Overlapping modularity at the critical point of k-clique percolation

B. Tóth [1], **T. Vicsek** [1] **and G. Palla** [2]

[1]Dept. of Biological Physics, Eötvös Univ., 1117 Budapest, Pázmány P. stny. 1/A
[2]Statistical and Biological Physics Research Group of HAS, 1117 Budapest, Pázmány P. stny. 1/A

E-mail: `btoth@hal.elte.hu`, `vicsek@hal.elte.hu`, `pallag@hal.elte.hu`

Abstract. One of the most remarkable social phenomena is the formation of communities in social networks corresponding to families, friendship circles, work teams, etc. Since people usually belong to several different communities at the same time, the induced overlaps result in an extremely complicated web of the communities themselves. Thus, uncovering the intricate community structure of social networks is a non-trivial task with great potential for practical applications, gaining a notable interest in the recent years. The Clique Percolation Method (CPM) is one of the earliest overlapping community finding methods, which was already used in the analysis of several different social networks. In this approach the communities correspond to k-clique percolation clusters, and the general heuristic for setting the parameters of the method is to tune the system just below the critical point of k-clique percolation. However, this rule is based on simple physical principles and its validity was never subject to quantitative analysis. Here we examine the quality of the partitioning in the vicinity of the critical point using recently introduced overlapping modularity measures. According to our results on real social- and other networks, the overlapping modularities show a maximum close to the critical point, justifying the original criteria for the optimal parameter settings.

1. Introduction

A widely used tool for the study of social phenomena is provided by networks, based on the fundamental concept of mapping the connections among people into a graph. Due to the developments in Information Technology, our social activities and relations generate various forms of data on the large scale. On the one hand this offers a gold mine for research, and on the other hand it can cause non-trivial data handling problems. The network approach has turned out to be very successful in the study of large scale social data, marked by investigations on mobile-phone networks [1, 2, 3, 4], e-mail networks [5, 6, 7, 8], co-authorship networks [9, 10, 11, 12, 13, 14] and online social networks [15, 16, 17, 18, 19]. We note that the idea of representing a complex system with a network is frequently used in various other fields as well, including biology, computer science, economy, etc. According to a very recent survey [20], the network approach can be useful also in the description of the collective motion of dynamically interacting agents. A highly interesting feature of real networks is that in spite of their independent origin, they show many universal features, characterised by a low average distance combined with a high average clustering coefficient, anomalous degree distributions, spreading phenomena and correlations [21, 22, 23, 24, 25].

One of the most widely studied area of complex network research is devoted to communities, (also called as modules, clusters, cohesive groups, etc.), associated with more highly interconnected parts [26, 27, 28, 29, 30, 31, 32, 33, 34, 35], (for a detailed review on communities see Ref.[36]). Such building blocks (functionally related proteins [37, 38], industrial sectors [39], interconnected Autonomous Systems in the Internet [40], similar blogs on the World Wide Web [41, 42], etc.) can play a crucial role in forming the structural and functional properties of the involved networks. Another field of growing interest in complex network theory is related to hierarchy [43, 44, 45, 46, 47, 48], and the presence of communities in networks is one of the relevant and informative signature of the hierarchical nature of complex systems [37, 49, 50].

Communities play a central role in social network research as well, where they can correspond to families, friendship circles, professional teams, or on a larger scale to fan clubs, institutions, etc. [51, 52]. These different types of modules show non-trivial behaviour from several aspects. E.g., the time evolution of smaller collaborative or friendship circles shows significant differences when compared to larger communities like institutions [53]. Another surprising result is the dissasortativity of the graph of communities in highschool friendship networks [54], especially in the light of the assortative nature of social networks in general. Very large scale social communities are also highly interesting, e.g., in a study concerning the mobile phone network of Belgian users the arising communities contained only adjacent municipalities, and the only community running across the "linguistic border" between the Walloon- and Flemish regions was the one related to the Brussels and surroundings [55].

Motivated by the importance of community finding in social networks, (and in complex networks in general), here we focus on a theoretical problem related to the

quality of partitioning. First of all, we point out that in case of social networks, allowing overlaps between the communities is crucial, as we are all members of our family, friendship circle, working group, etc., at the same time in parallel. (Several results suggest that overlaps between communities are important also in biology, where e.g., proteins can be part of more than one functional unit [56]). One of the first algorithms allowing shared members between the communities was given by the clique percolation method (CPM). Here the basic building blocks of the communities are given by k-cliques, (complete sub-graphs of k-nodes), and communities are associated with k-clique percolation clusters. The usual rule for finding the optimal partitioning in this approach is to tune the system to the critical point of k-clique percolation. (In case of e.g., weighted networks this can be achieved by applying an appropriate weight threshold). The reasoning behind this rule is based on simple "physical" principles: the emergence of a giant percolating community would merge (and make invisible) many smaller communities, thus, to find a community structure as highly structured as possible, one has to be at the critical point, where the rising of a giant k-clique percolation cluster is just avoided.

Although the above approach for setting the parameters of the CPM was successful in producing meaningful communities in many real networks, the quality of the partitioning obtained this way was never compared to the results for different parameter settings. Thus, in this paper we examine the quality of the communities in the vicinity of the critical point of k-clique percolation. For quantifying the quality of the community partitions, we rely on various recently introduced modularity measures [57, 58, 59, 60, 35, 61], all designed specially for overlapping communities. (The concept of maximising a real valued modularity function for finding the best community partition is a very popular approach in general, and is also used in non-overlaping clustering problems.) The motivation of our research is the following: Since the idea of tuning the k-clique percolation to the critical point and modularity maximisation are two independent principles aimed at the same target, (i.e., finding optimal community partitions), it is an interesting question whether they show any consistency with each other? I.e., if the partitioning is optimal at the critical point also from the point of modularity, we should observe a maximum in the modularity. We note however that this maximum should be treated as a "local" maximum, or more precisely as a maximum amongst the CPM partitions obtained at different stages of the k-clique percolation transition. The global maximum for the modularity may correspond to a partition (amongst all possible community partitions) which contains communities that are not k-clique percolation clusters.

The paper is organised as follows. In Sect.2. we describe the CPM in short, while in Sect.3. we overview the different overlapping modularity measures. In Sect.4. we examine the behaviour of the listed modularities in the vicinity of the critical point of k-clique percolation in real networks, and finally we conclude in Sect.5.

2. The Clique Percolation Method

As mentioned in Sect.1., the community definition in this approach is based on k-cliques. A k-clique is a sub-graph with maximal possible link density, (i.e., every member of a k-clique is connected to every other member), therefore, it is a good starting point for defining communities. However, a method accepting only complete sub-graphs as communities would be too restrictive. Therefore, k-cliques are "loosen up" in the following way. Two k-cliques are considered adjacent if they share $k-1$ nodes, and a community is defined as the union of k-cliques that can be reached from each other through a series of adjacent k-cliques. In other words, a community is equivalent to a k-clique percolation cluster. We note that a k-clique percolation cluster is very much like a regular edge percolation cluster in the k-clique adjacency graph, where the nodes represent the k-cliques of the original network, and there is a link between two nodes, if the corresponding two k-cliques are adjacent. The two main advantages of the community definition above is its local nature that it allows overlaps between the communities: a node can be part of several k-clique percolation clusters at the same time.

When applied to weighted networks the CPM method can have two parameters: the k-clique size k, and a weight threshold w^* (links weaker than w^* are ignored). When k and w^* are very high, only a few disintegrated community remain, while for low k and w^* in many cases we see a giant community arising, spreading over the majority of the network. When varying the weight threshold at a fixed k, the transition from the dispersed communities to the giant community is analogous to a percolation phase transition. (E.g., previous work has shown that the k-clique percolation transition in the Erdős-Rényi graph is a generalisation of the regular edge percolation transition [62, 63]). The criterion for finding the optimal value of w^* is based on the aim to find a community structure as highly as possible: when the threshold is high we neglect too many links (and communities), while the giant community appearing at low w^* values can smear out the details by merging (and making invisible) the smaller communities. Thus, former works suggested adjusting w^* close to the critical point of k-clique percolation, where we take into account as many links as possible without allowing the emergence of a giant community.

