# Form factor expansion for thermal correlators

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**Abstract**

We consider finite temperature correlation functions in massive integrable Quantum Field Theory. Using a regularization by putting the system in finite volume, we develop a novel approach (based on multi-dimensional residues) to the form factor expansion for thermal correlators. The first few terms are obtained explicitly in theories with diagonal scattering. We also discuss the validity of the LeClair-Mussardo proposal.

## 1 Introduction

One of the central tasks in many-body quantum physics is the calculation of correlation functions. They yield considerable amount of information about the dynamics of the system and their Fourier modes can be measured, for example with elastic neutron scattering experiments. In addition to the correlations in the ground state it is also important to calculate correlation functions at finite temperatures, in which case the system is populated by a number of excited states.

In this paper we consider finite temperature correlation functions in 1+1 dimensional integrable models. A common property of these theories is that their Hamiltonian can be diagonalized analytically using the Bethe Ansatz [1, 2]. Moreover, there are powerful methods available to obtain correlation functions, the most general being the so-called form factor expansion. The idea is to obtain the matrix elements of local operators (form factors) in the eigenstate basis of the Hamiltonian, and then to sum up the spectral series. The difference as compared to usual approaches (like perturbation theory) is that in integrable models both the spectrum and the form factors can be calculated exactly. This presents a unique opportunity to study strongly correlated quantum systems in situations where conventional methods break down. Interest in integrable models has recently been renewed, in large part due to the developments in recent years that made it possible to realize certain models with the help of optical and magnetic traps [3, 4, 5, 6].

The ideas of the form factor expansion are quite general, however the methods to obtain the form factors and to sum up the spectral series can be different. One framework is provided by the Algebraic Bethe Ansatz (ABA) [2, 7, 8], which was applied successfully to a number of models, most prominently the Heisenberg spin chains [9, 10, 11] and the 1D Bose gas [12, 13]. Correlation functions are obtained typically in the form of integral series [14, 15] or multiple integral formulas [10, 11], or one can resort to numerical summation schemes [16, 17]. Moreover, there are methods to handle the finite temperature situation, either through generalizations of the basic techniques [2] or developing an alternative description using the so-called Quantum Transfer Matrix [18, 19, 20].

Integrable Quantum Field Theory provides a different framework to obtain form factors and correlation functions. In these theories the basic object is the factorized S-matrix [21, 22], and the relation to a microscopic description is rather indirect. It can be shown, that the form factors satisfy a certain set of equations (the form factor bootstrap equations) which follow from general field theoretical arguments supplemented with the special analytic properties of the S-matrix [23, 24, 25, 26]. The idea is to provide a general solution to these equations, and then to identify those solutions which correspond to a given local operator [27]. The resulting form factor functions can then be used to construct correlation functions. An important feature is that the form factors are calculated in the infinite volume asymptotic state basis. This is to be contrasted with the situation in ABA, where one starts with a finite system and the infinite volume limit is only performed afterwards. Interesting connections between the form factor bootstrap and the ABA were pointed out recently in [28].

The problem of zero-temperature correlations in integrable QFT is well understood. Although an analytic summation of the spectral series is in general not possible (except in some simple models [29, 30, 31]), the series has very good convergence properties in massive models and can be evaluated numerically to a desired precision [25, 32]. On the other hand, the problem of finite temperature correlation functions is less understood and it has been subject to an active research in the last ten years [33, 34, 35, 36, 37, 38, 39, 40, 41]. The idea is to use the zero-temperature form factors in the spectral series, however, the thermal average is ill-defined in infinite volume. The problem is related to the appearance of disconnected terms in the expansion, which lead to formally divergent expressions. Following Balog it can be shown that the divergent parts cancel with contributions from the partition function [42], however it is a highly non-trivial task to obtain the finite left-over pieces.

There have been attempts to write down a regularized version of the spectral series. In particular, LeClair and Mussardo proposed an expansion for the one-point and two-point functions in terms of form factors dressed by appropriate occupation number factors containing the pseudo-energy function from the thermodynamical Bethe Ansatz [34]. Their proposal for the two-point function was questioned by Saleur [35] (see also [43]); on the other hand, he also gave a proof of the LeClair-Mussardo formula for one-point functions provided the operator considered is the density of some local conserved charge. However, it was demonstrated in the case of one-point functions that the results obtained by naive regularization are ambiguous; this is shown in particular by the difference [38] between the formulae proposed by LeClair and Mussardo and by Delfino [37]

This motivated the present authors to develop a regularization method based on finite volume form factors [44, 45] which was applied to one-point functions giving a confirmation of the LeClair-Mussardo formula. The central idea is to use a finite volume setting to regularize the divergences, and to compute the physical quantities in finite volume. At the end of the calculation, the volume is taken to infinity. If one computes only quantities meaningful in infinite volume, the divergences cancel and the end result is well-defined. Because finite volume is not an ad hoc, but a physical regulator (note that physically realizable systems are always of finite size) one is virtually guaranteed to obtain the correct result provided the calculations are performed correctly. The existence of a mass gap $m$ is essential in this approach: the Boltzmann-factor $e^{-m/T}$ provides a natural small parameter for the finite temperature expansion. The result is an integral series, where the $N$th term represents $N$-particle processes over the Fock-vacuum. The contributions with a low number of particles can be interpreted as disconnected terms of matrix elements calculated in a thermal state with a large number of particles [41, 46]. In this sense the approach is similar to the one used in the seminal works in ABA [14, 15]. However, a distinctive feature is that the calculations do not use any information about the form factors other than their singularity properties.

The finite volume form factor approach was extended to boundary operators as well [47], which was used to compute finite temperature one-point functions of boundary operators [48]. Another application of the bulk finite volume form factors is the construction of one-point functions of bulk operators on a finite interval [49]. One of the present authors have also used this formalism in [50] to construct the form factor perturbation expansion in non-integrable field theories (originally proposed by Delfino et al. [51]) beyond the lowest order. The finite volume regularization method was also exploited in a recent work by Essler and Konik [52, 53]. They computed the first nontrivial contribution to the dynamical spin-spin correlation function of a spin chain using the O(3) sigma model. However, the methods they use do not have any obvious extension to higher order, albeit it is reasonably clear that the finite volume regularization must work.

In this paper we develop a systematic method to compute the finite temperature form factor expansion to arbitrary orders, relying on an application of the residue theorem for multiple complex variables. We also demonstrate that the same method can be applied to computing the form factor perturbation theory contributions, and the zero-temperature three-point function.

The outline of the paper is as follows: section 2 summarizes the necessary background on form factor bootstrap and finite volume form factors. Also, it sets up the framework for the expansion of the thermal two-point function, and discusses the scope and validity of the approach. In section 3 we present the calculation of the (zero-temperature) 3-point functions as a warm-up example. The expansion for the finite temperature 2-point function is derived in section 4, except for some technical details that are relegated to appendices. The discussion of the LeClair-Mussardo proposal in light of our results is presented in subsection 4.5. We present an application of our method to form factor perturbation theory in section 5, and conclude in section 6.

# 2 Form factor expansion for the thermal two-point function

## 2.1 The form factor bootstrap

Here we give a very brief summary of the equations of the form factor bootstrap, in order to set up notations and to provide background for later arguments; the interested reader is referred to Smirnov's review [54] for more details. For the sake of simplicity let us suppose that the spectrum of the model consists of a single particle mass $m$. The energy and the momentum of an on-shell particle is parametrized by the rapidity variable as $E = m\cosh\theta$ and $p = m\sinh\theta$. Because of integrability, multi-particle scattering amplitudes factorize into the product of pairwise two-particle scatterings, which is described by a pure phase, which we denote by $S(\theta)$ where $\theta$ is the relative rapidity of the incoming particles. Incoming and outgoing asymptotic states can be distinguished by ordering of the rapidities:

$$|\theta_1,\dots,\theta_n\rangle = \begin{cases} |\theta_1,\dots,\theta_n\rangle^{in} & : \ \theta_1>\theta_2>\dots>\theta_n \\ |\theta_1,\dots,\theta_n\rangle^{out} & : \ \theta_1<\theta_2<\dots<\theta_n \end{cases} \tag{2.1}$$

and states which only differ in the order of rapidities are related by

$$|\theta_1,\dots,\theta_k,\theta_{k+1},\dots,\theta_n\rangle = S(\theta_k-\theta_{k+1})|\theta_1,\dots,\theta_{k+1},\theta_k,\dots,\theta_n\rangle \tag{2.2}$$

from which the $S$ matrix of any multi-particle scattering process can be obtained. The normalization of these states is specified by the following inner product for the one-particle states:

$$\langle\theta'|\theta\rangle = 2\pi\delta(\theta'-\theta) \tag{2.3}$$

The form factors of a local operator $\mathcal{O}(t,x)$ are defined as

$$F^{\mathcal{O}}_{mn}(\theta_1',\dots,\theta_m'|\theta_1,\dots,\theta_n) = \langle\theta_1',\dots,\theta_m'|\mathcal{O}(0,0)|\theta_1,\dots,\theta_n\rangle \tag{2.4}$$

With the help of the crossing relations

$$\begin{aligned} F^{\mathcal{O}}_{mn}(\theta_1',\dots,\theta_m'|\theta_1,\dots,\theta_n) &= F^{\mathcal{O}}_{m-1n+1}(\theta_1',\dots,\theta_{m-1}'|\theta_m'+i\pi,\theta_1,\dots,\theta_n) \\ &+\sum_{k=1}^{n}2\pi\delta(\theta_m'-\theta_k)\prod_{l=1}^{k-1}S(\theta_l-\theta_k)F^{\mathcal{O}}_{m-1n-1}(\theta_1',\dots,\theta_{m-1}'|\theta_1,\dots,\theta_{k-1},\theta_{k+1}\dots,\theta_n) \end{aligned} \tag{2.5}$$

all form factors can be expressed in terms of the elementary form factors

$$F^{\mathcal{O}}_n(\theta_1,\dots,\theta_n) = \langle 0|\mathcal{O}(0,0)|\theta_1,\dots,\theta_n\rangle \tag{2.6}$$

which satisfy the following equations:

I. Lorentz transformation:

$$F_n^{\mathcal{O}}(\theta_1+\Lambda,\theta_2+\Lambda,\dots,\theta_n+\Lambda)=\exp\left(s_{\mathcal{O}}\Lambda\right)F_n^{\mathcal{O}}(\theta_1,\theta_2,\dots,\theta_n) \tag{2.7}$$

where $s_{\mathcal{O}}$ denotes the Lorentz spin of the operator $\mathcal{O}$.

II. Exchange:

$$\begin{aligned}F_n^{\mathcal{O}}(\theta_1,\dots,\theta_k,\theta_{k+1},\dots,\theta_n)=&\\ S(\theta_k-\theta_{k+1})F_n^{\mathcal{O}}&(\theta_1,\dots,\theta_{k+1},\theta_k,\dots,\theta_n)\end{aligned} \tag{2.8}$$

III. Cyclic permutation:

$$F_n^{\mathcal{O}}(\theta_1+2i\pi,\theta_2,\dots,\theta_n)=F_n^{\mathcal{O}}(\theta_2,\dots,\theta_n,\theta_1) \tag{2.9}$$

IV. Kinematical singularity

$$-i\operatorname*{Res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta+i\pi,\theta',\theta_1,\dots,\theta_n)=\left(1-\prod_{k=1}^{n}S(\theta'-\theta_k)\right)F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n) \tag{2.10}$$

There is also a further equation related to bound states which we do not need in the sequel. These equations are supplemented by the assumption of maximum analyticity (i.e. that the form factors are meromorphic functions which only have the singularities prescribed by the equations) and possible further conditions expressing properties of the particular operator whose form factors are sought.

## 2.2 The thermal two-point function

Let us take a field theory at finite temperature $T$, which can be formulated in a periodic Euclidean time

$$t\equiv t+R\quad\text{where}\quad R=1/T \tag{2.11}$$

Our aim is to determine the following correlation function

$$\langle\mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle^R=\frac{\mathrm{Tr}\left(\mathrm{e}^{-RH}\mathcal{O}_1(x,t)\mathcal{O}_2(0)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-RH}\right)} \tag{2.12}$$

Naively, one can proceed by inserting two complete sets of states to obtain the spectral representation

$$\langle\mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle^R=\frac{\sum\limits_{m,n}\mathrm{e}^{-(R-t)E_n}\mathrm{e}^{-tE_m}\mathrm{e}^{ix(P_n-P_m)}\langle n|\mathcal{O}_1(0)|m\rangle\langle m|\mathcal{O}_2(0)|n\rangle}{\sum\limits_{n}\mathrm{e}^{-RE_n}} \tag{2.13}$$

where $E_m$ and $E_n$ denote the total energies, while $P_m$ and $P_n$ denote the total momenta of the states inserted. Using asymptotic completeness, the sets of states $\{|n\rangle\}$ and $\{|m\rangle\}$ can be chosen as the basis of asymptotic in (or out) states. In this case, the matrix elements appearing in the above formula are just the form factors of the operators $\mathcal{O}_1$ and $\mathcal{O}_2$.

However, the above expression is ill-defined because according to the crossing relation (2.5) any term in which there is at least one particle with the same rapidities in the states $|n\rangle$ and $|m\rangle$ both matrix elements contains a $\delta$ function term, and so the expression contains squares of $\delta$ functions. Similar divergences occur in the partition function in the denominator. A standard combinatorial consideration of disconnected terms shows that the divergent parts cancel out between the numerator and denominator [42]. The issue is therefore to compute the finite remainder.

The simplest idea is to put the system in a finite spatial volume $L$ with periodic boundary conditions

$$x\equiv x+L \tag{2.14}$$

in which case the expression becomes

$$\langle\mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_L^R = \frac{\mathrm{Tr}_L\left(\mathrm{e}^{-RH_L}\mathcal{O}_1(x,t)\mathcal{O}_2(0)\right)}{\mathrm{Tr}_L\left(\mathrm{e}^{-RH_L}\right)} \tag{2.15}$$

where $\mathrm{Tr}_L$ denotes the trace over the finite-volume states, $H_L$ is the Hamiltonian in volume $L$. This expression can be expanded inserting two complete sets of states

$$\mathrm{Tr}_L\left(\mathrm{e}^{-RH_L}\mathcal{O}_1(x,t)\mathcal{O}_2(0)\right) = \sum_{m,n}\mathrm{e}^{-RE_n(L)}\langle n|\mathcal{O}_1(x,t)|m\rangle_L\langle m|\mathcal{O}_2(0)|n\rangle_L \tag{2.16}$$

where the matrix elements of local operators are also taken in the finite volume system.

### 2.3 Form factors in finite volume

The next ingredient we need is the description of form factors in finite volume. Previously this was achieved using semi-classical techniques [55, 56], but that is not suitable for our purposes here. We need a formalism that gives the exact quantum form factors to all orders in $L^{-1}$ (i.e. up to corrections that decay exponentially with the volume). The relevant results were derived by us in [44, 45], Following our conventions in those papers, the finite volume multi-particle states can be denoted

$$|\{I_1,\dots,I_n\}\rangle_L \tag{2.17}$$

where the $I_k$ are momentum quantum numbers. We can order the momentum quantum numbers in a monotonically decreasing sequence: $I_1 \geq \dots \geq I_n$, which is just a matter of convention. The corresponding energy levels are determined by the Bethe-Yang equations

$$\mathrm{e}^{imL\sinh\tilde{\theta}_k}\prod_{l\neq k}S(\tilde{\theta}_k-\tilde{\theta}_l) = 1 \tag{2.18}$$

We define the two-particle phase shift $\delta(\theta)$ by the relation

$$S(\theta) = -\mathrm{e}^{i\delta(\theta)} \tag{2.19}$$

where the $-$ sign ensures that (due to the generic feature $S(0) = -1$ and the bootstrap relation $S(\theta)S(-\theta) = 1$)[1] the phase-shift can be defined as a continuous and odd function of $\theta$. In addition we introduce the following notation

$$\varphi(\theta) = \frac{\partial\delta(\theta)}{\partial\theta} \tag{2.20}$$

for the derivative of the phase shift. Using these definitions we can write

$$Q_k(\tilde{\theta}_1,\dots,\tilde{\theta}_n) = mL\sinh\tilde{\theta}_k + \sum_{l\neq k}\delta(\tilde{\theta}_k-\tilde{\theta}_l) = 2\pi I_k \quad , \quad k=1,\dots,n \tag{2.21}$$

where the quantum numbers $I_k$ take integer/half-integer values for odd/even numbers of particles respectively. Eqns. (2.21) must be solved with respect to the particle rapidities $\tilde{\theta}_k$, where the energy (relative to the finite volume vacuum state) can be computed as

$$\sum_{k=1}^{n} m\cosh\tilde{\theta}_k \tag{2.22}$$

up to corrections which decay exponentially with $L$. The density of $n$-particle states in rapidity space can be calculated as

$$\rho(\theta_1,\dots,\theta_n) = \det\mathcal{J}^{(n)} \quad , \quad \mathcal{J}^{(n)}_{kl} = \frac{\partial Q_k(\theta_1,\dots,\theta_n)}{\partial\theta_l} \quad , \quad k,l=1,\dots,n \tag{2.23}$$

[1] $S(0) = -1$ is valid in any known integrable model except for the free boson, which we do not consider here (as the issues of this paper can be solved trivially), while the other relation is a consequence of the unitarity and Hermitian analyticity of the $S$ matrix.

The finite volume behaviour of local matrix elements can be given as [44]

$$\langle\{I_1',\dots,I_m'\}|\mathcal{O}(0,0)|\{I_1,\dots,I_n\}\rangle_L=\frac{F^{\mathcal{O}}_{m+n}(\tilde\theta_m'+i\pi,\dots,\tilde\theta_1'+i\pi,\tilde\theta_1,\dots,\tilde\theta_n)}{\sqrt{\rho(\tilde\theta_1,\dots,\tilde\theta_n)\rho(\tilde\theta_1',\dots,\tilde\theta_m')}}+O(\mathrm{e}^{-\mu L}) \tag{2.24}$$

where $\tilde\theta_k$ ($\tilde\theta_k'$) are the solutions of the Bethe-Yang equations (2.21) corresponding to the state with the specified quantum numbers $I_1,\dots,I_n$ ($I_1',\dots,I_n'$) at the given volume $L$. The above relation is valid provided there are no disconnected terms i.e. the left and the right states do not contain particles with the same rapidity, i.e. the sets $\left\{\tilde\theta_1,\dots,\tilde\theta_n\right\}$ and $\left\{\tilde\theta_1',\dots,\tilde\theta_m'\right\}$ are disjoint. The lower limit on the exponent $\mu$ is independent of the operator and the states considered (it is related to the bound state pole structure of the infinite volume scattering theory).

It is easy to see that in the presence of nontrivial scattering there are only two cases when exact equality of (at least some of) the rapidities can occur [45]:

1. The two states are identical, i.e. $n=m$ and

$$\{I_1',\dots,I_m'\}=\{I_1,\dots,I_n\} \tag{2.25}$$

in which case the corresponding diagonal matrix element can be written as a sum over all bipartite divisions of the set of the $n$ particles involved (including the trivial ones when $A$ is the empty set or the complete set $\{1,\dots,n\}$)

$$\langle\{I_1\dots I_n\}|\mathcal{O}|\{I_1\dots I_n\}\rangle_L=\frac{\sum_{A\subset\{1,2,\dots n\}}\mathcal{F}(A)_L\rho(\{1,\dots,n\}\setminus A)_L}{\rho(\{1,\dots,n\})_L}+O(\mathrm{e}^{-\mu L}) \tag{2.26}$$

where $|A|$ denotes the cardinal number (number of elements) of the set $A$

$$\rho(\{k_1,\dots,k_r\})_L=\rho(\tilde\theta_{k_1},\dots,\tilde\theta_{k_r}) \tag{2.27}$$

is the $r$-particle Bethe-Yang Jacobi determinant (2.23) involving only the $r$-element subset $1\leq k_1<\dots<k_r\leq n$ of the $n$ particles, and

$$\begin{aligned}\mathcal{F}(\{k_1,\dots,k_r\})_L &= F^s_{2r}(\tilde\theta_{k_1},\dots,\tilde\theta_{k_r})\\ F^s_{2l}(\theta_1,\dots,\theta_l) &= \lim_{\epsilon\to 0}F^{\mathcal{O}}_{2l}(\theta_l+i\pi+\epsilon,\dots,\theta_1+i\pi+\epsilon,\theta_1,\dots,\theta_l)\end{aligned} \tag{2.28}$$

is the so-called symmetric evaluation of diagonal multi-particle matrix elements.

