On sets where lip f is finite

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Abstract

Given a function $f\colon \mathbb{R} \to \mathbb{R}$, the so-called "little lip" function $\operatorname{lip} f$ is defined as follows:

$$\operatorname{lip} f(x) = \liminf_{r \searrow 0} \sup_{|x-y| \le r} \frac{|f(y) - f(x)|}{r}.$$

We show that if f is continuous on $\mathbb{R}$, then the set where $\operatorname{lip} f$ is infinite is a countable union of a countable intersection of closed sets (that is an $F_{\sigma\delta}$ set). On the other hand, given a countable union of closed sets E, we construct a continuous function f such that $\operatorname{lip} f$ is infinite exactly on E. A further result is that for the typical continuous function f on the real line $\operatorname{lip} f$ vanishes almost everywhere.

1 Introduction

Throughout this section we will assume that f is a continuous real-valued function that is defined on $\mathbb{R}$. The so-called "big Lip" function, $\operatorname{Lip} f$, is defined as follows:

$$\operatorname{Lip} f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} = \limsup_{r \searrow 0} \sup_{|x-y| \le r} \frac{|f(y) - f(x)|}{r}.$$

According to the Rademacher–Stepanov Theorem, [7], f is differentiable almost everywhere on the set

$$L_f = \{x \in \mathbb{R} : \operatorname{Lip} f(x) < \infty\}.$$

More recently, the "little lip" function, $\operatorname{lip} f$, which is defined as follows, has been investigated:

$$\operatorname{lip} f(x) = \liminf_{r \searrow 0} \sup_{|x-y| \le r} \frac{|f(y) - f(x)|}{r}.$$

For example, $\operatorname{lip} f$ shows up in J. Cheeger's seminal paper [4], in which he shows that in quite general metric measure spaces a version of Rademacher's theorem holds. It also features prominently in [6], where it makes its appearance as part of a sufficient condition for a version of Rademacher's theorem. In [1] Balogh and Csörnyei show that the Rademacher–Stepanov Theorem does not remain true if we replace L_f with

$$l_f = \{x \in \mathbb{R} : \operatorname{lip} f(x) < \infty\}.$$

In fact, they produce an example where $\operatorname{lip} f(x) = 0$ almost everywhere, but f is nowhere differentiable. In [5] Hanson, the second author of this paper, sharpens their result to show that the exceptional set where lip $f(x) \neq 0$ can even be made to have Hausdorff dimension 0. On the other hand, Balogh and Csörnyei also show that if $\mathbb{R}\backslash l_f$ is countable, then every interval contains a set of positive measure on which f is differentiable.

Given the relationship between L_f and l_f and the set of differentiability of f, it is interesting to determine the possible structure of the sets L_f and l_f. It is not difficult to show that a set $E \subset \mathbb{R}$ is equal to L_f for some continuous function f if and only if E is an F_σ set, that is E is a countable union of closed sets (see Lemma 2.4 (b) and Theorem 3.35). On the other hand, characterizing the set l_f is more difficult. Recall that a $G_{\delta\sigma}$ set is a set that can be written as countable union of countable intersections of open sets. It is straightforward to show that every l_f is a $G_{\delta\sigma}$ set (see Lemma 2.4 (a)), and we conjecture that the converse is also true, namely that every $G_{\delta\sigma}$ set is equal to l_f for some continuous function f. Determining the truth of this conjecture appears to be quite difficult. The main result of this paper is to show that for every G_δ set E there is a continuous function f such that $l_f = E$, and already the proof of this result requires a bit of work.

On a related note, according to a result of Banach [2], for the typical continuous function $f \in C([0,1])$, we have Lip $f(x) = \infty$ for all $x \in [0,1]$. In contrast to this result, we show that, somewhat surprisingly, for the typical continuous function f we have lip f equal to 0 at points of a residual set of full measure.

The structure of the paper is as follows: In Section 2 we introduce some notation and present a few basic results about L_f and l_f. Section 3 contains the proof of the main result that for every G_δ set E there exists a continuous function f such that $l_f = E$. We also show that if E is an F_σ set, then $E = l_f$ for some continuous function f. Finally, in Section 4 we show that the typical continuous function has lip equal to 0 at points of a residual set of full measure.

2 Notation and Basic Results

We denote by $\overline{A}$ the closure of the set A.
A set E in a complete metric space X is of *first Baire category*, also called *meager*, if it is the union of countably many nowhere dense sets. We say that a property is *typical*, also known as *generic*, in X if the set of those $x \in X$ that do not have this property is meager.

After introducing some notation concerning the sets where lip and Lip are finite and infinite, respectively, we determine their places in the Borel hierarchy.

Definition 2.1 (The sets L_f, L_f^∞, l_f and l_f^∞)**.** For a continuous function $f\colon \mathbb{R}\to\mathbb{R}$, we set

$$\begin{aligned} L_f &= \{x\in\mathbb{R} : \operatorname{Lip} f(x) < \infty\},\\ L_f^\infty &= \{x\in\mathbb{R} : \operatorname{Lip} f(x) = \infty\},\\ l_f &= \{x\in\mathbb{R} : \operatorname{lip} f(x) < \infty\},\\ l_f^\infty &= \{x\in\mathbb{R} : \operatorname{lip} f(x) = \infty\}. \end{aligned}$$

2.1 Baire classes of l_f and L_f

Definition 2.2 (The functions $q_f(x,r)$, $l_f(x,r)$, and $L_f(x,r)$)**.** We assume that $f\colon I\to\mathbb{R}$ is a function, where $I\subset\mathbb{R}$ is a closed subinterval of $\mathbb{R}$ or $\mathbb{R}$ itself. We let r and R be positive numbers. Then we define the following quantities:

- $q_f(x,r) = \sup_{y\in[x-r,x+r]\cap I} \frac{|f(y)-f(x)|}{r}$,
- $l_f(x,R) = \inf_{r\in(0,R)} q_f(x,r)$,
- $L_f(x,R) = \sup_{r\in(0,R)} q_f(x,r)$.