3. Modularity measures

3.1. The modularity by Girvan and Newman

The most popular quality function for community partitions is given by the modularity of Newman and Girvan [64], comparing the fraction of links inside the communities to the expected fraction of links in a random graph where the individual node degrees are equal to the node degrees in the original network. The basic idea behind this approach is that the number of links inside well defined communities should be significantly larger than what we would except at random. The random null model serving as the reference point

is given by the configuration model [65], where the probability for having a connection between nodes i and j with degrees d_i and d_j is given by $d_i d_j / 4M^2$, where M denotes the total number of links. Accordingly, the expected fraction of links inside community α is expressed by first summing up the node degrees in α as $d_\alpha = \sum_{i \in \alpha} d_i$, and simply writing $(d_\alpha / 2M)^2$. Based on the above, the modularity can be given as

$$Q = \sum_{\alpha=1}^{K} Q_\alpha = \sum_{\alpha=1}^{K} \left[\frac{l_\alpha}{M} - \left(\frac{d_\alpha}{2M} \right)^2 \right], \tag{1}$$

where l_α denotes the number of links inside community α, (and l_α / M is simply the fraction of links in α). We note that (1) can be also rewritten in a form with summation over the individual nodes as

$$Q = \frac{1}{2M} \sum_{ij} \left(A_{ij} - \frac{d_i d_j}{2M} \right) \delta(\alpha_i, \alpha_j), \tag{2}$$

where A_{ij} stands for the adjacency matrix, ($A_{ij} = 1$ if i and j are linked, otherwise A_{ij} is zero), and $\delta(\alpha_i, \alpha_j)$ ensures the exclusion of terms where i and j are in different communities.

3.2. Fuzzy modularity

The original modularity (1-2) defined for "crisp" partitions can be generalised for overlapping communities in different ways. A straight forward solution was proposed by Nepusz et al.[57] defining a fuzzy partition matrix $u_{\alpha i}$ in the following way. The column i of $u_{\alpha i}$ is listing how the membership of node i is divided amongst the communities,

$$0 \leq u_{\alpha i} \leq 1, \tag{3}$$

$$\sum_{\alpha=1}^{K} u_{\alpha i} = 1, \tag{4}$$

i.e., $u_{\alpha i} = 0$ when it is not a member at all in community α, whereas a non-zero $u_{\alpha i}$ signs a belonging to α in some extent. In the limiting case of a crisp partition all entries except one in the column become zero, and the entry corresponding to the sole community of the node becomes one. In parallel, the row α of $u_{\alpha i}$ is listing the membership values of the nodes in community α. The row sum $\sum_{i=1}^{N} u_{\alpha i}$ can be treated as the generalisation of the community size.

By introducing a scalar product between the column vectors of $u_{\alpha i}$ we obtain a similarity measure between the nodes defined as

$$s_{ij} = \sum_{\alpha=1}^{K} u_{\alpha i} u_{\alpha j}, \tag{5}$$

where the summation is running over the communities. When nodes i and j have non-zero memberships in absolutely different communities, $s_{ij} = 0$, while larger s_{ij} values usually indicate more similar memberships vectors. In the limiting case of crisp partitions $s_{ij} = 1$ if and only if i and j belong to the same community, or in other

words, s_{ij} becomes equivalent to $\delta(\alpha_i, \alpha_j)$. This observation leads naturally to the idea of replacing $\delta(\alpha_i, \alpha_j)$ by s_{ij} in (2) for gaining an overlapping modularity measure as

$$Q_f = \frac{1}{2M} \sum_{ij} \left(A_{ij} - \frac{d_i d_j}{2M} \right) s_{ij}. \tag{6}$$

A very nice feature of the fuzzy modularity obtained in this way is that in case of crisp partitions it is equivalent to the original modularity by Newman given in (1-2).

However, not all overlapping community detection algorithms evaluate $u_{\alpha i}$ explicitly, instead they provide only the list of members in each community. In this case several possibilities open up for calculating $u_{\alpha i}$. The simplest idea is to divide the membership values of the nodes equally amongst their communities independently of the underlying network topology [66] as

$$u_{\alpha i} = \frac{1}{q_i}, \tag{7}$$

where q_i denotes the number of communities i participates in. To take into account the number of links between the community members and the communities Chen et al.[58] instead proposed

$$u_{\alpha i} = \frac{\sum_{j \in \alpha} A_{ij}}{\sum_{\alpha'} \sum_{j \in \alpha'} A_{ij}}. \tag{8}$$

An even more sophisticated approach is suggested by Shen et al.[59], considering the maximal cliques in the network and summing over all neighbours of a given member i inside the community α as

$$u_{\alpha i} = \frac{1}{u_i} \sum_{j \in \alpha} \frac{C_{ij}^{\alpha}}{C_{ij}} A_{ij}, \tag{9}$$

where C_{ij}^{α} denotes the number of maximal cliques in α containing the link (i, j), and C_{ij} stands for the total number of maximal cliques in the network containing the link (i, j). The pre-factor $1/u_i$ is for normalisation and can be calculated as

$$u_i = \sum_{\alpha=1}^{K} \sum_{j \in \alpha} \frac{C_{ij}^{\alpha}}{C_{ij}} A_{ij}. \tag{10}$$

The most general formulation of the fuzzy modularity (6) was given by Nicosia et al.[60], where the concept of comparing the observed number of links between community members to expectation values based on random null-models was extended to overlapping communities.

3.3. Alternative ideas for modularity

Instead of generalising the terms in the original modularity (1-2), another option for constructing a measure for the quality of overlapping partitions is to build up a formula based on "first principles", i.e., combining terms expressing various criteria for well behaving communities with possible overlaps. Here we overview two different approaches along this line.

3.3.1. Partition density by Ahn et al. A very interesting approach for revealing overlapping communities was suggested by Ahn et al. [35], based on clustering the links instead of the nodes. In this approach link pairs sharing a node are ordered according to the similarity between the neighbourhoods of their other end points. By using a single-linkage hierarchical clustering based on this similarity, we obtain a link dendrogram, and cutting this dendrogram at some threshold yields overlapping communities for the nodes. For determining the optimal cut, Ahn et al. defined the partition density for an individual community α as

$$D_\alpha = \frac{M_\alpha - (N_\alpha - 1)}{N_\alpha(N_\alpha - 1)/2 - (N_\alpha - 1)}, \tag{11}$$

where N_α and M_α denote the number of nodes and links inside α respectively. Assuming that α is connected, $0 \leq D_\alpha \leq 1$, i.e., when α is tree-like, $D_\alpha = 0$, whereas a fully connected community receives $D_\alpha = 1$. The partition density D for the whole system is given by the average of D_α, weighted by the fraction of links inside the communities [35]:

$$D = \sum_{\alpha=1}^{K} \frac{M_\alpha}{M} D_\alpha = \frac{2}{M} \sum_{\alpha=1}^{K} M_\alpha \frac{M_\alpha - (N_\alpha - 1)}{(N_\alpha - 2)(N_\alpha - 1)}. \tag{12}$$

A very nice feature of (12) compared to e.g., (1-2) is its local nature, preventing the emergence of the resolution limit observed in case of the original modularity [67].

3.3.2. The overlapping modularity by Lázár et al. Another alternative for the overlapping modularity was proposed by Lázár et al.[61]. The first criterion for obtaining a well defined community in this approach is that the members should devote the majority of their links to the community rather than other parts of the network. To quantify this aspect, the contribution of member i to the modularity of its community α is calculated by comparing the number of its neighbours inside α to the number of its neighbours outside α as

$$\frac{\sum_{j \in \alpha} A_{ij} - \sum_{j \notin \alpha} A_{ij}}{d_i}. \tag{13}$$

Thus, the contribution becomes negative when i has more neighbours outside α. In case of overlapping nodes belonging to multiple communities we also have to divide the formula above by q_i, corresponding to the number of communities of i.

A further simple criterion for decent communities is to have a relatively large link density. Thus, the modularity of community α is given by the average of (13) over the community members, multiplied by the link density inside α as

$$Q_\alpha^{\text{ov}} = \left[\frac{1}{N_\alpha} \sum_{i \in \alpha} \frac{\sum_{j \in \alpha} A_{ij} - \sum_{j \notin \alpha} A_{ij}}{d_i \cdot q_i} \right] \frac{M_\alpha}{\binom{N_\alpha}{2}}, \tag{14}$$

where N_α and M_α denote the number of nodes and links inside α respectively. The overall modularity of a given partition is simply the average of Q_α^{ov} over the communities given by

$$Q^{\text{ov}} = \frac{1}{K}\sum_{\alpha=1}^{K} Q_\alpha^{\text{ov}} = \frac{1}{K}\sum_{\alpha=1}^{K}\frac{\sum_{i\in\alpha}\frac{\sum_{j\in\alpha}A_{ij}-\sum_{j\notin\alpha}A_{ij}}{d_i\cdot q_i}}{N_\alpha}\frac{M_\alpha}{\binom{N_\alpha}{2}}. \tag{15}$$

Since $\binom{1}{2}$ is not defined, Q_α^{ov} for communities corresponding to single nodes is zero by definition. However, in order to avoid partitions having only a few communities with very high Q_α^{ov} values, Lázár et al. suggested collecting all unclassified nodes and communities corresponding to single nodes into a separate community. A nice feature of (13-15) is that it does not require normalised membership values for the nodes, which makes it possible that e.g., high degree members can contribute more to the modularity of their community compared to low degree nodes.