2. Both states are parity symmetric states in the spin zero sector, i.e.

$$\begin{aligned}\{I_1,\dots,I_n\} &\equiv \{-I_n,\dots,-I_1\}\\ \{I_1',\dots,I_m'\} &\equiv \{-I_m',\dots,-I_1'\}\end{aligned} \tag{2.29}$$

Furthermore, both states must contain one (or possibly more, in a theory with more than one species) particle of quantum number 0, whose rapidity is then exactly 0 for any value of the volume $L$ due to the symmetric assignment of quantum numbers. Writing $m=2k+1$ and $n=2l+1$ and defining

$$\begin{aligned}&\mathcal{F}_{k,l}(\theta_1',\dots,\theta_k'|\theta_1,\dots,\theta_l)=\\ &\lim_{\epsilon\to 0}F_{2k+2l+2}(i\pi+\theta_1'+\epsilon,\dots,i\pi+\theta_k'+\epsilon,i\pi-\theta_k'+\epsilon,\dots,i\pi-\theta_1'+\epsilon,\\ &i\pi+\epsilon,0,\theta_1,\dots,\theta_l,-\theta_l,\dots,-\theta_1)\end{aligned} \tag{2.30}$$

the formula for the finite-volume matrix element takes the form

$$\begin{aligned}&\langle\{I_1',\dots,I_k',0,-I_k',\dots,-I_1'\}|\mathcal{O}|\{I_1,\dots,I_l,0,-I_l,\dots,-I_1\}\rangle_L \\ =&\left(\rho_{2k+1}(\tilde\theta_1',\dots,\tilde\theta_k',0,-\tilde\theta_k',\dots,-\tilde\theta_1')\rho_{2l+1}(\tilde\theta_1,\dots,\tilde\theta_l,0,-\tilde\theta_l,\dots,-\tilde\theta_1)\right)^{-1/2}\\ &\times\Big[\mathcal{F}_{k,l}(\tilde\theta_1',\dots,\tilde\theta_k'|\tilde\theta_1,\dots,\tilde\theta_l)\\ &+mL\,F_{2k+2l}(i\pi+\tilde\theta_1',\dots,i\pi+\tilde\theta_k',i\pi-\tilde\theta_k',\dots,i\pi-\tilde\theta_1',\tilde\theta_1,\dots,\tilde\theta_l,-\tilde\theta_l,\dots,-\tilde\theta_1)\Big]\\ &+O(\mathrm{e}^{-\mu L})\end{aligned} \tag{2.31}$$

### 2.4 The form factor expansion using finite volume regularization

Using the finite volume description introduced in subsection 2.3 we can write

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_L^R = \frac{1}{Z}\sum_{N,M} C_{NM} \tag{2.32}$$

where

$$\begin{aligned} C_{NM} \quad = \quad & \sum_{I_1\dots I_N}\sum_{J_1\dots J_M} \langle\{I_1\dots I_N\}|\mathcal{O}_1(0)|\{J_1\dots J_M\}\rangle_L \times \\ & \langle\{J_1\dots J_M\}|\mathcal{O}_2(0)|\{I_1\dots I_N\}\rangle_L e^{i(P_1-P_2)x} e^{-E_1(R-t)} e^{-E_2 t} \end{aligned} \tag{2.33}$$

and $E_{1,2}$ and $P_{1,2}$ are the total energies and momenta of the multi-particle states $|\{I_1\dots I_N\}\rangle_L$ and $|\{J_1\dots J_M\}\rangle$. The task is to calculate the sum in finite volume and then take the limit $L\to\infty$.

It is easy to see, that the terms $N=0, M=0\dots\infty$ and $N=0\dots\infty, M=0$ add up to zero-temperature correlation functions

$$\lim_{L\to\infty}\left(\sum_{M=0}^{\infty} C_{0M}\right) = \langle\mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle \qquad \lim_{L\to\infty}\left(\sum_{N=0}^{\infty} C_{N0}\right) = \langle\mathcal{O}_1(x,R-t)\mathcal{O}_2(0)\rangle \tag{2.34}$$

In particular

$$\lim_{L\to\infty} C_{0M} = \frac{1}{M!}\int\frac{d\theta_1}{2\pi}\dots\frac{d\theta_M}{2\pi}F^{\mathcal{O}_1}(\theta_1,\dots,\theta_M)F^{\mathcal{O}_2}(\theta_M,\dots,\theta_1)e^{-im(\sum_j\sinh\theta_j)x-m(\sum_j\cosh\theta_j)t} \tag{2.35}$$

and similarly for $C_{N0}$.

Let us introduce two auxiliary variables $u$ and $v$ to keep track of the orders of $e^{-mt}$ and $e^{-m(R-t)}$ (at the end both will be set to 1). Then (2.32) takes the form

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_L^R = \frac{1}{Z}\sum_{N,M} u^N v^M C_{NM} \tag{2.36}$$

We define a similar expansion for the partition function

$$Z = \sum_N (uv)^N Z_N \tag{2.37}$$

with $Z_N$ denoting the $N$-particle contribution to the partition function. The first few terms are given by

$$Z_0 = 1 \qquad Z_1 = \sum_I e^{-E_I R} \qquad Z_2 = \sum_{I\neq J} e^{-(E_I+E_J)R} \tag{2.38}$$

The inverse of the partition function is expanded as

$$Z^{-1} = \sum_N (uv)^N \bar{Z}_N \tag{2.39}$$

where

$$\bar{Z}_0 = 1 \qquad \bar{Z}_1 = -Z_1 \qquad \bar{Z}_2 = Z_1^2 - Z_2 \tag{2.40}$$

Putting this together we can rewrite the expansion as

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_L^R = \sum u^N v^N \tilde{D}_{NM} \tag{2.41}$$

with

$$\tilde{D}_{NM} = \sum_l C_{N-l,M-l}\bar{Z}_l \tag{2.42}$$

The first few nontrivial terms are given by

$$\begin{aligned}\tilde{D}_{1M} &= C_{1M} - Z_1 C_{0,M-1} \\ \tilde{D}_{2M} &= C_{2M} - Z_1 C_{1,M-1} + (Z_1^2 - Z_2) C_{0,M-2}\end{aligned} \tag{2.43}$$

In this way we produce a double series expansions in powers of the variables $e^{-mt}$ and $e^{-m(R-t)}$. Since these variables are independent, each quantity $\tilde{D}_{NM}$ must have a well-defined $L \to \infty$ limit which we denote as

$$D_{NM} = \lim_{L\to\infty} \tilde{D}_{NM} \tag{2.44}$$

and we obtain that

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle^R = \lim_{L\to\infty} \langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_L^R = \sum_{N,M} D_{NM} \tag{2.45}$$

The reordering of the series in (2.41) using the coefficients $\tilde{D}_{NM}$ is also an integral part of Essler and Konik's calculation in [53]; we used a similar reordering for the expansion of the one-point function in powers of $e^{-mR}$ [45].

Note that individual terms contributing in (2.42) to $\tilde{D}_{NM}$ may contain divergent pieces which scale with positive powers of $L$. Similarly to the considerations for the one-point function in [45], it turns out that the $N$-particle terms which are most singular in the large-$L$ limit carry a factor of $(mLe^{-mR})^N$. Since it is also necessary that the exponential corrections to the finite volume form factors in eqns. (2.24) and (2.26) are small, the finite-volume expansion is valid in the domain

$$1 \ll mL \ll e^{mR} \tag{2.46}$$

For the limit (2.44) to exist the positive powers of $L$ must drop out, therefore one can understand the $L \to \infty$ limit of the series as an analytic continuation to very large values of $L$ outside the domain (2.46). Eventually, the condition that the coefficients $\tilde{D}_{nm}$ must have a finite large volume limit can be used as a nontrivial check to verify our calculations.

It is evident from (2.33) that the quantities $D_{NM}$ with $N > M$ can be obtained from those with $N < M$ after a trivial exchange of $t$ and $R - t$. Therefore we will only consider the case $N \leq M$.

# 3 Warm-up example: the zero-temperature three-point function

Before tackling the central issue of the paper, we consider a simpler problem which allows us to introduce the central ideas without too many complications. Let us consider the three-point function

$$\langle 0|\mathcal{O}_1(t_1,x_1)\mathcal{O}_2(t_2,x_2)\mathcal{O}_3(0)|0\rangle \tag{3.1}$$

in the Euclidean theory with non-compact time direction (i.e. $T = 0$). Suppose that $t_1 > t_2$ and to shorten the formulae we also omit the dependence on $x_1$ and $x_2$ (it can be reintroduced easily). The spectral decomposition takes the form

$$\begin{aligned}&\langle 0|\mathcal{O}_1(t_1,x_1)\mathcal{O}_2(t_2,x_2)\mathcal{O}(0,0)|0\rangle \\ &= \sum_{m,n} \langle 0|\mathcal{O}_1(0)|m\rangle\langle m|\mathcal{O}_2(0)|n\rangle\langle n|\mathcal{O}_3(0)|0\rangle \mathrm{e}^{-E_m(t_1-t_2)}\mathrm{e}^{-E_n t_2}\end{aligned} \tag{3.2}$$

What makes this example simpler is that the disconnected terms appear linearly since they only enter from the $\mathcal{O}_2$ matrix element. Following the example of the three point function, we can introduce a finite volume regularization

$$\langle 0|\mathcal{O}_1(t_1,x_1)\mathcal{O}_2(t_2,x_2)\mathcal{O}(0,0)|0\rangle_L = \sum_{N,M} C^{(3)}_{NM} \tag{3.3}$$

where

$$C_{NM}^{(3)} = \sum_{I_1 \dots I_N} \sum_{J_1 \dots J_M} \langle 0 | \mathcal{O}_1(0) | \{I_1 \dots I_N\} \rangle_L \langle \{I_1 \dots I_N\} | \mathcal{O}_2(0) | \{J_1 \dots J_M\} \rangle_L \tag{3.4}$$

$$\times \langle \{J_1 \dots J_M\} | \mathcal{O}_3(0) | \rangle_L \, \mathrm{e}^{-E_1(L)(t_1-t_2)} \mathrm{e}^{-E_2(L)t_2} \tag{3.5}$$

There is no denominator $Z$ to supply counter terms for the $L$ dependence, therefore each of these expressions must have a finite limit as $L \to \infty$:

$$D_{NM}^{(3)} = \lim_{L\to\infty} C_{NM}^{(3)} \tag{3.6}$$

and

$$\langle 0 | \mathcal{O}_1(t_1,x_1) \mathcal{O}_2(t_2,x_2) \mathcal{O}(0,0) | 0 \rangle = \sum_{N,M} D_{NM}^{(3)} \tag{3.7}$$

Eventually, it is trivial to write down some terms of the expansion:

$$\begin{aligned} D_{0M}^{(3)} &= \langle \mathcal{O}_1 \rangle \frac{1}{M!} \int \frac{d\theta_1}{2\pi} \dots \int \frac{d\theta_M}{2\pi} F_M^{\mathcal{O}_2}(\theta_1,\dots,\theta_M) F_M^{\mathcal{O}_3}(\theta_M + i\pi,\dots,\theta_1+i\pi) \\ &\times \exp\left(-mt_2 \sum_{i=1}^M \cosh\theta_i\right) \\ D_{N0}^{(3)} &= \frac{1}{N!} \int \frac{d\theta_1}{2\pi} \dots \int \frac{d\theta_N}{2\pi} F_N^{\mathcal{O}_1}(\theta_1,\dots,\theta_N) F_N^{\mathcal{O}_2}(\theta_N + i\pi,\dots,\theta_1+i\pi) \\ &\times \exp\left(-m(t_1-t_2) \sum_{i=1}^N \cosh\theta_i\right) \langle \mathcal{O}_3 \rangle \end{aligned} \tag{3.8}$$

and also

$$D_{11}^{(3)} = \int \frac{d\theta_1}{2\pi} \int \frac{d\theta_1'}{2\pi} F_1^{\mathcal{O}_1} F_2^{\mathcal{O}_2}(\theta_1 + i\pi, \theta_1') F_1^{\mathcal{O}_3} \mathrm{e}^{-m(t_2-t_1)\cosh\theta_1} \mathrm{e}^{-mt_1\cosh\theta_1'} \tag{3.9}$$

since the two-particle form factor has no kinematical singularities.

### 3.1 The contribution $D_{12}^{(3)}$

The first nontrivial contribution is $D_{12}^{(3)}$, for which the finite volume expression is

$$C_{12}^{(3)} = \sum_{I_1} \sum_{J_1<J_2} \frac{F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i\pi, \theta_1', \theta_2') F_2^{\mathcal{O}_3}(\theta_2', \theta_1')}{\rho_1(\theta_1)\rho_2(\theta_1',\theta_2')} \mathrm{e}^{-m(t_1-t_2)\cosh\theta_1} \mathrm{e}^{-mt_2(\cosh\theta_1' + \cosh\theta_2')} \tag{3.10}$$

where the rapidities satisfy the appropriate Bethe-Yang quantization relations. We can make a choice whether to perform first the one-particle or two-particle summation. The latter proceeds by an application of the multi-dimensional residue theorem (A.2) and illustrates one of the central ideas that make the expansion of the thermal correlator feasible.

#### 3.1.1 Summing over one-particle states first

We can substitute the sum over $I_1$

$$\sum_{I_1} \to \sum_{I_1} \oint_{C_{I_1}} \frac{d\theta_1}{2\pi} \frac{\rho_1(\theta_1)}{\mathrm{e}^{iQ_1(\theta_1)} - 1} \tag{3.11}$$

where

$$Q_1(\theta_1) = mL\sinh\theta_1 \qquad \rho_1(\theta_1) = Q'(\theta_1) \tag{3.12}$$

and $C_{I_1}$ are small closed curves surrounding the solution of

$$Q_1(\theta_1) = 2\pi I_1 \tag{3.13}$$

in the complex $\theta_1$ plane. Now we open these circles and join them to obtain the contour

$$C = C_+ + C_- \tag{3.14}$$

the $C_+$ running from $\infty + i\epsilon$ to $-\infty + i\epsilon$ (i.e. backward in $\Re e\,\theta_1$) while $C_-$ running from $-\infty - i\epsilon$ to $+\infty - i\epsilon$ (forward in $\Re e\,\theta_1$). However, by this operation we also include the contribution of two poles at $\theta_1 = \theta_1'$ and $\theta_1 = \theta_2'$ which must be subtracted. Using (2.10), the singularity of the integrand at $\theta_1 \sim \theta_1'$ can be written as

$$\frac{1}{\theta_1 - \theta_1'} \frac{iF_1^{\mathcal{O}_1}(1 - S(\theta_1' - \theta_2'))F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_2', \theta_1')}{\rho_2(\theta_1', \theta_2')\left(\mathrm{e}^{iQ_1(\theta_1')} - 1\right)} \mathrm{e}^{-mt_1 \cosh\theta_1'} \mathrm{e}^{-mt_2 \cosh\theta_2'} \tag{3.15}$$

The quantization relation of the two-particle state can also be written as

$$\mathrm{e}^{imL\sinh\theta_1'} S(\theta_1' - \theta_2') = 1 \tag{3.16}$$

and so we can rewrite (3.15) as

$$\frac{1}{\theta_1 - \theta_1'} \frac{iF_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1', \theta_2')}{\rho_2(\theta_1', \theta_2')} \mathrm{e}^{-mt_1 \cosh\theta_1'} \mathrm{e}^{-mt_2 \cosh\theta_2'} \tag{3.17}$$

The contribution of this singularity can be evaluated as

$$\oint_{\theta_1'} \frac{d\theta_1}{2\pi} \frac{1}{\theta_1 - \theta_1'} \frac{iF_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1', \theta_2')}{\rho_2(\theta_1', \theta_2')} \mathrm{e}^{-mt_1 \cosh\theta_1'} \mathrm{e}^{-mt_2 \cosh\theta_2'} =$$
$$-\frac{F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1', \theta_2')}{\rho_2(\theta_1', \theta_2')} \mathrm{e}^{-mt_1 \cosh\theta_1'} \mathrm{e}^{-mt_2 \cosh\theta_2'} \tag{3.18}$$

The contribution of the $\theta_1 = \theta_2'$ pole can be obtained in a similar way. These must be subtracted from the $\theta_1$ integral and therefore we obtain

$$\begin{aligned} C_{12}^{(3)} = \sum_{J_1, J_2} \Bigg[ & \oint_C \frac{d\theta_1}{2\pi} \frac{F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i\pi, \theta_1', \theta_2') F_2^{\mathcal{O}_3}(\theta_2', \theta_1')}{\rho_2(\theta_1', \theta_2')\left(\mathrm{e}^{iQ_1(\theta_1)} - 1\right)} \mathrm{e}^{-m(t_1 - t_2)\cosh\theta_1} \mathrm{e}^{-mt_2(\cosh\theta_1' + \cosh\theta_2')} \\ & + \frac{F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1', \theta_2')}{\rho_2(\theta_1', \theta_2')} \mathrm{e}^{-mt_1 \cosh\theta_1'} \mathrm{e}^{-mt_2 \cosh\theta_2'} \\ & + \frac{F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_2', \theta_1')}{\rho_2(\theta_1', \theta_2')} \mathrm{e}^{-mt_1 \cosh\theta_2'} \mathrm{e}^{-mt_2 \cosh\theta_1'} \Bigg] \end{aligned} \tag{3.19}$$

Taking the large $L$ limit, we can substitute the discrete sum with an integral

$$\sum_{J_1 < J_2} \to \frac{1}{2} \iint \frac{d\theta_1'}{2\pi} \frac{d\theta_2'}{2\pi} \rho_2(\theta_1', \theta_2') \tag{3.20}$$

and so we obtain

$$\begin{aligned} C_{12}^{(3)} = \frac{1}{2} \iint \frac{d\theta_1'}{2\pi} \frac{d\theta_2'}{2\pi} \Bigg[ & \\ & - \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \frac{F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i(\pi + \epsilon), \theta_1', \theta_2') F_2^{\mathcal{O}_3}(\theta_2', \theta_1')}{\mathrm{e}^{iQ_1(\theta_1 + i\epsilon)} - 1} \mathrm{e}^{-m(t_1 - t_2)\cosh(\theta_1 + i\epsilon) - mt_2(\cosh\theta_1' + \cosh\theta_2')} \\ & + \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \frac{F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i(\pi - \epsilon), \theta_1', \theta_2') F_2^{\mathcal{O}_3}(\theta_2', \theta_1')}{\mathrm{e}^{iQ_1(\theta_1 - i\epsilon)} - 1} \mathrm{e}^{-m(t_1 - t_2)\cosh(\theta_1 - i\epsilon) - mt_2(\cosh\theta_1' + \cosh\theta_2')} \\ & + F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1', \theta_2') \mathrm{e}^{-mt_1 \cosh\theta_1' - mt_2 \cosh\theta_2'} \\ & + F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_2', \theta_1') \mathrm{e}^{-mt_1 \cosh\theta_2' - mt_2 \cosh\theta_1'} \Bigg] \end{aligned} \tag{3.21}$$

Note that

$$iQ_1(\theta_1 \pm i\epsilon) = \mp mL\cosh\theta_1 \sin\epsilon + imL\sinh\theta_1\cos\epsilon \tag{3.22}$$

and therefore the $L \to \infty$ limit yields

$$\begin{aligned} D_{12}^{(3)} = \frac{1}{2}\iint \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}\Bigg[ & \int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi} F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i(\pi+\epsilon), \theta_1', \theta_2') F_2^{\mathcal{O}_3}(\theta_2', \theta_1') \\ & \times \mathrm{e}^{-m(t_1-t_2)\cosh(\theta_1+i\epsilon) - mt_2(\cosh\theta_1' + \cosh\theta_2')} \\ & + F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1', \theta_2')\mathrm{e}^{-mt_1\cosh\theta_1' - mt_2\cosh\theta_2'} \\ & + F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_2', \theta_1')\mathrm{e}^{-mt_1\cosh\theta_2' - mt_2\cosh\theta_1'}\Bigg] \end{aligned} \tag{3.23}$$

### 3.1.2 Evaluating the two-particle summation first

Using the multi-dimensional residue theorem (A.2) we can represent the two-particle sum as

$$\sum_{J_1>J_2} \frac{1}{\rho_2(\theta_1', \theta_2')} \to \sum_{J_1>J_2} \oint\oint_{C_{J_1J_2}} \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} \frac{1}{\left(\mathrm{e}^{iQ_1(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_2(\theta_1',\theta_2')}+1\right)} \tag{3.24}$$

where $C_{J_1J_2}$ is a multi-contour (a direct product of two curves in the variables $\theta_1'$ and $\theta_2'$) surrounding the solution of

$$\begin{aligned} Q_1(\theta_1', \theta_2') &= mL\sinh\theta_1' + \delta(\theta_1' - \theta_2') = 2\pi J_1 \\ Q_2(\theta_1', \theta_2') &= mL\sinh\theta_2' + \delta(\theta_2' - \theta_1') = 2\pi J_2 \end{aligned} \tag{3.25}$$

where due to the definition

$$S = -\mathrm{e}^{i\delta}$$

$J_1$ and $J_2$ take half-integer values. Since the form factors vanish when any two of their arguments coincide, we can extend the sum by adding the diagonal

$$\sum_{J_1>J_2} \to \frac{1}{2}\sum_{J_1,J_2}$$

and so

$$C_{12}^{(3)} = \sum_{I_1} \frac{\tilde{C}_{12}(\theta_1)}{\rho_1(\theta_1)} \tag{3.26}$$

where

$$\begin{aligned} \tilde{C}_{12}(\theta_1) \;=\; & \frac{1}{2}\sum_{J_1,J_2}\oint\oint_{C_{J_1J_2}} \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} \frac{F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i\pi, \theta_1', \theta_2') F_2^{\mathcal{O}_3}(\theta_2', \theta_1')}{\left(\mathrm{e}^{iQ_1(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_2(\theta_1',\theta_2')}+1\right)} \\ & \times \mathrm{e}^{-m(t_1-t_2)\cosh\theta_1}\mathrm{e}^{-mt_2(\cosh\theta_1' + \cosh\theta_2')} \end{aligned} \tag{3.27}$$

