Small amplitude quasi-breathers and oscillons

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Quasi-breathers (QB) are time-periodic solutions with weak spatial localization introduced in G. Fodor et al. in Phys. Rev. D. **74**, 124003 (2006). QB's provide a simple description of oscillons (very long-living spatially localized time dependent solutions). The small amplitude limit of QB's is worked out in a large class of scalar theories with a general self-interaction potential, in D spatial dimensions. It is shown that the problem of small amplitude QB's is reduced to a universal elliptic partial differential equation. It is also found that there is the critical dimension, $D_{\rm crit}=4$, above which no small amplitude QB's exist. The QB's obtained this way are shown to provide very good initial data for oscillons. Thus these QB's provide the solution of the complicated, nonlinear time dependent problem of small amplitude oscillons in scalar theories.

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I. INTRODUCTION

Static scalar lumps (finite energy particle-like solutions in scalar field theories) are known to be absent in more than two spatial dimensions ($D>2$), however very long lived oscillating lumps have been observed in scalar theories with rather general self-interaction potentials in spatial dimensions $D<7$ [1, 2, 3, 4, 5]. These states (baptized oscillons in Ref. [3]) are of quite some interest, in spite of the fact that they eventually decay, since oscillons evolve from rather generic initial data in a remarkably large class of theories.

In 2 spatial dimensions, extremely long living breather-type objects have been found in the sine-Gordon (sG) model [6], and more recently in Ref. [7] generic oscillons, with lifetimes of the order of 10^6 in natural time units have been exhibited both in the sG and in a ϕ^4 scalar model. In 3 spatial dimensions spherically symmetric oscillons in the ϕ^4 scalar model with lifetimes $\approx 10^4$ have been found and investigated in detail [8, 9, 10]. Recently oscillons have been numerically observed starting from rather generic initial data in spontaneously broken SU(2) gauge theories (for the case when the mass ratio of the Higgs field to the W boson is ≈ 2), [11]. More recently 3 dimensional numerical simulations have shown [12] that long living oscillons are present in the SU(2)×U(1) model, i.e. in the full bosonic sector of the Standard model of electroweak interactions, which is clearly of great potential interest. According to the findings of Ref. [13] non-symmetric oscillons evolve towards symmetric ones (at least in $1+2$ dimensions), indicating that the long time evolutions will be dominated by spherically symmetric configurations.

Oscillons are already interesting in their own right as very long living lumps, appearing in many physical theories, whose existence is due to the non-linearities. They are expected to have important effects on the dynamics of various systems (including the Early Universe), since they retain a considerable amount of energy. Importantly, oscillons can easily form in physical processes such as the QCD phase transition, where oscillon like objects in the axion field have been observed [14], in vortex-antivortex annihilation [15] and in domain collapse [16]. In [17] oscillating field configurations (I-balls) were found in potentials where the quadratic term dominates. Once formed, oscillons could considerably influence the dynamics of the system as has been suggested for the case of the bubble nucleation process [18]. Oscillon formation has been reported after supersymmetric hybrid inflation [19]. A slightly different mechanism for the formation of long lived objects (quasi-lumps) during first order phase transitions has been investigated in [20]. The persistence of oscillons in one spatial dimension in an expanding background metric has been reported in Ref. [21]. Most recently in a study of semiclassical decay of topological defects possible oscillons formation has been reported [22].

Infinitely long lived oscillons with finite energy, commonly known under the name of breathers, are rather exceptional. Simple heuristic arguments indicate that spatially localized time periodic solutions (breathers) do not exist in generic theories [23], and that under a general perturbation breathers are unstable [24]. In 1 spatial dimension the absence of small amplitude breathers in a scalar theory with ϕ^4 interaction has been demonstrated in Refs. [25] [26]. More generally it has been shown in Ref. [27] that the only non-trivial $1+1$ dimensional scalar theory with real analytic self-interaction potential which admits breather solutions is the sine-Gordon (sG) theory. Interestingly in a recent work [28] a whole family non-radiating solutions of the $1+1$ dimensional 'signum-Gordon' model has been found. This illustrates that one can expect new and surprising phenomena in theories where the interaction potential is not smooth.

On the other hand, bounded, time periodic solutions of nonlinear wave equations in $\mathbf{R}^D$ are abundant. By performing a Fourier decomposition in time, the problem of finding bounded, time periodic solutions is reduced to solve an infinite set of coupled nonlinear partial differential equations (PDE's). Even the simplest cases, such as $D = 1$ or spherical symmetry, when the Fourier amplitudes satisfy an infinite set of ordinary differential equations (ODE's), are non-trivial to analyze, but it is clear that a plethora of bounded solutions exist. The class of bounded, time periodic solutions in $\mathbf{R}^D$ contains families of breather-like objects, which have a well defined core, outside of which the fields fall of rapidly, but barring exceptional cases, in the far field region there is also a radiative tail, which corresponds to a standing wave. Because of the asymptotically standing wave asymptotics, such objects are only weakly localized in space. The energy density of the core is much larger than that of the tail, however, the total energy contained in the tail is infinite. Considering such a weakly localized object in a finite volume, V, its total energy, E, is proportional to $E \propto V^{1/D}$. Such objects can be thought of as radiating lumps, whose energy loss is compensated by a flux of radiation coming from infinity, rendering the system time periodic . In [10] a special class has been singled out in the huge phase space of bounded, time periodic solutions, obtained by minimizing the energy in the radiative tail. Solutions belonging to this class have been called as quasi-breathers (QB's). Even if the QB's are not physical objects by themselves in the whole space, $\mathbf{R}^D$, because of containing an infinite amount of energy, nevertheless a finite piece of them (containing the core and even part of the tail) constitutes a very good approximation to oscillons in a $\mathbf{R}^D$, as demonstrated in detail in the ϕ^4 scalar theory in $D = 3$ [10].

In this paper we carry out a systematic analysis of bounded, time dependent solutions in a D dimensional scalar theory with a rather general class of self-interaction potential in the limit when the amplitude, ε, of the solution goes to zero. This way we obtain a rather general method to find small amplitude QB's. Following Refs. [23, 25, 27] we derive a formal series solution in ε whose terms are all bounded both in time and space. We show that demanding boundedness in time necessarily leads to periodicity, and that the time and space dependence of the solution separates. We derive a single master equation determining the spatial dependence of the leading term in the series, which turns out to be universal for the class of scalar theories we consider. This equation is a nonlinear elliptic PDE with a cubic nonlinearity. It turns out that exponentially localized solutions of the master equation exist in spatial dimension, $D < 4$. While in $D = 1$ the solution is unique, for $D > 1$ there is an infinite family of exponentially localized solutions. In the case of spherical symmetry, members of this family can be characterized by the number of their nodes. Solutions with nodes contain considerably higher energies than the fundamental one, nevertheless they also correspond to oscillons. The higher order terms in the small amplitude series can be obtained from linear inhomogeneous PDE's whose source terms are determined by the localized solution of the master equation. Each term of the small amplitude expansion obtained this way is exponentially localized, and one could think that it represents a breather. In general this series is, however, not convergent, it is rather an asymptotic one. This fact reflects the absence of 'genuine' breathers with spatial localization. One can think of this series as an excellent approximation of QB's whose radiative tail is smaller than any power of ε. Therefore the small amplitude series provides only the exponentially localized core part of the QB. Since the amplitude of the radiative tail is so small this does not really matter and in fact the oscillon states corresponding to such initial data for small values of ε do have very long lifetimes. Even the first few terms in the small amplitude expansion yield quite good initial data for long living oscillon states, as demonstrated by our numerical simulations for the standard ϕ^4 theory for the spherically symmetric case in $D = 2$ and $D = 3$. As already mentioned, one of the interests of the small amplitude QB's, is that they can be identified with the core part of small amplitude oscillons, which radiate very weakly, and hence they have a very long lifetime. We have verified this by numerical simulations, namely we shown in dimensions $D = 1\,, 2\,, 3$ that small amplitude QB's do provide excellent initial data for long lived oscillons.

We have also computed the energy of the QB's approximated by the small amplitude series, which corresponds to the energy content of their core and this is of course always finite. We have shown that the energy of the QB core is a monotonously decreasing function of the frequency near the mass threshold in dimensions $D \le 2$. This implies the absence of a critical frequency minimizing the energy in dimensions $D \le 2$, in agreement with known numerical results. For $D > 2$ the energy of the small amplitude QB's increases without bound as their frequency approaches the mass threshold. This implies the existence of a critical frequency where the energy is minimized.

As it has been already mentioned, well (exponentially) localized QB's with a finite energy core exist only for $D < 4$. In higher dimensions, $D \ge 4$ the exponential localization property of the core of the QB's is lost and in fact the very existence of a well defined core is problematic. In Ref. [5] it has been already suggested that oscillons cease to exist, for dimensions greater than 5 or 6 depending on the details of the potential. According to the findings of Ref. [29] the lifetimes of oscillons decreases rapidly as the dimensionality of space is increased, and the QB picture does not give a good description. We have pushed further the analysis of small amplitude oscillons in higher dimensions to understand the situation better. We shall present our results in dimensions $D \ge 4$ in a sequel to this paper [30]. Without going into too much details we can state the following. Small amplitude oscillons do exist in dimensions $D \ge 4$, without any apparent limitation for D. These small amplitude oscillons in dimensions $D \ge 4$ are, however, qualitatively different from their lower dimensional counterparts. In particular, they are *not well (exponentially)* *localized*, and they cannot be described by QB's in the sense of Ref. [10]. The energy of higher dimensional small amplitude oscillons also becomes quite large, therefore they are probably less interesting physically than genuine QB's, however, by choosing suitable initial data, they can still have very long lifetimes. Their large energy content may explain that such oscillons in $D > 4$ are somewhat more difficult to be found.

The plan of our paper is the following: In Section II first the class of scalar models to be studied is introduced, then the small amplitude expansion in D spatial dimension is carried out. We derive the master equation without any symmetry assumptions, and calculate the QB solution to order 4 in the ε expansion as well. The energy of the QB's is also computed, and in the last subsection the existence of the critical dimension $D = 4$ is derived above which no small amplitude QB's exist. Section III is devoted to a detailed numerical analysis of the solutions of the spherically symmetric master equation and the explicit computations of the higher order terms in the ε expansion in spatial dimensions $D = 2\,,3$. In Section IV results on the numerical time evolution of QB initial data up 6th order in ε is presented for $D = 2\,,3$.

II. THE SMALL AMPLITUDE EXPANSION

In this Section we carry out a detailed analysis of the small amplitude limit of QB's of the NLWE (2) in D spatial dimensions, without any symmetry assumptions. It turns out that bounded non-trivial small amplitude solutions are periodic in time, and that the time and spatial dependence completely separates. In subsection II B we derive a universal elliptic PDE, referred to as the master eq. governing the behaviour of the solutions. Next, in subsection II C the solution is obtained up to order 4 in ε in a general class potentials, and up to order 6 for theories with a symmetric potential, such as the sine-Gordon model.

A. The class of theories considered

We consider a scalar theory in a $1 + D$ dimensional flat Minkowski space-time, with a general self-interaction potential, whose action can be written as

$$A = \int dt\, d^D x \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_i \phi)^2 - U(\phi) \right] , \tag{1}$$

where ϕ is a real scalar field, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x^i$ and $i = 1, 2, \ldots, D$. The equation of motion following from (1) is a non-linear wave equation (NLWE) which is given as

$$-\phi_{,tt} + \Delta\phi = U'(\phi) = \phi + \sum_{k=2}^{\infty} g_k \phi^k \,, \quad \text{where} \quad \Delta = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2} \,. \tag{2}$$

In Eq. (2) the mass of the field is chosen to be 1, and it has been assumed that the potential, $U(\phi)$, can be written as a power series in ϕ, where the g_k are real constants. For the standard ϕ^4 theory this interaction potential is simply

$$U(\phi) = \frac{1}{8}\phi^2(\phi - 2)^2 \,, \quad U'(\phi) = \phi - \frac{3}{2}\phi^2 + \frac{1}{2}\phi^3 \,, \tag{3}$$

i.e. $g_2 = -\frac{3}{2}$, $g_3 = \frac{1}{2}$ and $g_i = 0$ for $i \geq 4$ in this case. Note that in our previous paper, Ref. [10], a different scaling of the ϕ^4 potential has been used, making the value of the mass to be $m = \sqrt{2}$ instead of the present value $m = 1$ used in this paper. For the sine-Gordon potential $U(\phi) = 1 - \cos(\phi)$, we have $g_{2i} = 0$ and $g_{2i+1} = (-1)^i/(2i + 1)!$.

The energy corresponding to the action (1) can be written as

$$E = \int d^D x\, \mathcal{E} \,, \quad \mathcal{E} = \frac{1}{2}\left(\partial_t \phi\right)^2 + \frac{1}{2}\left(\partial_i \phi\right)^2 + U(\phi) \,, \tag{4}$$

where $\mathcal{E}$ denotes the energy density.