   We note that a slight drawback of (15) is that it does not take into account the size of the communities, every community is treated equally when calculating the average. This can cause problems when a giant community has emerged spreading over the whole system in the following way. The individual modularity for the giant community given by (14) is almost surely low, since the link density inside cannot be significantly larger than the overall link density in the network. However, as long as there are still at least a few small good quality communities around, the modularity of the whole system can remain high if the contributions from the small communities to (15) suppress the single low quality contribution from the giant community. In order to prevent very large communities from "hiding" their contribution in the overall modularity in this manner we propose an alternative version for for Q^{ov}. Instead of treating the communities equally, we weight them by the fraction of contained links obtaining

$$\widehat{Q}^{\text{ov}} = \sum_{\alpha=1}^{K}\frac{M_\alpha}{M}Q_\alpha^{\text{ov}}. \tag{16}$$

The weighted average above is very similar in nature to (12), used for calculating the overall partition density D from the D_α defined for the individual communities.

4. Applications

We studied the behaviour of the overlapping modularities described in Sect.3. for partitions obtained by the CPM in a couple of real networks. As mentioned in Sect.1., the question of main interest here is whether we find a (local or even global) maximum in the modularity in the vicinity of the critical point of k-clique percolation. Since the networks we studied were all weighted, our method for tuning the system to the critical point was the application of a weight threshold, as explained in Sect.2. However, due to the different origin, the total range and the distribution of the link weights was varying from system to system. To treat all networks we investigated in the same framework, we

ordered the links in each system according to their weights, and subsequently removed them in this order starting from the lowest link weights. For this deterministic removal process the control parameter of the phase transition is given by the fraction of removed links, f. (A considerable advantage of this approach is that it can be used also for un-weighted networks with random link removal processes).

For each given value of f, the communities were extracted with the help of CFinder [68], a freely downloadable implementation of the CPM. To monitor the k-clique percolation transition, we calculated the relative size of the largest community α_G given by $S_G \equiv N_{\alpha_G}/N$, corresponding to the order parameter, (which is 1 if the largest community includes all nodes, and is of the order of $1/N$ when $f \to 1$.) However, probably the most widely used method for determining the critical fraction of removed links f_c is via the susceptibility, χ. This quantity can be defined as the expected change in the size of the largest community α_G when merging with another community chosen at random with a probability proportional to the community size:

$$\chi = \sum_{\alpha \neq \alpha_G} \frac{N_\alpha^2}{\sum\limits_{\alpha \neq \alpha_G} N_\alpha}. \tag{17}$$

At the critical point χ is diverging in the thermodynamic limit, which is manifested in a sharp peak for finite size systems.

Beside calculating S_G and χ, for a given value of the removed fraction of links f we also evaluated the overlapping modularity measures discussed in Sect.3., (taking into account only the remaining links and omitting the already removed ones). For clarity, in Table 1. we summarise their notion and defining formulae used in this paper.

The family of fuzzy modularities, Q^f, Q^C and Q^S are generalisations of the original

Q^f	Fuzzy modularity by Nepusz et al. [57] given in Eq.(6), where the partition matrix $u_{\alpha i}$ is evaluated according to Eq.(7)
Q^C	Fuzzy modularity by Chen et al. [58], given in Eq.(6), where the partition matrix $u_{\alpha i}$ is evaluated according to Eq.(8)
Q^S	Fuzzy modularity by Shen et al. [59], given in Eq.(6), where the partition matrix $u_{\alpha i}$ is evaluated according to Eq.(9)
D	Partition density by Ahn et al. [35], given in Eq.(12)
Q^{ov}	Overlapping modularity by Lázár et al. [61], given in Eq.(15)
$\widehat{Q}^{\mathrm{ov}}$	Overlapping modularity by Lázár et al. modified, given in Eq.(16)

Table 1. Summary of the examined overlapping modularity measures. The three fuzzy modularities, Q^f, Q^C and Q^S are very similar in nature, originating from the Q introduced by Girvan and Newman [64] and formulated in Eq.(6). The only difference between them is in the evaluation of the partition matrices. The partition density D was introduced for "link-based" communities, and hence, it is based on the internal link density of communities. Finally, the overlapping modularity by Lázár et al., Q^{ov}, and also its modified version $\widehat{Q}^{\mathrm{ov}}$ proposed here provide a third alternative approach by combining different requirements for overlapping communities.

non-overlapping modularity by Girvan and Newman [64]. The sole difference between these three quantities lies in the evaluation of the partition matrices, $u_{\alpha i}$, needed for the calculation of the similarity s_{ij} between the nodes in Eq.(5). The obtained s_{ij} are then plugged into Eq.(6) for all three quantities. The partition density, D, was introduced by Ahn et al.[35] for measuring the quality of overlapping "link-communities", and hence, it relies heavily on the link density inside the communities. Finally, the original- and slightly modified version of the overlapping modularity by Lázár et al. [61] were formulated by mixing multiple requirements towards meaningful overlapping communities.

Furthermore, we also evaluated the modularities when the giant percolating community was omitted in the calculation. A community spreading over the vast majority of the nodes is a singular object. In contrast, the modularity measures were designed for quantifying meaningful partitions, and are not expected to handle singular communities like the giant k-clique percolation cluster correctly. A very simple idea to get around this problem is to consistently disregard α_G when calculating the modularity measures.

4.1. The studied networks

The list of studied networks was the following:

- The social network between students of the University of California, (UNICAL), constructed from an online message record [69]. The database contained 1899 users forming altogether 13833 connections, where the weight of a link corresponded to the total number of characters sent between the two endpoints.

- The social network between scientists based on co-authorship given by publications in the Los-Alamos e-print archive under "astro-ph" (Astrophysics), from 1995 to 1999 [70]. Here the network was obtained by projecting the bipartite graph of authors and articles onto the single mode graph of authors. The link weights were calculated according to

$$w_{ij} = \sum_k \frac{\widehat{A}_{ik}\widehat{A}_{jk}}{\sum_l \widehat{A}_{lk} - 1},\tag{18}$$

where $\widehat{A}_{ik}$ denotes the adjacency matrix of the bipartite graph, (i.e., $\widehat{A}_{ik} = 1$ if scientist i is a co-author of article k). The resulting co-authorship network contained 16046 nodes and 121251 links.

- The word association network obtained from the South Florida Free Association norms list (containing 10617 nodes and 63788 links), where the weight of a link from one word to another indicated the frequency that the people in the survey associated the end point of the link with its start point [71]. Since we were interested in un-directed networks, the final weight of the links corresponded to the sum of the weights in the two opposite directions.

4.2. Results

In this section we show the results obtained by omitting the giant community in the calculation of the fuzzy modularities. The same figures for the "full" modularities including also α_G are given in the Appendix. We present our findings only for $k = 3$ or $k = 4$, since the low number and small size of the communities at larger k values made the precise location of the critical point impossible in the systems we studied. In all of our experiments, the resolution in the removed fraction of links, f, was set to 0.005.

We begin with the results for the word association network in Fig.1., since the critical behaviour of the k-clique percolation transition is a far more transparent and articulate here compared to the other systems investigated in this paper.