Now we open the contours to surround the whole of the real $\theta_1'$ and $\theta_2'$ axes (but close enough so as to avoid all singularities of the $S$ matrix). Just as before it is necessary to subtract the contributions of any singularities encountered in the process. There are two classes of such singularities:

- $\theta_1' = \theta_1$ and $\mathrm{e}^{iQ_2} + 1 = 0$
- $\theta_2' = \theta_1$ and $\mathrm{e}^{iQ_1} + 1 = 0$

There are no triple singularities because $\theta_1$ satisfies

$$mL\sinh\theta_1 = 2\pi I_1 \qquad I_1 \in \mathbb{Z} \tag{3.28}$$

and it is impossible for the three Bethe-Yang conditions $Q_1$, $Q_2$ and $Q_3$ to be satisfied simultaneously. As before, it is enough to evaluate the first case; the second can be obtained by swapping $\theta_1'$ and $\theta_2'$. In the large $L$ limit

$$\mathrm{e}^{iQ_{1,2}(\theta_1'\pm i\epsilon_1,\theta_2'\pm i\epsilon_2)} \to \begin{cases} 0 & + \text{ sign} \\ \infty & - \text{ sign} \end{cases} \tag{3.29}$$

therefore we obtain

$$\begin{aligned}
&\tilde{C}_{12}(\theta_1) = \\
&\frac{1}{2}\iint_{C_{++}} \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_1',\theta_2') F_2^{\mathcal{O}_3}(\theta_2',\theta_1') \mathrm{e}^{-m(t_1-t_2)\cosh\theta_1 - mt_2(\cosh(\theta_1')+\cosh(\theta_2'))} \\
&-\frac{1}{2}\Bigg\{ \sum_{J_2\in\mathbb{Z}+1/2} i^2 \operatorname*{Res}_{\substack{\theta_1'=\theta_1\\ Q_2=J_2}} \Bigg[ \frac{F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_1',\theta_2') F_2^{\mathcal{O}_3}(\theta_2',\theta_1')}{\left(\mathrm{e}^{iQ_1(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_2(\theta_1',\theta_2')}+1\right)} \\
&\times \mathrm{e}^{-m(t_1-t_2)\cosh\theta_1 - mt_2(\cosh\theta_1'+\cosh\theta_2')} \Bigg] + \left(\theta_1' \leftrightarrow \theta_2'\right) \Bigg\}
\end{aligned} \tag{3.30}$$

where $C_{++}$ denotes the part of the two-particle multi-contour that survives in the $L\to\infty$ limit; it corresponds to an integration parallel to the real axes in $\theta_1'$ and $\theta_2'$ with a shift in the positive imaginary direction, i.e.

$$\iint_{C_{++}} \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} f(\theta_1',\theta_2') = \int_{\mathbb{R}} \frac{d\theta_1'}{2\pi} \int_{\mathbb{R}} \frac{d\theta_2'}{2\pi} f(\theta_1'+i\epsilon_1,\theta_2'+i\epsilon_2) \tag{3.31}$$

Recalling (3.25) we get

$$\begin{aligned}
&\operatorname*{Res}_{\substack{\theta_1'=\theta_1\\ Q_3=I_3}} \frac{F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_1',\theta_2') F_2^{\mathcal{O}_3}(\theta_2',\theta_1') \mathrm{e}^{-m(t_1-t_2)\cosh\theta_1 - mt_2(\cosh\theta_1'+\cosh\theta_2')}}{\left(\mathrm{e}^{iQ_1(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_2(\theta_1',\theta_2')}+1\right)} \\
&= \frac{-i\left(1-S(\theta_1-\theta_2')\right) F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_2',\theta_1) \mathrm{e}^{-mt_1\cosh\theta_1 - mt_2\cosh\theta_2'}}{\left(1-S(\theta_1-\theta_2')\right)(-i)\left(mL\cosh\theta_2'+\varphi(\theta_2'-\theta_1)\right)}
\end{aligned} \tag{3.32}$$

Substituting these into the expression for $C_{12}^{(3)}$ and taking $L\to\infty$ we obtain

$$\begin{aligned}
D_{12}^{(3)} &= \frac{1}{2}\iint_{C_{++}} \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}\Bigg[\int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi} F_1^{\mathcal{O}_3} F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_1',\theta_2') F_2^{\mathcal{O}_1}(\theta_2',\theta_1') \\
&\times \; \mathrm{e}^{-m(t_1-t_2)\cosh\theta_1} \mathrm{e}^{-mt_1(\cosh(\theta_1')+\cosh(\theta_2'))} \\
&+ \; F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1',\theta_2') \mathrm{e}^{-mt_2\cosh\theta_1'} \mathrm{e}^{-mt_1\cosh\theta_2'} \\
&+ \; F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_2',\theta_1') \mathrm{e}^{-mt_2\cosh\theta_2'} \mathrm{e}^{-mt_1\cosh\theta_1'} \Bigg]
\end{aligned} \tag{3.33}$$

After substituting $\theta_{1,2}' \to -\theta_{1,2}'$ and using

$$F_2^{\mathcal{O}_3}(-\theta_1',-\theta_2') = F_2^{\mathcal{O}_1}(\theta_2,'\theta_1')$$

we can make a combined shift of the three contours in the triple integral term to obtain

$$\begin{aligned}
D_{12}^{(3)} &= \frac{1}{2}\iint \frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}\Bigg[\int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi} F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1+i(\pi-\epsilon),\theta_1',\theta_2') F_2^{\mathcal{O}_3}(\theta_2',\theta_1') \\
&\times \; \mathrm{e}^{-m(t_1-t_2)\cosh(\theta_1-i\epsilon)} \mathrm{e}^{-mt_1(\cosh\theta_1'+\cosh\theta_2')} \\
&+ \; F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_1',\theta_2') \mathrm{e}^{-mt_2\cosh\theta_1'} \mathrm{e}^{-mt_1\cosh\theta_2'} \\
&+ \; F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} F_2^{\mathcal{O}_3}(\theta_2',\theta_1') \mathrm{e}^{-mt_2\cosh\theta_2'} \mathrm{e}^{-mt_1\cosh\theta_1'} \Bigg]
\end{aligned} \tag{3.34}$$

It can easily be shown that this expression agrees with (3.23); the difference due to the $\epsilon \to -\epsilon$ change drops out:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i(\pi+\epsilon), \theta'_1, \theta'_2) F_2^{\mathcal{O}_3}(\theta'_2, \theta'_1) \mathrm{e}^{-m(t_2-t_1)\cosh(\theta_1+i\epsilon)} \mathrm{e}^{-mt_1(\cosh\theta'_1+\cosh\theta'_2)} - \\
- & \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i(\pi-\epsilon), \theta'_1, \theta'_2) F_2^{\mathcal{O}_3}(\theta'_2, \theta'_1) \mathrm{e}^{-m(t_2-t_1)\cosh(\theta_1-i\epsilon)} \mathrm{e}^{-mt_1(\cosh\theta'_1+\cosh\theta'_2)} \\
= & \; i \left( \operatorname*{Res}_{\theta_1=\theta'_1} + \operatorname*{Res}_{\theta_1=\theta'_2} \right) F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\theta_1 + i\pi, \theta'_1, \theta'_2) F_2^{\mathcal{O}_3}(\theta'_2, \theta'_1) \mathrm{e}^{-m(t_2-t_1)\cosh\theta_1} \mathrm{e}^{-mt_1(\cosh\theta'_1+\cosh\theta'_2)} \\
= & \; - \left( F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} \left( F_2^{\mathcal{O}_1}(\theta'_2, \theta'_1) - F_2^{\mathcal{O}_1}(\theta'_1, \theta'_2) \right) \mathrm{e}^{-mt_2\cosh\theta'_1} \mathrm{e}^{-mt_1\cosh\theta'_2} - (\theta'_1 \leftrightarrow \theta'_2) \right)
\end{aligned} \tag{3.35}$$

and the integral of the last expression over $\theta'_{1,2}$ vanishes by symmetry.

## 3.2 $D_{22}^{(3)}$

The new aspect in this case is that the contribution must be split into two parts: one with the two two-particle states being different and the "diagonal" when these states are the same. Using the finite volume form factor formulae (2.24) and (2.26) we can write

$$\begin{aligned}
C_{22}^{(3)} = & \sum_{\{I_1,I_2\} \neq \{J_1,J_2\}} \frac{F_2^{\mathcal{O}_1}(\theta_1,\theta_2) F_4^{\mathcal{O}_2}(\theta_2+i\pi, \theta_1+i\pi, \theta_3, \theta_4) F_2^{\mathcal{O}_3}(\theta'_2, \theta'_1)}{\rho_2(\theta_1,\theta_2)\rho_2(\theta'_1,\theta'_2)} \\
& \times \; \mathrm{e}^{-m(t_1-t_2)(\cosh\theta_1+\cosh\theta_2)} \mathrm{e}^{-mt_2(\cosh\theta'_1+\cosh\theta'_2)} \\
& + \sum_{\{I_1,I_2\}} \frac{F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2) + F_{2s}^{\mathcal{O}_2}(\rho_1(\theta_1)+\rho_1(\theta_2)) + \langle \mathcal{O}_2 \rangle \rho_2(\theta_1,\theta_2)}{\rho_2(\theta_1,\theta_2)^2} \\
& \times \; F_2^{\mathcal{O}_1}(\theta_1,\theta_2) F_2^{\mathcal{O}_3}(\theta_2,\theta_1) \mathrm{e}^{-mt_1(\cosh\theta_1+\cosh\theta_2)}
\end{aligned} \tag{3.36}$$

The diagonal part can be written as

$$\begin{aligned}
C_{22}^{(3)\mathrm{diag}} = & \sum_{\{I_1,I_2\}} \frac{F_{4,s}^{\mathcal{O}_2}(\theta_1,\theta_2) + F_{2,s}^{\mathcal{O}_2}(\rho_1(\theta_1)+\rho_1(\theta_2)) + \langle \mathcal{O}_2 \rangle \rho_2(\theta_1,\theta_2)}{\rho_2(\theta_1,\theta_2)^2} \\
& \times \; F_2^{\mathcal{O}_1}(\theta_1,\theta_2) F_2^{\mathcal{O}_3}(\theta_2,\theta_1) \mathrm{e}^{-mt_1(\cosh\theta_1+\cosh\theta_2)} \\
\underset{L\to\infty}{\longrightarrow} & \; \frac{1}{2} \iint \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} F_2^{\mathcal{O}_1}(\theta_1,\theta_2) \langle \mathcal{O}_2 \rangle F_2^{\mathcal{O}_3}(\theta_2,\theta_1) \mathrm{e}^{-mt_1(\cosh\theta_1+\cosh\theta_2)}
\end{aligned} \tag{3.37}$$

For the non-diagonal part we need to evaluate

$$\tilde{C}(\theta'_1,\theta'_2) = \frac{1}{2} \sum_{I_1,I_2} \frac{F_4^{\mathcal{O}_2}(\theta_2+i\pi, \theta_1+i\pi, \theta'_1, \theta'_2)}{\rho_2(\theta_1,\theta_2)} K_{t_1,t_2}(\theta_1,\theta_2,\theta'_1,\theta'_2) \tag{3.38}$$

where

$$K_{t_1,t_2}(\theta_1,\theta_2,\theta'_1,\theta'_2) = F_2^{\mathcal{O}_3}(\theta_1,\theta_2) F_2^{\mathcal{O}_1}(\theta'_1,\theta'_2) \mathrm{e}^{-m(t_1-t_2)(\cosh\theta_1+\cosh\theta_2)} \mathrm{e}^{-mt_2(\cosh\theta'_1+\cosh\theta'_2)} \tag{3.39}$$

(again we extended the $I_1 < I_2$ summation by symmetry and included the diagonal $I_1 = I_2$ where the form factors vanish). Using the residue trick it can be represented as

$$\frac{1}{2} \sum_{I_1,I_2} \oint \oint_{C_{I_1 I_2}} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{F_4^{\mathcal{O}_2}(\theta_2+i\pi, \theta_1+i\pi, \theta'_1, \theta'_2)}{\left(\mathrm{e}^{iQ_1(\theta_1,\theta_2)}+1\right)\left(\mathrm{e}^{iQ_2(\theta_1,\theta_2)}+1\right)} K_{t_1,t_2}(\theta_1,\theta_2,\theta'_1,\theta'_2) \tag{3.40}$$

where

$$\begin{aligned}
Q_1(\theta_1,\theta_2) &= mL\sinh\theta_1 + \delta(\theta_1-\theta_2) \\
Q_2(\theta_1,\theta_2) &= mL\sinh\theta_2 + \delta(\theta_2-\theta_1)
\end{aligned} \tag{3.41}$$

To open the contour we need to find the singularities that do not result as solutions of $Q_{1,2} = 2\pi I_{1,2}$. There are the following possibilities:

- $Q_2 = 2\pi I_2$ and $\theta_1 = \theta'_1$ or $\theta_1 = \theta'_2$
- $Q_1 = 2\pi I_1$ and $\theta_2 = \theta'_1$ or $\theta_2 = \theta'_2$
- $\theta_1 = \theta'_1$ and $\theta_2 = \theta'_2$ or $\theta_2 = \theta'_1$ and $\theta_1 = \theta'_2$. Albeit the form factor is regular at this point, the denominator has a double zero due to the quantization condition satisfied by $\theta'_1$ and $\theta'_2$:

$$\begin{aligned} mL\sinh\theta'_1 + \delta(\theta'_1 - \theta'_2) &= 2\pi J_1 \\ mL\sinh\theta'_2 + \delta(\theta'_2 - \theta'_1) &= 2\pi J_2 \end{aligned} \tag{3.42}$$

These were excluded and their contribution calculated in the diagonal part $C_{22}^{\mathrm{diag}}$.

As an example we consider the contribution from the singularity $Q_2 = 2\pi I_2$ and $\theta_1 = \theta'_1$. Using (3.42) we can evaluate

$$\mathrm{e}^{iQ_1(\theta'_1,\theta_2)} = -\mathrm{e}^{imL\sinh\theta'_1}S(\theta'_1-\theta_2) = -S(\theta'_2-\theta'_1)S(\theta'_1-\theta_2) \tag{3.43}$$

and the appropriate residue takes the form

$$\begin{aligned} &\operatorname*{Res}_{\substack{Q_2=2\pi I_2\\ \theta_1=\theta'_1}} \frac{1}{2}\frac{F_4^{\mathcal{O}_2}(\theta_2+i\pi,\theta_1+i\pi,\theta'_1,\theta'_2)}{\left(\mathrm{e}^{iQ_1(\theta_1,\theta_2)}+1\right)\left(\mathrm{e}^{iQ_2(\theta_1,\theta_2)}+1\right)}K_{t_1,t_2}(\theta_1,\theta_2,\theta'_1,\theta'_2) = \\ &\frac{1}{2}i^2\frac{i(1-S(\theta_2-\theta'_1)S(\theta'_1-\theta'_2))F_2^{\mathcal{O}_2}(\theta_2+i\pi,\theta'_2)}{(1-S(\theta'_2-\theta'_1)S(\theta'_1-\theta_2))(-i)(mL\cosh\theta_2+\varphi(\theta_2-\theta'_1))}K_{t_1,t_2}(\theta'_1,\theta_2,\theta'_1,\theta'_2) \\ = \;& -\frac{1}{2}\frac{F_2^{\mathcal{O}_2}(\theta_2+i\pi,\theta'_2)}{(mL\cosh\theta_2+\varphi(\theta_2-\theta'_1))}K_{t_1,t_2}(\theta_2,\theta'_1,\theta'_2,\theta'_1) \end{aligned} \tag{3.44}$$

When we sum over $I_2$ we must exclude $I_2 = J_2$ i.e. the term $\theta_1 = \theta'_1$ and $\theta_2 = \theta'_2$ (this singularity was taken into account in $C_{22}^{(3)\mathrm{diag}}$). However, its contribution to the $I_2$ sum is

$$-\frac{1}{2}\frac{F_{2s}^{\mathcal{O}_2}}{(mL\cosh\theta'_2+\varphi(\theta'_2-\theta'_1))}K_{t_1,t_2}(\theta'_2,\theta'_1,\theta'_2,\theta'_1) \tag{3.45}$$

which vanishes when $L\to\infty$.

One can calculate the contribution of all other singularities in a similar way. Taking $L\to\infty$ the final result is

$$\begin{aligned} D_{22}^{(3)} &= \frac{1}{2}\iint_{C_{++}}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\iint\frac{d\theta'_1}{2\pi}\frac{d\theta'_2}{2\pi}F_4^{\mathcal{O}_2}(\theta_2+i\pi,\theta_1+i\pi,\theta'_1,\theta'_2)F_2^{\mathcal{O}_1}(\theta_1,\theta_2)F_2^{\mathcal{O}_3}(\theta'_2,\theta'_1) \\ &\quad\times\mathrm{e}^{-m(t_1-t_2)(\cosh\theta_1+\cosh\theta_2)}\mathrm{e}^{-mt_2(\cosh\theta'_1+\cosh\theta'_2)} \\ &+ \int\frac{d\theta_2}{2\pi}\iint\frac{d\theta'_1}{2\pi}\frac{d\theta'_2}{2\pi}F_2^{\mathcal{O}_1}(\theta_2,\theta'_1)F_2^{\mathcal{O}_2}(\theta_2+i\pi,\theta'_2)F_2^{\mathcal{O}_3}(\theta'_2,\theta'_1) \\ &\quad\times\mathrm{e}^{-m(t_1-t_2)\cosh\theta_2}\mathrm{e}^{-mt_1\cosh\theta'_1}\mathrm{e}^{-mt_2\cosh\theta'_2} \\ &- \frac{1}{2}\iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}F_2^{\mathcal{O}_1}(\theta_1,\theta_2)\left\langle\mathcal{O}_2\right\rangle F_2^{\mathcal{O}_3}(\theta_2,\theta_1)\mathrm{e}^{-mt_1(\cosh\theta_1+\cosh\theta_2)} \end{aligned} \tag{3.46}$$

where the $(\theta_1,\theta_2)$ contour $C_{++}$ is specified in (3.31)

# 4 Evaluating the thermal correlator

Now we show how evaluate the series (2.45). Since according to (2.34) the contributions $D_{0N}$ and $D_{M0}$ are identical to terms contributing to the zero-temperature two-point function, the first nontrivial temperature correction is given by $D_{11}$, which is evaluated in the next subsection. The contributions $D_{12}$, $D_{1n}$ for arbitrary $n>2$, and $D_{22}$ are calculated in subsections 4.2, 4.3 and 4.4 respectively. The final expressions (which constitute the main results of this work) are given by equations (4.8), (4.22), (4.47) and (4.89), respectively.

### 4.1 The $D_{11}$ correction

According to (2.43) and (2.44)

$$D_{11} = \lim_{L\to\infty} \tilde{D}_{11}$$
$$\tilde{D}_{11} = C_{11} - Z_1 C_{00} \tag{4.1}$$

where

$$C_{00} = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \tag{4.2}$$

and

$$C_{11} = \sum_{I,J} \langle \{I\} | \mathcal{O}_1(0) | \{J\} \rangle_L \, \langle \{J\} | \mathcal{O}_2(0) | \{I\} \rangle_L \, e^{i(p_1-p_2)x} e^{-E_1(R-t)} e^{-E_2 t} \tag{4.3}$$

where $E_1$, $E_2$ and $p_1$, $p_2$ are the finite size energies and momenta of the one-particle states. Using the Bethe-Yang quantization conditions (2.21) we have

$$mL\sinh\theta = 2\pi I \quad , \quad mL\sinh\theta' = 2\pi J \tag{4.4}$$

and

$$E_1 = m\cosh\theta \quad , \quad p_1 = m\sinh\theta$$
$$E_2 = m\cosh\theta' \quad , \quad p_2 = m\sinh\theta'$$

According to (2.24) and (2.26), the two-particle matrix elements are given by

$$\langle \{I\} | \mathcal{O}_1(0) | \{J\} \rangle_L = \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta')}{\sqrt{\rho_1(\theta)\rho_1(\theta')}} + \delta_{IJ} \langle \mathcal{O}_1 \rangle \tag{4.5}$$

$$\langle \{J\} | \mathcal{O}_2(0) | \{I\} \rangle_L = \frac{F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta)}{\sqrt{\rho_1(\theta)\rho_1(\theta')}} + \delta_{IJ} \langle \mathcal{O}_2 \rangle \tag{4.6}$$

Substituting the above formulas into (4.3) one obtains

$$\begin{aligned} C_{11} = & \sum_{I,J} \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta') F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta)}{\rho_1(\theta)\rho_1(\theta')} e^{i(p_1-p_2)x} e^{-E_1(R-t)} e^{-E_2 t} \\ & + \langle \mathcal{O}_1 \rangle \sum_J \frac{F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta')}{\rho_1(\theta')} e^{-E_2 R} + \langle \mathcal{O}_2 \rangle \sum_J \frac{F_2^{\mathcal{O}_1}(\theta + i\pi, \theta)}{\rho_1(\theta)} e^{-E_1 R} \\ & + \sum_I \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle e^{-E_1 R} \end{aligned} \tag{4.7}$$

The last term in (4.7) is $O(L)$, but it is canceled in $\tilde{D}_1 1$ by the term $Z_1 C_{00}$. All the other terms have a finite limit as $L \to \infty$ which can be written in the form

$$\begin{aligned} D_{11} = & \int \frac{d\theta}{2\pi} \int \frac{d\theta'}{2\pi} F_2^{\mathcal{O}_1}(\theta + i\pi, \theta') F_2^{\mathcal{O}_2}(\theta' + i\pi, \theta) e^{i(\sinh\theta - \sinh\theta')mx} e^{-m(R-t)\cosh\theta - mt\cosh\theta'} \\ + & \left( \langle \mathcal{O}_1 \rangle F_{2s}^{\mathcal{O}_2} + \langle \mathcal{O}_2 \rangle F_{2s}^{\mathcal{O}_1} \right) \int \frac{d\theta}{2\pi} e^{-mR\cosh\theta} \end{aligned} \tag{4.8}$$

Note that according to (2.10) the two-particle form factor does not have kinematical singularities. The result (4.8) was first obtained in [57].