B. Derivation of the master equation

We are looking for small amplitude solutions, therefore we expand the scalar field, ϕ, in terms of a parameter ε as

$$\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_n \,. \tag{5}$$

In order to obtain non-trivial solutions of Eq. (2) their characteristic scale must also become ε-dependent. The size of smooth configurations is expected to increase for decreasing values of ε, therefore it is natural to introduce new spatial coordinates by the following rescaling

$$\zeta^i = \varepsilon x^i \,. \tag{6}$$

One must also allow for the ε dependence of the time-scale of the configurations, therefore a new time coordinate is introduced as

$$\tau = \omega(\varepsilon) t \,. \tag{7}$$

$\omega(\varepsilon)$ is assumed to be analytic near the threshold, $\omega = 1$, and it is expanded as

$$\omega^2(\varepsilon) = 1 + \sum_{k=1}^{\infty} \varepsilon^k \omega_k \,. \tag{8}$$

After these rescalings Eq. (2) takes the following form

$$-\omega^2 \ddot{\phi} + \varepsilon^2 \Delta\phi = \phi + \sum_{k=2}^{\infty} g_k \phi^k \,. \tag{9}$$

In equation (9) and in the rest of this section an overdot stands for the derivative with respect to the rescaled time coordinate, τ, and all spatial derivatives are taken with respect to the rescaled coordinates ζ^i. Substituting the ε expansion of the scalar field, ϕ, and of ω^2 into (9) the equations determining the first three lowest order terms are:

$$\ddot{\phi}_1 + \phi_1 = 0 \,, \tag{10}$$

$$\ddot{\phi}_2 + \phi_2 + g_2\phi_1^2 + \omega_1\ddot{\phi}_1 = 0 \,, \tag{11}$$

$$\ddot{\phi}_3 + \phi_3 + 2g_2\phi_1\phi_2 + g_3\phi_1^3 - \ddot{\phi}_1 - \Delta\phi_1 + \omega_1\ddot{\phi}_2 + \omega_2\ddot{\phi}_1 = 0 \,. \tag{12}$$

As it is clear from Eqs. (10)-(12) the time dependence has been separated from the spatial one, and we have obtained a set of harmonic oscillator equations. Now the solution of Eq. (10) is clearly given by

$$\phi_1 = p_1 \cos(\tau + \alpha) \,, \tag{13}$$

where p_1 and α are functions of the spatial variables ζ^i. That is, the lowest order term of the solution is just a harmonic oscillator in time, with frequency $\omega = 1$ (note that this is the same with respect to both time coordinates, τ and t at this order). This distinguished value of the frequency, $\omega = 1$, corresponds to the threshold determined by the mass of the scalar field.

As Eq. (11) is a linear inhomogeneous equation, its solution is easily obtained:

$$\phi_2 = p_2 \cos(\tau+\alpha) + q_2 \sin(\tau+\alpha) + \frac{g_2}{6} p_1^2 [\cos(2\tau + 2\alpha) - 3] + \frac{\omega_1}{4} p_1 \left[2\tau \sin(\tau+\alpha) + \cos(\tau+\alpha)\right] . \tag{14}$$

Since we are looking for bounded solutions, it is necessary to impose $\omega_1 = 0$. This amounts to demanding the absence of the resonance term $\omega_1\ddot{\phi}_1$ in eq. (11). Substituting the solutions for ϕ_1 and ϕ_2 into Eq. (12) one obtains yet another forced oscillator equation for the time dependence of ϕ_3:

$$\begin{aligned}&\ddot{\phi}_3 + \phi_3 + (p_1\Delta\alpha + 2\nabla\alpha\nabla p_1)\sin(\tau+\alpha) - \left[\Delta p_1 + \omega_2 p_1 + \lambda p_1^3 - p_1(\nabla\alpha)^2\right]\cos(\tau+\alpha)\\ &+\frac{1}{12}p_1^3(2g_2^2 + 3g_3)\cos(3\tau + 3\alpha) + g_2 p_1\left[q_2\sin(2\tau+2\alpha) + p_2\cos(2\tau+2\alpha) + p_2\right] = 0 \,,\end{aligned} \tag{15}$$

where we have introduced the combination

$$\lambda = \frac{5}{6}g_2^2 - \frac{3}{4}g_3 \,, \tag{16}$$

which will play an important rôle in the following. For the standard ϕ^4 theory (3), this parameter takes the value $\lambda = 3/2$. For the sine-Gordon potential $\lambda = 1/8$.

As already explained we are looking for bounded solutions, therefore it is necessary to guarantee the absence of resonance terms also in Eq. (15). The vanishing of the coefficient of the $\sin(\tau + \alpha)$ terms implies

$$\nabla(p_1^2 \nabla\alpha) = 0 \,, \tag{17}$$

and from this equation one immediately derives the following condition

$$\int_\Omega \alpha\nabla(p_1^2\nabla\alpha) = \int_{\partial\Omega} \alpha p_1^2 n\cdot\nabla\alpha - \int_\Omega p_1^2(\nabla\alpha)^2 = 0\,. \tag{18}$$

Assuming that the integrand of the boundary term vanishes sufficiently fast we conclude that $\nabla\alpha = 0$. Our assumption is quite reasonable since we are looking for bounded solutions in time and localized in space. Therefore α must be a constant which can be absorbed by a shift in the time variable. From now on we set $\alpha = 0$. Then the vanishing of the coefficient of the resonance term proportional to $\cos\tau$ implies

$$\Delta p_1 + \omega_2 p_1 + \lambda p_1^3 = 0\,. \tag{19}$$

A necessary condition that this equation admit exponentially localized solutions is $\omega_2 < 0$, which we shall assume from now on. In this case we can set $\omega_2 = -1$ by a simultaneous rescaling of ζ^i and p_1. This rescaling corresponds to choosing a different parametrization ε for a solution with a specific frequency ω.

By an analytic redefinition of the expansion parameter, ε, and by rescalings, all coefficients ω_i can be made to vanish for $i > 2$. This means that by a suitable transformation of the expansion parameter, ε, one can always achieve that the following relation between the frequency, ω, and the expansion parameter, ε, holds:

$$\omega = \sqrt{1-\varepsilon^2}\,. \tag{20}$$

In terms of the physical time coordinate, t, the configuration oscillates with frequency ω. This is the physically important frequency characterizing these periodic solutions. Apart from the leading order behaviour, the precise choice of how ε depends on ω is physically irrelevant.

After setting $\omega_2 = -1$, it is easy to see (multiplying (19) with p_1 and integrating) that $\lambda > 0$ is a necessary condition for the existence of bounded solutions vanishing at infinity. Assuming $\lambda > 0$, by rescaling p_1 one obtains

$$\Delta S - S + S^3 = 0\,, \quad S = p_1\sqrt{\lambda}\,. \tag{21}$$

This master equation constitutes an important result equation of our paper. Quite remarkably Eq.(21) is universal for the class of theories considered, and the dependence on the parameters of the interaction potential enters only through the combination λ when reconstructing ϕ_1.

C. Higher orders in the ε expansion

Let us now turn to the determination of some higher order terms in the ε expansion. Armed with the simple form of the solution for p_1 it is now easy to obtain the explicit time dependence of ϕ_3 by integrating Eq. (15):

$$\phi_3 = q_3\sin\tau + p_3\cos\tau + \frac{p_1}{3}\left\{\frac{1}{8}\left(\frac{4}{3}g_2^2 - \lambda\right)p_1^2\cos(3\tau) + g_2\left[q_2\sin(2\tau) + p_2(\cos(2\tau)-3)\right]\right\}\,. \tag{22}$$

In ϕ_3 two new functions, p_3, q_3 have appeared. The absence of resonances at fourth order yields two conditions:

$$\lambda q_2 p_1^2 - q_2 + \Delta q_2 = 0\,, \tag{23}$$

which is the vanishing condition of coefficient of the term of $\sin\tau$, and another one

$$3\lambda p_2 p_1^2 - p_2 + \Delta p_2 = 0\,, \tag{24}$$

which ensures the vanishing of the coefficient of $\cos\tau$.

Let us first note, that quite remarkably $q_2 \propto p_1$ actually solves Eq. (23). Therefore by shifting the time coordinate by a suitable term of order ε we can eliminate q_2. In higher orders of the ε expansion, we have verified that up to order ε^9 the vanishing of the coefficient of the terms proportional to $\sin\tau$ leads to equations which are equivalent to Eq. (23). We conjecture that this is in fact true to all orders, i.e. by a suitable choice of the origin of the time coordinate τ one eliminate all terms proportional to $\sin\tau$. This observation is quite important because it implies that all small amplitude QB-type solutions necessarily possess time reflection symmetry. This is of course what one would expect based on simple physical intuition. Let us point out here, that all long-lived oscillon configurations observed in time evolution simulations appear to show this symmetry to a very high degree. Of course oscillons are not *exactly* time reflection symmetric because they radiate some energy to infinity. Moreover, time reflection symmetry has been usually implicitly assumed when performing Fourier decomposition in order to find time periodic states. For all these reasons it would be of interest to find a mathematical proof of the validity of time reflection symmetry for periodic solutions of NLWE's.

There is no reason to expect that Eq. (24) admits bounded solutions vanishing at infinity apart from those corresponding to the translational and rotational symmetries, therefore from now on we set $p_2 \equiv 0$. The vanishing of the coefficient of the $\cos\tau$ term in the fifth order equation yields

$$\Delta p_3 - p_3 + 3\lambda p_1^2 p_3 + \frac{g_2^2}{9} p_1 \left(17p_1^2 + 19(\nabla p_1)^2\right) + \frac{p_1^5}{216}(378 g_4 g_2 + 36\lambda g_2^2 - 280 g_2^4 - 9\lambda^2 - 135 g_5) = 0 \,. \tag{25}$$

This is a linear, inhomogeneous equation. It can be brought to a much simpler form by introducing a new variable Z instead of p_3:

$$p_3 = \frac{1}{\lambda^2\sqrt{\lambda}} \left[\left(\frac{1}{24}\lambda^2 - \frac{1}{6}\lambda g_2^2 + \frac{5}{8} g_5 - \frac{7}{4} g_2 g_4 + \frac{35}{27} g_2^4 \right) Z - \frac{1}{54}\lambda g_2^2 S(32 + 19S^2) \right] . \tag{26}$$

Then Eq. (25) takes the compact form

$$\Delta Z - Z + 3S^2 Z - S^5 = 0 \,. \tag{27}$$

For the specific example of the standard ϕ^4 theory this relation is simply

$$p_3 = \frac{\sqrt{2}}{3\sqrt{3}} \left(\frac{65}{8} Z - \frac{8}{3} S - \frac{19}{12} S^3 \right) . \tag{28}$$

Integrating the corresponding equation for ϕ_5 a new unknown function, p_5 appears, in analogy to the third order case.

To summarize, we have obtained the solution of the NLWE (2) in the small amplitude expansion up to order four. All terms have harmonic time dependence, and the spatial part is determined by the two universal elliptic PDE's, eqs. (21,27). There is no obstacle to continue the computation to higher orders, the general formulae become then quite complicated of course. The small amplitude expansion of the solution of Eq. (9) up to order four for general interaction potentials can be written as:

$$\phi_1 = p_1 \cos\tau \tag{29}$$

$$\phi_2 = \frac{1}{6} g_2 p_1^2 \left(\cos(2\tau) - 3\right) \tag{30}$$

$$\phi_3 = p_3 \cos\tau + \frac{1}{72}(4g_2^2 - 3\lambda) p_1^3 \cos(3\tau) \tag{31}$$

$$\begin{aligned} \phi_4 = {} & \frac{1}{360} p_1^4 \left(3g_4 - 5g_2\lambda + 5g_2^3\right) \cos(4\tau) \\ & - \frac{1}{72}\left(8g_2(\nabla p_1)^2 - 12 g_4 p_1^4 + 16 g_2^3 p_1^4 - 24 g_2 p_1 p_3 - 23 g_2 \lambda p_1^4 - 8 g_2 p_1^2\right) \cos(2\tau) \\ & - g_2 p_1^2 - g_2 p_1 p_3 + \frac{1}{6} g_2 \lambda p_1^4 - g_2 (\nabla p_1)^2 + \frac{31}{72} g_2^3 p_1^4 - \frac{3}{8} g_4 p_1^4 \,. \end{aligned} \tag{32}$$

A considerable simplification occurs when the scalar self-interaction potential, $U(\phi)$, is symmetric around its minimum $\phi = 0$. In this case $g_{2i} = 0$ for all $i = 1, \ldots$, and all even power terms in the ε expansion vanish, i.e. $\phi_{2i} = 0$ for $i = 1, \ldots$. Since ϕ_{2n} contains only terms of the form $\cos(2k\tau)$ with $k = 1, \ldots n$, and ϕ_{2n+1} contains only terms proportional to $\cos((2k+1)\tau)$, with $k = 1, \ldots n$, this also implies that for such symmetric potentials no even terms in the Fourier expansion arise. In this case p_3 is proportional to Z and the equation determining the function p_5 becomes reasonably simple, it can be written as

$$\begin{aligned} & \Delta p_5 - p_5 + 3S^2 p_5 + \frac{SZ}{576\sqrt{\lambda}}(3Z - 5S^3)\left(\frac{15 g_5}{\lambda^2} + 1\right)^2 \\ & + \frac{S^3}{32\sqrt{\lambda}}\left[(\nabla S)^2 - S^2\right] - \frac{S^7}{576\sqrt{\lambda}}\left(\frac{315 g_7}{\lambda^3} - \frac{60 g_5}{\lambda^2} + 1\right) = 0 \,. \end{aligned} \tag{33}$$

Then

$$\begin{aligned} & \phi_5 = p_5 \cos\tau + \frac{S^5}{1152\sqrt{\lambda}}\left(\frac{3g_5}{\lambda^2} + 2\right)\cos(5\tau) \\ & - \frac{S}{384\sqrt{\lambda}}\left[\left(\frac{30 g_5}{\lambda^2} + 2\right) SZ + 12 S^2 - 12(\nabla S)^2 - \left(\frac{15 g_5}{\lambda^2} - 2\right) S^4\right] \cos(3\tau) \,. \end{aligned} \tag{34}$$

These expressions encompass for example the case of the sine-Gordon model. The corresponding equations for the ϕ^4 theory, in the case of spherical symmetry, will be listed in Section III.