In Fig.1a we plot the relative size of the largest community, S_G, (corresponding to the order parameter of the k-clique percolation transition), as a function of f, at $k = 3$. In this panel we also show the relative size of the total coverage of communities in the network, denoted by S_C. (Due to the community definition originating in k-cliques, a part of the nodes may not belong to any communities in case of the CPM, and the size of this fraction is given by $1 - S_C$.) Starting with a value very close to 1 at $f = 0$, both S_G and S_C show a slowly decreasing tendency, turning into a steep decay in the vicinity of the critical point, where the two curves separate from each other due to the faster change in S_G. To pinpoint the critical point of the k-clique percolation more precisely, in Fig.1b we display the susceptibility χ as a function of f, showing a very sharp peak at f_c. The behaviour of the various overlapping modularities can be followed in Fig.1c, where each modularity measure is rescaled by its maximal value and is plotted as a function of f. The fuzzy modularities Q^f, Q^C and Q^S show rather smooth curves with single (global) maximums very close to f_c. Although we can observe an unimodal shape for also the partition density D, in this case the position of the maximum is shifted slightly towards higher f values. Nevertheless, this maximum is also consistent with f_c. Interestingly, the modularity by Lázár et al., Q^{ov} shows a monotonously increasing tendency as a function of f, with no maximum in the vicinity of the critical point. In our opinion, this is due to the equal treatment of the communities irrespectively of the community size, "hiding" the giant community with low quality among the better quality communities of normal size. When switching from Q^{ov} to $\widehat{Q}^{\text{ov}}$, also taking into account the community sizes in the averaging, we regain the unimodal shape with a maximum very close to the maximum of D, in consistency with the position of the critical point.

In Fig.2. we display our results for the co-authorship network at $k = 4$. The percolation transition here is far less pronounced compared to the word association network. E.g., the relative size of the giant community, S_G is far below 1 even at $f = 0$, as shown in Fig.2a. Although we observe a peak in the susceptibility, χ, (Fig.2b), its width compared to its magnitude reveals a significantly broader nature compared to the very sharp peak seen in Fig.1b. The corresponding overlapping modularities (scaled by their maximal values) are depicted in Fig.2c, as functions of f. Interestingly, Q^S and the partition density, D have global maximums consistent with the critical f_c, and
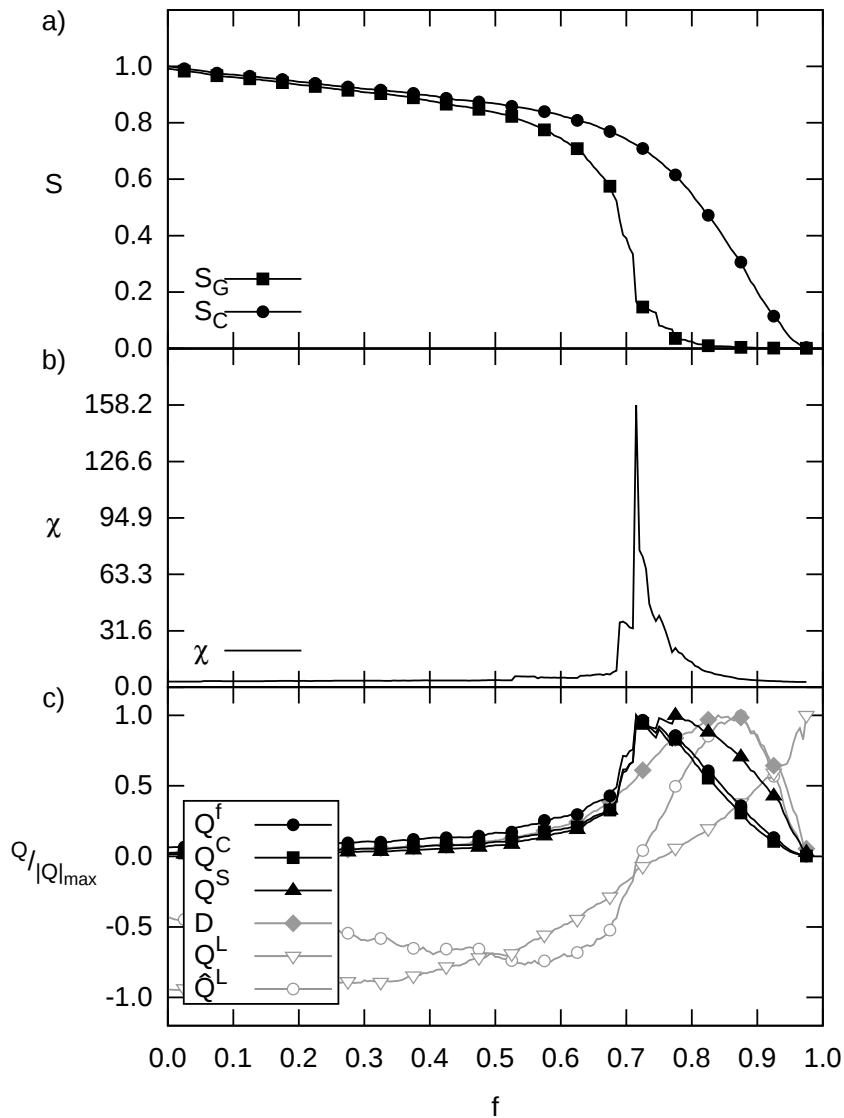
Figure 1. Results for the word association network at $k = 3$, with a resolution in the fraction of removed links, f, set to 0.005. (Please note that symbols on the plots appear at a smaller frequency, as they are intended only for making the different curves more distinguishable). a) The relative size of the largest community, S_G, and the relative size of the total coverage of the communities, S_C, as functions of f. b) The susceptibility, χ, as a function of f, showing a significant sharp peak at the critical point. c) The different overlapping modularities, each scaled by its maximal value, as functions of f. The Q^f, Q^C and Q^S have maximums very close to the critical point, and these maximums are quite very pronounced as well. The partition density D and the modified modularity by Lázár et al., $\widehat{Q}^{\mathrm{ov}}$, have also significant maximums close to the critical point, however their position is slightly shifted towards higher f. The original modularity by Lázár et al., Q^{ov} shows a more or less monotonously increasing tendency as a function of f.
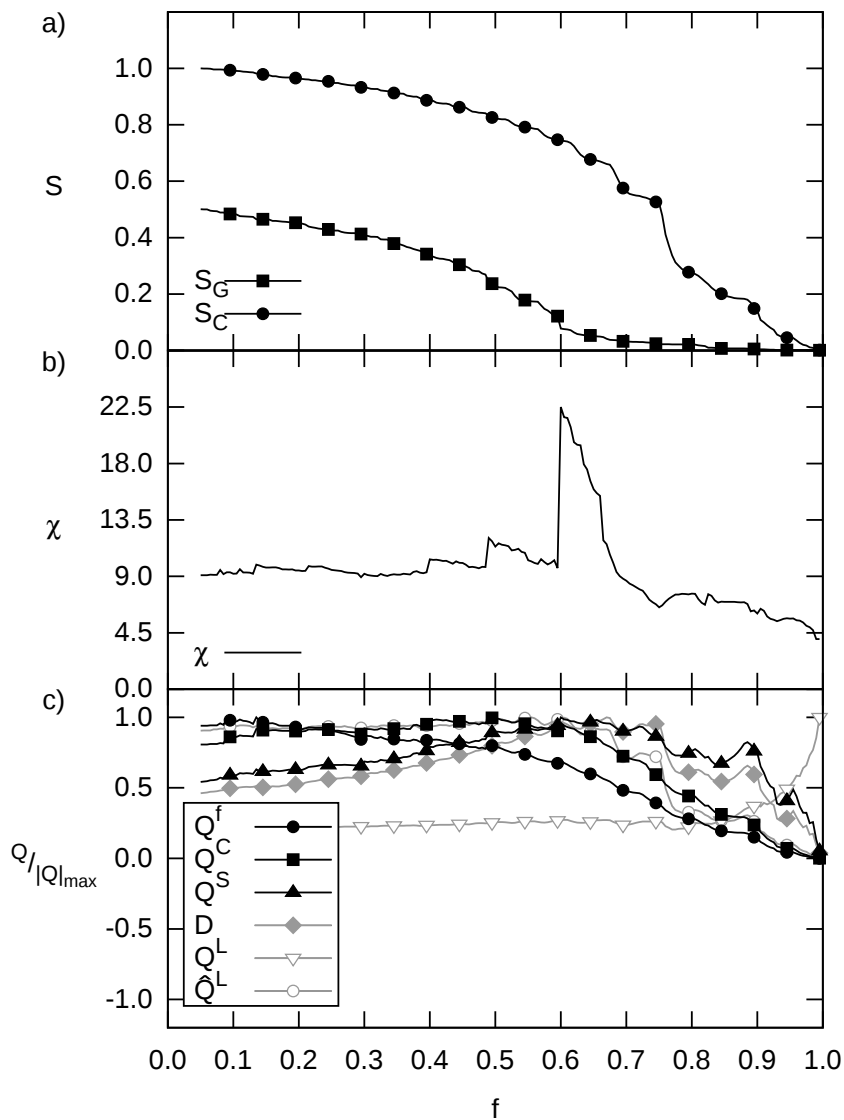
Figure 2. Results for the co-authorship network at $k = 4$, with a resolution in the fraction of removed links, f, set to 0.005. a) The relative size of the largest community, S_G and the relative size of the total coverage of the communities, S_C in the network as functions of f. Note that in this case S_G remains significantly lower than 1 in the whole range of f. b) The susceptibility, χ, as a function of f. The peak signalling the critical point is far less significant compared to the case of the word association network shown in Fig.1. c) The different overlapping modularities, each scaled by its maximal value, as functions of f. The Q^f shows a decreasing tendency in the entire f range, while Q^C has a very weak and protracted maximum in the vicinity of f_c. This maximum is more prominent (and is closer to the critical point) for Q^S and also for the partition density D. Finally, the original modularity by Lázár et al. does not show any relevant maximum, while in case of $\widehat{Q}^L$ we can observe a very weak maximum similar to that of Q^C.

also a very weak global maximum can be observed slightly below f_c for Q^C and $\widehat{Q}^L$. In contrast, the rest of the modularities show no relevant maximum. (Extrema at either $f = 0$ or $f = 1$ are discarded).