### 4.2 More than just a warm-up: $D_{12}$

According to (2.43) and (2.44)

$$D_{12} = \lim_{L\to\infty} \left(C_{12} - Z_1 C_{01}\right)$$

where

$$\begin{aligned} C_{12} &= \sum_I \sum_{J_1 J_2} \langle \{I\}|\mathcal{O}_1(0)|\{J_1,J_2\}\rangle_L \times \langle \{J_1,J_2\}|\mathcal{O}_2(0)|\{I\}\rangle_L \mathrm{e}^{i(P_1-P_2)x}\mathrm{e}^{-E_1(R-t)}\mathrm{e}^{-E_2 t} \\ &= \sum_I \sum_{J_1 J_2} \frac{F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2')F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_2',\theta_1')}{\rho_1(\theta_1)\rho_2(\theta_1',\theta_2')} K_{t,x}^{(R)}(\theta_1,\theta_1',\theta_2') \end{aligned} \tag{4.9}$$

$$K_{t,x}^{(R)}(\theta_1,\theta_1',\theta_2') \;=\; \mathrm{e}^{imx(\sinh\theta_1 - \sinh\theta_1' - \sinh\theta_2')}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt(\cosh\theta_1'+\cosh\theta_2')} \tag{4.10}$$

and

$$Z_1 C_{01} = \left(\sum_I \mathrm{e}^{-ER}\right)\left(\int \frac{d\theta}{2\pi} F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} \mathrm{e}^{-imx\,\mathrm{sh}\,\theta - mt\,\mathrm{ch}\,\theta}\right)$$

#### 4.2.1 First summation: one-particle states

We first perform the summation over $I$. The quantization condition reads

$$Q_1(\theta_1) = mL\sinh\theta_1 = 2\pi I \qquad\qquad \rho_1 = \frac{\partial Q_1}{\partial\theta_1} = mL\cosh\theta_1$$

with $I \in \mathbb{N}$. Therefore it is possible to convert the summation into a sum over contour integrals

$$\sum_{J_1 J_2}\sum_I \oint \frac{d\theta_1}{2\pi} \frac{F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2')F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_2',\theta_1')}{\rho_2(\theta_1',\theta_2')} K_{t,x}^{(R)}(\theta_1,\theta_1',\theta_2') \frac{1}{\mathrm{e}^{iQ_1(\theta_1)}-1} \tag{4.11}$$

In order to open up the contours one has to calculate the surplus singularities of the integrand, which appear at $\theta_1 \to \theta_1'$ and at $\theta_1 \to \theta_2'$. Each of the form factors have first order poles, therefore the singularity is a second order pole. In the following we calculate the residue at $\theta_1 \to \theta_1'$; the case $\theta_1 \to \theta_2'$ will be given by a change of variables.

The residue of the form factors for $\theta_1 \to \theta_1'$ read

$$F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2') = i\Big(1 - S(\theta_1'-\theta_2')\Big)\frac{F_1^{\mathcal{O}_1}}{\theta_1-\theta_1'} + \dots \tag{4.12}$$

Let us introduce the connected part of the three-particle form factor as

$$F_{3sc}^{\mathcal{O}_1}(\theta_1'|\theta_1',\theta_2') = \lim_{\theta_1\to\theta_1'}\left(F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2') - i\Big(1-S(\theta_1'-\theta_2')\Big)\frac{F_1^{\mathcal{O}_1}}{\theta_1-\theta_1'}\right) \tag{4.13}$$

The connected form factor defined above still has a pole at $\theta_1' = \theta_2'$. In fact, the singularity structure of the original form factor near $\theta_1 = \theta_1' = \theta_2'$ is given by

$$F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2') = 2iF_1^{\mathcal{O}_1}\left(\frac{1}{\theta_1-\theta_1'} - \frac{1}{\theta_1-\theta_2'}\right) \tag{4.14}$$

and after subtracting the first pole there remains the second one leading to

$$F_{3sc}^{\mathcal{O}_1}(\theta_1'|\theta_1',\theta_2') = -2iF_1^{\mathcal{O}_1}\frac{1}{\theta_1'-\theta_2'} + \dots \tag{4.15}$$

Also, it can be proven that

$$F_{3sc}^{\mathcal{O}_1}(\theta_1'|\theta_1',\theta_2') = S(\theta_1'-\theta_2')F_{3sc}^{\mathcal{O}_1}(\theta_2'|\theta_2',\theta_1') \tag{4.16}$$

In the case of the crossed form factor one has

$$F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_2',\theta_1') = -i\Big(1-S(\theta_2'-\theta_1')\Big)\frac{F_1^{\mathcal{O}_2}}{\theta_1-\theta_1'} + S(\theta_2'-\theta_1')F_{3sc}^{\mathcal{O}_2}(\theta_1'|\theta_1',\theta_2') + \dots \tag{4.17}$$

With these notations the residue of (4.11) at $\theta_1=\theta_1'$ is expressed as

$$\begin{aligned} K_{t,x}^{(R)}(\theta_1',\theta_1',\theta_2')\Big\{\Big((S(\theta_1'-\theta_2')-1)(im\operatorname{ch}\theta_1' x - m\operatorname{sh}\theta_1'(R-t)) + imL\operatorname{ch}\theta_1'\Big)F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2} \\ + iF_{3sc}^{\mathcal{O}_1}(\theta_1'|\theta_1',\theta_2')F_1^{\mathcal{O}_2} + iF_{3sc}^{\mathcal{O}_2}(\theta_1'|\theta_1',\theta_2')F_1^{\mathcal{O}_1}\Big\} \end{aligned} \tag{4.18}$$

There is a similar contribution at $\theta_1=\theta_2'$, with the role of $\theta_1'$ and $\theta_2'$ exchanged. After integrating over $\theta_1',\theta_2'$ one could make a change of variables to obtain the same contribution twice. However, one has to keep both residues separately because of the poles of the quantities $F_{3sc}$. Making the change of variables only in the regular terms one obtains the two contributions

$$\begin{aligned} \mathrm{dsing}_{FF} =& 2K_{t,x}^{(R)}(\theta_1',\theta_1',\theta_2')\Big((S(\theta_1'-\theta_2')-1)(im\operatorname{ch}\theta_1' x - m\operatorname{sh}\theta_1'(R-t)) + imL\operatorname{ch}\theta_1'\Big)F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2} \\ \mathrm{ssing}_{FF} =& iK_{t,x}^{(R)}(\theta_1',\theta_1',\theta_2')\left(F_{3sc}^{\mathcal{O}_1}(\theta_1'|\theta_1',\theta_2')F_1^{\mathcal{O}_2} + F_{3sc}^{\mathcal{O}_2}(\theta_1'|\theta_1',\theta_2')F_1^{\mathcal{O}_1}\right) + \\ & iK_{t,x}^{(R)}(\theta_2',\theta_2',\theta_1')\left(F_{3sc}^{\mathcal{O}_1}(\theta_2'|\theta_2',\theta_1')F_1^{\mathcal{O}_2} + F_{3sc}^{\mathcal{O}_2}(\theta_2'|\theta_2',\theta_1')F_1^{\mathcal{O}_1}\right) \end{aligned} \tag{4.19}$$

Now is is possible to perform the summations over $\theta_1',\theta_2'$. The $\mathcal{O}(L)$ term of (4.19) can be transformed in the $L\to\infty$ limit into

$$\begin{aligned} & mLF_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}\left(\int\frac{d\theta_2'}{2\pi}\mathrm{e}^{-im\operatorname{sh}\theta_2' x - m\operatorname{ch}\theta_2' t}\right)\left(\int\frac{d\theta_1'}{2\pi}\operatorname{ch}\theta_1'\mathrm{e}^{-m\operatorname{ch}\theta_1' R}\right) - \\ & - F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}\int\frac{d\theta_2'}{2\pi}\mathrm{e}^{-m\operatorname{ch}\theta_2'(R+t)-im\operatorname{sh}\theta_2' x} \end{aligned} \tag{4.20}$$

The first term gets exactly canceled by $Z_1C_{01}$ leaving only the finite contribution

$$-F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}\int\frac{d\theta_2'}{2\pi}\mathrm{e}^{-m\operatorname{ch}\theta_2'(R+t)-im\operatorname{sh}\theta_2' x} \tag{4.21}$$

The remaining terms of (4.19) are regular, therefore it is allowed to replace the summation over $\theta_1',\theta_2'$ with the appropriate integral. The final result is

$$\begin{aligned} D_{12} =& \frac{1}{2}\int_{C_+}\frac{d\theta_1}{2\pi}\int\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2')F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_2',\theta_1')K_{t,x}^{(R)}(\theta_1,\theta_1',\theta_2') \\ & + \int\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}K_{t,x}^{(R)}(\theta_1',\theta_1',\theta_2')(S(\theta_1'-\theta_2')-1)(m\operatorname{ch}\theta_1' x + im\operatorname{sh}\theta_1'(R-t))F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2} \\ & + \frac{1}{2}\int\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}\Big\{K_{t,x}^{(R)}(\theta_1',\theta_1',\theta_2')\Big(F_{3sc}^{\mathcal{O}_1}(\theta_1'|\theta_1',\theta_2')F_1^{\mathcal{O}_2} + F_{3sc}^{\mathcal{O}_2}(\theta_1'|\theta_1',\theta_2')F_1^{\mathcal{O}_1}\Big) + (\theta_1'\leftrightarrow\theta_2')\Big\} \\ & - F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}\int\frac{d\theta_2'}{2\pi}\mathrm{e}^{-m\operatorname{ch}\theta_2'(R+t)-im\operatorname{sh}\theta_2' x} \end{aligned} \tag{4.22}$$

### 4.2.2 Performing the two-particle summation first

We can express $C_{12}$ as

$$\sum_I\frac{1}{\rho_1(\theta_1)}\sum_{J_1J_2}\oint\oint_{C_{J_1J_2}}\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}\frac{F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2')F_3^{\mathcal{O}_2}(\theta_2'+i\pi,\theta_1'+i\pi,\theta_1)}{\left(\mathrm{e}^{iQ_{1'}(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1',\theta_2')}+1\right)}K_{t,x}^{(R)}(\theta_1,\theta_1',\theta_2') \tag{4.23}$$

$$K^{(R)}_{t,x}(\theta_1,\theta'_1,\theta'_2) \quad = \quad \mathrm{e}^{imx(\sinh\theta_1-\sinh\theta'_1-\sinh\theta'_2)}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt(\cosh\theta'_1+\cosh\theta'_2)}$$

Now we open the multi-contour to surround the real axes in $\theta'_1$ and $\theta'_2$; however, we encounter some "surplus" singularities:

- QF poles, where the singularity in one of the variables arise from a Q-denominator while in the other from a form factor:

$$\begin{aligned}\theta'_1=\theta_1 \quad &, \quad \mathrm{e}^{iQ_{2'}(\theta'_1,\theta'_2)}+1=0\\ \theta'_2=\theta_1 \quad &, \quad \mathrm{e}^{iQ_{1'}(\theta'_1,\theta'_2)}+1=0\end{aligned}$$

- FF poles, where the singularity in both variables comes from the form factors:

$$\theta'_1=\theta_1 \quad , \quad \theta'_2=\theta_1$$

(note that positions where there is only a singularity in one of the variables do not contribute, as the contour in the other variable can be shrunk to a point). In the following we calculate the contributions of these singularities.

**The FF singularity**

We can write

$$F^{\mathcal{O}_1}_3(\theta_1+i\pi,\theta'_1,\theta'_2)=\frac{1}{\theta_1-\theta'_1}i\left(1-S(\theta'_1-\theta'_2)\right)F^{\mathcal{O}_1}_1+\frac{1}{\theta_1-\theta'_2}i\left(S(\theta'_2-\theta'_1)-1\right)F^{\mathcal{O}_1}_1 \\ +F^{\mathcal{O}_1}_{3cc}(\theta_1|\theta'_1,\theta'_2) \tag{4.24}$$

where $F^{\mathcal{O}_1}_{3cc}$ is the regular part of the form factor around the singularity. The pole contribution is then

$$\begin{aligned}&\frac{1}{2}\oint_{\theta_1}\frac{d\theta'_1}{2\pi}\oint_{\theta_1}\frac{d\theta'_2}{2\pi}\frac{K^{(R)}_{t,x}(\theta_1,\theta'_1,\theta'_2)}{\left(\mathrm{e}^{iQ_{1'}(\theta'_1,\theta'_2)}+1\right)\left(\mathrm{e}^{iQ_{2'}(\theta'_1,\theta'_2)}+1\right)}S(\theta'_1-\theta'_2)\\ &\times\left(\frac{1}{\theta_1-\theta'_1}i\left(1-S(\theta'_1-\theta'_2)\right)F^{\mathcal{O}_1}_1+\frac{1}{\theta_1-\theta'_2}i\left(S(\theta'_2-\theta'_1)-1\right)F^{\mathcal{O}_1}_1+F^{\mathcal{O}_1}_{3cc}(\theta_1|\theta'_1,\theta'_2)\right)\\ &\times\left(\frac{1}{\theta_1-\theta'_1}i\left(1-S(\theta'_1-\theta'_2)\right)F^{\mathcal{O}_2}_1+\frac{1}{\theta_1-\theta'_2}i\left(S(\theta'_2-\theta'_1)-1\right)F^{\mathcal{O}_2}_1+F^{\mathcal{O}_2}_{3cc}(\theta_1|\theta'_1,\theta'_2)\right)\end{aligned}$$

$F_{3cc}$ does not contribute since then either the $\theta'_1$ or the $\theta'_2$ integration contour can be contracted to a point. For similar reasons, the only terms that could give a nonzero contribution are the "cross-terms"

$$\begin{aligned}&\frac{1}{2}\oint_{\theta_1}\frac{d\theta'_1}{2\pi}\oint_{\theta_1}\frac{d\theta'_2}{2\pi}\frac{K^{(R)}_{t,x}(\theta_1,\theta'_1,\theta'_2)}{\left(\mathrm{e}^{iQ_{1'}(\theta'_1,\theta'_2)}+1\right)\left(\mathrm{e}^{iQ_{2'}(\theta'_1,\theta'_2)}+1\right)}2S(\theta'_1-\theta'_2)\\ \times&\left(\frac{1}{\theta_1-\theta'_1}i\left(1-S(\theta'_1-\theta'_2)\right)F^{\mathcal{O}_1}_1\times\frac{1}{\theta_1-\theta'_2}i\left(S(\theta'_2-\theta'_1)-1\right)F^{\mathcal{O}_2}_1\right)\end{aligned}$$

We need the residue at $\theta'_1=\theta'_2=\theta_1$. Then

$$S(\theta'_1-\theta'_2)=S(0)=-1$$

and

$$\mathrm{e}^{iQ_{1'}(\theta_1,\theta_1)}=\mathrm{e}^{iQ_{2'}(\theta_1,\theta_1)}=\mathrm{e}^{imL\sinh\theta_1}=1$$

therefore the contribution of the pole is given by

$$-F^{\mathcal{O}_1}_1F^{\mathcal{O}_2}_1\mathrm{e}^{-imx\sinh\theta_1-m(R+t)\cosh\theta_1}$$

After performing the $\theta_1$ sum converted to an integral we find

$$S_{FF} = -\int \frac{d\theta}{2\pi} F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} \mathrm{e}^{-imx\sinh\theta_1 - m(R+t)\cosh\theta_1} \tag{4.25}$$

which correctly reproduces the last term of $D_{12}$ (4.22).

**The QF pole at $\theta_1' = \theta_1$**

The singular contribution is

$$\frac{1}{2}\oint_{\theta_1} \frac{d\theta_1'}{2\pi} \oint_{C_{J_2 I}} \frac{d\theta_2'}{2\pi} \frac{F_3^{\mathcal{O}_1}(\theta_1 + i\pi, \theta_1', \theta_2') F_3^{\mathcal{O}_2}(\theta_1 + i\pi, \theta_2', \theta_1')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1',\theta_2')} + 1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1',\theta_2')} + 1\right)} K_{t,x}^{(R)}(\theta_1, \theta_1', \theta_2') \tag{4.26}$$

where $C_{J_2 I}$ surrounds the $\theta_2'$ solution of

$$Q_{2'}(\theta_1, \theta_2') = mL\sinh\theta_2' + \delta(\theta_2' - \theta_1) = 2\pi J_2 \tag{4.27}$$

where $\theta_1$ is the solution of

$$Q_1(\theta_1) = mL\sinh\theta_1 = 2\pi I \tag{4.28}$$

The behaviours of the form factors are given by (4.13) and (4.17). We can separate the integrand into two terms according to the order of the $\theta_1' = \theta_1$ singularity. The first order term has the form

$$\begin{aligned} S_{QF}^1 &= -\tfrac{1}{2} \oint_{\theta_1} \frac{d\theta_1'}{2\pi} \oint_{C_{J_2 I}} \frac{d\theta_2'}{2\pi} \frac{K_{t,x}^{(R)}(\theta_1, \theta_1', \theta_2')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1',\theta_2')} + 1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1',\theta_2')} + 1\right)} \frac{i}{\theta_1' - \theta_1} \left(S(\theta_2' - \theta_1') - 1\right) \\ &\times \left(F_1^{\mathcal{O}_2} F_{3c}^{\mathcal{O}_1}(\theta_1|\theta_1', \theta_2') + F_1^{\mathcal{O}_1} F_{3c}^{\mathcal{O}_2}(\theta_1|\theta_1', \theta_2')\right) \end{aligned}$$

This can be easily evaluated:

$$-\frac{1}{2}\sum_{\theta_2'} \frac{K_{t,x}^{(R)}(\theta_1, \theta_1, \theta_2')}{(1 - S(\theta_1 - \theta_2'))\,\bar{\rho}_3(\theta_2'|\theta_1')} \left(S(\theta_2' - \theta_1) - 1\right) \left(F_1^{\mathcal{O}_2} F_{3c}^{\mathcal{O}_1}(\theta_1|\theta_1, \theta_2') + F_1^{\mathcal{O}_1} F_{3c}^{\mathcal{O}_2}(\theta_1|\theta_1, \theta_2')\right)$$

where

$$\bar{\rho}_3(\theta_2'|\theta_1) = \frac{\partial}{\partial\theta_2'} Q_{2'}(\theta_1, \theta_2') = mL\cosh\theta_2' + \varphi(\theta_2' - \theta_1)$$

is the density of $\theta_2'$ solutions for a given $\theta_1$. This gives

$$S_{QF}^1 = -\frac{1}{2}\sum_{\theta_2'} \frac{K_{t,x}^{(R)}(\theta_1, \theta_1, \theta_2')}{\bar{\rho}_3(\theta_2'|\theta_1)} S(\theta_2' - \theta_1) \left(F_1^{\mathcal{O}_2} F_{3c}^{\mathcal{O}_1}(\theta_1|\theta_1, \theta_2') + F_1^{\mathcal{O}_1} F_{3c}^{\mathcal{O}_2}(\theta_1|\theta_1, \theta_2')\right) \tag{4.29}$$

Together with a similar term $S_{QF}^2$ obtained by swapping $\theta_1' \leftrightarrow \theta_2'$ this gives the contribution

$$\begin{aligned} &\frac{1}{2}\int \frac{d\theta_1'}{2\pi} \int \frac{d\theta_2'}{2\pi} \Big[ K_{t,x}^{(R)}(\theta_1', \theta_1', \theta_2') S(\theta_2' - \theta_1') \left(F_1^{\mathcal{O}_1} F_{3c}^{\mathcal{O}_2}(\theta_1'|\theta_1', \theta_2') + F_1^{\mathcal{O}_2} F_{3c}^{\mathcal{O}_1}(\theta_1'|\theta_1', \theta_2')\right) \\ &+ (\theta_1' \leftrightarrow \theta_2') \Big] \end{aligned} \tag{4.30}$$

The second order term reads

$$\begin{aligned} D_{QF}^1 &= -\frac{1}{2}\oint_{\theta_1} \frac{d\theta_1'}{2\pi} \oint_{C_{J_2 I}} \frac{d\theta_2'}{2\pi} \frac{K_{t,x}^{(R)}(\theta_1, \theta_1', \theta_2')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1',\theta_2')} + 1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1',\theta_2')} + 1\right)} \\ &\times \frac{(1 - S(\theta_1' - \theta_2'))\,(S(\theta_2' - \theta_1') - 1)}{(\theta_1 - \theta_1')^2} F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} \end{aligned} \tag{4.31}$$

The contribution of the double pole can be evaluated by taking the derivative with respect to $\theta'_1$. The result reads

$$D^1_{QF} = -\frac{1}{2}i\oint_{C_{J_2I}}\frac{d\theta'_2}{2\pi}\frac{F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}K^{(R)}_{t,x}(\theta_1,\theta_1,\theta'_2)}{\left(\mathrm{e}^{iQ_{2'}(\theta_1,\theta'_2)}+1\right)}\Big[\left(-imx\cosh\theta_1-mt\sinh\theta_1\right)\left(S(\theta'_2-\theta_1)-1\right)$$
$$+imL\cosh\theta_1-i\varphi(\theta_1-\theta'_2)S(\theta'_2-\theta_1)\Big]$$
$$+\frac{1}{2}\oint_{C_{J_2I}}\frac{d\theta'_2}{2\pi}\frac{F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}K^{(R)}_{t,x}(\theta_1,\theta_1,\theta'_2)}{\left(\mathrm{e}^{iQ_{2'}(\theta_1,\theta'_2)}+1\right)^2}\left(S(\theta'_2-\theta_1)-1\right)\varphi(\theta'_2-\theta_1)$$

The last term with the double pole yields zero for $L\to\infty$ since it is proportional to $L^{-2}$ and becomes $L^{-1}$ after including the density. Explicitly it evaluates to

$$\frac{1}{2}\oint_{C_{J_2I}}\frac{d\theta'_2}{2\pi}\frac{F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}K^{(R)}_{t,x}(\theta_1,\theta_1,\theta'_2)}{-Q'_{2'}(\theta_1,\theta_{3*})^2(\theta'_2-\theta_{3*})^2}\left(S(\theta'_2-\theta_1)-1\right)\varphi(\theta'_2-\theta_1)(-S(\theta'_2-\theta_1))$$

where $\theta_{3*}$ is the location of the solution of

$$Q_{2'}(\theta_1,\theta'_2)=2\pi J_2 \tag{4.32}$$

but

$$Q'_{2'}(\theta_1,\theta_{3*})^2=(mL\cosh\theta_{3*}+\varphi(\theta_{3*}-\theta_1))^2=O(L^{-2}) \tag{4.33}$$

Even after multiplying this by the density $Q'_{2'}(\theta_1,\theta_{3*})$ when converting the summation over $\theta_{3*}$ to integral a suppression $O(L^{-1})$ remains, resulting in zero large volume limit.