As we have already stressed several times, it is by now well understood that spatially localized breathers of the NLWE (2) do not exist in $\mathbf{R}^D$ for general analytic potentials, even if a general mathematical proof is known only in $D = 1$. Let us note here, that a remarkable example in $D = 1$ admitting non-radiating breather-type solutions in the framework of "V"-shaped (non-differentiable) potentials evades this theorem [28]. In the case of analytic potentials, where the theorem applies there is still a point to be stressed. Assuming that exponentially decreasing solutions of the master equation exist, all higher order terms in the small amplitude expansion are also exponentially localized, and they are periodic in time. As we have learned from the example of the one dimensional ϕ^4 theory [25] the series solution in powers of ε does not converge to a breather, it is an asymptotic series. Nevertheless, to a given order in the expansion for sufficiently small values of ε the corresponding sum yields a configuration with a spatially well localized core. This time periodic configuration corresponds to a QB whose standing wave tail is smaller than ε^n for any $n > 0$. As it will be shown in Section IV such QB's constitute an excellent approximation to an oscillon.

D. The energy

In this subsection we evaluate the energy of small amplitude QB's, in D dimension. In the rescaled coordinate system, $\tau\,,\zeta$, the energy of a configuration, Eq. (4) can be written as

$$E = \frac{1}{\varepsilon^D}\int d^D\zeta\,\mathcal{E}\,, \quad \text{where} \quad \mathcal{E} = \frac{1}{2}(1-\varepsilon^2)\left(\partial_\tau\phi\right)^2 + \varepsilon^2\frac{1}{2}\left(\partial_i\phi\right)^2 + U(\phi)\,. \tag{35}$$

Because of the periodic time dependence we shall compute the energy density averaged over a period,

$$\bar{E} = \frac{1}{2\pi}\int_0^{2\pi} d\tau E\,, \tag{36}$$

and in this subsection the bar over a quantity will denote its time average. Using the results of the ε expansion, Eqs. (29) - (31), the time averaged energy density, $\bar{\mathcal{E}}$, up to fourth order in ε can be written as

$$\begin{aligned}\bar{\mathcal{E}} \;&=\; \frac{\varepsilon^2}{2\lambda}S^2 - \frac{\varepsilon^4}{216\lambda^3}\Big[\lambda S^2(64g_2^2+27\lambda)(S^2+2) - 54\lambda^2(\nabla S)^2 \\ &\quad -SZ(135g_5 - 378g_2g_4 + 280g_2^4 - 36\lambda g_2^2 + 9\lambda^2)\Big]\,. \end{aligned}\tag{37}$$

For the ϕ^4 theory the time averaged energy density takes a much simpler form:

$$\bar{\mathcal{E}} = \frac{\varepsilon^2}{3}S^2 + \varepsilon^4\left[\frac{1}{6}(\nabla S)^2 - \frac{41}{108}S^2(S^2+2) + \frac{65}{36}SZ\right]S^4\,. \tag{38}$$

Using the above results, the time averaged total energy, $\bar{E}$, has the following ε expansion in D dimension:

$$\bar{E} = \varepsilon^{2-D}\frac{E_0}{2\lambda} + \varepsilon^{4-D}E_1\,, \quad \text{where} \quad E_0 = \int d^D\zeta\, S^2\,, \tag{39}$$

and E_1 denotes the integral of the 4-th order term in the energy density, (37) in $\mathbf{R}^D$. As one sees from Eq. (39) the leading order behaviour of the time averaged total energy is $\bar{E} \propto \varepsilon^{2-D}$. This implies that the ε-dependence of $\bar{E}$ changes essentially at $D = 2$. In dimensions $D > 2$ the total energy *increases* without any bound for decreasing values of ε. In $D = 2$ $\bar{E}$ tends to a constant, and for $D < 2$ it goes to zero as $\varepsilon \to 0$. This also implies that the core energy of a QB in dimensions $D > 2$ should exhibit a minimum for some frequency $\omega_{\rm m}$. In fact, from Eq. (39) one immediately finds

$$\omega_{\rm m}^2 = 1 - \varepsilon_{\rm m}^2 = 1 - \frac{1}{2\lambda}\frac{(D-2)E_0}{(4-D)E_1}\,. \tag{40}$$

The above result can only be taken as an indication of the minimum even if $\varepsilon_{\rm m} \ll 1$. The numerical values of E_0 and E_1 will be given for the fundamental solutions in $D = 2$ and $D = 3$ in case of spherical symmetry in Section III. We note that in $D = 1$ $S(\zeta) = \sqrt{2}\text{sech}(\zeta)$, therefore $E_0 = 4$, and the leading order ε dependence of the energy is given as $\bar{E} = 2\varepsilon/\lambda + \mathcal{O}(\varepsilon^3)$.

E. Critical dimension $D = 4$

In the following we present some simple, although important results concerning the existence of spatially localized solutions of the master equation, (21). First spatially localized solutions of (21) which have a limit for $|\vec{x}| \to \infty$ should decrease exponentially, since those tending to a constant exhibit oscillatory behaviour. Next we show that exponentially localized solutions of Eq. (21) cannot exist for $D \geq 4$, implying that small amplitude QB's exist only in dimensions $D < 4$. To see this, consider the following virial identity derived from equation (21):

$$\langle(\vec{\nabla}S)^2\rangle + \langle S^2\rangle - \langle S^4\rangle = 0\,, \tag{41}$$

where $\langle f\rangle := \int d^D x f(x)$. Furthermore, another virial identity can be found from the scaling transformation ($\vec{x} \to \mu\vec{x}$) of the action corresponding to (21), $\int d^D x[(\vec{\nabla}S)^2 + S^2 - S^4/2]$:

$$(D-2)\langle(\vec{\nabla}S)^2\rangle + D\langle S^2\rangle - \frac{D}{2}\langle S^4\rangle = 0\,. \tag{42}$$

From Eqs. (41) and (42) one immediately finds

$$2\langle S^2\rangle + \frac{1}{2}(D-4)\langle S^4\rangle = 0\,, \tag{43}$$

which equality can only be satisfied if $D < 4$.

The absence of small amplitude QB's in more than 3 spatial dimensions does not imply per se that oscillons would be also absent if $D \geq 4$. As a matter of fact we have found that small amplitude oscillons exist in dimensions $D \geq 4$, without any apparent limitation on D. These higher dimensional ($D \geq 4$) small amplitude oscillons do not have a well defined, exponentially localized core, and they cannot be described by small amplitude QB's. Even the total energy of the core in $D \geq 4$ is not well defined. Interestingly by choosing suitable initial data, for increasing energy content one can achieve that they have very long lifetimes. Various arguments and numerical studies of spherically symmetric oscillons in D-dimensions by Gleiser [5] led him to conjecture the existence of a critical value of D ($D_c \gtrsim 6$) above which no long lived oscillon states would exist. The existence of higher dimensional small amplitude oscillons contradict this conjecture, however, since these contain very large amount of energies it may have been less obvious to start with such initial data. This might explain why such objects have been missed. Also for a fixed amount of energy, the lifetime of oscillons exhibits a significant decrease for $D > 3$. The results of the recent work [29] show that in $D = 5$ the lifetimes becomes as small as a few 100 (in natural units). Let us mention here another interesting point. In Ref. [5] a very long lived oscillon state has been exhibited in $D = 6$. This object is not in the class of small amplitude oscillons, and should be understood better. In any case this provides another example how rich the phase space of time dependent solutions of a simple non-linear wave equation can be.

III. SOLUTION OF THE MASTER EQUATION FOR SPHERICAL SYMMETRY

In this Section we consider spherically symmetric configurations, in which case the PDE's determining the functions S, Z, etc. reduce to ODE's. This simplifies of course significantly the problem of solving both the master equation and the associated inhomogeneous ones. We exhibit some numerical solutions of these equations in $D = 2$ and in $D = 3$. We present the solution of the ε expansion in the ϕ^4 theory up to 6th order.

For spherically symmetric configurations the master equation (21) takes the form

$$\frac{d^2S}{d\rho^2} + \frac{D-1}{\rho}\frac{dS}{d\rho} - S + S^3 = 0\,, \tag{44}$$

where S is a function of the rescaled radial coordinate $\rho = \varepsilon r$. In 1 spatial dimension the solution of (44) vanishing at infinity is unique, it is given explicitly by $S = \sqrt{2}\,\mathrm{sech}\,\rho$. In contradistinction to $D = 1$, for higher dimensions, $1 < D < 4$, the solution vanishing at $\rho \to \infty$ is not unique. Our numerical analysis indicates, that for $1 < D < 4$ there is a family of localized solutions of Eq. (44) indexed by the number of zeros (nodes) of $S(\rho)$. On Figures 1 and 2 the first few members of this solution family are exhibited in dimensions $D = 2$ and $D = 3$. As it has been shown in the previous Section, there are no solutions of Eq. (44) for $D \geq 4$, which tend to zero for $\rho \to \infty$. The values of S at the origin $\rho = 0$ are tabulated in Table I. It can be already expected that the fundamental solution (without nodes), S_0, is physically the most important. Indeed, as we shall show later, oscillons corresponding to solutions of the master equation with nodes contain more energy and have significantly smaller lifetimes than those corresponding to nodeless ones, and they are also less stable.

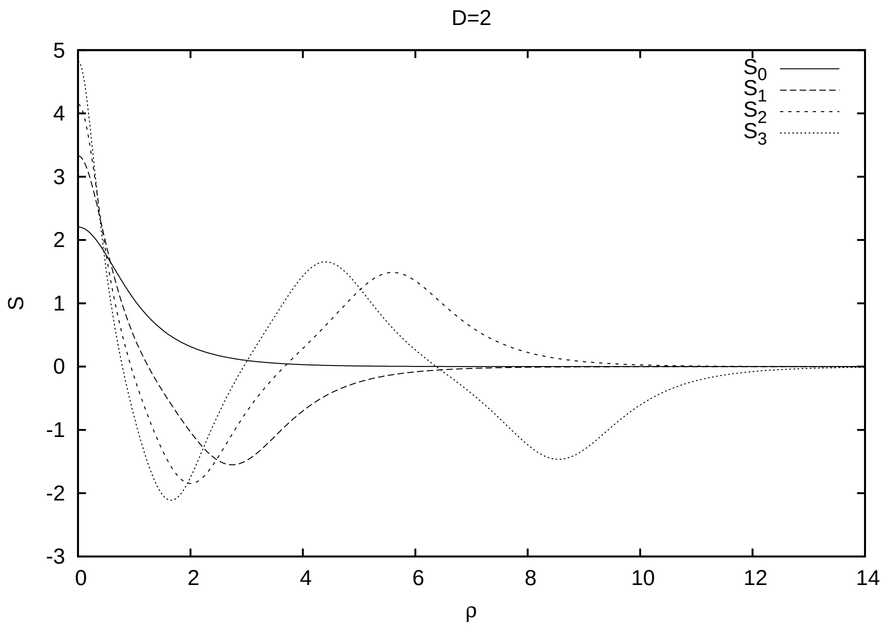

FIG. 1: Solutions of the master equation (44) in $D = 2$ with $0, 1, 2$ and 3 nodes.