Finally, in Fig.3. we show the results for the UNICAL network. Similarly to the co-authorship network, the k-clique percolation transition is far less pronounced compared to the word-association network. Although the relative size of the giant community, S_G is reaching 1 at $f = 0$, (Fig3a), the overall shape of the curve shows a constant decay between $f = 0$ and f_c, in contrast to the very slow decay turning into a sudden drop at the critical point seen in Fig.1a. Furthermore, the susceptibility, χ displayed in Fig.3b shows a peak with a height smaller by an order of magnitude acompanied by roughly the same width compared to the peak in Fig.1b for the word association network. In Fig.3c we plotted the corresponding overlapping modularities as functions of f. All three fuzzy modularities (Q^f, Q^C and Q^S) and also the partition density D show slightly varying curves with an overall unimodal shape, and their maximums is very close to the critical point. In contrast, Q^{ov} and $\widehat{Q}^{\text{ov}}$ display no relevant maximum.

According to our results detailed in the Appendix, when including also the giant community in the evaluation of the modularities, the maximum for most of the measures is shifted either to $f = 0$ or to $f = 1$. These correspond to trivial optima, i.e., either all of the links have to be kept, or all of them has to be deleted to achieve maximal modularity. An interesting exception is provided by the partition density, D, for which we observed maximums in full consistency with the critical point for all networks we investigated.

In summary, the overall behaviour of the various different overlapping modularities support the basic assumption that the optimal partitioning for the CPM is obtained in the vicinity of the critical point of k-clique percolation. However, the consistency between the position of the critical point and the maximums of the modularities can be best observed for systems showing a sharp, fully fledged phase transition. Furthermore, one has to take into account how the modularities are actually evaluated. The most reassuring results were given by the partition density, D, providing a global maximum always in consistency with the critical point. When omitting the giant percolating community from the calculation, the fuzzy modularities, Q^f, Q^C and Q^S also showed a prominent maximums very close to f_c in case of the word association network and the UNICAL network, while Q^C and Q^S showed consistency with the critical point also for the co-authorship network. The original overlapping modularity by Lázár et al., Q^{ov} showed a rather monotonous behaviour in all systems we investigated, lacking any relevant maximum other than $f = 0$ or $f = 1$. This is mainly due to the fact that the averaging over the community-vise individual modularities in (15) does not take into account the community size. When switching to $\widehat{Q}^{\text{ov}}$ incorporating size dependent weights, we observed a significant maximum close to the critical point in case of the word association network, and this maximum remained at its place even when including also the giant community in the calculation.
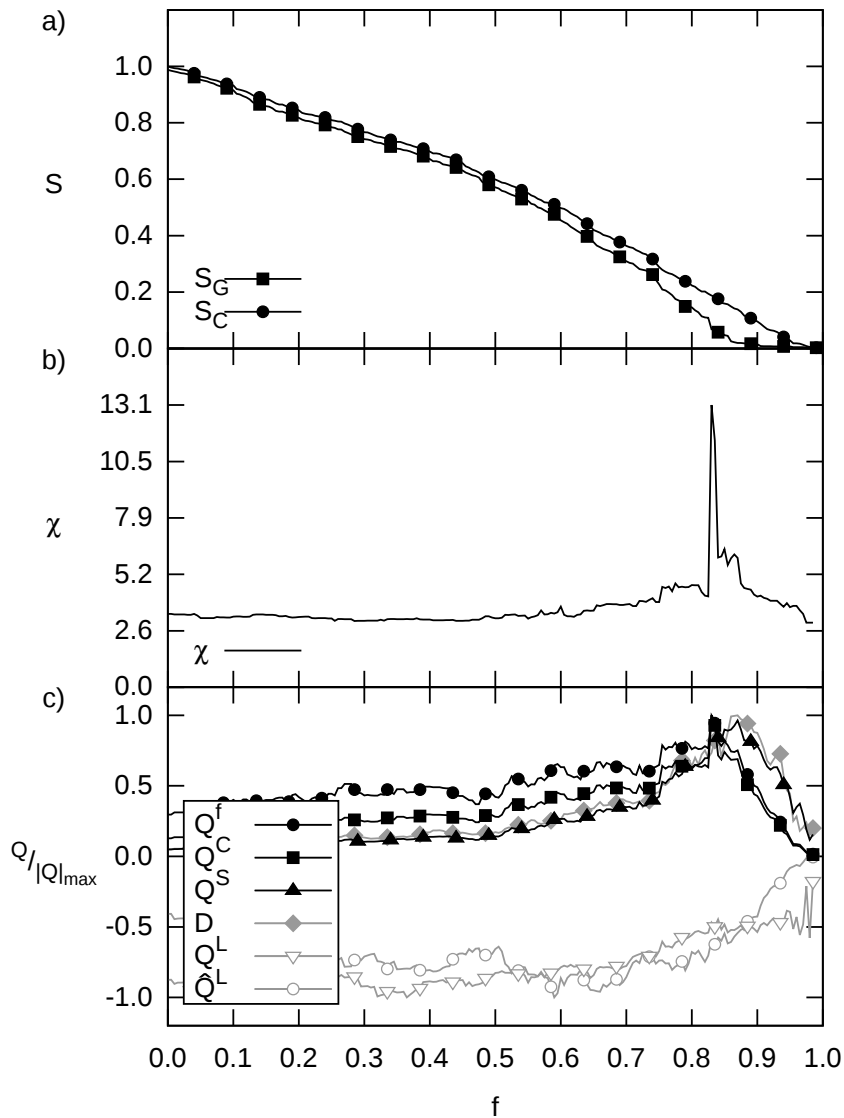
Figure 3. Results for the UNICAL network at $k = 3$, with a resolution in the fraction of removed links, f, set to 0.005. a) The relative size of the largest community, S_G and the relative size of the total coverage of the communities, S_C in the network as functions of f. b) The susceptibility, χ, as a function of f. Similarly to the co-authorship network, the peak at the critical point is far less pronounced here compared to word-association network shown in Fig.1. c) The different overlapping modularities, each scaled by its maximal value, as functions of f. In this case the fuzzy modularities Q^f, Q^C, Q^S, and also the partition density have a roughly unimodal shape with a global maximum very close to the critical point. The Q^{ov} and $\widehat{Q}^{ov}$ have rather varying shapes without any relevant global maximum.

5. Conclusions

Motivated by the importance of overlapping community finding methods in social networks, we studied the behaviour of various overlapping modularity measures in the vicinity of the critical point of k-clique percolation. According to our analysis of real social- and other networks, the overlapping modularities showed large maximums close to the critical point when the critical behaviour of the phase transition was prominent and articulate. However in some of the networks we could observe only blurred, slightly ambiguous phase transitions. Nevertheless, a part of the involved modularity measures still displayed a maximum in consistency with the likely position of the critical point, while the others showed no relevant maximums at all. These findings provide a strong quantitative validation for the former heuristic for setting the parameters of the CPM, suggesting that the quality of the partitioning is best in the vicinity of the critical point of k-clique percolation.

Appendix

Here we show the results for the modularities presented in Figs.1-3. when the giant percolating community is also included in the calculation. For simplicity we also re-plotted the relative size of the giant component, S_G, and the susceptibility, χ, as functions of f for each system.