Putting in the $\theta_1$ summation and a factor 2 to account for the contribution obtained by exchanging $\theta'_1$ with $\theta'_2$, plus a minus sign since this is to be subtracted in the end, and adding the $-Z_1C_{01}$ term gives

$$-\int\frac{d\theta_1}{2\pi}\int\frac{d\theta'_2}{2\pi}F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}K^{(R)}_{t,x}(\theta_1,\theta_1,\theta'_2)\Big[\left(mx\cosh\theta_1-imt\sinh\theta_1\right)\left(S(\theta'_2-\theta_1)-1\right)$$
$$-mL\cosh\theta_1+\varphi(\theta_1-\theta'_2)S(\theta'_2-\theta_1)\Big]$$
$$-mL\int\frac{d\theta}{2\pi}\cosh\theta\mathrm{e}^{-mR\cosh\theta}F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}\int\frac{d\theta'}{2\pi}\mathrm{e}^{-imx\sinh\theta'-mt\cosh\theta'}$$

The $O(L)$ term cancels as expected, and after a partial integration one obtains

$$D_{QF}=-\int\frac{d\theta_1}{2\pi}\int\frac{d\theta'_2}{2\pi}F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}K^{(R)}_{t,x}(\theta_1,\theta_1,\theta'_2)(mx\cosh\theta_1+im(R-t)\sinh\theta_1)\left(S(\theta'_2-\theta_1)-1\right)$$

**End result**

Putting together everything

$$D_{12}=\frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\int_{C_{++}}\frac{d\theta'_1}{2\pi}\frac{d\theta'_2}{2\pi}F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta'_1,\theta'_2)F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta'_2,\theta'_1)K^{(R)}_{t,x}(\theta_1,\theta'_1,\theta'_2)$$
$$+\frac{1}{2}\int\frac{d\theta'_1}{2\pi}\int\frac{d\theta'_2}{2\pi}\Big[K^{(R)}_{t,x}(\theta'_1,\theta'_1,\theta'_2)S(\theta'_2-\theta'_1)\left(F_1^{\mathcal{O}_1}F_{3c}^{\mathcal{O}_2}(\theta'_1|\theta'_1,\theta'_2)+F_1^{\mathcal{O}_2}F_{3c}^{\mathcal{O}_1}(\theta'_1|\theta'_1,\theta'_2)\right)$$
$$+(\theta'_1\leftrightarrow\theta'_2)\Big]$$
$$-\int\frac{d\theta_1}{2\pi}\int\frac{d\theta'_2}{2\pi}F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2}K^{(R)}_{t,x}(\theta_1,\theta_1,\theta'_2)(mx\cosh\theta_1+im(R-t)\sinh\theta_1)\left(S(\theta'_2-\theta_1)-1\right)$$
$$-\int\frac{d\theta}{2\pi}\mathrm{e}^{-m(R+t)\cosh\theta-imx\sinh\theta}F_1^{\mathcal{O}_1}F_1^{\mathcal{O}_2} \tag{4.34}$$

It is a straightforward, although somewhat tedious exercise to show that the above expression can be transformed in the form (4.22). First of all observe, that shifting all three variables the first term of (4.34) can be written as

$$\frac{1}{2}\int_{C_-}\frac{d\theta_1}{2\pi}\int\frac{d\theta_1'}{2\pi}\int\frac{d\theta_2'}{2\pi}F_3^{\mathcal{O}_1}(\theta_1+i\pi,\theta_1',\theta_2')F_3^{\mathcal{O}_2}(\theta_1+i\pi,\theta_2',\theta_1')K_{t,x}^{(R)}(\theta_1,\theta_1',\theta_2')$$

This differs from the corresponding term in (4.22) in the contour for $\theta_1$, which in this case runs below the real axis. Shifting this contour to run above the real axis one picks up the poles of the integrand, which can be evaluated using standard techniques. It can be shown that the resulting contributions are

$$\begin{aligned}&-\int\frac{d\theta_1'}{2\pi}\int\frac{d\theta_2'}{2\pi}(-2+S(\theta_1'-\theta_2')+S(\theta_2'-\theta_1'))(mx\cosh\theta_1'+im(R-t)\sinh\theta_1')K_{t,x}^{(R)}(\theta_1',\theta_1',\theta_2')\\&+\frac{1}{2}\int\frac{d\theta_1'}{2\pi}\int\frac{d\theta_2'}{2\pi}\Big[\left(S(\theta_2'-\theta_1')-1\right)\left(F_1^{\mathcal{O}_1}F_{3c}^{\mathcal{O}_2}(\theta_1'|\theta_1',\theta_2')+F_1^{\mathcal{O}_2}F_{3c}^{\mathcal{O}_1}(\theta_1'|\theta_1',\theta_2')\right)K_{t,x}^{(R)}(\theta_1',\theta_1',\theta_2')\\&+(\theta_1'\leftrightarrow\theta_2')\Big]\end{aligned}$$

Adding these terms to the second and third lines of (4.34) one recovers (4.22).

We wish to note, that if instead of (4.23) we had started with a similar formula including the factors

$$\frac{1}{\left(\mathrm{e}^{-iQ_{1'}(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{-iQ_{2'}(\theta_1',\theta_2')}+1\right)}$$

we would have arrived immediately at the result (4.22). However, the calculation presented above is a non-trivial cross-check of our methods.

### 4.3 The contribution $D_{1n}$ for $n>2$

Based on the previous subsection it is now straightforward to evaluate the contribution $D_{1n}$ for arbitrary $n$. It is given by

$$D_{1n}=\lim_{L\to\infty}\left(C_{1n}-Z_1C_{0,n-1}\right)$$

where

$$\begin{aligned}C_{1n}&=\frac{1}{n!}\sum_I\sum_{J_1\dots J_n}\langle\{I\}|\mathcal{O}_1(0)|\{J_1,\dots,J_n\}\rangle_L\langle\{J_1,\dots,J_n\}|\mathcal{O}_2(0)|\{I\}\rangle_L\\&\quad\times\mathrm{e}^{i(P_1-P_2)x}\mathrm{e}^{-E_1(R-t)}\mathrm{e}^{-E_2t}\\&=\frac{1}{n!}\sum_I\sum_{J_1\dots J_n}\frac{F_{n+1}^{\mathcal{O}_1}(\theta+i\pi,\theta_1',\dots,\theta_n)F_{n+1}^{\mathcal{O}_2}(\theta+i\pi,\theta_n',\dots,\theta_1')}{\rho_1(\theta)\rho_n(\theta_1',\dots,\theta_n')}\\&\quad\times\mathrm{e}^{imx(\sinh\theta-mx\sum_j\sinh\theta_j)-m(R-t)\cosh\theta-mt\sum_j\cosh\theta_j}\end{aligned}\tag{4.35}$$

There are additional disconnected terms in the case of $n$ being odd, according to the rule explained in subsection 2.3. This happens in the presence of zero-momentum particles, which requires $I=0$ and the set $\{J_1,\dots,J_n\}$ to be parity symmetric; the disconnected term is given by formula (2.31). It is easy to show using the constrained density of states, that all contributions associated to these disconnected terms scales with negative powers of $L$, therefore we neglect them in the following.

We consider (4.35) and we first perform the summation over $I$. The quantization condition is

$$Q_1(\theta)=mL\sinh\theta=2\pi I\qquad\qquad\rho_1=Q_1'\tag{4.36}$$

with $I\in\mathbb{N}$. The converting the sum over $I$ to contour integrals we get

$$\begin{aligned}&\sum_{J_1\dots J_n}\sum_I\oint_{C_I}\frac{d\theta}{2\pi}\frac{F_{n+1}^{\mathcal{O}_1}(\theta+i\pi,\theta_1',\dots,\theta_n)F_{n+1}^{\mathcal{O}_2}(\theta+i\pi,\theta_n',\dots,\theta_1')}{\rho_n(\theta_1',\dots,\theta_n')}\times\\&\times\mathrm{e}^{imx(\sinh\theta-mx\sum_j\sinh\theta_j)-m(R-t)\cosh\theta-mt\sum_j\cosh\theta_j}\frac{1}{\mathrm{e}^{iQ_1(\theta)}-1}\end{aligned}\tag{4.37}$$

where the contour $C_I$ surrounds the solution of (4.36). When opening the contour to surround the real axis in $\theta$, we get the following contribution in the $L \to \infty$ limit:

$$\frac{1}{n!}\int_{C_+}\frac{d\theta}{2\pi}\int\frac{d\theta'_1}{2\pi}\dots\frac{d\theta'_n}{2\pi}F^{\mathcal{O}_1}_{n+1}(\theta+i\pi,\theta'_1,\dots,\theta_n)F^{\mathcal{O}_2}_{n+1}(\theta+i\pi,\theta'_n,\dots,\theta'_1)\times \\ \times\,\mathrm{e}^{imx(\sinh\theta-\sum_j\sinh\theta'_j)x-m(R-t)\cosh\theta-mt\sum_j\cosh\theta'_j} \tag{4.38}$$

where the contour $C_+$ is defined as

$$\int_{C_+}\frac{d\theta}{2\pi}f(\theta)=\int_{\mathbb{R}}\frac{d\theta}{2\pi}f(\theta+i\epsilon) \tag{4.39}$$

However, there are additional poles of the integrand for $\theta=\theta'_j$ for $j=1\dots n$, whose contribution must be subtracted.

First we calculate the residue at $\theta\to\theta'_1$. The behaviour of the form factors is given by the kinematical residue equation (2.10). Let us introduce the (partially) connected part of the form factor as

$$F^{\mathcal{O}_1}_{n+1,sc}(\theta'_1|\theta'_1,\dots,\theta'_n)=\lim_{\theta\to\theta'_1}\Bigg[F^{\mathcal{O}_1}_{n+1}(\theta+i\pi,\theta'_1,\dots,\theta'_n) \\ -i\Big(1-\prod_{j=2}^{n}S(\theta-\theta_j)\Big)\frac{F^{\mathcal{O}_1}_{n-1}(\theta'_1,\dots,\theta'_n)}{\theta-\theta'_1}\Bigg] \tag{4.40}$$

The form factor $F_{sc}$ defined above is only "partially" connected since only one of the singularities is subtracted and so it still has poles at $\theta'_1=\theta_j$ for $j=2\dots n$. In fact, the singularity structure of the original form factor near $\theta=\theta'_1=\theta'_1$ is given by

$$F^{\mathcal{O}_1}_{n+1}(\theta+i\pi,\theta'_1,\theta'_1,\dots)=i(1+\prod_{k=3}^{n}S(\theta'_1-\theta_k))F^{\mathcal{O}_1}_{n-1}(\theta_2,\dots,\theta'_n)\left(\frac{1}{\theta-\theta'_1}-\frac{1}{\theta-\theta_2}\right) \tag{4.41}$$

and after subtracting the first pole there remains the second one leading to

$$F^{\mathcal{O}_1}_{n+1,sc}(\theta'_1|\theta_2,\dots,\theta'_n)=-i(1+\prod_{k=3}^{n}S(\theta'_1-\theta_k))F^{\mathcal{O}_1}_{n-1}(\theta_2,\dots,\theta'_n)\frac{1}{\theta'_1-\theta_2}+\dots \tag{4.42}$$

The connected part satisfies the exchange equation

$$F^{\mathcal{O}_1}_{n+1,sc}(\theta'_1|\theta'_1,\dots,\theta'_j,\theta'_k\dots,\theta'_n)=S(\theta'_j-\theta'_k)F^{\mathcal{O}_1}_{n+1,sc}(\theta'_1|\theta'_1,\dots,\theta'_k,\theta'_j\dots,\theta'_n) \tag{4.43}$$

In the case of the crossed form factor one has

$$F^{\mathcal{O}_2}_{n+1}(\theta+i\pi,\theta'_n,\dots,\theta'_1)= \\ -i\Big(1-\prod_{j=2}^{n}S(\theta_j-\theta)\Big)\frac{F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta_2)}{\theta-\theta'_1}+\Big(\prod_{j=2}^{n}S(\theta_j-\theta'_1)\Big)F^{\mathcal{O}_2}_{n+1,sc}(\theta'_1|\theta'_n,\dots,\theta_2)+\dots$$

The residue of the integrand at $\theta=\theta'_1$ is then expressed as

$$\mathrm{e}^{-imx\sum_{j=2}^{n}\sinh\theta'_j-mR\cosh\theta'_1-mt\sum_{j=2}^{n}\cosh\theta'_j}\Big\{F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n)F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2) \\ \times\Big[(\prod_{j=2}^{n}S(\theta'_1-\theta'_j)-1)(imx\cosh\theta'_1-m(R-t)\sinh\theta'_1)+imL\cosh\theta'_1\Big] \\ +iF^{\mathcal{O}_1}_{n+1,c}(\theta'_1|\theta'_1,\dots,\theta'_n)F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2)+iF^{\mathcal{O}_2}_{n+1,c}(\theta'_1|\theta'_n,\dots,\theta'_1)F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n)\Big\} \tag{4.44}$$

There are similar contributions at $\theta = \theta_j$ for some $j \geq 2$, with the role of $\theta_j$ and $\theta'_1$ exchanged. After integrating over all the $\theta_j$ one could make a change of variables to obtain the same contribution $n$ times. However, one has to keep these residues separately because of the poles of the connected form factors. Making the change of variables only in the regular terms one obtains

$$\begin{aligned}
&n\mathrm{e}^{-imx\sum_{j=2}^n \sinh\theta'_j - mR\cosh\theta'_1 - mt\sum_{j=2}^n\cosh\theta'_j} F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n) F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2) \\
&\times \Big[\big(\prod_{j=2}^n S(\theta'_1-\theta'_j) - 1\big)\big(imx\cosh\theta'_1 - m(R-t)\sinh\theta'_1\big) + imL\cosh\theta'_1\Big]
\end{aligned} \tag{4.45}$$

The $O(L)$ term of (4.45) term can be transformed in the $L \to \infty$ limit into

$$\begin{aligned}
&\frac{1}{(n-1)!} mL \int \frac{d\theta'_1}{2\pi} \cosh\theta'_1 \mathrm{e}^{-mR\cosh\theta'_1} \\
&\times \Big(\int \frac{d\theta'_2}{2\pi}\dots\frac{d\theta'_n}{2\pi} F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n) F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2) \mathrm{e}^{-imx\sum_{j=2}^n\sinh\theta'_j - mt\sum_{j=2}^n\cosh\theta'_j}\Big) \\
&- \frac{1}{(n-1)!}\int \frac{d\theta'_2}{2\pi}\dots\frac{d\theta'_n}{2\pi} F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n) F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2)\Big(\sum_{j=2}^n \mathrm{e}^{-mR\cosh\theta'_j}\Big) \\
&\times \mathrm{e}^{-imx\sum_{j=2}^n\sinh\theta'_j - mt\sum_{j=2}^n\cosh\theta'_j}
\end{aligned} \tag{4.46}$$

The subtraction of the last term takes into account the exclusion principle $\theta'_1 \neq \theta'_j$ for $j = 2\dots n$, which is already present at the level of quantum numbers. The first term in (4.46) gets exactly canceled by $Z_1 C_{0,n-1}$ leaving only the second one which is finite as $L \to \infty$.

The $O(L^0)$ terms of (4.45) are regular, therefore it is allowed to replace the summation over the rapidities with the appropriate integral.

Putting everything together, the net result is

$$\begin{aligned}
D_{1n} =& \frac{1}{n!}\int_{C_+}\frac{d\theta}{2\pi}\int\frac{d\theta'_1}{2\pi}\dots\frac{d\theta'_n}{2\pi} F^{\mathcal{O}_1}_{n+1}(\theta+i\pi,\theta'_1,\dots,\theta_n) F^{\mathcal{O}_2}_{n+1}(\theta+i\pi,\theta'_n,\dots,\theta'_1) \\
&\times \mathrm{e}^{imx(\sinh\theta - \sum_j\sinh\theta'_j) - m(R-t)\cosh\theta - mt\sum_j\cosh\theta'_j} \\
&+ \frac{1}{(n-1)!}\int\frac{d\theta'_1}{2\pi}\dots\frac{d\theta'_n}{2\pi}\mathrm{e}^{-imx\sum_{j=2}^n\sinh\theta'_j - mR\cosh\theta'_1 - mt\sum_{j=2}^n\cosh\theta'_j}\big(\prod_{j=2}^n S(\theta'_1-\theta'_j) - 1\big) \\
&\times (mx\cosh\theta'_1 + im(R-t)\sinh\theta'_1) F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n) F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2) \\
&+ \frac{1}{n!}\int\frac{d\theta'_1}{2\pi}\dots\frac{d\theta'_n}{2\pi}\Bigg\{\Big[\mathrm{e}^{-imx\sum_{j=2}^n\sinh\theta'_j - mR\cosh\theta'_1 - mt\sum_{j=2}^n\cosh\theta'_j} \\
&\times \big(F^{\mathcal{O}_1}_{n+1,sc}(\theta'_1|\theta'_1,\dots,\theta'_n) F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2) + F^{\mathcal{O}_2}_{n+1,sc}(\theta'_1|\theta'_n,\dots,\theta'_1) F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n)\big)\Big] \\
&+ \Big[\theta'_1 \leftrightarrow \theta'_j \text{ for } j = 2..n\Big]\Bigg\} \\
&- \frac{1}{(n-1)!}\int\frac{d\theta'_2}{2\pi}\dots\frac{d\theta'_n}{2\pi} F^{\mathcal{O}_1}_{n-1}(\theta'_2,\dots,\theta'_n) F^{\mathcal{O}_2}_{n-1}(\theta'_n,\dots,\theta'_2)\Big(\sum_{j=2}^n\mathrm{e}^{-mR\cosh\theta'_j}\Big) \\
&\times \mathrm{e}^{-imx\sum_{j=2}^n\sinh\theta'_j - mt\sum_{j=2}^n\cosh\theta'_j}
\end{aligned} \tag{4.47}$$

### 4.4 Life is not that simple: $D_{22}$

Now we turn to the evaluation of the 2-particle – 2-particle contribution to the thermal correlator. The novel feature of this contribution is that the diagonal terms must be separated from the non-diagonal ones, since the four-particle form factors (in contrast to the two-particle one that appears in $D_{11}$) have nonzero residues for the kinematical poles according to (2.10). When evaluated at the diagonal, these singularities are eliminated but result in an ambiguity of the diagonal matrix element, which was discussed in much detail in [45]. Once this complication is attended to, the evaluation proceeds similarly to that of $D_{12}$.