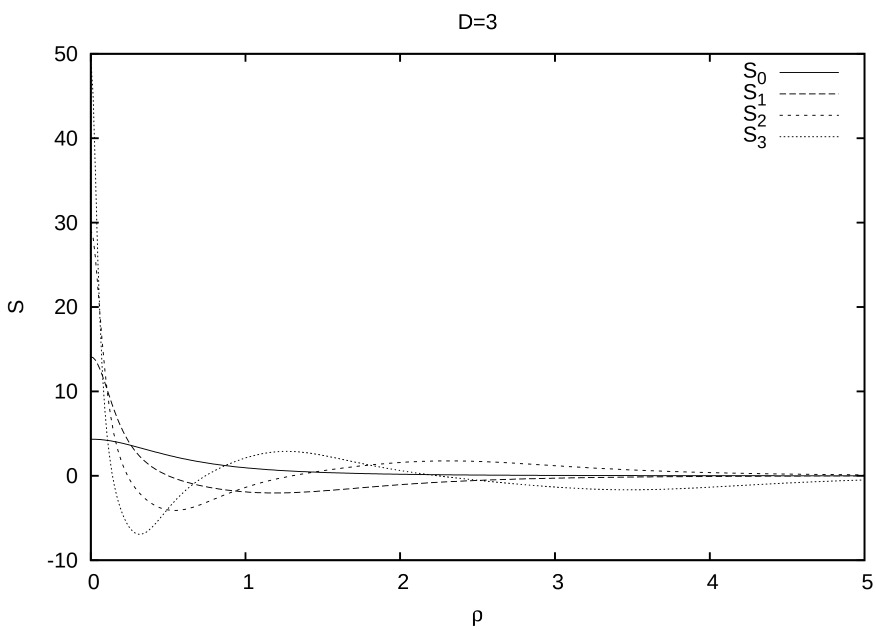

FIG. 2: Solutions of the master equation (44) in $D = 3$ dimensions with $0, 1, 2$ nodes.

number of nodes	$S(\rho = 0)$ $D = 2$	$S(\rho = 0)$ $D = 3$
0	2.20620086	4.33738768
1	3.33198927	14.10358440
2	4.15009404	29.13121158
3	4.82960282	49.36070988

TABLE I: Central values of S for two and three dimensions.

In the case of spherical symmetry, equation (27) determining the third and fourth order terms in ε takes the form

$$\frac{d^2Z}{d\rho^2} + \frac{D-1}{\rho}\frac{dZ}{d\rho} - Z + 3S^2Z - S^5 = 0\,. \tag{45}$$

Since Eq. (45) is linear for the unknown, Z, with inhomogeneity S^5 it admits a globally regular solution for any S regular at $\rho = 0$ and vanishing for $\rho \to \infty$. Some numerical solutions of Eq. (45) are plotted on Figs. 3 and 4 in two and three dimensions.

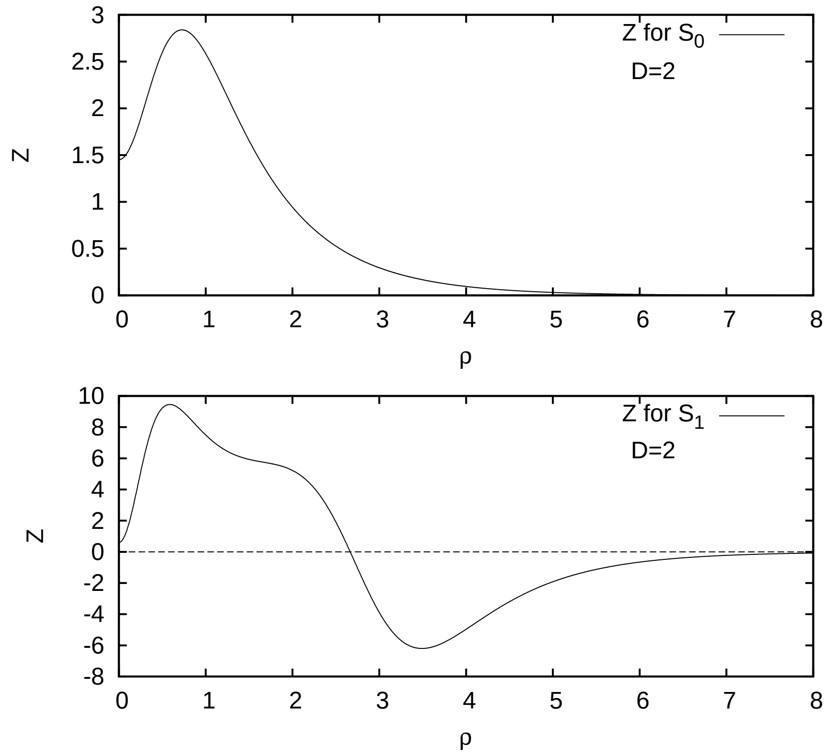

FIG. 3: Solutions of Eq. (45) for Z corresponding to S without and with 1 node in two dimensions. The central values are 1.45076 and 0.6018575, respectively.

A. Solution for the ϕ^4 theory up to 6th order

As already discussed, the equation determining the function, Z, is universal for any choice of the potential $U(\phi)$. Of course the terms in the small amplitude expansion do depend on the potential $U(\phi)$, and the reconstruction of ϕ_3 from Z using Eqs. (26) and (31) depends of the values of the coefficients g_i. In the rest of this paper we shall concentrate on the ϕ^4 theory given by Eq. (3) and provide the results of numerical simulations only for this case.

The equation determining the fifth order term in the small amplitude expansion, ϕ_5, through p_5 can be written as

$$\frac{d^2Y}{d\rho^2} + \frac{D-1}{\rho}\frac{dY}{d\rho} - Y + 3S^2Y + \frac{4225}{64}SZ(3Z-5S^3)$$
$$+\frac{53D(D-1)}{\rho^2}S\left(\frac{dS}{d\rho}\right)^2 + \frac{106(D-1)}{\rho}S^2(S^2-1)\frac{dS}{d\rho} + \frac{8287}{48}S^7 = 0\,, \tag{46}$$

where Y is defined by

$$p_5 = \frac{\sqrt{2}}{9\sqrt{3}}\left(Y - \frac{1235}{32}S^2Z + \frac{1503}{16}Z - 24S - \frac{17}{3}S^3 + \frac{11525}{384}S^5\right)\,. \tag{47}$$

The solution of Eq. (46) corresponding to the fundamental solution of the master Eq. S_0, has the central value $Y_0(0) = -87.78183$ in two dimensions and $Y_0(0) = 60356.38$ in $D = 3$. These actual values themselves have no

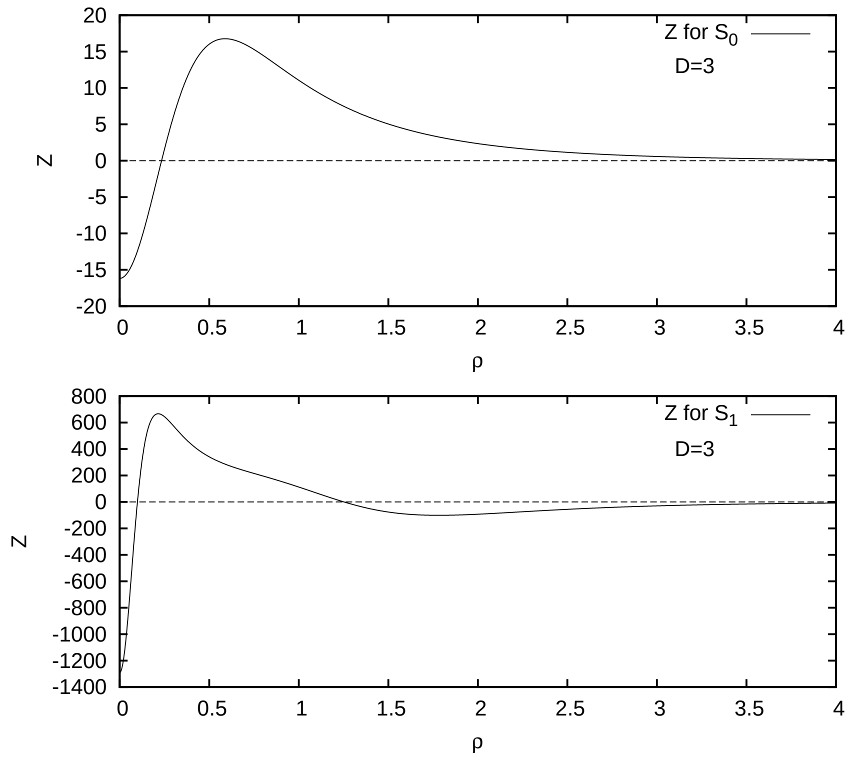

FIG. 4: Solutions of Eq. (45) in three dimensions. The central values of Z are -16.17403 and -1290.021.

physical significance in view of the scaling freedom in the definition of the function Y. Clearly, any constant could have been included in front of the term Y in Eq. (47) without modifying the final results for the magnitude of the ϕ_i's in the ε expansion.

In the following we list the values of the terms of the small amplitude expansion, ϕ_i, up to order six in the ϕ^4 theory, at the moment of time reflection symmetry, $\tau = 0$:

$$\phi_1^{(\tau=0)} = \sqrt{\frac{2}{3}}S \tag{48}$$

$$\phi_2^{(\tau=0)} = \frac{1}{3}S^2 \tag{49}$$

$$\phi_3^{(\tau=0)} = \frac{1}{9}\sqrt{\frac{2}{3}}\left(\frac{195}{8}Z - 8S - \frac{35}{8}S^3\right) \tag{50}$$

$$\phi_4^{(\tau=0)} = \frac{1}{9}\left[\frac{65}{4}SZ + 10\left(\frac{dS}{d\rho}\right)^2 + \frac{8}{3}S^2 - \frac{125}{12}S^4\right] \tag{51}$$

$$\phi_5^{(\tau=0)} = \frac{1}{9}\sqrt{\frac{2}{3}}\left[Y - \frac{2275}{64}S^2Z + \frac{1503}{16}Z - \frac{15}{32}S\left(\frac{dS}{d\rho}\right)^2 - 24S - \frac{595}{96}S^3 + \frac{11285}{384}S^5\right] \tag{52}$$

$$\begin{aligned}\phi_6^{(\tau=0)} = \frac{2}{27}\Bigg[&SY + \frac{325}{4}\frac{dS}{d\rho}\frac{dZ}{d\rho} + \frac{4225}{128}Z^2 - \frac{8125}{48}S^3Z + \frac{6589}{48}SZ - \frac{9223}{32}S^2\left(\frac{dS}{d\rho}\right)^2 + \frac{88}{3}\left(\frac{dS}{d\rho}\right)^2 \\ &+ \frac{26D(D-1)}{\rho^2}\left(\frac{dS}{d\rho}\right)^2 + \frac{52(D-1)}{\rho}S(S^2-1)\frac{dS}{d\rho} + \frac{92}{9}S^2 - \frac{35417}{288}S^4 + \frac{21467}{144}S^6\Bigg].\end{aligned} \tag{53}$$

These expressions are presented because they are needed to provide good initial data for numerical time evolution simulations. They will be actually used in the next Section IV. Since the range of $\phi_k^{(\tau=0)}$ increases very much with k, we depict the product $\varepsilon^k\phi_k^{(\tau=0)}$ for some chosen values of ε on Figs. 5, 6, 7 and 8. On these four figures S_0 and S_1 are depicted in spatial dimensions $D = 2$ and $D = 3$. In a given order in the expansion, the value of ε which brings the contribution of lower order terms approximately to the same order of magnitude will be used as an upper

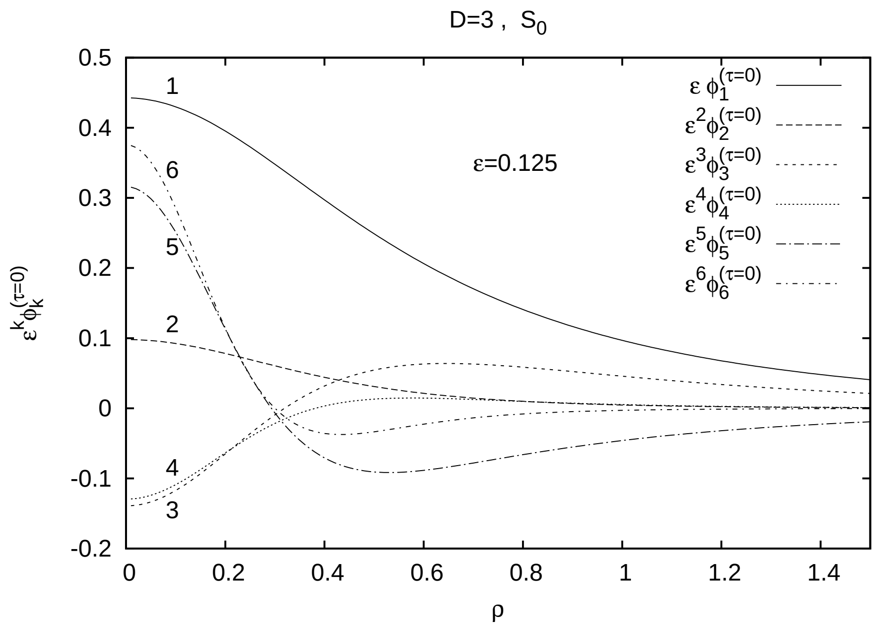

FIG. 5: Contributions of the various ε^k order terms corresponding to S_0 to the scalar field, ϕ, at the moment of time symmetry $\tau = 0$ in $D = 3$. The value of ε has been chosen to be 0.125, which brings the contribution of different terms to the same order. This value of ε is an obvious upper limit for the range of validity of our expansion.

