = \sum_{i=1}^{2} \omega_i \ . \qquad (4.24)
$$

In our case the perturbing term turns out to be the field $\Phi(11|12)$, which leads to an integrable perturbation [24]. The weight of this field is

$$
h(11|12) = \frac{4}{3}\frac{p}{p'} - 1 \ . \qquad (4.25)
$$

To get a massive field theory, we require the perturbing field to be a relevant one, which means that its weight is less than one. Then we obtain the following condition:

$$
\frac{p}{p'} < \frac{3}{2} \text{ or } \beta^2 < 6\pi \qquad (4.26)
$$

This condition is always satisfied for unitary minimal models of WA_2 ($|p-p'|=1$).

For the above choice of β the parameter q becomes a root of unity. We therefore expect that there exists a consistent restriction of the affine Toda field theory to the perturbed minimal model, which is described by the corresponding restriction of the representation theory of $\mathcal{U}_q(a_2)$ at this value of q. This is the point where the a_2 homogeneous gradation enters the game, since in this gradation the generators of $\mathcal{U}_q(a_2)$ are Lorentz invariant and therefore the space of states can be restricted under the action of $\mathcal{U}_q(a_2)$. In fact, changing the gradation can be seen as redefining the energy-momentum tensor by a Feigin-Fuchs term, which is a crucial step before RSOS restriction [25]. One must have a new energy-momentum tensor commuting with the $\mathcal{U}_q(a_2)$ charges in order to get a consistent relativistic theory after the restriction.

Eqn. (4.26) can also be viewed as the condition $\xi > 0$. In fact, using an argument similar to [12], for $\xi < 0$ we expect that the theory becomes trivial in the infrared in analogy with sine-Gordon theory. Then the unrestricted theory describes two free massless fields and the RSOS restriction must coincide with the free field realization of WA_2 minimal models [23].

5 Conclusion

In this paper we have constructed the R-matrix of the $\mathcal{U}_q\left(d_4^{(3)}\right)$ algebra in the 8-dimensional fundamental representation. The construction is made difficult by the occurrence of multiplicities in the tensor product. It can be seen that the result is essentially nondiagonal in the multiplicity space (i.e. in the space of multiplicity labels of the subspace $\mathbf{8}+\mathbf{8}$) which means that it can not be diagonalized in a basis independent of the spectral parameter x. This is analogous to a phenomenon observed by the author in connection with the fundamental soliton S-matrix in the $\Phi_{(1,5)}$ perturbation of the Virasoro minimal model $Vir(3,16)$ [27]. Therefore there seems to be no way to fit this R-matrix into the usual scheme of the tensor product graph method, although the TPG applies in the subspace which is multiplicity-free. It was also pointed out that the degeneration of the R-matrix at its pole singularities follows closely the pattern observed in the case of $\mathcal{U}_q\left(g_2^{(1)}\right)$ in [12].

Using the R-matrix, the scattering amplitudes for the fundamental soliton of imaginary coupled $g_2^{(1)}$ affine Toda field theory were constructed in the a_2 gradation. The scattering amplitude of the first breather was computed and shown to correspond to the S-matrix of the second particle of the real coupling theory. It is proposed that the first particle must come from the other type of solitonic multiplets, but unfortunately it is not possible to verify this statement due to technical complications. This is an open question for future investigation.

Another interesting open question is the application of an RSOS restriction procedure to this S-matrix to get the scattering amplitudes of $\Phi(11|12)$ perturbations of WA_2 minimal models. These models are more interesting than the $\Phi(11|14)$ perturbations studied in [12] because they include unitary integrable quantum field theories as well.

The solution of these questions would help us in understanding more of the physics behind affine Toda field theories. Some of these issues were already mentioned in [12]. In particular, it would be interesting to see more evidence for the validity of the approach based on the quantum symmetry algebra. It is not obvious that this approach is correct and the main issue is that classically the soliton solutions fail to fill up the affine algebra multiplets completely [26], while the quantum symmetry approach takes this for granted at the quantum level. It would also be interesting to see how a unitary restriction can emerge from a theory which is strongly nonunitary, taking into account especially the instability of classical solitonic solutions [6] and the strong unitarity violation at the quantum level that may persist even after an RSOS restriction [27].

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