According to the general formalism outlined in section 2.4 we can write

$$D_{22} = \lim_{L\to\infty} \tilde{D}_{22} \tag{4.48}$$

where

$$\tilde{D}_{22} = C_{22} - Z_1 C_{11} + (Z_1^2 - Z_2)C_{00} \tag{4.49}$$

Using

$$\tilde{D}_{11} = C_{11} - Z_1 C_{00} \tag{4.50}$$

gives

$$\tilde{D}_{22} = C_{22} - Z_1 \tilde{D}_{11} - Z_2 C_{00} \tag{4.51}$$

The new contribution is

$$\begin{aligned} C_{22} &= \sum_{I_1>I_2}\sum_{J_1>J_2} \langle \{I_1,I_2\}\,|\mathcal{O}_1|\,\{J_1,J_2\}\rangle_L \langle \{J_1,J_2\}\,|\mathcal{O}_2|\,\{I_1,I_2\}\rangle_L \\ &\times\ K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2') \end{aligned} \tag{4.52}$$

where

$$\begin{aligned} K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2') &= \mathrm{e}^{imx(\sinh\theta_1+\sinh\theta_2-\sinh\theta_1'-\sinh\theta_2')}\mathrm{e}^{-m(R-t)(\cosh\theta_1+\cosh\theta_2)} \\ &\times\ \mathrm{e}^{-mt(\cosh\theta_1'+\cosh\theta_2')} \end{aligned} \tag{4.53}$$

We can separate the sum into diagonal and non-diagonal part:

$$\sum_{I_1>I_2}\sum_{J_1>J_2} = \sum_{I_1>I_2} (\text{terms with } \{J_1,J_2\}=\{I_1,I_2\}) + \sum_{I_1>I_2}\sum_{J_1>J_2}{}' \tag{4.54}$$

where the prime means that $\{J_1,J_2\}\neq\{I_1,I_2\}$.

### 4.4.1 Evaluating $Z_2$

First of all, we need the two-point contribution to the partition function. This is easy to obtain:

$$Z_2 = \sum_{I_1<I_2} \mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)} \left(\frac{1}{2}\sum_{I_1,I_2} - \frac{1}{2}\sum_{I_1=I_2}\right) \mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)} \tag{4.55}$$

$$\tag{4.56}$$

To convert the sums to integrals, we need the two-particle density of states

$$\rho_2(\theta_1,\theta_2) = m^2L^2\cosh\theta_1\cosh\theta_2 + mL(\cosh\theta_1+\cosh\theta_2)\varphi(\theta_1-\theta_2) \tag{4.57}$$

and also the density of states on the diagonal $I_1 = I_2$ which can be obtained as the derivative of the degenerate ($\theta_1 = \theta_2$) Bethe-Yang quantization condition as follows[2]

$$\begin{aligned} Q_d(\theta_1) &= mL\sinh\theta_1 = 2\pi I_1 \\ Q_d'(\theta_1) &= mL\cosh\theta_1 = \rho_1(\theta_1) \end{aligned} \tag{4.58}$$

i.e. it coincides with the one-particle density $\rho_1$. The result is

$$Z_2 = \frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\rho_2(\theta_1,\theta_2)\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)} - \frac{1}{2}\int\frac{d\theta}{2\pi}\rho_1(\theta)\mathrm{e}^{-2mR\cosh\theta} \tag{4.59}$$

[2] Note that due to (2.19) $\delta(0) = 0$.

### 4.4.2 The diagonal sum

Using (2.26), the diagonal matrix element is

$$\langle\{I_1,I_2\}|\mathcal{O}|\{I_1,I_2\}\rangle_L = \frac{F_{4s}^{\mathcal{O}}(\theta_1,\theta_2)+\rho_1(\theta_1)F_{2c}^{\mathcal{O}}+\rho_1(\theta_2)F_{2c}^{\mathcal{O}}+\rho_2(\theta_1,\theta_2)\langle\mathcal{O}\rangle}{\rho_2(\theta_1,\theta_2)} \tag{4.60}$$

Substituting these into the diagonal sum

$$\sum_{I_1>I_2}\langle\{I_1,I_2\}|\mathcal{O}_1|\{I_1,I_2\}\rangle_L\langle\{I_1,I_2\}|\mathcal{O}_2|\{I_1,I_2\}\rangle_L\,\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)} \tag{4.61}$$

we have terms that can be ordered by the number of $F_{4s}$ factors they contain. The term which contains two $F_{4s}$ factors can be written as

$$\begin{aligned}&\sum_{I_1>I_2}\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)^2}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)=\\&\left(\frac{1}{2}\sum_{I_1,I_2}-\sum_{I_1=I_2}\right)\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)^2}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)=\\&\frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)\\&-\int\frac{d\theta_1}{2\pi}\frac{\mathrm{e}^{-2mR(\cosh\theta_1)}mL\cosh\theta_1}{\rho_2(\theta_1,\theta_1)^2}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_1)F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_1)\end{aligned} \tag{4.62}$$

where we added and subtracted the diagonal $\theta_1=\theta_2$. We used that the density of diagonal ($\theta_1=\theta_2$) two-particle states is given by $mL\cosh\theta_1$. The first term is $O(L^{-2})$ while the second is $O(L^{-3})$ and so they vanish as $L\to\infty$.

There are two terms containing a single $F_{4s}$. One of them is

$$\frac{1}{2}\sum_{I_1,I_2}\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)^2}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)\left(\rho_1(\theta_1)F_{2c}^{\mathcal{O}_2}+\rho_1(\theta_2)F_{2c}^{\mathcal{O}_2}+\rho_2(\theta_1,\theta_2)\langle\mathcal{O}_2\rangle\right) \tag{4.63}$$

and the other can be obtained by interchanging $\mathcal{O}_1$ and $\mathcal{O}_2$. In writing the above formula we already included the $I_1=I_2$ diagonal, using again that it is suppressed by an $L^{-1}$ factor. For $L\to\infty$ we get

$$\frac{1}{2}\int\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)\langle\mathcal{O}_2\rangle \tag{4.64}$$

Similarly, the other term yields

$$\frac{1}{2}\int\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)\langle\mathcal{O}_1\rangle \tag{4.65}$$

The terms without $F_{4s}$ give

$$\begin{aligned}&\sum_{I_1>I_2}\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)^2}\left(\rho_1(\theta_1)F_{2c}^{\mathcal{O}_1}+\rho_1(\theta_2)F_{2c}^{\mathcal{O}_1}+\rho_2(\theta_1,\theta_2)\langle\mathcal{O}_1\rangle\right)\\&\times\left(\rho_1(\theta_1)F_{2c}^{\mathcal{O}_2}+\rho_1(\theta_2)F_{2c}^{\mathcal{O}_2}+\rho_2(\theta_1,\theta_2)\langle\mathcal{O}_2\rangle\right)\end{aligned} \tag{4.66}$$

We can replace

$$\sum_{I_1>I_2}\to\frac{1}{2}\sum_{I_1,I_2}-\frac{1}{2}\sum_{I_1=I_2} \tag{4.67}$$

and after converting the sums to integrals we obtain

$$
\begin{aligned}
& \frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\frac{\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}}{\rho_2(\theta_1,\theta_2)}\Big(\rho_1(\theta_1)F_{2c}^{\mathcal{O}_1}+\rho_1(\theta_2)F_{2c}^{\mathcal{O}_1}+\rho_2(\theta_1,\theta_2)\left\langle\mathcal{O}_1\right\rangle\Big) \\
\times & \Big(\rho_1(\theta_1)F_{2c}^{\mathcal{O}_2}+\rho_1(\theta_2)F_{2c}^{\mathcal{O}_2}+\rho_2(\theta_1,\theta_2)\left\langle\mathcal{O}_2\right\rangle\Big) - \text{diagonal term} \\
= & \int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\Bigg[\frac{1}{2}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}F_{2c}^{\mathcal{O}_1}F_{2c}^{\mathcal{O}_2}\frac{(\cosh\theta_1+\cosh\theta_2)^2}{\cosh\theta_1\cosh\theta_2} \\
+ & mL\cosh\theta_1\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\Big(\left\langle\mathcal{O}_1\right\rangle F_{2c}^{\mathcal{O}_2}+\left\langle\mathcal{O}_2\right\rangle F_{2c}^{\mathcal{O}_1}\Big) \\
+ & \frac{1}{2}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\left(m^2L^2\cosh\theta_1\cosh\theta_2+mL(\cosh\theta_1+\cosh\theta_2)\varphi(\theta_1-\theta_2)\right)\left\langle\mathcal{O}_1\right\rangle\left\langle\mathcal{O}_2\right\rangle\Bigg] \\
- & \frac{1}{2}\int\frac{d\theta}{2\pi}\mathrm{e}^{-2mR\cosh\theta}\Big[mL\cosh\theta\left\langle\mathcal{O}_1\right\rangle\left\langle\mathcal{O}_2\right\rangle+2\Big(\left\langle\mathcal{O}_1\right\rangle F_{2c}^{\mathcal{O}_2}+\left\langle\mathcal{O}_2\right\rangle F_{2c}^{\mathcal{O}_1}\Big)\Big] \qquad (4.68)
\end{aligned}
$$

where we dropped terms that vanish as $L\to\infty$. This has terms which diverge in the limit; however, we must now add the "counter terms"

$$
\begin{aligned}
-Z_2C_{00} = & \frac{1}{2}\int\frac{d\theta}{2\pi}mL\cosh\theta\mathrm{e}^{-2mR\cosh\theta}\left\langle\mathcal{O}_1\right\rangle\left\langle\mathcal{O}_2\right\rangle \qquad (4.69) \\
& -\frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\big[m^2L^2\cosh\theta_1\cosh\theta_2 \\
& +mL(\cosh\theta_1+\cosh\theta_2)\varphi(\theta_1-\theta_2)\big]\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\left\langle\mathcal{O}_1\right\rangle\left\langle\mathcal{O}_2\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
-Z_1D_{11} = & -Z_1\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}F_2^{\mathcal{O}_1}(\theta_1+i\pi,\theta_2)F_2^{\mathcal{O}_2}(\theta_1,\theta_2+i\pi) \\
& \times\mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2)}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2} \\
& -Z_1\int\frac{d\theta}{2\pi}\Big(F_{2c}^{\mathcal{O}_1}\left\langle\mathcal{O}_2\right\rangle+F_{2c}^{\mathcal{O}_2}\left\langle\mathcal{O}_1\right\rangle\Big)\mathrm{e}^{-mR\cosh\theta} \qquad (4.70)
\end{aligned}
$$

These cancel all the divergences leaving us with the final result for the diagonal contribution:

$$
\begin{aligned}
\mathrm{D}_{22}^{(\mathrm{diag})} = & \frac{1}{2}\int\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\Big(F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)\left\langle\mathcal{O}_2\right\rangle+F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)\left\langle\mathcal{O}_1\right\rangle\Big) \\
+ & \frac{1}{2}\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}F_{2c}^{\mathcal{O}_1}F_{2c}^{\mathcal{O}_2}\frac{(\cosh\theta_1+\cosh\theta_2)^2}{\cosh\theta_1\cosh\theta_2} \\
- & \int\frac{d\theta}{2\pi}\mathrm{e}^{-2mR\cosh\theta}\Big(\left\langle\mathcal{O}_1\right\rangle F_{2c}^{\mathcal{O}_2}+\left\langle\mathcal{O}_2\right\rangle F_{2c}^{\mathcal{O}_1}\Big) \qquad (4.71)
\end{aligned}
$$

However, the first piece of $-Z_1D_{11}$:

$$
-Z_1\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}F_2^{\mathcal{O}_1}(\theta_1+i\pi,\theta_2)F_2^{\mathcal{O}_2}(\theta_1,\theta_2+i\pi)\mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2)}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2} \qquad (4.72)
$$

is not canceled by the diagonal part. We now turn to the evaluation of the non-diagonal contribution, which does eliminate this last divergence, as explicitly demonstrated in appendix C.2.

### 4.4.3 Evaluating the non-diagonal part

Now we must evaluate

$$
\begin{aligned}
C_{22}^{(\mathrm{nondiag})} = & \sum_{I_1>I_2}\sum_{J_1>J_2}{}'\left\langle\{I_1,I_2\}\right|\mathcal{O}_1\left|\{J_1,J_2\}\right\rangle_L\left\langle\{J_1,J_2\}\right|\mathcal{O}_2\left|\{I_1,I_2\}\right\rangle_L \\
& \times K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2') \qquad (4.73)
\end{aligned}
$$

where

$$K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2') = \mathrm{e}^{imx(\sinh\theta_1+\sinh\theta_2-\sinh\theta_1'-\sinh\theta_2')}\mathrm{e}^{-m(R-t)(\cosh\theta_1+\cosh\theta_2)} \times\mathrm{e}^{-mt(\cosh\theta_1'+\cosh\theta_2')} \tag{4.74}$$

Using (2.24), the matrix elements are of the form

$$\begin{aligned}\langle\{I_1,I_2\}|\mathcal{O}_1|\{J_1,J_2\}\rangle_L &= \frac{F_4^{\mathcal{O}_1}(\theta_2+i\pi,\theta_1+i\pi,\theta_1',\theta_2')}{\rho_2(\theta_1,\theta_2)^{1/2}\rho_2(\theta_1',\theta_2')^{1/2}}\\ \langle\{J_1,J_2\}|\mathcal{O}_2|\{I_1,I_2\}\rangle_L &= \frac{F_4^{\mathcal{O}_2}(\theta_2'+i\pi,\theta_1'+i\pi,\theta_1,\theta_2)}{\rho_2(\theta_1,\theta_2)^{1/2}\rho_2(\theta_1',\theta_2')^{1/2}}\end{aligned} \tag{4.75}$$

The quantization conditions read

$$\begin{aligned}Q_1(\theta_1,\theta_2) &= mL\sinh\theta_1+\delta(\theta_1-\theta_2)=2\pi I_1\\ Q_2(\theta_1,\theta_2) &= mL\sinh\theta_2+\delta(\theta_2-\theta_1)=2\pi I_2\end{aligned} \tag{4.76}$$

and

$$\begin{aligned}Q_{1'}(\theta_1',\theta_2') &= mL\sinh\theta_1'+\delta(\theta_1'-\theta_2')=2\pi J_1\\ Q_{2'}(\theta_1',\theta_2') &= mL\sinh\theta_2'+\delta(\theta_2'-\theta_1')=2\pi J_2\end{aligned} \tag{4.77}$$

Now we can write

$$\langle\{I_1,I_2\}|\mathcal{O}_1|\{J_1,J_2\}\rangle_L\langle\{J_1,J_2\}|\mathcal{O}_2|\{I_1,I_2\}\rangle_L K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2') \tag{4.78}$$

$$= \oint\oint_{C_{J_1J_2}}\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}F_4^{\mathcal{O}_1}(\theta_2+i\pi,\theta_1+i\pi,\theta_1',\theta_2')F_4^{\mathcal{O}_2}(\theta_2'+i\pi,\theta_1'+i\pi,\theta_1,\theta_2) \times\frac{K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1',\theta_2')}+1\right)} \tag{4.79}$$

where $C_{J_1J_2}$ is a multi-contour surrounding the solution of (4.77). When we open the multi-contours to surround the real axes we encounter new singularities. These can be classified as follows:

1. $QF$-singularities: partly from the $Q$s, partly from the $F$s:

$$\theta_1'=\theta_1 \quad\text{and}\quad Q_{2'}(\theta_1',\theta_2')=2\pi J_2 \tag{4.80}$$
$$\theta_1'=\theta_2 \quad\text{and}\quad Q_{2'}(\theta_1',\theta_2')=2\pi J_2 \tag{4.81}$$
$$\theta_2'=\theta_1 \quad\text{and}\quad Q_{1'}(\theta_1',\theta_2')=2\pi J_1 \tag{4.82}$$
$$\theta_2'=\theta_2 \quad\text{and}\quad Q_{1'}(\theta_1',\theta_2')=2\pi J_1 \tag{4.83}$$

2. $FF$-singularities: come from the $F$s

$$\theta_1' = \theta_2'=\theta_1 \tag{4.84}$$
$$\theta_1' = \theta_2'=\theta_2 \tag{4.85}$$

3. (Spurious) $QQ$-singularities: these result from

$$\theta_1'=\theta_1 \quad\text{and}\quad \theta_2'=\theta_2 \tag{4.86}$$
$$\theta_1'=\theta_2 \quad\text{and}\quad \theta_2'=\theta_1 \tag{4.87}$$

(it turns out that eventually these do not give any contributions in the $L\to\infty$ limit).

Since the evaluation of these contributions is the same as for $D_{12}$, the details are relegated to appendix C. The upshot is that

$$\sum_{J_1>J_2}{}'\oint\oint_{C_{J_1J_2}}\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} = \frac{1}{2}\oint\oint_C\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi} - (FF\text{ terms}) - (QF\text{ terms}) \tag{4.88}$$

where $C$ is the open multi-contour. Here we also used the fact that there are no singularities at $J_1=J_2$ because the form factors vanish.

### 4.4.4 End result for $D_{22}$

Putting together the results (C.15), (C.22) and (C.29) of appendix C with (4.71) one obtains

$$
\begin{aligned}
& D_{22} = \qquad\qquad (4.89)\\
& \frac{1}{4}\iint \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\iint_{C_{++}}\frac{d\theta_1'}{2\pi}\frac{d\theta_2'}{2\pi}F_4^{\mathcal{O}_1}(\theta_2+i\pi,\theta_1+i\pi,\theta_1',\theta_2')F_4^{\mathcal{O}_2}(\theta_2'+i\pi,\theta_1'+i\pi,\theta_1,\theta_2)\\
& \times K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2')\\
+\ & \frac{1}{2}\iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}\left(F_{4s}^{\mathcal{O}_1}(\theta_1,\theta_2)\left\langle\mathcal{O}_2\right\rangle+F_{4s}^{\mathcal{O}_2}(\theta_1,\theta_2)\left\langle\mathcal{O}_1\right\rangle\right)\\
+\ & \frac{1}{2}\iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\mathrm{e}^{-mR(\cosh\theta_1+\cosh\theta_2)}F_{2c}^{\mathcal{O}_1}F_{2c}^{\mathcal{O}_2}\frac{(\cosh\theta_1+\cosh\theta_2)^2}{\cosh\theta_1\cosh\theta_2}\\
-\ & \int\frac{d\theta}{2\pi}\mathrm{e}^{-2mR\cosh\theta}\left(\left\langle\mathcal{O}_1\right\rangle F_{2c}^{\mathcal{O}_2}+\left\langle\mathcal{O}_2\right\rangle F_{2c}^{\mathcal{O}_1}\right)\\
+\ & \iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\int\frac{d\theta_2'}{2\pi}F_2^{O_1}(\theta_2+i\pi,\theta_2')F_2^{O_2}(\theta_2'+i\pi,\theta_2)\\
& \times\mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_2')}\mathrm{e}^{-mR\cosh\theta_1}\mathrm{e}^{-m(R-t)\cosh\theta_2}\mathrm{e}^{-mt\cosh\theta_2'}\\
& \times\big((mx\cosh\theta_1-imt\sinh\theta_1)(1-S(\theta_2'-\theta_1)S(\theta_1-\theta_2))+\varphi(\theta_1-\theta_2')S(\theta_2'-\theta_1)S(\theta_1-\theta_2)\big)\\
+\ & \frac{1}{2}\iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\int\frac{d\theta_2'}{2\pi}\Big[\mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_2')}\mathrm{e}^{-mR\cosh\theta_1}\mathrm{e}^{-m(R-t)\cosh\theta_2}\mathrm{e}^{-mt\cosh\theta_2'}\\
& \times\left(F_2^{O_1}(\theta_2+i\pi,\theta_2')F_{4sc}^{\mathcal{O}_2}(\theta_2',\theta_1|\theta_1,\theta_2)+F_2^{O_2}(\theta_2'+i\pi,\theta_2)F_{4sc}^{\mathcal{O}_1}(\theta_2',\theta_1|\theta_1,\theta_2)\right)\\
& +(\theta_1\leftrightarrow\theta_2)\Big]\\
-\ & \iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\Big[\mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_1)}\mathrm{e}^{-m(R-t)(\cosh\theta_1+\cosh\theta_2)}\mathrm{e}^{-2mt\cosh\theta_1}\\
& \times S(\theta_1-\theta_2)F_2^{O_1}(\theta_2+i\pi,\theta_1)F_2^{O_2}(\theta_1+i\pi,\theta_2)\\
& +(\theta_1\leftrightarrow\theta_2)\Big]\\
-\ & \iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}F_2^{O_1}(\theta_1+i\pi,\theta_2)F_2^{O_2}(\theta_2+i\pi,\theta_1)\mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2)-m(2R-t)\cosh\theta_1-mt\cosh\theta_2}
\end{aligned}
$$

where the function $F_{4sc}$ is defined in (C.16), and $C_{++}$ denotes the integration contour specified in (3.31). All the other integrals are taken over real values of their variables.

## 4.5 Discussion of the proposal of LeClair and Mussardo

In [34] LeClair and Mussardo introduced a regularization scheme for finite temperature correlation functions. The two main assumptions of the proposal are that the spectral expansion should be built using the zero-temperature form factors, and that the only effect of finite temperature is an appropriate modification (dressing) of the statistical weight functions and the one-particle energies and momenta. In the case of one-point function the proposed formula was proven to be correct up to the third order in [45]; an all-orders proof is also possible [46]. However, the two-point function seems to be more problematic [35, 43].