estimate for the range of ε, below which our expansion can still be expected to yield an acceptable approximation. Generally speaking, in an asymptotic expansion for a given (small) value of the expansion parameter one can only sum terms up to such an order until which all terms decrease. A simple comparison of a QB corresponding to the

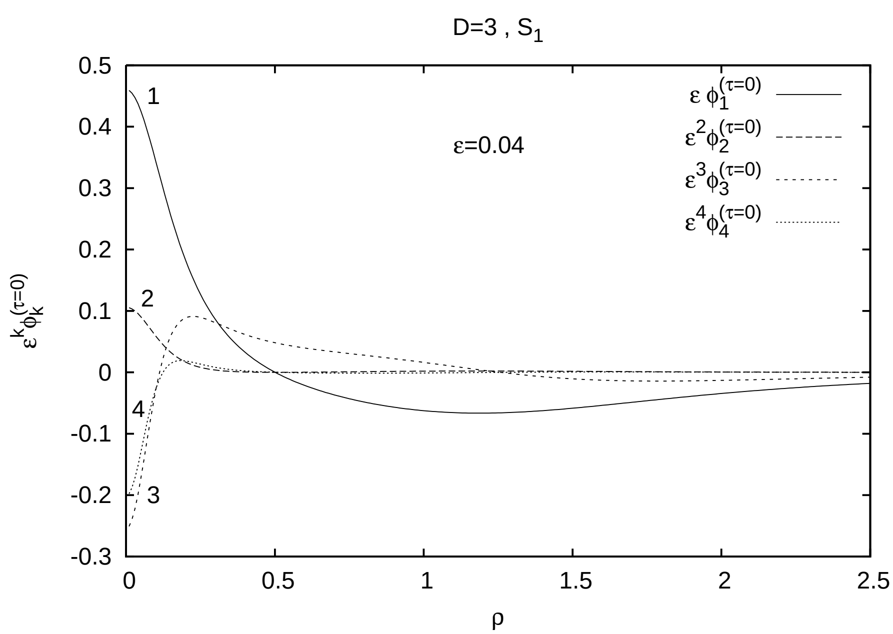

FIG. 6: $\varepsilon^k \phi_k^{(\tau=0)}$ corresponding to the solution S_1, with one node in the three dimensional case. Because of the sharp increase of the higher order functions the value of ε was chosen to be a much smaller as for S_0.

fundamental solution, S_0, with another one corresponding to a solution with a single node, S_1, (compare Figures 5 and 6), makes one to guess that oscillons containing fundamental QB's are likely to have better stability properties and longer lifetimes, than those containing QB's based on solutions with nodes, at least in $D = 3$. Our numerical simulations show that this is indeed the case (see Section IV). In two dimensions the difference between the longevity and stability properties of oscillons containing QB's corresponding to solutions of the S equation (21) with nodes is much less pronounced than in three spatial dimensions. It is also apparent that the ε expansion is valid for significantly larger values of ε in the two dimensional case than in the three dimensional one.

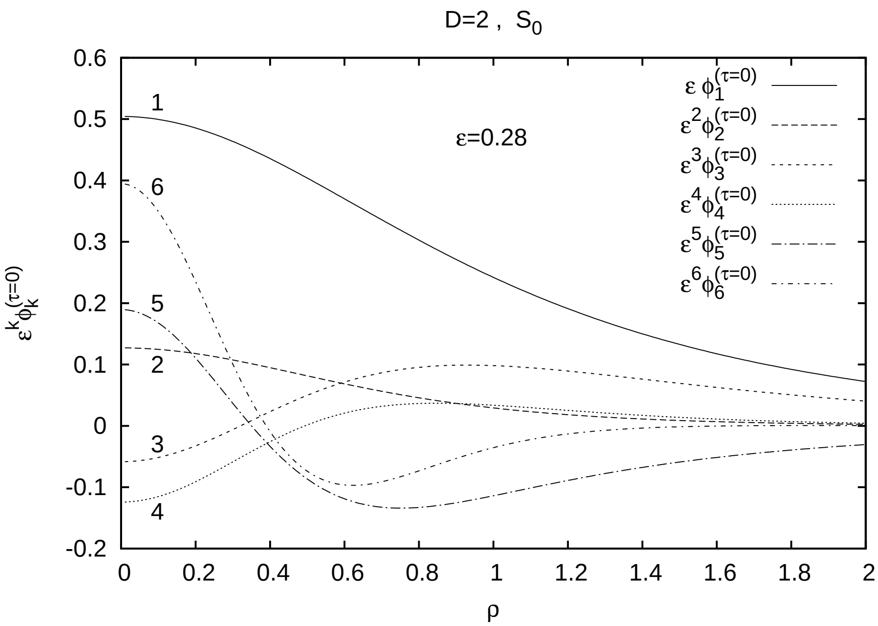

FIG. 7: $\varepsilon^k \phi_k^{(\tau=0)}$ corresponding to S_0 in the two dimensional case. It can be seen that the value of the expansion parameter ε can be chosen here significantly larger than for $D = 3$.

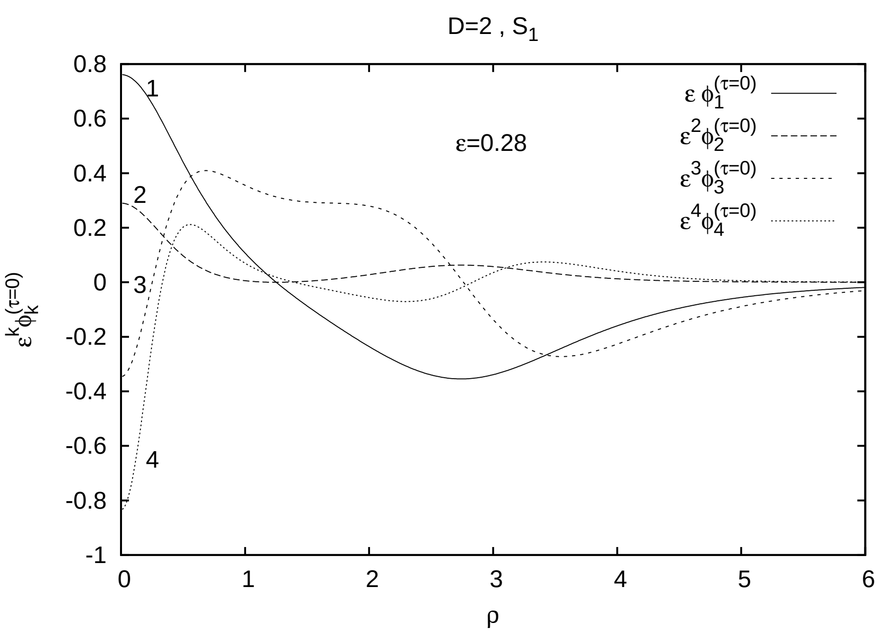

FIG. 8: $\varepsilon^k \phi_k^{(\tau=0)}$ for $D = 2$ and with S_1. The value of ε is the same as on Fig. 7. The contribution of the higher order terms is bigger than for S_0, nevertheless it is smaller than for $D = 3$.

In order to get an independent check on the validity of the small amplitude expansion, we have compared some QB's up to order 6, with time periodic QB's obtained previously by solving the NLWE (2) directly by Fourier mode decomposition [10]. On Figures 9 and 10 we depict ϕ computed to various orders in the ε-expansion, and also the QB obtained in Ref. [10] by Fourier mode decomposition using very precise spectral methods provided by the LORENE library [31]. The chosen frequencies correspond to two states investigated in detail in [10]. Note that in Ref. [10] the interaction potential had a different scale, and the resulting threshold frequency was $\sqrt{2}$ as opposed to the value 1 in the present paper. The periodic quasi-breather solutions chosen from Ref. [10] have frequencies $\tilde{\omega} = 1.412033$ and $\tilde{\omega} = 1.398665$ (see Figs 5. and 19. in [10]). In the present conventions these values correspond to frequencies $\omega = 0.9984581$ and $\omega = 0.9890055$. The corresponding values of ε are: $\varepsilon = 0.05551039$ and $\varepsilon = 0.1478787$. It can

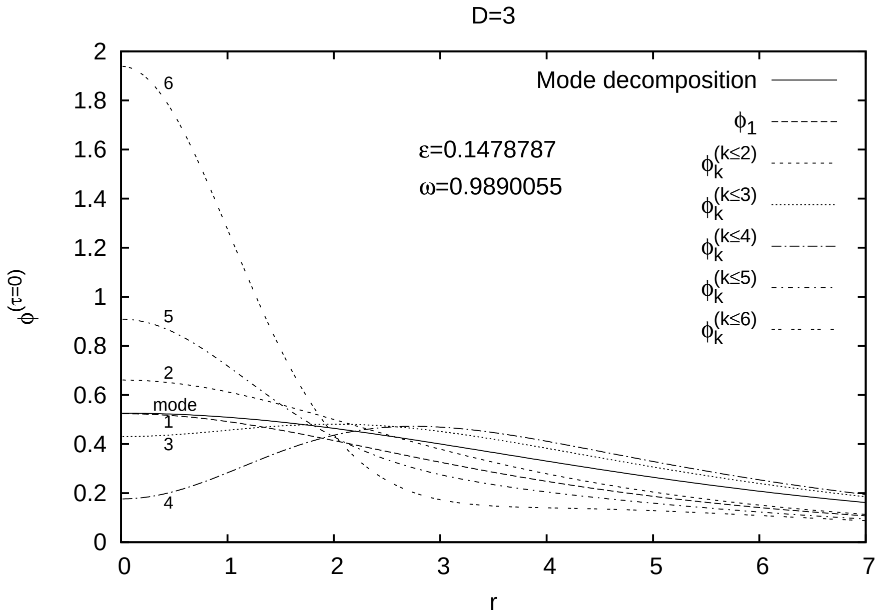

FIG. 9: Comparison of the value of the field at $\tau = 0$ obtained by the ε expansion to a very precise value obtained by a high order Fourier mode decomposition for $\omega = 0.9890055$, i.e. for $\varepsilon = 0.1478787$ in $D = 3$. For such a large value of ε the first order approximation gives a remarkably good estimate in the neighbourhood of the central region. The contributions of the higher order approximations make this agreement increasingly worse in the neighbourhood of the origin. Farther away from the center, however, the third order approximation gives the best, although not a very precise, result.

be seen on Figures 9, 10 that for higher values of ε, only the leading term or eventually the first two orders in the ε expansion give meaningful results. The relatively big error in these approximation cannot be decreased because of the asymptotic nature of the expansion. As ε gets smaller and the frequency ω gets closer to the basis frequency 1, more and more higher order terms in the ε expansion can be used, and then the error also decreases significantly. It can be seen on Fig. 10 that for the smaller value ε although the second, third and fifth order expansion gives an improvement on the lower order values, the fourth and sixth order expansion turns out to be less precise than the third and fifth order expressions. We think that this is related to the fact that the signature of the various order contributions changes in pairs, i.e. $\phi_k^{(\tau=0)}$ is positive at the center $r = 0$ for $k = 1, 2, 5, 6...$ and negative for $k = 3, 4, 7, 8...$, as can be seen on Fig. 5. This alternating improving and not improving behaviour for odd and even orders happens only for intermediate values of ε. For even smaller ε, e.g. for $\varepsilon = 0.01$, the error decreases monotonically when increasing the order of the expansion.

Let us now come back to the ε-expansion of the energy (39) computed in subsection II D. From our results it is not difficult to calculate numerically E_0 and E_1 in $D = 2$ and $D = 3$. We find that the first two terms of the time averaged energy in the ε-expansion are given as:

$$\bar{E} \approx 3.9003 + 26.9618\varepsilon^2 \,, \quad \text{for } D = 2 \,, \quad \bar{E} \approx 6.29908/\varepsilon + 264.262\varepsilon \,, \quad \text{for } D = 3 \,. \tag{54}$$

This simple estimate Eq. (54) gives for the minimal value of $\varepsilon_{\rm m} \approx 0.15428$, which is unfortunately already too large to be trusted. Nevertheless it can still be accepted as the indication that such a minimal value, $\omega_{\rm m}$ exists. For spherically symmetric oscillons in $D = 3$ it has been found that $\omega_{\rm m} \approx 0.9659$ [10]. This value of $\omega_{\rm m}$ corresponds to $\varepsilon_{\rm m} \approx 0.2588$ which value is way too large for us.