In Fig.4. we show the results for the word association network. The three fuzzy modularities, (Q^f, Q^C and Q^S), show an overall decreasing tendency with a strong change in the slope at the critical point for Q^C and Q^S. In contrast, the partition density D and the two variations for the overlapping modularity by Lázár et al., Q^L and $\widehat{Q}^L$ behave similarly to the case shown in Fig.1c: D and $\widehat{Q}^L$ have prominent maximums in consistency with f_c, while Q^L shows an increasing tendency as a function of f.

In Fig.5. we display the results for the co-authorship network. Here the fuzzy modularities, (Q^f, Q^C and Q^S), show a smoothly and slowly decreasing curve in the entire f range. Similarly to the case of the word association network, the shape of the curves for D, Q^L and $\widehat{Q}^L$ are less affected by the inclusion of α_G in the calculation: D has a global maximum in the vicinity of f_c and Q^L shows an almost constant behaviour with a sudden increase for large f. For $\widehat{Q}^L$, the very weak maximum in Fig.2c has been flattened, however, a steeply decreasing function can still be observed above f_c.

Finally, in Fig.6. we show the results for the UNICAL network. Again, the fuzzy modularities, (Q^f, Q^C and Q^S), display a smoothly decreasing tendency. However, for Q^C and Q^S the slope becomes higher close to the critical point, similarly to the case shown in Fig.4c. The partition density, D, has an overall unimodal shape with smaller fluctuations and a global maximum quite close to the critical point. The Q^{ov} and $\widehat{Q}^{ov}$ display no relevant maximum, with $\widehat{Q}^{ov}$ actually arriving to a minimum close to f_c. The explanation of this effect is yet unresolved and waits for future work.
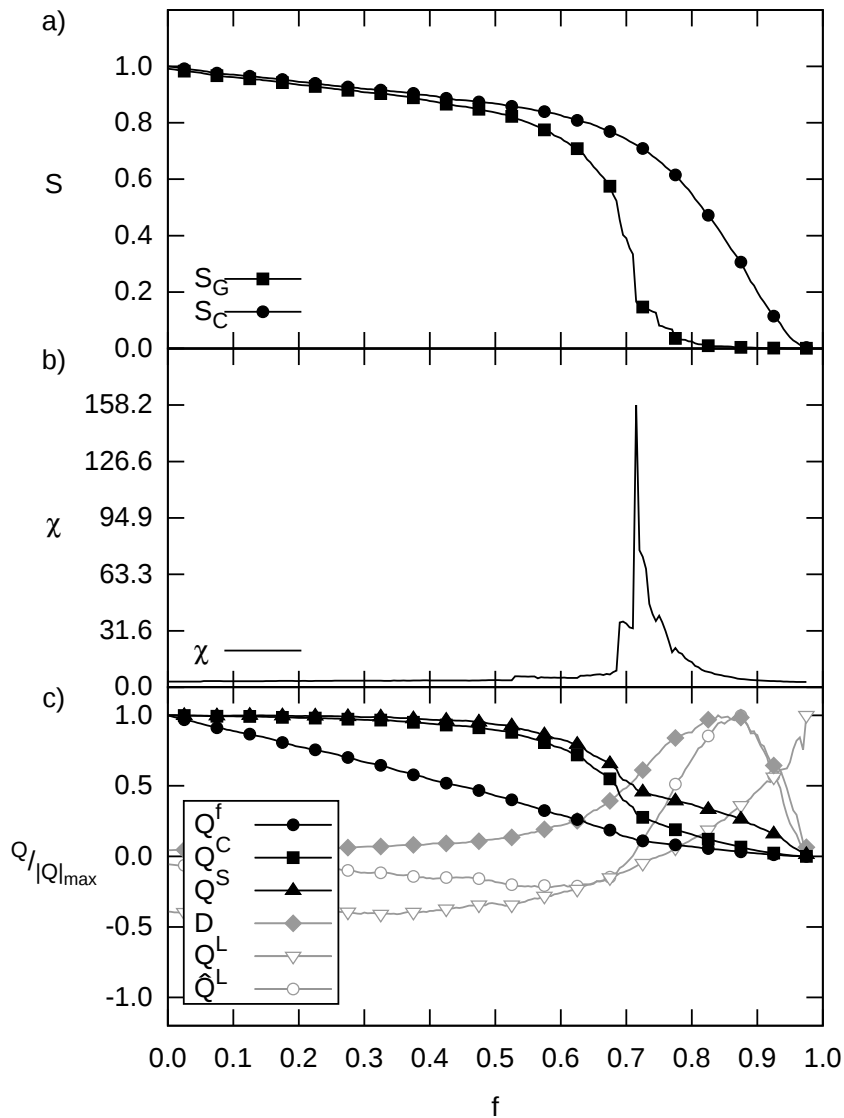
Figure 4. Results for the word association network at $k = 3$ when including also α_G in the evaluation of the modularities. a) The relative size and relative coverage of α_G, as functions of the fraction of removed links, f, (same as in Fig.1a). b) The susceptibility, χ, as a function of f (same as in Fig.1b). c) The different overlapping modularities, each scaled by its maximal value, as functions of f. The Q^f, Q^C and Q^S show a decreasing tendency, with a more steep drop at f_c in case of Q^C and Q^S. The partition density D and the modified modularity by Lázár et al., $\widehat{Q}^{ov}$, still have significant maximums at a position consistent with the critical point. The original modularity by Lázár et al., Q^{ov} shows an increasing tendency as a function of f, similarly as in case of Fig.1c.
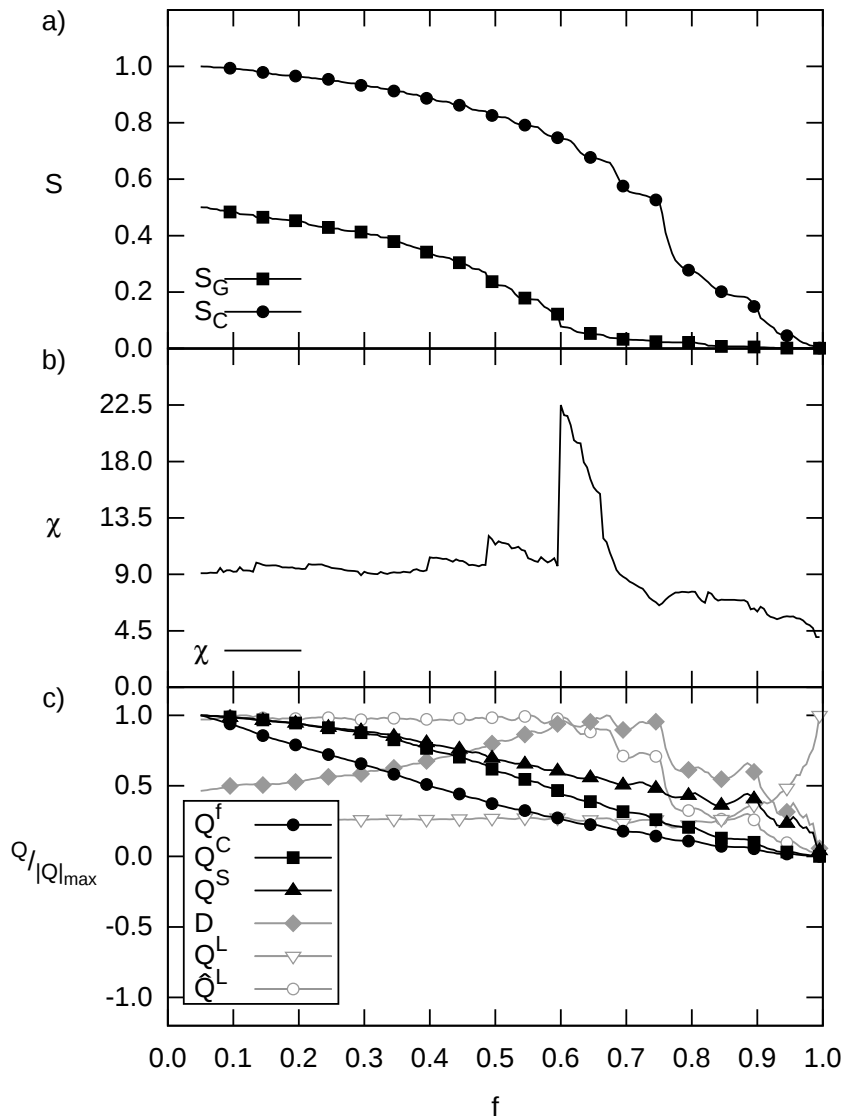
Figure 5. Results for the co-authorship network at $k = 4$. a) The relative size and relative coverage of α_G, as functions of the fraction of removed links, f, (same as in Fig.2a). b) The susceptibility, χ, as a function of f (same as in Fig.2b) c) The different overlapping modularities, each scaled by its maximal value, as functions of f. The fuzzy modularities, Q^f, Q^C and Q^S show a slowly decreasing tendency, while the partition density D has a global maximum relatively close to the critical point. The modularity Q^L is more or less constant except for the very large f region where it becomes increasing. In contrast, $\widehat{Q}^L$ is close to 1 below f_c, and starts decreasing above.
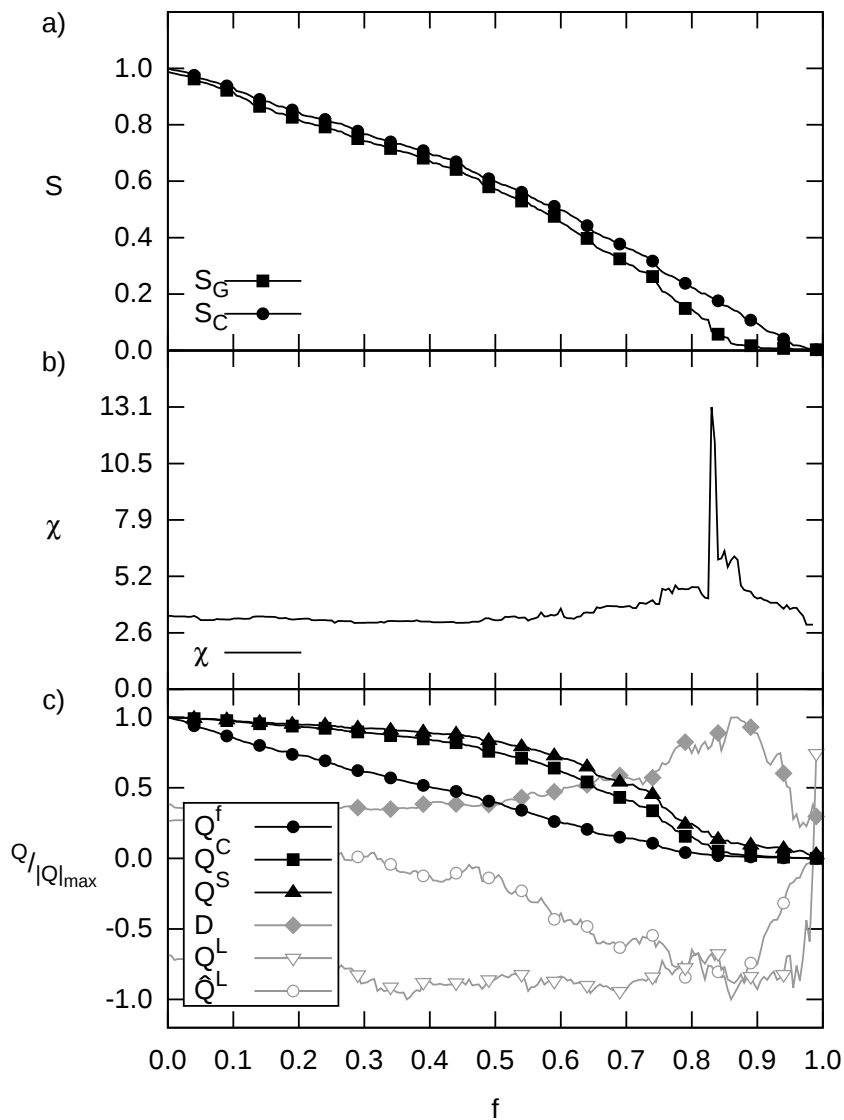
Figure 6. Results for the UNICAL network at $k = 3$. a) The relative size and relative coverage of α_G, as functions of the fraction of removed links, f, (same as in Fig.3a). b) The susceptibility, χ, as a function of f (same as in Fig.3b). c) The different overlapping modularities, each scaled by its maximal value, as functions of f. Similarly to Fig.4c, the fuzzy modularities Q^f, Q^C and Q^S show a decreasing tendency with a steeper drop in Q^C and Q^S at the critical point. The partition density, D has retained its overall unimodal shape with a global maximum in consistency with f_c. The original- and modified modularity by Lázár et al. show rather varying curves with no relevant maximum.