In the following we compare our results to the proposal of [34]. For the two-point functions their formula reads

$$
\begin{aligned}
\left\langle\mathcal{O}(x,t)\mathcal{O}(0,0)\right\rangle^R &= \left(\left\langle\mathcal{O}\right\rangle^R\right)^2+\sum_{N=1}^{\infty}\frac{1}{N!}\sum_{\sigma_i=\pm1}\int\frac{d\theta_1}{2\pi}\dots\frac{d\theta_N}{2\pi}\left[\prod_{j=1}^N f_{\sigma_j}(\theta_j)\mathrm{e}^{-\sigma_j(t\varepsilon_j+ixk_j)}\right]\\
&\quad\times\left|\left\langle 0|\mathcal{O}|\theta_1\dots\theta_N\right\rangle_{\sigma_1\dots\sigma_N}\right|^2 \qquad (4.90)
\end{aligned}
$$

where $f_{\sigma_j}(\theta_j)=1/(1+e^{-\sigma_j\varepsilon(\theta_j)})$, $\varepsilon_j=\varepsilon(\theta_j)/R$ and $k_j=k(\theta_j)$ with $\varepsilon(\theta)$ being the solution of the

TBA equation

$$\epsilon(\theta) = mR\cosh\theta - \int \frac{d\theta'}{2\pi}\varphi(\theta-\theta')\log(1+\mathrm{e}^{-\epsilon(\theta')}) \tag{4.91}$$

and $k(\theta)$ is given by

$$\begin{aligned} k(\theta) &= m\sinh\theta + \int d\theta' \delta(\theta-\theta')\rho_1(\theta') \\ 2\pi\rho_1(\theta)(1+\mathrm{e}^{\varepsilon(\theta)}) &= m\cosh\theta + \int d\theta' \varphi(\theta-\theta')\rho_1(\theta') \end{aligned} \tag{4.92}$$

The form factors appearing in (4.90) are defined by

$$\langle 0|\mathcal{O}|\theta_1\dots\theta_N\rangle_{\sigma_1\dots\sigma_N} = F_N^{\mathcal{O}}(\theta_1 - i\pi\tilde{\sigma}_1,\dots,\theta_N - i\pi\tilde{\sigma}_N) \qquad \tilde{\sigma}_j = (1-\sigma_j)/2 \in \{0,1\}$$

The interpretation of the series (4.90) is as follows: the excitations with $\sigma_j = +1$ or $\sigma_j = -1$ represent particles or holes over an infinite volume thermal state. Therefore the statistical weight functions and the one-particle energies and momenta are given by the dressed values as calculated in TBA.

If the (4.90) series expression were correct, then a systematic double expansion in terms of $\mathrm{e}^{-mt}$ and $\mathrm{e}^{-m(R-t)}$ should reproduce our results. Indeed, the first few terms indicate that this might be true. It was already pointed out in [57] that to the lowest order in $\mathrm{e}^{-mR}$ the term $N=2$ reproduces our $D_{02}+D_{11}+D_{20}$. Moreover, in the case of our $D_{12}$, the last line of formula (4.22) suggests the dressing

$$e^{-mt\cosh\theta + imx\sinh\theta} \quad \to \quad \frac{e^{-mt\cosh\theta+imx\sinh\theta}}{1+e^{-mR\cosh\theta}} + \mathcal{O}(e^{-2mR})$$

for the exponential factors in $D_{01}$. This pattern repeats itself and similar contributions can be found in $D_{1n}$ (4.47), which suggest a dressing of the factors in $D_{0,n-1}$. However, the situation is more complicated as we consider higher order terms.

First of all observe, that the formula (4.90) is not well-defined for $N \geq 3$. There appear second order poles whenever the rapidity of a particle approaches the rapidity of a hole, and in the original work [34] it is not explained how to integrate over these singularities. Note that it was the evaluation of these ill-defined terms which required a lot of effort in our evaluation of the two-point function.

Based on the form of our results (4.22), (4.47) and (4.89) it seems unlikely, that any regularized form of (4.90) would be correct. However, at present we cannot make any definitive statement about this issue. The inspection of higher order terms ($D_{nm}$ with $n,m>2$) might decide whether there exists a neat formula for the two-point function, possibly with a structure similar to (4.90) but with different dressing prescriptions. This problem is left for future work.

## 5 Second order form factor perturbation theory

As a further application of the framework presented here, we show how to derive the main results of the paper [50] on second order form factor perturbation theory using the present formalism. We simplify the presentation by considering a theory with a single massive particle in its spectrum instead of the double sine-Gordon theory treated in [50]. Consider modifying the Hamiltonian of an integrable model as follows:

$$H_{\text{nonintegrable}} = H_{\text{integrable}} + \lambda \int dx \Psi(t,x) \tag{5.1}$$

where $\Psi$ denotes a local (Lorentz scalar) field which breaks integrability. Corrections that are first order in $\lambda$ were derived in [51], but when evaluating the second order one encounters the same difficulties with disconnected terms as in the case of the thermal two-point function. The principle of the solution to this problem is the same as for the thermal correlator: we perform perturbation theory in finite volume, express the quantities we are interested in and then take the limit $L \to \infty$. In the approach of [50] it was necessary to compute some part of the discrete sum over the finite volume quantum numbers explicitly; we show that this can be greatly simplified by applying the residue methods of the present work.

The general perturbation theory formula for second order corrections to energy levels is

$$\delta E_i = \sum_{k\neq i} \frac{|\langle i\,|H_1|\,k\rangle|^2}{E_i^{(0)} - E_k^{(0)}} \quad , \quad H_1 = \lambda \int dx \Psi(t,x) \tag{5.2}$$

therefore the correction to the vacuum level can be written as

$$\delta E_0 = -\lambda^2 L^2 \sum_{k\neq 0} \frac{\left|\left\langle 0 \left| : \exp i\frac{\beta}{2}\varphi(0,0) : \right| k \right\rangle_L\right|^2}{E_k^{(0)} - E_0^{(0)}} \tag{5.3}$$

The summation goes over all excited states in the spectrum (with zero total momentum selected for by translational invariance), which can be described using the Bethe-Yang picture of section 2.3. The leading contribution is given by the state containing a single stationary particle, and can be written as

$$\delta E_0(L) = -\lambda^2 L^2 \frac{\left|\left\langle 0 \left| : \exp i\frac{\beta}{2}\varphi(0,0) : \right| \{0\} \right\rangle_L\right|^2}{m} + O\left(\mathrm{e}^{-\mu L}\right) \tag{5.4}$$

Using the relation (2.24) we obtain

$$\delta E_0(L) = -\lambda^2 L^2 \frac{\left|F_1^\Psi\right|^2}{\rho_1(0) m} + O\left(\mathrm{e}^{-\mu L}\right) = -\lambda^2 L \frac{\left|F_1^\Psi\right|^2}{m^2} + O\left(\mathrm{e}^{-\mu L}\right) \tag{5.5}$$

which results in the following shift of the bulk energy density

$$\delta \mathcal{E} = -\lambda^2 \frac{\left|F_1^\Psi\right|^2}{m^2} \tag{5.6}$$

Next we are interested in the correction to the particle mass. This can be obtained by evaluating the correction to the first zero-momentum excited level in the finite volume system and then taking the limit

$$\delta m = \lim_{L\to\infty} \delta E_1(L) - \delta E_0(L) \tag{5.7}$$

The correction to the first level can be written as

$$\begin{aligned} \delta E_1(L) &= \lambda^2 L^2 \frac{\left|\left\langle \{0\} \left| : \exp i\frac{\beta}{2}\varphi(0,0) : \right| 0 \right\rangle_L\right|^2}{m} + \lambda^2 L^2 \sum_I \frac{\left|\left\langle \{0\} \left| : \exp i\frac{\beta}{2}\varphi(0,0) : \right| \{I,-I\} \right\rangle_L\right|^2}{m - 2m\cosh\theta} \\ &= \lambda^2 L^2 \frac{|F_1|^2}{\rho_1(0) m} - \lambda^2 L^2 \sum_\theta \frac{F_3(i\pi,\theta,-\theta) F_3(\theta,-\theta,i\pi)}{\rho_1(0)\rho_2(\theta,-\theta)(2m\cosh\theta - m)} \end{aligned} \tag{5.8}$$

(where we omitted the states with three or more particles) where $\theta$ is the solution of (cf. subsection 2.3).

$$Q(\theta) = mL\sinh\theta + \delta(2\theta) = 2\pi I \qquad , \qquad I \in \mathbb{N} + \frac{1}{2} \tag{5.9}$$

and

$$\rho_2(\theta,-\theta) = mL\cosh\theta(mL\cosh\theta + 2\varphi(2\theta)) \tag{5.10}$$

Extending the sum over $\theta$ to negative values and performing the residue trick we get

$$\begin{aligned} \delta E_1(L) &= \lambda^2 L \frac{|F_1|^2}{m^2} \\ &\quad + \frac{\lambda^2 L}{2} \sum_{I\in\mathbb{Z}+\frac{1}{2}} \oint_{C_I} \frac{d\theta}{2\pi} \frac{1}{\mathrm{e}^{iQ(\theta)}+1} \frac{\tilde{\rho}_2(\theta)}{\rho_2(\theta,-\theta)} \frac{F_3(i\pi,\theta,-\theta) F_3(-\theta+i\pi,\theta+i\pi,0)}{m(2m\cosh\theta - m)} \\ &= \lambda^2 L \frac{|F_1|^2}{m^2} + \frac{\lambda^2}{2} \sum_{I\in\mathbb{Z}+\frac{1}{2}} \oint_{C_I} \frac{d\theta}{2\pi} \frac{1}{\mathrm{e}^{iQ(\theta)}+1} \frac{F_3(i\pi,\theta,-\theta) F_3(-\theta+i\pi,\theta+i\pi,0)}{m^3(2\cosh\theta - 1)\cosh\theta} \end{aligned} \tag{5.11}$$

where

$$\tilde{\rho}_2(\theta) = mL\cosh\theta + 2\varphi(2\theta) \tag{5.12}$$

is nothing else than the density of two-particle states with zero total momentum. Using (2.10), the form factor has the following singularity at $\theta = 0$

$$|F_3(i\pi,\theta,-\theta)|^2 \sim \frac{16\,|F_1|^2}{\theta^2} + O(\theta^0) \tag{5.13}$$

where we also used $S(0) = -1$. Subtracting and adding the singular term at the origin results in

$$\begin{aligned}\delta E_1(L) \;=\; & \lambda^2 L\frac{|F_1|^2}{m^2} + \lambda^2\frac{1}{2}\sum_{I\in\mathbb{Z}+\frac{1}{2}}\oint_{C_I}\frac{d\theta}{2\pi}\frac{1}{\mathrm{e}^{iQ(\theta)}+1}\Bigg[\frac{F_3(i\pi,\theta,-\theta)F_3(-\theta+i\pi,\theta+i\pi,0)}{m^3(2\cosh\theta-1)\cosh\theta} \\ & -\frac{16\,|F_1|^2}{m^3\sinh^2\theta\cosh\theta}\Bigg] + \lambda^2\frac{1}{2}\sum_{I\in\mathbb{Z}+\frac{1}{2}}\oint_{C_I}\frac{d\theta}{2\pi}\frac{1}{\mathrm{e}^{iQ(\theta)}+1}\frac{16\,|F_1|^2}{m^3\sinh^2\theta\cosh\theta}\end{aligned} \tag{5.14}$$

We can then open the contours to surround the real axis. In the first term, only the upper contour contributes in the infinite volume limit, and one can also take $\epsilon \to 0$. In the second term, we can open the contour, but we must also subtract the contribution of the double pole at the origin since that is not included in the original sum:

$$\begin{aligned}&\frac{\lambda^2}{2}\sum_{I\in\mathbb{Z}+\frac{1}{2}}\oint_{C_I}\frac{d\theta}{2\pi}\frac{1}{\mathrm{e}^{iQ(\theta)}+1}\frac{16\,|F_1|^2}{m^3\sinh^2\theta\cosh\theta} \\ &-\frac{\lambda^2}{2}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\frac{16\,|F_1|^2}{m^3\sinh^2(\theta+i\epsilon)\cosh(\theta+i\epsilon)} - \frac{\lambda^2}{2}\oint_{C_0}\frac{d\theta}{2\pi}\frac{1}{\mathrm{e}^{iQ(\theta)}+1}\frac{16\,|F_1|^2}{m^3\sinh^2\theta\cosh\theta}\end{aligned} \tag{5.15}$$

The first integral is

$$-\frac{\lambda^2}{2}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\frac{16\,|F_1|^2}{m^3\sinh^2(\theta+i\epsilon)\cosh(\theta+i\epsilon)} = \lambda^2\frac{16\,|F_1|^2}{4m^3} \tag{5.16}$$

while the second integral is given by the residue theorem as

$$-\frac{\lambda^2}{2}i\frac{\partial}{\partial\theta}\left(\frac{1}{\mathrm{e}^{iQ(\theta)}+1}\frac{16\,|F_1|^2}{m^3\cosh\theta}\right)\Bigg|_{\theta=0} = -\frac{\lambda^2}{2}\frac{16\,|F_1|^2}{m^3}\frac{\mathrm{e}^{iQ(0)}Q'(0)}{(1+\mathrm{e}^{iQ(0)})^2} \tag{5.17}$$

Using

$$Q(0) = 0 \qquad , \qquad Q'(0) = mL + 2\varphi(0) \tag{5.18}$$

we get

$$\begin{aligned}\delta E_1(L) \;=\; & -\lambda^2 L\frac{|F_1|^2}{m^2} - \lambda^2\int_0^{\infty}\frac{d\theta}{2\pi}\left(\frac{|F_3(i\pi,\theta,-\theta)|^2}{m^3(2\cosh\theta-1)\cosh\theta} - \frac{16\,|F_1|^2}{m^3\sinh^2\theta\cosh\theta}\right) \\ + \; & \lambda^2\frac{16\,|F_1|^2}{4m^3}(1-\varphi(0))\end{aligned} \tag{5.19}$$

Finally, using (5.7) and (5.5) we obtain the mass correction

$$\delta m = -\lambda^2\int_0^{\infty}\frac{d\theta}{2\pi}\left(\frac{|F_3(i\pi,\theta,-\theta)|^2}{m^3(2\cosh\theta-1)\cosh\theta} - \frac{16\,|F_1|^2}{m^3\sinh^2\theta\cosh\theta}\right) + \lambda^2\frac{16\,|F_1|^2}{4m^3}(1-\varphi(0)) \tag{5.20}$$

which agrees with the result in [50]. Note that the leading bulk term drops out from the difference of the energy levels as it indeed should.

The contributions of higher-particle states to the spectral sum can be computed analogously. It is straightforward to verify that the $n$ particle state term always contains a double pole part analogous to the one treated above, which exactly cancels the $n-1$ particle contribution to the vacuum level in (5.3).

## 6 Conclusions and outlook

First let us sum up what has been achieved in this paper. Using the idea of finite volume regularization and multi-dimensional residue techniques we have developed a systematic technique to evaluate the form factor expansion for the finite-temperature two-point function in integrable field theories. In fact, as the examples of the zero-temperature three-point function and of form factor perturbation theory show, the approach can be applied to many problems involving spectral sums with singularities coming from the presence of disconnected terms. Albeit it was expected on general physical grounds, it is an important fact that our calculation demonstrated that the resulting expressions for the correlators (and for the mass gap in the case of FFPT) are well-defined when removing the regulator by taking the infinite volume limit.

For the three-point function it is apparent that the resulting formula is just the proper way of separating the disconnected pieces and can in fact be written down directly by inspection of the infinite volume expression. This is due to the disconnected pieces appearing linearly. However, the other two cases involve the disconnected terms squared, similarly to the case of one-point functions of bulk operators with boundaries [49]. While it is well-known that the resulting terms, naively containing squares of Dirac $\delta$ functions can be regularized in a finite box, the correct result can only be obtained by carefully taking into account that the finite volume spectrum is different from that of a non-interacting system. This was pointed out for the one-point functions in our previous paper [45], and is manifested by the explicit dependence of the results (4.22), (4.89) and (5.20) on the $S$ matrix (directly or via the derivative $\varphi$ of the phase-shift). The LeClair-Mussardo proposal for the one and two-point functions tries to capture this feature by a TBA dressing of the energy and momentum of the finite-temperature quasi-particles. Contrary to the one-point case [45], our result for the two-point function does not confirm their conjectured expression (which is, in any case, eventually ill-defined). Despite some partial indications of resummation, it is not obvious whether the interaction dependence can be summed up to yield some simple dressing prescription.

This leads us to one of the main open questions, namely, to investigate the possibility of such resummation and find out whether there is a way to introduce some sort of dressing prescription to simplify the expansion by systematically combining contributions. At this point this seems to require the evaluation of higher orders, which is in principle straightforward, but an extremely tedious task. Therefore another important (albeit technical) issue is to simplify the method of evaluating the contributions to the expansion.

It is also very important, especially in view of potential applications, to extend the method to non-diagonal theories. While this is in principle straightforward (for the basic ideas cf. [53] in the framework of $O(3)$ model), it would be desirable to have some efficient approach to characterizing the finite-volume form factors of non-diagonal models for general number of particles. Work in this direction is in progress. This is even more important, since at present the most we can show for testing the expansion for the thermal correlator is its internal consistency. Consistency is shown by two facts: (1) that terms divergent for large volumes cancel in the final result order-by-order and (2) that different orders of performing the summation lead to identical results, as demonstrated in for $D_{12}$. These are indeed very nontrivial tests of the calculation, but a physical application of the method would be much better.

Another important issue, especially in view of potential applications to non-relativistic systems along the lines of [58], is the extension to include a nonzero chemical potential. At present it is not entirely clear how to do that, but rewriting the expansion through some partial resummation/dressing procedure could be helpful (in analogy to the way the dependence is introduced into the LeClair-Mussardo formula for the one-point function [34]).

Finally we comment on the relation of our results to the recent work by Essler and Konik [53]. Their finite-volume calculation is essentially the evaluation of $D_{12}$ using the one-particle summation, with an explicit summation of the discrete part instead of a residue trick. However, this approach is very limited: it can only be applied to $D_{1n}$ since all other contributions require summation over states with two (or more) particles and there is no obvious way to perform the discrete sums directly. This is where the multi-dimensional residue method presented here is so powerful since it makes the evaluation of such sums a mechanical (albeit somewhat tedious) exercise.

The infinite-volume regularization method of [53], on the other hand, is plagued by (at least potential, but most likely actual) ambiguities: for states containing more than one particle, regularization by point-splitting in rapidity space is ambiguous (direction-dependent) at the locations in rapidity space where the form factors have either of the two types of disconnected contributions described in subsection 2.3. These ambiguities were analyzed in much detail in our previous paper [45]. Therefore (at least at the present state of art) the only safe method to evaluate multi-particle contributions is by finite volume regularization, and the only systematic way to perform the summations is by using the multi-dimensional residue theorem, i.e. in the framework presented here. As mentioned above, however, it is an important goal to simplify the method of calculation, which could potentially lead to dispensing with these technical requirements in the end.

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# A Multi-dimensional residue formula

Suppose that we have functions $g(z)$, $f_1(z), \dots, f_n(z)$ of $n$ complex variables $z = (z_1, \dots, z_n)$. Let us take a multi-contour $C$ in $\mathbb{C}^n$ (i.e. a direct sum of elementary multi-contours that are defined as direct products of $n$ one-dimensional contours; without loss of generality – due to the linearity of integration – we may suppose that it is a single (i.e. monomial) product contour $C = C_1 \times \dots \times C_n$). Let us suppose that the equations

$$f_k(z) = 0 \qquad k = 1, \dots, n \tag{A.1}$$

have a single solution $z_* = (z_{*1}, \dots, z_{*n})$ such that for each $k$, $z_{k*}$ is inside $C_k$. Then we have the formula

$$\oint_C \frac{dz_1}{2\pi i} \dots \frac{dz_n}{2\pi i} \frac{g(z)}{f_1(z) \dots f_n(z)} = \frac{g(z_*)}{\det \left( \frac{\partial f_k}{\partial z_l} \right) \Big|_{z=z_*}} \tag{A.2}$$

provided that the determinant does not vanish (this will always be the case in our calculations). Note that if any of the $f_k$ is nonzero everywhere inside its contour $C_k$ then the integral vanishes since $C_k$ can be shrunk to a point. Therefore, all the usual contour deformation arguments work as long as the contour deformations take place away from the analytic variety defined by

$$\det \left( \frac{\partial f_k}{\partial z_l} \right) = 0 \tag{A.3}$$

# B Finite volume FF and phase conventions

One has to evaluate the products of finite volume form factors. It follows from the crossing formula (2.5) that

$$\begin{aligned} &\langle \{I_1 \dots I_N\} | \mathcal{O}_1(0) | \{J_1 \dots J_M\} \rangle_L \langle \{J_1 \dots J_M\} | \mathcal{O}_2(0) | \{I_1 \dots I_N\} \rangle_L = \\ &= \frac{F^{\mathcal{O}_1}_{N+M}(\theta_1 + i\pi, \dots, \theta_N + i\pi, \theta'_M, \dots, \theta'_1) F^{\mathcal{O}_2}_{N+M}(\theta'_1 + i\pi, \dots, \theta'_M + i\pi, \theta_N, \dots, \theta_1)}{\rho_N(\theta_1, \dots, \theta_N) \rho_M(\theta'_1, \dots, \theta'_M)} \end{aligned} \tag{B.1}$$

In the following we show that in unitary models the crossing procedure described by (B.1) reproduces the usual complex conjugation of the matrix element. In unitary models the phase of the form factor is given by (allow for an extra sign ambiguity)

$$F_N(\theta_1, \dots, \theta_N) = \Big| F_N(\theta_1, \dots, \theta_N) \Big| \times \sqrt{\prod_{i<j} S(\theta_i - \theta_j)} \qquad \theta_i \in \mathbb{R} \tag{B.2}$$

In particular $F_1$ is always real. The extension to include also "bra" vectors reads

$$F_{N+M}(\theta'_1+i\pi,\dots,\theta'_M+i\pi,\theta_1,\dots,\theta_N) = \left|F_{N+M}(\theta'_1+i\pi,\dots,\theta'_M+i\pi,\theta_1,\dots,\theta_N)\right| \times \sqrt{\prod_{i<j} S(\theta_i-\theta_j)} \times \sqrt{\prod_{k<l} S(\theta'_k-\theta'_l)} \qquad \theta_i,\theta'_k \in \mathbb{R} \tag{B.3}$$

The complex conjugation property is then given by

$$\Big(F_{N+M}(\theta'_1+i\pi,\dots,\theta'_M+i\pi,\theta_1,\dots,\theta_N)\Big)^* = F_{N+M}(\theta'_M+i\pi,\dots,\theta'_1+i\pi,\theta_N,\dots,\theta_1) \tag{B.4}$$

This way one can avoid the operation of complex conjugation and one can work with analytic functions in the complex plane.