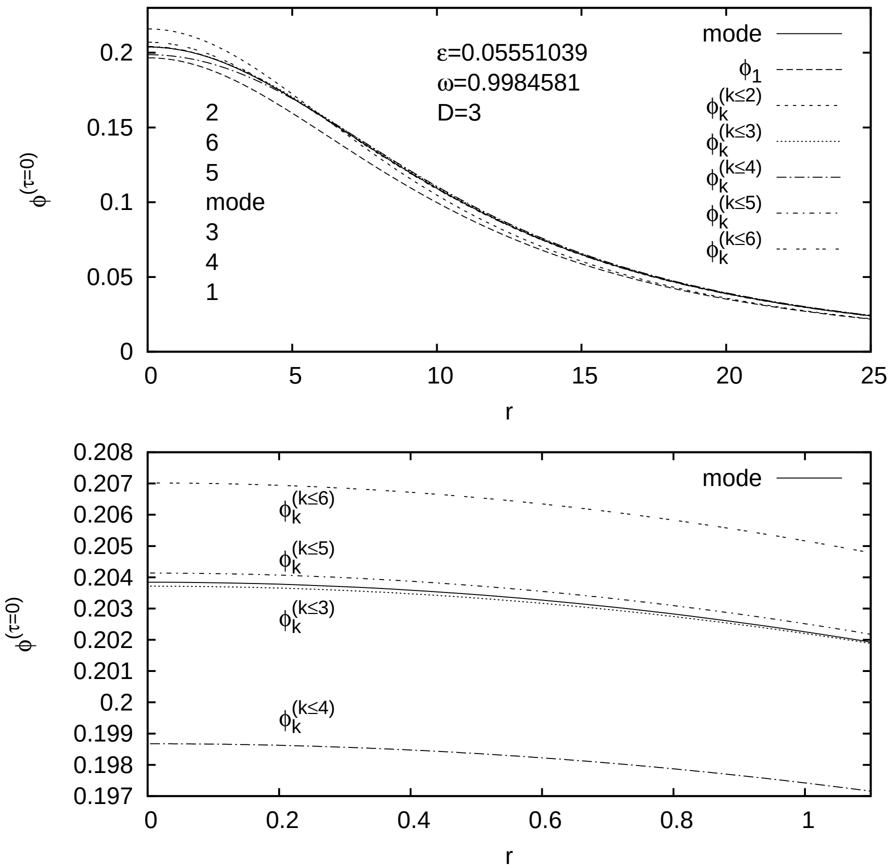

FIG. 10: Comparison of various order ε expansions of ϕ at $\tau = 0$ to the value obtained by Fourier mode decomposition for $\omega = 0.9984581$ i.e. for $\varepsilon = 0.05551039$ in the $D = 3$ case.

IV. TIME EVOLUTION

The precision and applicability of the ε expansion can be checked by using the field value obtained by the expansion as initial data for a numerical time evolution code applied for our spherically symmetric scalar field system. The field value given by Eqs. (48) - (53) and (5) at $\tau = t = 0$ has been used as initial data for various ε values in $D = 2$ and $D = 3$ spatial dimensions.

The applied numerical evolution code is a slightly modified version of the fourth order method of line code used in [10] for studying oscillons and developed in [32] for the study of spherically symmetric magnetic monopole configurations. The spatial grid is chosen to be uniform in the compactified radial coordinate R defined by

$$r = \frac{2R}{\kappa(1-R^2)}\,, \tag{55}$$

where κ is a constant which may be chosen differently though for each choice of initial data. The whole range $0 \le r < \infty$ of the physical radial coordinate r is mapped to the interval $0 \le R < 1$, avoiding the need for explicitly describing boundary conditions at some large but finite radius. Since the characteristic size of the obtained oscillon states is inversely proportional to ε we chose κ to be proportional to ε, keeping the oscillon occupying approximately the same region of the R coordinate range. For the actual calculations we used $\kappa = 5\varepsilon$.

Using the radial coordinate R the field equation (2) takes the form

$$\phi_{,tt} = \frac{\kappa^2(1-R^2)^3}{2(1+R^2)}\left[\frac{(1-R^2)}{2(1+R^2)}\phi_{,RR} - \frac{R(3+R^2)}{(1+R^2)^2}\phi_{,R} + \frac{(D-1)}{2R}\phi_{,R}\right] - U'(\phi)\,. \tag{56}$$

Introducing the new variables

$$\phi_t = \phi_{,t} \tag{57}$$

$$\phi_R = \phi_{,R} \tag{58}$$

the problem can be interpreted as a system of first order differential equations comprising (57), (56), and $\phi_{R,t} = \phi_{t,R}$ for the three variables ϕ, ϕ_t and ϕ_R. If equation (58) holds at $t = 0$ it is preserved by the evolution equations, thereby it can be considered as a constraint. The third term inside the bracket on the right hand side of (56) cannot be directly evaluated numerically at the center $R = 0$. At the grid point corresponding to the center this term is calculated using the identity

$$\lim_{r\to 0} \frac{\phi_{,R}}{R} = \lim_{r\to 0} \phi_{,RR} \, . \tag{59}$$

Since the evolution of the initial data provided by the ε expansion show different characteristics in two and three dimensions we discuss these cases in different subsections.

A. $D = 3$

In the three dimensional case, there are two different types of oscillons, a stable and an unstable type. For the frequency range $\omega < \omega_c \approx 0.967$, (i.e. for $\varepsilon > 0.255$), oscillons are essentially stable. They slowly radiate energy while their frequency, $\omega(t)$, increases towards a critical frequency ω_c. When they reach the critical frequency these oscillons quickly disintegrate. Oscillons with $\omega > \omega_c$ have one unstable decay mode, which can be suppressed by fine tuning the initial data. Close to the critical value of the parameter in the initial data there can be two types of decay mechanisms. One with a uniform outwards motion of the energy, and another through a temporary collapse to a small central region (see Fig. 4 of [10]). In order to find such oscillons a very precise fine tuning of the initial data is necessary. For example in Ref. [10] this fine tuning corresponded to the classical bisection procedure between two values of a suitable parameter in the initial data yielding the two different decay modes. This way one obtains very long living oscillon states. The frequency of these unstable oscillons decreases slowly towards $\omega(t) \to \omega_c$.

Remarkably the energy of QB's as a function of their frequency exhibits a minimum at $\omega \approx \omega_c$, (see Fig. 3 in [8], Fig. 17 of [10] and Fig. 4 of [29]). Therefore it is natural to assume that the two types of oscillons are also distinguished by the same behaviour of their energy as function of their oscillon frequency.

Unfortunately, for values of $\varepsilon \gtrsim 0.25$ initial data obtained by the ε expansion are well outside the domain of validity of the expansion. Using such initial data gives decaying states which are unrelated to the stable oscillons with the intended frequency. On the other hand the ε expansion yields good initial data for small amplitude unstable oscillons. As a matter of fact the initial data obtained this way makes a fine-tuning procedure of the initial data unnecessary. For sufficiently small values of $\varepsilon \lesssim 0.1$ the first few terms (at least to order $\mathcal{O}(\varepsilon^3)$) of the series expansion (48) - (53) already yield sufficiently good initial data which evolve directly into long living oscillon states.

For the three dimensional case we present the results of the time evolution of the initial data obtained up to order six by the ε expansion method for two choices of ε. First we consider initial data obtained from the basic solution S_0 of (21) without nodes. As we will see shortly, those with nodes provide initial data that leads to states of significantly shorter lifetimes. Figure 11 shows the upper envelope of the central value of ϕ for the initial data belonging to $\varepsilon = 0.05551039$. The decay method changes with the order of the initial data. The amplitude peak on the evolution of the third and fourth order initial data reflects the collapsing decay mode of the oscillon state. Although, in general, lifetimes get longer for higher order approximations, this increase is not monotonic. In accordance with Fig. 10, approximations of order 4 and 6 do not bring any improvement on the the functions of order 3 and 5. The time evolution of the frequency of the oscillations is plotted on Fig. 12. It can be seen that for this relatively high value of ε the first and second order approximation yields a shorter living state quite different from the expected oscillon state with frequency $\omega = 0.9984581$. On Figure 13 the evolution of a higher frequency initial data with $\varepsilon = 0.01$ is shown. It can be seen that initial data with smaller ε provide evolutions with significantly longer lifetimes. Fig. 14 shows the initial stage of the evolution. It can be seen that although for a short time the error of the solution decreases monotonically with the order of the ε expansion, for longer time intervals the fourth and sixth order approximations do not improve on the previous order expansions.

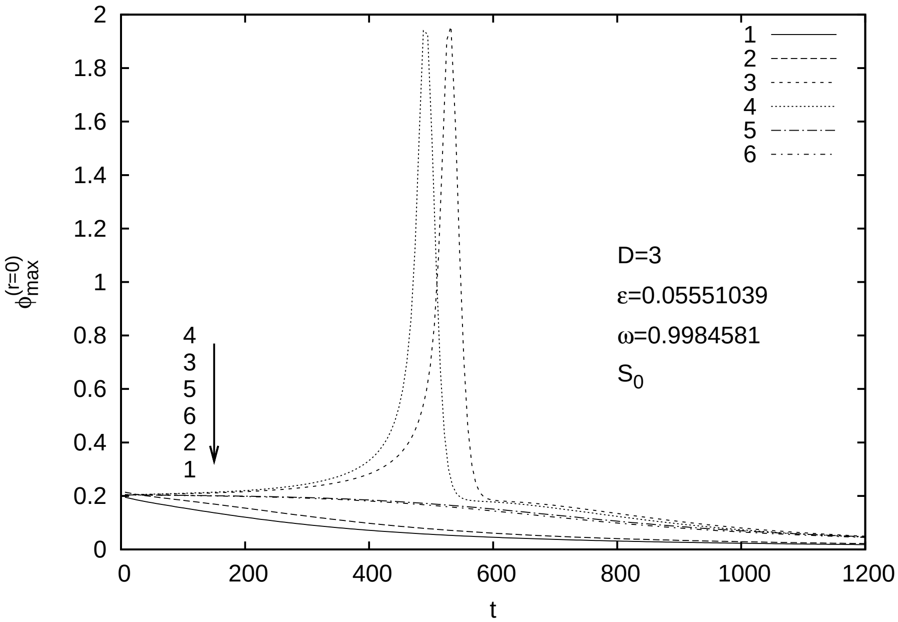

FIG. 11: Time evolution of the initial data obtained by the ε expansion method up to order six for $\varepsilon = 0.05551039$ in three spatial dimensions.

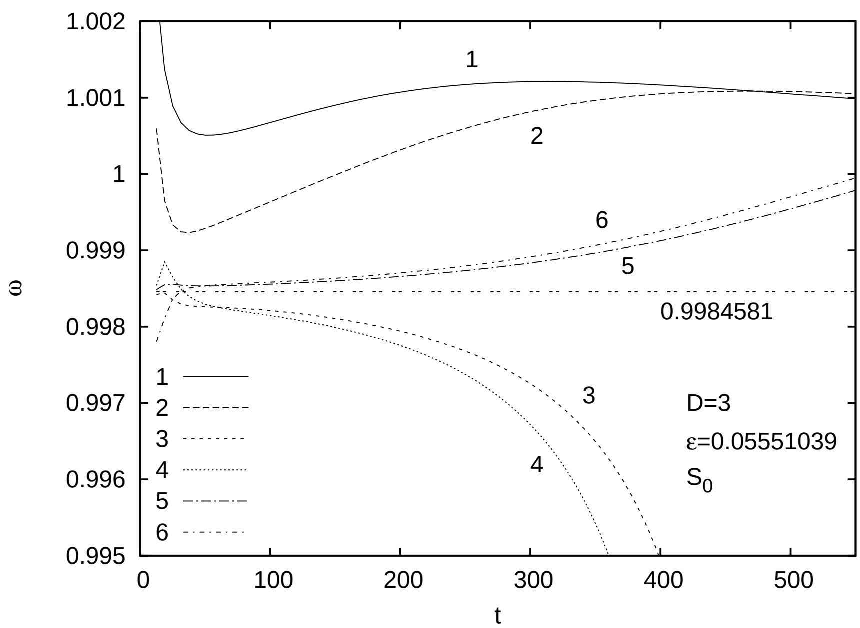

FIG. 12: Oscillation frequency as a function of time for the states shown on Fig. 11.