[1] J.-P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, D. Lazer, K. Kaski, J. Kertész, and A.-L. Barabási. Structure and tie strengths in mobile communication networks. *Proc. Natl. Acad. Sci. USA*, 104:7332–7336, 2007.

[2] J.-P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, M. A. de Menezes, K. Kaski, A. L. Barabási, and J. Kertész. Analysis of a large-scale weighted network of one-to-one human communication. *New Journal of Physics*, 9:179, 2007.

[3] R. Lambiotte, V. D. Blondel, C. de Kerchove, E. Huens, C. Prieur, Z. Smoreda, and P. Van Dooren. Geographical dispersal of mobile communication networks. *Physica A: Statistical Mechanics and its Applications*, 387:5317–5325, 2008.

[4] M. Seshadri, S. Machiraju, A. Sridharan, J. Bolot, C. Faloutsos, and J. Leskovec. Mobile call graphs: beyond power-law and lognormal distributions. In *Proceedings of the 14th ACM SIGKDD international conference on Knowledge discovery and data mining*, KDD '08, pages 596–604, New York, NY, USA, 2008. ACM.

[5] H. Ebel, L.-I. Mielsch, and S. Bornholdt. Scale-free topology of e-mail networks. *Phys. Rev. E*, 66:035103(R), 2002.

[6] M. E. J. Newman, S. Forrest, and J. Balthrop. Email networks and the spread of computer viruses. *Phys. Rev. E*, 66:035101(R), 2002.

[7] C. Zhu, A. Kuh, J. Wang, and P. De Wilde. Analysis of an evolving email network. *Phys. Rev. E*, 74:046109, 2006.

[8] G. Kossinets and D. J. Watts. Empirical analysis of an evolving social network. *Science*, 311:88–90, 2006.

[9] A.-L. Barabási, H. Jeong, Z. Néda, E. Ravasz, A. Schubert, and T. Vicsek. Evolution of the social network of scientific collaborations. *Physica A: Statistical Mechanics and its Applications*, 311:590–614, 2002.

[10] M. E. J. Newman. Coauthorship networks and patterns of scientific collaboration. *Proc. Natl. Acad. Sci. USA*, 101(SUPPL. 1):5200–5205, 2004.

[11] C. S. Wagner and L. Leydesdorff. Network structure, self-organization, and the growth of international collaboration in science. *Research Policy*, 34:1608–1618, 2005.

[12] J. J. Ramasco, S. N. Dorogovtsev, and R. Pastor-Satorras. Self-organization of collaboration networks. *Phys. Rev. E*, 70:036106, 2004.

[13] K. Börner, J. T. Maru, and R. L. Goldstone. The simultaneous evolution of author and paper networks. *Proc. Natl. Acad. Sci. USA*, 101(SUPPL. 1):5266–5273, 2004.

[14] C. S. Wagner. Six case studies of international collaboration in science. *Scientometrics*, 62:3–26, 2005.

[15] A. Mislove, M. Marcon, K. P. Gummadi, P. Druschel, and B. Bhattacharjee. Measurement and analysis of online social networks. In *Proceedings of the 7th ACM SIGCOMM conference on Internet measurement*, IMC '07, pages 29–42, New York, NY, USA, 2007. ACM.

[16] A. Nazir, S. Raza, and C.-N. Chuah. Unveiling facebook: a measurement study of social network based applications. In *Proceedings of the 8th ACM SIGCOMM conference on Internet measurement*, IMC '08, pages 43–56, New York, NY, USA, 2008. ACM.

[17] R. Kumar, J. Novak, and A. Tomkins. Structure and evolution of online social networks. In P. S. S. Yu, J. Han, and C. Faloutsos, editors, *Link Mining: Models, Algorithms, and Applications*, pages 337–357. Springer New York, 2010.

[18] R. Schifanella, A. Barrat, C. Cattuto, B. Markines, and F. Menczer. Folks in folksonomies: social link prediction from shared metadata. In *WSDM '10 Proceedings of the third ACM international conference on Web search and data mining*, pages 271–280, 2010.

[19] L. M. Aiello, A. Barrat, C. Cattuto, G. Ruffo, and R. Schifanella. Link creation and profile alignment in the aNobii social network. In *Proceedings of the Second IEEE International Conference on Social Computing SocialCom 2010*, pages 249–256, 2010.