# C Evaluating the subtractions in the non-diagonal part of $D_{22}$

We first substitute

$$\sum_{I_1>I_2} \to \frac{1}{2}\sum_{I_1,I_2} - \frac{1}{2}\sum_{I_1=I_2} \tag{C.1}$$

## C.1 Spurious QQ singularities

Let us consider the first family of such singularities, which is when

$$\theta'_1=\theta_1 \quad \text{and} \quad \theta'_2=\theta_2 \tag{C.2}$$

The evaluation of such terms is complicated by the fact that the diagonal limit of $F_4$ is undefined (i.e. direction dependent). Fortunately, after converting the sums to integrals a factor of $1/\rho_2$ remains therefore all such terms are of order $\mathcal{O}(L^{-2})$. Similar considerations apply to the other case

$$\theta'_1=\theta_2 \quad \text{and} \quad \theta'_2=\theta_1 \tag{C.3}$$

## C.2 QF singularities

We consider the case

$$\theta'_1=\theta_1 \quad \text{and} \quad Q_{2'}(\theta'_1,\theta'_2)=2\pi J_2 \tag{C.4}$$

Just as in the case of $D_{12}$, the contribution can be split into a double and a single pole part.

### C.2.1 The double pole part

The double pole part is given by

$$\begin{aligned} \mathrm{D}^{11}_{QF} &= \sum_{I_1>I_2} \frac{1}{\rho_2(\theta_1,\theta_2)} \oint_{C_{J_2I_1}} \frac{d\theta'_2}{2\pi} \oint_{\theta_1} \frac{d\theta'_1}{2\pi} \frac{K^{(R)}_{t,x}(\theta_1,\theta_2;\theta'_1,\theta'_2)}{\left(\mathrm{e}^{iQ_{1'}(\theta'_1,\theta'_2)}+1\right)\left(\mathrm{e}^{iQ_{2'}(\theta'_1,\theta'_2)}+1\right)} \\ &\times \frac{i}{\theta_1-\theta'_1}(1-S(\theta_2-\theta_1)S(\theta'_1-\theta'_2))F^{O_1}_2(\theta_2+i\pi,\theta'_2) \\ &\times \frac{i}{\theta'_1-\theta_1}(1-S(\theta'_2-\theta'_1)S(\theta_1-\theta_2))F^{O_2}_2(\theta'_2+i\pi,\theta_2) \end{aligned} \tag{C.5}$$

We have

$$\mathrm{e}^{iQ_{1'}(\theta_1,\theta'_2)} = -S(\theta_1-\theta'_2)\mathrm{e}^{imL\sinh\theta_1} \tag{C.6}$$

On the other hand

$$\mathrm{e}^{imL\sinh\theta_1}S(\theta_1-\theta_2)=1 \tag{C.7}$$

i.e

$$\mathrm{e}^{iQ_{1'}(\theta_1,\theta_2')} = -S(\theta_1-\theta_2')S(\theta_2-\theta_1) \tag{C.8}$$

Using $S(\theta)S(-\theta)=1$ we can rearrange the contribution as

$$\begin{aligned}
\mathrm{D}^{11}_{QF} &= \sum_{I_1>I_2}\frac{1}{\rho_2(\theta_1,\theta_2)}\oint_{C_{J_2 I_1}}\frac{d\theta_2'}{2\pi}F_2^{O_1}(\theta_2+i\pi,\theta_2')F_2^{O_2}(\theta_2'+i\pi,\theta_2)\\
&\times\oint_{\theta_1}\frac{d\theta_1'}{2\pi}\frac{K^{(R)}_{t,x}(\theta_1,\theta_2;\theta_1',\theta_2')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1',\theta_2')}+1\right)}\\
&\times\frac{1}{(\theta_1-\theta_1')^2}\left((1-S(\theta_2-\theta_1)S(\theta_1'-\theta_2'))+(1-S(\theta_2'-\theta_1')S(\theta_1-\theta_2))\right)
\end{aligned} \tag{C.9}$$

We now use

$$\oint_{\theta_1}\frac{d\theta_1'}{2\pi}\frac{1}{(\theta_1-\theta_1')^2}f(\theta_1) = if'(\theta_1'=\theta_1) \tag{C.10}$$

and after manipulations similar to those in subsection 4.2.2 we arrive at

$$\begin{aligned}
\mathrm{D}^{11}_{QF} &= -\sum_{I_1>I_2}\frac{1}{\rho_2(\theta_1,\theta_2)}\oint_{C_{J_2 I_1}}\frac{d\theta_2'}{2\pi}F_2^{O_1}(\theta_2+i\pi,\theta_2')F_2^{O_2}(\theta_2'+i\pi,\theta_2)\\
&\times\frac{K^{(R)}_{t,x}(\theta_1,\theta_2;\theta_1,\theta_2')}{\left(\mathrm{e}^{iQ_{2'}(\theta_1,\theta_2')}+1\right)}\Big((mx\cosh\theta_1-imt\sinh\theta_1)(1-S(\theta_2'-\theta_1)S(\theta_1-\theta_2))\\
&+\varphi(\theta_1-\theta_2')S(\theta_2'-\theta_1)S(\theta_1-\theta_2)+mL\cosh\theta_1\Big)
\end{aligned} \tag{C.11}$$

The full contribution is then obtained by adding the three other double pole terms pertaining to the other QF-singularities in (4.83), which can also be obtained by suitably permuting the rapidity variables. Converting the sum over $\theta_1$, $\theta_2$ to integrals, taking care to subtract the diagonal $\theta_1=\theta_2$ we obtain for the full double pole contribution the expression:

$$\begin{aligned}
\mathrm{D}_{QF} &= \iint\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\int\frac{d\theta_2'}{2\pi}F_2^{O_1}(\theta_2+i\pi,\theta_2')F_2^{O_2}(\theta_2'+i\pi,\theta_2)\\
&\times\ \mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_2')}\mathrm{e}^{-mR\cosh\theta_1}\mathrm{e}^{-m(R-t)\cosh\theta_2}\mathrm{e}^{-mt\cosh\theta_2'}\\
&\times\ \Big((mx\cosh\theta_1-imt\sinh\theta_1)(1-S(\theta_2'-\theta_1)S(\theta_1-\theta_2))\\
&\quad+\varphi(\theta_1-\theta_2')S(\theta_2'-\theta_1)S(\theta_1-\theta_2)+mL\cosh\theta_1\Big)\\
&-\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2'}{2\pi}F_2^{O_1}(\theta_1+i\pi,\theta_2')F_2^{O_2}(\theta_2'+i\pi,\theta_1)\\
&\times\ \mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2')}\mathrm{e}^{-m(2R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2'}
\end{aligned} \tag{C.12}$$

Now we recall the leftover counter term from (4.72)

$$-Z_1\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}F_2^{\mathcal{O}_1}(\theta_1+i\pi,\theta_2)F_2^{\mathcal{O}_2}(\theta_1,\theta_2+i\pi)\mathrm{e}^{imx(\sinh\theta_1-\sinh\theta_2)}\mathrm{e}^{-m(R-t)\cosh\theta_1}\mathrm{e}^{-mt\cosh\theta_2} \tag{C.13}$$

with

$$Z_1 = mL\int\frac{d\theta}{2\pi}\cosh\theta\,\mathrm{e}^{-mR\cosh\theta} \tag{C.14}$$

and see that it exactly cancels the $O(L)$ part, leaving us with the finite expression

$$
\begin{aligned}
\mathrm{D}_{QF}^{\text{finite}} &= \int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2}{2\pi} \int \frac{d\theta_2'}{2\pi} F_2^{O_1}(\theta_2 + i\pi, \theta_2') F_2^{O_2}(\theta_2' + i\pi, \theta_2) \\
&\times \ \mathrm{e}^{imx(\sinh\theta_2 - \sinh\theta_2')} \mathrm{e}^{-mR\cosh\theta_1} \mathrm{e}^{-m(R-t)\cosh\theta_2} \mathrm{e}^{-mt\cosh\theta_2'} \\
&\times \ \Big( (mx\cosh\theta_1 - imt\sinh\theta_1)(1 - S(\theta_2' - \theta_1) S(\theta_1 - \theta_2)) \\
&\quad + \varphi(\theta_1 - \theta_2') S(\theta_2' - \theta_1) S(\theta_1 - \theta_2) \Big) \\
&- \int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2'}{2\pi} F_2^{O_1}(\theta_1 + i\pi, \theta_2') F_2^{O_2}(\theta_2' + i\pi, \theta_1) \\
&\times \ \mathrm{e}^{imx(\sinh\theta_1 - \sinh\theta_2')} \mathrm{e}^{-m(2R-t)\cosh\theta_1} \mathrm{e}^{-mt\cosh\theta_2'} \qquad \text{(C.15)}
\end{aligned}
$$

### C.2.2 Single pole contributions

Once again, we consider the $\theta_1' = \theta_1$ case and introduce the notation:

$$
\begin{aligned}
F_4^{\mathcal{O}_1}(\theta_2 + i\pi, \theta_1 + i\pi, \theta_1', \theta_2') &= \frac{i}{\theta_1 - \theta_1'} (1 - S(\theta_2 - \theta_1) S(\theta_1' - \theta_2')) F_2^{O_1}(\theta_2 + i\pi, \theta_2') \\
&+ F_{4sc}^{\mathcal{O}_1}(\theta_2, \theta_1 | \theta_1', \theta_2') \qquad \text{(C.16)}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
F_4^{\mathcal{O}_2}(\theta_2' + i\pi, \theta_1' + i\pi, \theta_1, \theta_2) &= \frac{i}{\theta_1' - \theta_1} (1 - S(\theta_2' - \theta_1') S(\theta_1 - \theta_2)) F_2^{O_2}(\theta_2' + i\pi, \theta_2) \\
&+ F_{4sc}^{\mathcal{O}_2}(\theta_2', \theta_1' | \theta_1, \theta_2) \qquad \text{(C.17)}
\end{aligned}
$$

The contribution has the following form

$$
\begin{aligned}
\mathrm{S}_{QF}^{11} &= -\frac{1}{2} \oint_{\theta_1} \frac{d\theta_1'}{2\pi} \oint_{C_{J_2 I_1}} \frac{d\theta_2'}{2\pi} \frac{K_{t,x}^{(R)}(\theta_1, \theta_2; \theta_1', \theta_2')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1', \theta_2')} + 1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1', \theta_2')} + 1\right)} \\
&\times \ \Big[ \frac{i}{\theta_1 - \theta_1'} (1 - S(\theta_2 - \theta_1) S(\theta_1' - \theta_2')) F_2^{O_1}(\theta_2 + i\pi, \theta_2') F_{4sc}^{\mathcal{O}_2}(\theta_2', \theta_1' | \theta_1, \theta_2) \\
&\quad + \frac{i}{\theta_1' - \theta_1} (1 - S(\theta_2' - \theta_1') S(\theta_1 - \theta_2)) F_2^{O_2}(\theta_2' + i\pi, \theta_2) F_{4sc}^{\mathcal{O}_1}(\theta_2, \theta_1 | \theta_1', \theta_2') \Big] \qquad \text{(C.18)}
\end{aligned}
$$

Recalling (C.8) and evaluating the residue integrals

$$
\begin{aligned}
\mathrm{S}_{QF}^{11} &= +\frac{1}{2} \oint_{C_{J_2 I_1}} \frac{d\theta_2'}{2\pi} \frac{K_{t,x}^{(R)}(\theta_1, \theta_2; \theta_1, \theta_2')}{(1 - S(\theta_1 - \theta_2') S(\theta_1 - \theta_2))\left(\mathrm{e}^{iQ_{2'}(\theta_1, \theta_2')} + 1\right)} \\
&\times \ \Big[ (1 - S(\theta_2 - \theta_1) S(\theta_1 - \theta_2')) F_2^{O_1}(\theta_2 + i\pi, \theta_2') F_{4sc}^{\mathcal{O}_2}(\theta_2', \theta_1 | \theta_1, \theta_2) \\
&\quad - (1 - S(\theta_2' - \theta_1) S(\theta_1 - \theta_2)) F_2^{O_2}(\theta_2' + i\pi, \theta_2) F_{4sc}^{\mathcal{O}_1}(\theta_2, \theta_1 | \theta_1, \theta_2') \Big] \qquad \text{(C.19)}
\end{aligned}
$$

Note that this vanishes when $\theta_1 = \theta_2$ due to the form factors vanishing, so when putting in the $\theta_1, \theta_2$ summation we can include the diagonal $\theta_1 = \theta_2$. Converting the summation to integrals we obtain

$$
\begin{aligned}
\mathrm{S}_{QF}^{11} &= \frac{1}{4} \iint \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \int \frac{d\theta_2'}{2\pi} \mathrm{e}^{imx(\sinh\theta_2 - \sinh\theta_2')} \mathrm{e}^{-mR\cosh\theta_1} \mathrm{e}^{-m(R-t)\cosh\theta_2} \mathrm{e}^{-mt\cosh\theta_2'} \\
&\times \Big( F_2^{O_1}(\theta_2 + i\pi, \theta_2') F_{4sc}^{\mathcal{O}_2}(\theta_2', \theta_1 | \theta_1, \theta_2) \\
&+ S(\theta_2' - \theta_1) S(\theta_1 - \theta_2) F_2^{O_2}(\theta_2' + i\pi, \theta_2) F_{4sc}^{\mathcal{O}_1}(\theta_2, \theta_1 | \theta_1, \theta_2') \Big) \qquad \text{(C.20)}
\end{aligned}
$$

or, using the definition of $F_{4sc}$ and the form factor equation (2.8)

$$\begin{aligned} \mathrm{S}^{11}_{QF} &= \frac{1}{4}\iint \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\int\frac{d\theta_2'}{2\pi}\mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_2')}\mathrm{e}^{-mR\cosh\theta_1}\mathrm{e}^{-m(R-t)\cosh\theta_2}\mathrm{e}^{-mt\cosh\theta_2'} \\ &\times\Big(F_2^{O_1}(\theta_2+i\pi,\theta_2')F_{4sc}^{\mathcal{O}_2}(\theta_2',\theta_1|\theta_1,\theta_2) \\ &+F_2^{O_2}(\theta_2'+i\pi,\theta_2)F_{4sc}^{\mathcal{O}_1}(\theta_2',\theta_1|\theta_1,\theta_2)\Big) \end{aligned} \tag{C.21}$$

The full contribution is then obtained by adding the three other single pole terms pertaining to the other QF-singularities in (4.83), which can also be obtained by suitably permuting the rapidity variables:

$$\begin{aligned} \mathrm{S}_{QF} &= \frac{1}{2}\iint \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\int\frac{d\theta_2'}{2\pi}\Big[\mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_2')}\mathrm{e}^{-mR\cosh\theta_1}\mathrm{e}^{-m(R-t)\cosh\theta_2}\mathrm{e}^{-mt\cosh\theta_2'} \\ &\times\Big(F_2^{O_1}(\theta_2+i\pi,\theta_2')F_{4sc}^{\mathcal{O}_2}(\theta_2',\theta_1|\theta_1,\theta_2)+F_2^{O_2}(\theta_2'+i\pi,\theta_2)F_{4sc}^{\mathcal{O}_1}(\theta_2',\theta_1|\theta_1,\theta_2)\Big) \\ &+(\theta_1\leftrightarrow\theta_2)\Big] \end{aligned} \tag{C.22}$$

### C.3 FF singularities

First let us consider the $\theta_1'=\theta_2'=\theta_1$ case. We need to separate the singular terms from the form factors, fro which we introduce a new function $F_{4dc}$ defined by

$$\begin{aligned} F_4^{\mathcal{O}_1}(\theta_2+i\pi,\theta_1+i\pi,\theta_1',\theta_2') &= \frac{i}{\theta_1-\theta_1'}(1-S(\theta_2-\theta_1)S(\theta_1'-\theta_2'))F_2^{O_1}(\theta_2+i\pi,\theta_2') \\ &+ \frac{i}{\theta_1-\theta_2'}(S(\theta_1'-\theta_2')-S(\theta_2-\theta_1))F_2^{O_1}(\theta_2+i\pi,\theta_1') \\ &+ F_{4dc}^{\mathcal{O}_1}(\theta_2,\theta_1|\theta_1',\theta_2') \end{aligned} \tag{C.23}$$

and similarly

$$\begin{aligned} F_4^{\mathcal{O}_2}(\theta_2'+i\pi,\theta_1'+i\pi,\theta_1,\theta_2) &= \frac{i}{\theta_1'-\theta_1}(1-S(\theta_2'-\theta_1')S(\theta_1-\theta_2))F_2^{O_2}(\theta_2'+i\pi,\theta_2) \\ &+ \frac{i}{\theta_2'-\theta_1}(S(\theta_2'-\theta_1')-S(\theta_1-\theta_2))F_2^{O_2}(\theta_1'+i\pi,\theta_2) \\ &+ F_{4dc}^{\mathcal{O}_2}(\theta_2',\theta_1'|\theta_1,\theta_2) \end{aligned} \tag{C.24}$$

Only the cross terms can contribute, otherwise at least one of the contour integrals can be shrunk to a point. We obtain

$$\begin{aligned} \mathrm{S}^1_{FF} &= -\sum_{I_1>I_2}\frac{1}{\rho_2(\theta_1,\theta_2)}\frac{1}{2}\oint_{\theta_1}\frac{d\theta_2'}{2\pi}\oint_{\theta_1}\frac{d\theta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1',\theta_2')}{\left(\mathrm{e}^{iQ_{1'}(\theta_1',\theta_2')}+1\right)\left(\mathrm{e}^{iQ_{2'}(\theta_1',\theta_2')}+1\right)} \\ &\times\frac{1}{\theta_1-\theta_1'}\frac{1}{\theta_1-\theta_2'}F_2^{\mathcal{O}_1}(\theta_2+i\pi,\theta_2')F_2^{\mathcal{O}_2}(\theta_2'+i\pi,\theta_2) \\ &\times 2(1+S(\theta_1-\theta_2))(1+S(\theta_2-\theta_1)) \end{aligned} \tag{C.25}$$

We have

$$K_{t,x}^{(R)}(\theta_1,\theta_2;\theta_1,\theta_1)=\mathrm{e}^{imx(\sinh\theta_2-\sinh\theta_1)}\mathrm{e}^{-m(R-t)(\cosh\theta_1+\cosh\theta_2)}\mathrm{e}^{-2mt\cosh\theta_1} \tag{C.26}$$

and

$$\mathrm{e}^{iQ_{1'}(\theta_1,\theta_1)} = \mathrm{e}^{imL\sinh\theta_1}=S(\theta_2-\theta_1) \tag{C.27}$$

$$\mathrm{e}^{iQ_{2'}(\theta_1,\theta_1)} = \mathrm{e}^{imL\sinh\theta_1}=S(\theta_2-\theta_1) \tag{C.28}$$

The contribution $\mathrm{S}_{FF}^2$ from $\theta_1' = \theta_2' = \theta_2$ can be obtained by interchanging $\theta_1$ and $\theta_2$. Putting in the $\theta_1$ and $\theta_2$ integrals we obtain

$$\begin{aligned} \mathrm{S}_{FF} &= -\int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2}{2\pi} \Big[ \mathrm{e}^{imx(\sinh\theta_2 - \sinh\theta_1)} \mathrm{e}^{-m(R-t)(\cosh\theta_1+\cosh\theta_2)} \mathrm{e}^{-2mt\cosh\theta_1} \\ &\times \; S(\theta_1-\theta_2) F_2^{\mathcal{O}_1}(\theta_2+i\pi,\theta_1) F_2^{\mathcal{O}_2}(\theta_1+i\pi,\theta_2) + (\theta_1 \leftrightarrow \theta_2) \Big] \end{aligned} \quad (C.29)$$

(in this case the diagonal subtraction is $\mathcal{O}(L^{-1})$, so it does not give a contribution in the infinite volume limit).

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