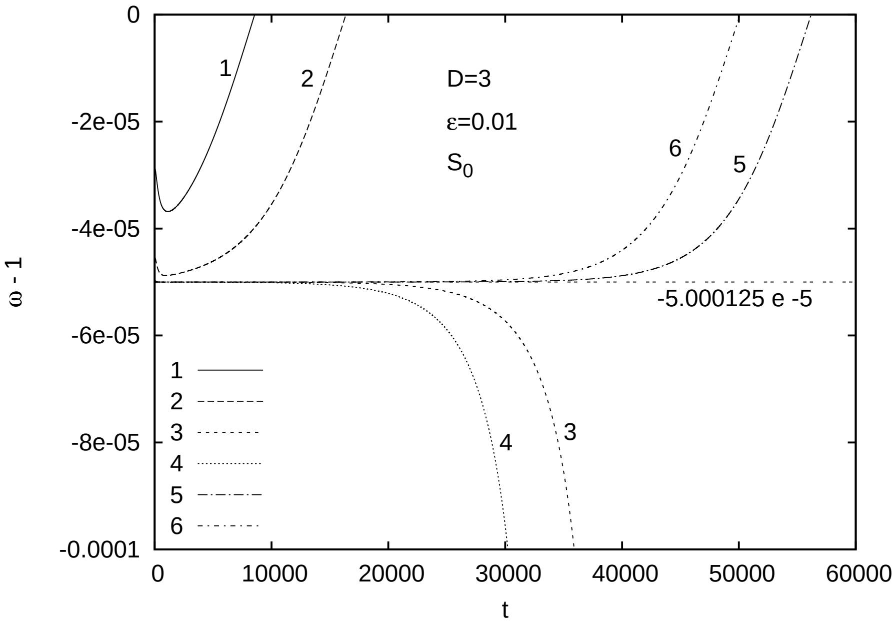

FIG. 13: Time evolution of the frequency of the oscillations of ϕ evolving from initial data with $\varepsilon = 0.01$ in the $D = 3$ case. Since the frequency is very close to one, the value $\omega - 1$ is plotted instead of ω.

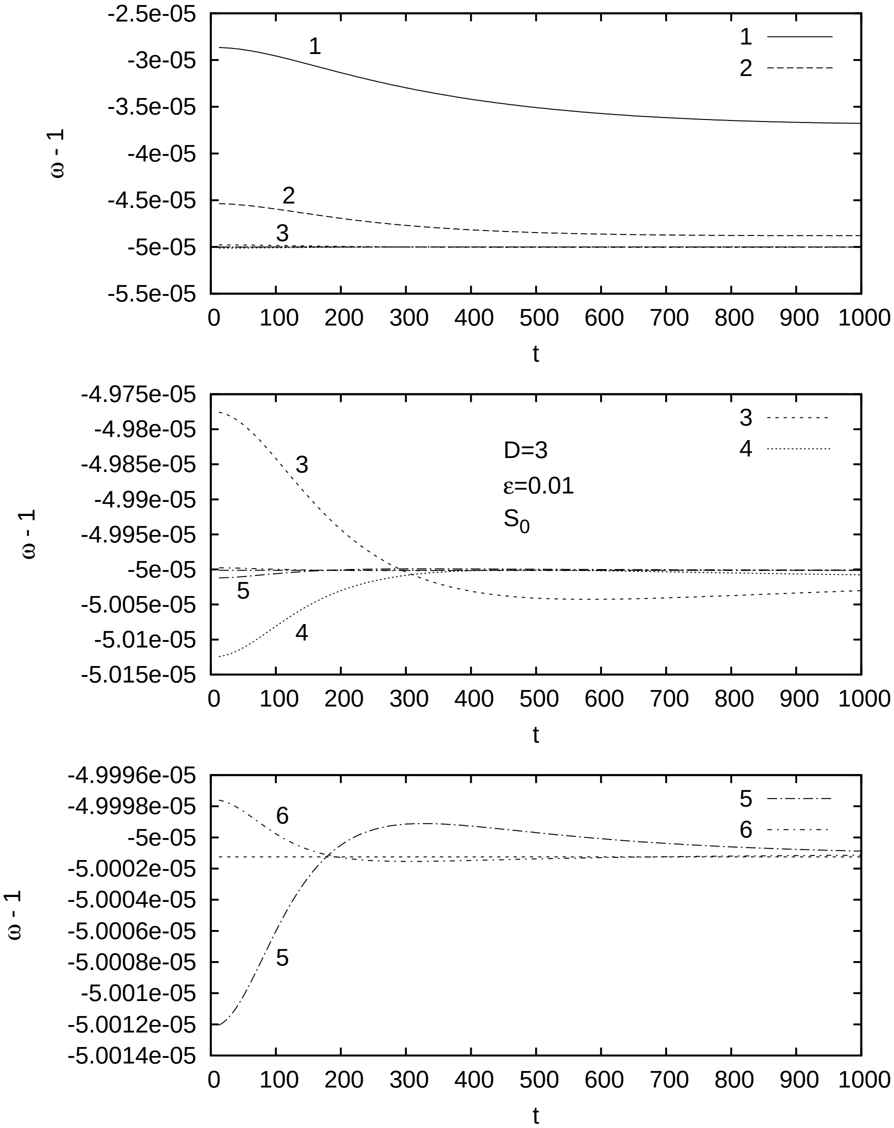

FIG. 14: Beginning of the time evolution simulations shown on Fig. 13. For this shorter time interval the difference from the expected solution with $\omega = 0.99994999875$ decreases monotonically with the order of the ε approximation.

Figure 15 shows the evolution of various order initial data with the same ε as on the previous two figures but obtained by using the solution S_1 of (21) with one node. These are localized, although big size, high energy states.

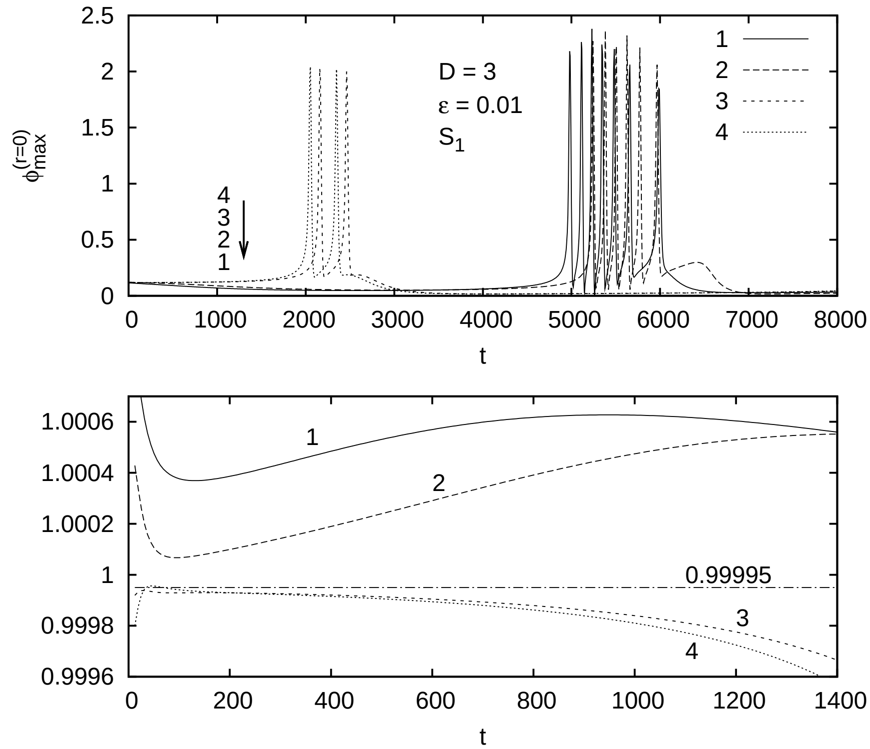

FIG. 15: Time evolution of the initial data obtained by the ε expansion up to order four using the solution S_1 of (21). Space is three dimensional and $\varepsilon = 0.01$. The top graph shows the upper envelope of ϕ at $r = 0$, while the lower graph plots the evolution of the oscillation frequency. It can be seen that the first and second order initial data still evolve to states very different from the one with the expected frequency.

The time dependence of these states is rather complex; there is an interior part ($r \lesssim 50$) which after an initial time interval (≈ 1200) shows a complicated time dependence, with large amplitude variations. The time dependence of these states is close to being periodic for recurrent time intervals. In the initial stages these initial data still evolve close to a periodic configuration with the expected frequency, although it can stay near this state much shorter time than the evolution obtained using S_0. Decreasing the value of ε increases the lifetime of these states as well, but they still remain less stable and shorter living than the basic states obtained by using S_0.

On Figure 16 the value of ϕ as a function of the radial coordinate r for subsequent time slices is plotted during a half period of oscillation. For comparison, the corresponding configuration obtained from initial data generated with S_0 is also presented. In case of S_1 initial data the value of ϕ remains very close to zero at $r = 51.3$. The energy density $\mathcal{E}$ also remains very small at this radius, as can be seen on Fig. 17. The total energy of the S_0 configuration is $E = 632.536$, while the S_1 configuration contains significantly more energy, $E = 4061.88$.

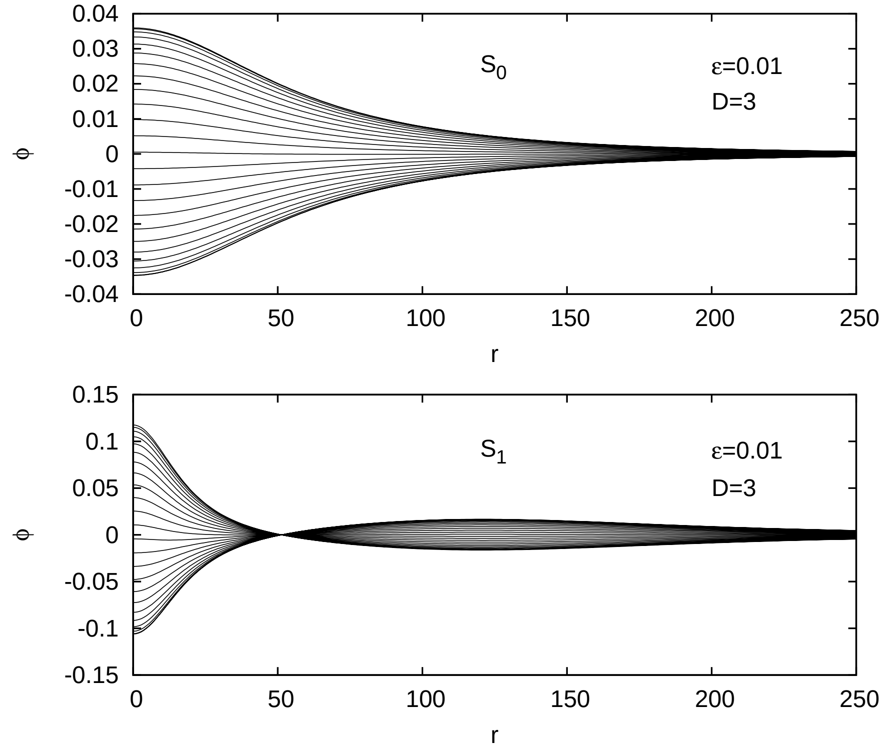

FIG. 16: Time evolution of ϕ during a half period of oscillation, plotted as a function of the radial distance r. The upper plot corresponds to initial data obtained from S_0, the lower from S_1. The absolute value of ϕ at $r = 51.3$ remains below 10^{-5}. Both plots contain lines corresponding to uniform time steps between the first maximum and minimum after $t = 300$, although the plot would remain very similar in a large time interval. The initial data in the S_0 case was generated by a sixth order ε expansion, while a third order expansion was used in the S_1 case.

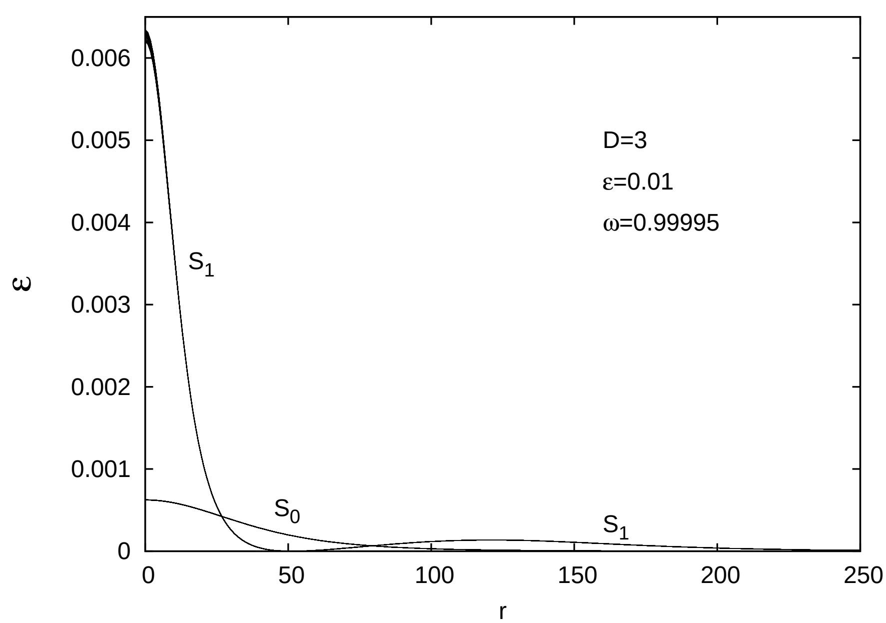

FIG. 17: Energy density $\mathcal{E}$ of oscillon configurations evolved from S_0 and S_1 initial data, using $\varepsilon = 0.01$. The energy density is plotted on the same uniformly placed moments of time as on Fig. 16. The minimum of $\mathcal{E}$ in the S_1 case remains below 10^{-7} near $r = 51.3$.