[20] T. Vicsek and A. Zafeiris. Collective motion. *Physics Reports*, 517:71–140, 2012.

[21] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–

442, 1998.

[22] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286:509–512, 1999.

[23] R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. *Rev. Mod. Phys.*, 74:47–97, 2002.

[24] J. F. F. Mendes and S. N. Dorogovtsev. *Evolution of Networks: From Biological Nets to the Internet and WWW*. Oxford University Press, Oxford, 2003.

[25] A. Barrat, M. Barthelemy, and A. Vespignani. Weighted evolving networks: Coupling topology and weight dynamics. *Phys. Rev. Lett.*, 92:228701, 2004.

[26] M. Girvan and M. E. J. Newman. Community structure in social and biological networks. *Proc. Natl. Acad. Sci. USA*, 99:7821–7826, 2002.

[27] D. M. Wilkinson and B. A. Huberman. A method for finding communities of related genes. *Proc. Natl. Acad. Sci. USA*, 101:5241, 2004.

[28] L. Donetti and M. A. Muñoz. Detecting network communities: a new systematic and efficient algorithm. *J. Stat. Mech.*, (2004):P10012, 2004.

[29] J. Reichardt and S. Bornholdt. Detecting fuzzy community structures in complex networks with a potts model. *Phys. Rev. Lett.*, 93:218701, 2004.

[30] G. Palla, I. Derényi, I. Farkas, and T. Vicsek. Uncovering the overlapping community structure of complex networks in nature and society. *Nature*, 435:814–818, 2005.

[31] M. Rosvall and C. T. Bergstrom. An information-theoretic framework for resolving community structure in complex networks. *Proc. Natl. Acad. Sci. USA*, 104:7327, 2007.

[32] V.D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre. Fast unfolding of community hierarchies in large networks. *J. Stat. Mech.*, (2008):P10008, 2008.

[33] P. Ronhovde and Z. Nussinov. Multiresolution community detection for megascale networks by information-based replica correlations. *Phys. Rev. E*, 80:016109, 2009.

[34] A. Lancichinetti, S. Fortunato, and J. Kertész. Detecting the overlapping and hierarchical community structure in complex networks. *New Journal of Physics*, 11:033015, 2009.

[35] Y.-Y. Ahn, J. P. Bagrow, and S. Lehmann. Link communities reveal multiscale complexity in networks. *Nature*, 466:761–764, 2010.

[36] S. Fortunato. Community detection in graphs. *Physics Reports*, 486:75–174, 2010.

[37] E. Ravasz, A. L. Somera, D. A. Mongru, Z. Oltvai, and A.-L. Barabási. Hierarchical organization of modularity in metabolic networks. *Science*, 297:1551, 2002.

[38] V. Spirin and L. A. Mirny. Protein complexes and functional modules in molecular networks. *Proc. Natl. Acad. Sci. USA*, 100:12123, 2003.

[39] J. P. Onnela, A. Chakraborti, K. Kaski, J. Kertész, and A. Kanto. Dynamics of market correlations: Taxonomy and portfolio analysis. *Phys. Rev. E*, 68:056110, 2003.

[40] K. A. Eriksen, I. Simonsen, S. Maslov, and K. Sneppen. Modularity and extreme edges of the internet. *Phys. Rev. Lett.*, 90:148701, 2003.

[41] M. Chau and J. Xu. Mining communities and their relationships in blogs: A study of online hate groups. *International Journal of Human-Computer Studies*, 65:57–70, 2007.

[42] Y. Zhou and J. Davis. Discovering web communities in the blogspace. In *Proceedings of the 40th Annual Hawaii International Conference on System Sciences*, HICSS'07, page 85c, Los Alamitos, CA, USA, 2007. IEEE Computer Society.

[43] A.-L. Barabási, E. Ravasz, and T. Vicsek. Deterministic scale-free networks. *Physica A*, 299:559–564, 2001.

[44] C. M. Song, S. Havlin, and H. A. Makse. Self-similarity of complex networks. *Nature*, 433:392–395, 2005.

[45] A. Clauset, C. Moore, and M. E. J. Newman. Hierarchical structure and the prediction of missing links in networks. *Nature*, 453:98–101, 2008.

[46] V. A. Avetisov, A. V. Chertovich, S. K. Nechaev, and O. A. Vasil yev. On scale-free and poly-scale behaviors of random hierarchical network. *J. Stat. Mech.*, (2009):P07008, 2009.

[47] G. Palla, L. Lovász, and T. Vicsek. Multifractal network generator. *Proc. Natl. Acad. Sci. USA*, 107:7640–7645, 2010.

[48] G. Palla, P. Pollner, and T. Vicsek. Rotated multifractal network generator. *J. Stat. Mech.*, (2011):P02003, 2011.

[49] T. Vicsek. The bigger picture. *Nature*, 418:131, 2002.

[50] R. Guimerá, L. Danon, A. Díaz-Guilera, F. Giralt, and A. Arenas. Self-similar community structure in a network of human interactions. *Phys. Rev. E*, 68:065103, 2003.

[51] J. Scott. *Social Network Analysis: A Handbook*. Sage Publications, London, 2nd edition, 2000.

[52] D. J. Watts, P. S. Dodds, and M. E. J. Newman. Identity and search in social networks. *Science*, 296:1302, 2002.

[53] G. Palla, A.-L. Barabási, and T. Vicsek. Quantifying social group evolution. *Nature*, 446:664–667, 2007.

[54] M. C. González, H. J. Herrmann, J. Kertész, and T. Vicsek. Community structure and ethnic preferences in school friendship networks. *Physica A-Statistical Mechenics and its Applications*, 379:307–316, 2007.

[55] V. D. Blondel, G. M. Krings, and I. Thomas. Regions and borders of mobile telephony in belgium and in the brussels metropolitan zone. *Brussels Studies*, 2010. ISSN 2031-0293.

[56] A. C. Gavin *et al.* Functional organization of the yeast proteome by systematic analysis of protein complexes. *Nature*, 415:141, 2002.

[57] T. Nepusz, A. Petróczi, L. Négyessy, and F. Bazsó. Fuzzy communities and the concept of bridgeness in complex networks. *Phys. Rev. E*, 77:016107, 2008.

[58] D. Chen, M. Shang, Z. Lv, and Y. Fu. Detecting overlapping communities of weighted networks via a local algorithm. *Physica A*, 389:4177–4187, 2010.

[59] H.-W. Shen, X.-Q. Cheng, and J.-F. Guo. Quantifying and identifying the overlapping community structure in networks. *J. Stat. Mech.*, (2009):P07042, 2009.

[60] V. Nicosia, G. Mangioni, V. Carchiolo, and M. Malgeri. Extending the definition of modularity to directed graphs with overlapping communities. *J. Stat. Mech.*, (2009):P03024, 2009.

[61] A. Lázár, D. Ábel, and T. Vicsek. Modularity measure of networks with overlapping communities. *Europhys. Lett.*, 90:18001, 2010.

[62] I. Derényi, G. Palla, and T. Vicsek. Clique percolation in random networks. *Phys. Rev. Lett.*, 94:160202, 2005.

[63] G. Palla, I. Derényi, and T. Vicsek. The critical point of k-clique percolation in the erds-rnyi graph. *J. Stat. Phys.*, 128:219–227, 2007.

[64] M. E. J. Newman and M. Girvan. Finding and evaluating community structure in networks. *Phys. Rev. E*, 69:026113, 2004.

[65] M. Molloy and B. Reed. A critical point for random graphs with a given degree sequence. *Random Structures & Algorithms*, 6:161–180, 1995.

[66] H. Shen, X. Cheng, K. Cai, and M.-B. Hu. Detect overlapping and hierarchical community structure in networks. *Physica A*, 388:1706–1712, 2009.

[67] S. Fortunato and M. Barthelemy. Resolution limit in community detection. *Proc. Natl. Acad. Sci. USA*, 104:36–41, 2007.

[68] B. Adamcsek, G. Palla, I. J. Farkas, I. Derényi, and T. Vicsek. CFinder: Locating cliques and overlapping modules in biological networks. *Bioinformatics*, 22:1021–1023, 2006.

[69] T. Opsahl and P Panzarasa. Clustering in weighted networks. *Social Networks*, 31:155–163, 2009.

[70] M. E. J. Newman. The structure of scientific collaboration networks. *Proc. Natl. Acad. Sci. USA*, 98:404–409, 2001.

[71] D. L. Nelson, C. L. McEvoy, and T. A. Schreiber. The university of south florida word association, rhyme, and word fragment norms. 1998.