B. $D = 2$

In two spatial dimensions there seems to be a single type of oscillon, which is stable. Once formed, oscillons in $D = 2$ are not observed to disintegrate. Their energy is a monotonically decreasing function of the frequency. This explains the observation that all oscillon states are stable. When they slowly emit energy by radiation they gradually evolve through oscillon states with increasing frequency, $\omega(t) \to 1$. Oscillons evolve from a wide range of initial data, by shedding most of the surplus energy quickly during an initial state. However, in general, a slow periodic change can be seen on the amplitude and on the frequency of the oscillations, indicating a breathing type oscillation of the oscillon as a whole. The amplitude of this low frequency ringing depends on how closely the initial data approaches a given pure oscillon state.

Since, similarly to the $D = 3$ case, evolutions from initial data obtained using solutions S of (21) with nodes produce less stable and shorter living states, in the following we present only numerical simulations corresponding to the nodeless solution S_0. In two dimensions the validity domain of the expansion extends to significantly higher values of ε. On Fig. 18 we present time evolution results for $\varepsilon = 0.218632$ corresponding to $\omega = 0.9758073$. It can

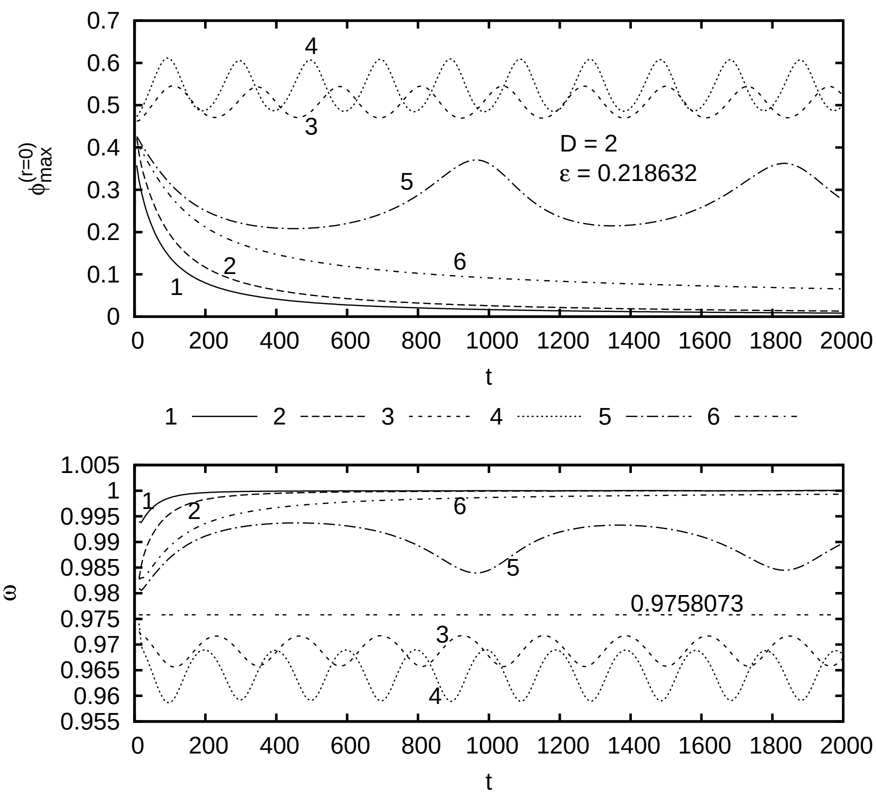

FIG. 18: Time evolution of the initial data produced by various order ε expansions for $\varepsilon = 0.218632$ in the two dimensional case. The top graph shows the upper envelope of ϕ at $r = 0$, while the bottom graph the time dependence of the frequency. The evolution of the third and fourth order initial data clearly corresponds to some oscillon state deformed by a low frequency ringing mode. The evolution of the fifth order initial data corresponds to a similar state with extreme high amplitude ringing. The first, second and sixth order initial data evolve into decaying modes, indicating a large error at these orders of the ε expansion.

be seen that at this high ε value the approximation improves up to order tree in the expansion, and then starts to deteriorate in line with the asymptotic nature of the expansion.

On Fig. 19 the $\varepsilon = 0.05551039$ case is presented. This value of ε with the corresponding frequency $\omega = 0.9984581$ has been also studied in $D = 3$ dimensions. In this intermediate frequency case the first and second order initial data still lead to decaying evolutions. Higher order expansions tend to give improving approximations of an oscillon state ringing with a low frequency. Increasing the order of the expansion gives smaller amplitude ringings, thereby approaching a pure, very closely periodic, oscillon state. Similarly to the three dimensional case, initial data of order four and six yield larger error than order three and five, which, however, does not change the overall improving

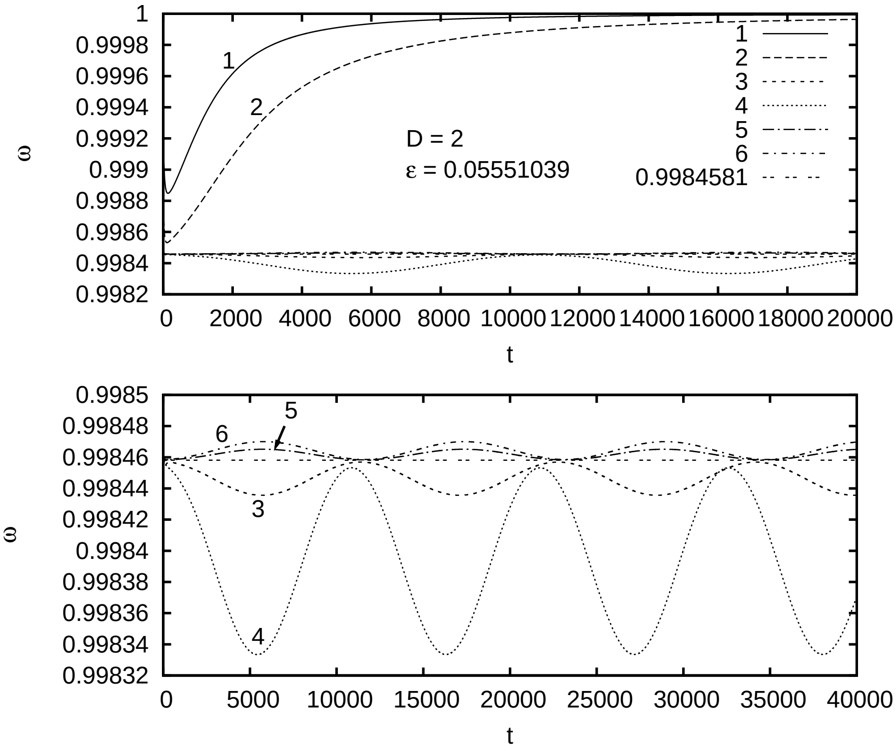

FIG. 19: The frequency as a function of time for the evolution of initial data with $\varepsilon = 0.05551039$ in case of $D = 2$.

tendency of the initial data.

The average amplitude and average frequency of the presented states show extremely little change even for much longer time periods than the ones presented on the figures. The amplitude of the low frequency ringing decreases very slowly too. To determine the rate of change of the amplitude and the corresponding slow energy loss by radiation would require very high resolution numerical runs requiring excessive processor time.

V. CONCLUSIONS

Small amplitude oscillons represent an important subset of time dependent long-living lumps. We have shown that they can be very well approximated by an asymptotic series of localized, time-periodic breather-like objects (quasi-breathers). We have developed a general framework to derive the asymptotic series expansion of small amplitude quasi-breathers, in D spatial dimensions in general scalar theories. We have derived a 2nd order elliptic PDE with a cubic non-linearity, universal for scalar models, which determines these quasi-breathers. Our numerical investigations in ϕ^4-theories show that the small amplitude quasi-breathers obtained by the asymptotic expansion, provide excellent initial data for long-living oscillons in $D = 2$ and $D = 3$. We have found that small amplitude QB's do not exist for $D \geq 4$.

VI. ACKNOWLEDGMENTS

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[1] A. E. Kudryavtsev, *JETP Letters* **22**, 82 (1975).
[2] I. L. Bogolyubskii, and V. G. Makhan'kov, *JETP Letters* **24**, 12 (1976), ibid. **25**, 107 (1977).

[3] M. Gleiser, *Phys. Rev.* **D49**, 2978 (1994).
[4] E. J. Copeland, M. Gleiser and H.-R. Müller, *Phys. Rev.* **D52**, 1920 (1995).
[5] M. Gleiser *Phys. Lett.* **B600**, 126 (2004).
[6] B. Piette, W. J. Zakrzewski, *Nonlinearity* **11**, 1103 (1998).
[7] M. Hindmarsh and P. Salmi, *Phys. Rev.* **D74**, 105005 (2006).
[8] R. Watkins, Theory of oscillons, DART-HEP-96/03 preprint (1996).
[9] E. P. Honda and M. W. Choptuik, *Phys. Rev.* **D65**, 084037 (2002).
[10] G. Fodor, P. Forgács, P. Grandclément and I. Rácz,*Phys. Rev.* **D74**, 124003 (2006).
[11] E. Farhi, N. Graham, V. Khemani, R. Markov and R. Rosales, *Phys. Rev.* **D72**, 101701(R) (2005).
[12] N. Graham *Phys. Rev.Lett.* **98**, 101801 (2007); *Phys. Rev.* **D76**, 085017 (2007).
[13] A. B. Adib, M. Gleiser, C. A. S. Almeida, *Phys. Rev.* **D66,** 085011 (2002).
[14] E. W. Kolb and I. I. Tkachev, *Phys. Rev.* **D49**, 5040 (1994).
[15] M. Gleiser and J. Thorarinson, *Phys. Rev.* **D76**, 041701(R) (2007).
[16] M. Hindmarsh and P. Salmi, arXiv:0712.0614 [hep-th] (2007).
[17] S. Kasuya, M. Kawasaki, F. Takahashi, *Phys. Lett.* **B559** 99-106 (2003).
[18] M. Gleiser, B. Rogers and J. Thorarinson, *Phys. Rev.* **D77**, 023513 (2008).
[19] M. Broadhead and J. McDonald, *Phys. Rev.* **D72**, 043519 (2005).
[20] I. Dymnikova, L. Koziel, M. Khlopov, S. Rubin, *Gravitation and Cosmology* **6**, 311 (2000).
[21] N. Graham and N. Stamatopoulos, *Phys. Lett.* **B639**, 541 (2006).
[22] Sz. Borsanyi, M. Hindmarsh, arXiv:0712.0300 [hep-ph] (2007).
[23] V. M. Eleonski, N. E. Kulagin, N. S. Novozhilova and V. P. Silin, *Teor. Mat. Fiz* **60**, 395 (1984).
[24] Y.A. Kivshar, B.A. Malomed, *Rev. Mod. Phys* **61**, 763 (1989).
[25] H. Segur and M. D. Kruskal, *Phys. Rev. Lett.* **58**, 747 (1987).
[26] P. A. Vuillermot, *Comment. Math. Helvetici* **64**, 573 (1987).
[27] S. Kichenassamy, *Comm. Pur. Appl. Math.* **44**, 789 (1991).
[28] H. Arodź, P. Klimas and T. Tyranowski, arXiv:0710.2244 (2007).
[29] P. M. Saffin and A. Tranberg, *JHEP* 01(2007)030 (2007).
[30] G. Fodor, P. Forgács, Z. Horváth, Á. Lukács, in preparation.
[31] P. Grandclément, S. Bonazzola, E. Gourgoulhon and J.-A. Marck, *J. Comput. Phys.* **170**, 231 (2001); E. Gourgoulhon, P. Grandclément, K. Taniguchi, J.-A. Marck and S. Bonazzola, *Phys. Rev.* **D63**, 064029 (2001).
[32] G. Fodor and I. Rácz, *Phys. Rev. Lett.* **92**, 151801 (2004).