Observation 2.3. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function. Then the following statements hold.

(a) Let $r>0$ be fixed. Then $q_f(\cdot,r)$ is a continuous function.

(b) With $R>0$ fixed, $l_f(\cdot,R)$ is upper semi-continuous and $L_f(\cdot,R)$ is lower semi-continuous.

(c) $\operatorname{lip} f(x) = \lim_{R\searrow 0} l_f(x,R) = \lim_{n\to\infty} l_f(x,1/n) = \sup_{n\in\mathbb{N}} l_f(x,1/n)$.

(d) $\operatorname{Lip} f(x) = \lim_{R\searrow 0} L_f(x,R) = \lim_{n\to\infty} L_f(x,1/n) = \inf_{n\in\mathbb{N}} L_f(x,1/n)$.

Proof. The proofs of (a), (c) and (d) are trivial from the definitions. The proof of (b) uses (a) and the fact that upper semi-continuous functions are closed under taking infima, and lower semi-continuous functions are closed under taking suprema. □

Lemma 2.4. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function. In this case, the following statements are valid.*

(a) The set l_f is a $G_{\delta\sigma}$ set.

(b) *The set* L_f *is an* F_σ *set.*

(c) *The set* $\{x : \operatorname{lip} f(x) = 0\}$ *is a* G_δ *set.*

Proof. To prove (a), we note that:

$$\begin{aligned}\operatorname{lip} f(x) = \infty \quad &\iff \quad \forall k \in \mathbb{N}: \quad \operatorname{lip} f(x) > k \\ &\iff \quad \forall k \in \mathbb{N} \quad \exists n \in \mathbb{N}: \quad l_f\left(x, \frac{1}{n}\right) \geq k \\ &\iff \quad \forall k \in \mathbb{N} \quad \exists n \in \mathbb{N} \quad \forall r \in \left(0, \frac{1}{n}\right): \quad q_f(x,r) \geq k \\ &\iff \quad x \in \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{r\in(0,1/n)} \{z \in \mathbb{R} : q_f(z,r) \geq k\}.\end{aligned}$$

It now follows from the continuity of q_f that l_f^∞ is an $F_{\sigma\delta}$ set, establishing (a). The proofs of (b) and (c) are similar and left to the reader. □

3 The Main Result

Theorem 3.1. *Let* $F \subset \mathbb{R}$ *be an* F_σ *set. Then there exists a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $l_f^\infty = F$.

The proof of Theorem 3.1 is rather involved and will be accomplished in a sequence of steps. We start with supposing that F is countable. In the second step, we handle a nowhere dense perfect set, which prepares us for the case where F is the countable union of nowhere dense perfect sets. Having established this case, we next look at the case where the sets are nowhere dense and merely closed. In the final step, we consider the general case: a countable union of closed sets.

3.1 The countable case

Theorem 3.2. *Given any countable set of points* S *in* $\mathbb{R}$*, there exists a continuous, increasing function* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$l_f^\infty = S \tag{3.1}$$

and

$$\operatorname{Lip} f(x) < \infty \text{ for all } x \notin \overline{S}. \tag{3.2}$$

If S is a finite set, it is trivial to verify the theorem, so we will assume throughout this section that S is countably infinite, and we write henceforth $S = \{s_1, s_2, \ldots\}$. We also assume without loss of generality that $S \subset (0, 1)$.

We begin with a few definitions.

Definition 3.3 (acceptable for)**.** Given x_0 in $(0,1)$, we say that the sequence $\{x_n\} \subset (0,1)$ is *acceptable for* x_0 if

$$x_n \notin S \text{ for } n = 2, 3, \dots, \tag{3.3}$$

$$x_n \searrow x_0, \tag{3.4}$$

and

$$x_{n+1} - x_{n+2} < \frac{1}{3}(x_n - x_{n+1}) \text{ for all } n \in \mathbb{N}. \tag{3.5}$$

We note that given any $0 < x_0 < y < 1$, we can easily choose a sequence $\{x_n\}$ that is acceptable for x_0 such that $x_1 = y$.

Definition 3.4 (The function $f_{\{x_n\}}$)**.** Given a sequence $\{x_n\}$ that is acceptable for x_0, we define $f = f_{\{x_n\}}$ on $[x_0, x_1]$ as follows:

$$f(x_0) = 0, \tag{3.6}$$

$$f(x_n) = 2^{-n+1}(x_1 - x_0) \text{ for all } n \in \mathbb{N}, \tag{3.7}$$

$$f \text{ is linear on each } I_n = [x_{n+1}, x_n] \text{ for all } n \in \mathbb{N}. \tag{3.8}$$

Remark 3.5. Note that by (3.7) the slope m_n of f on I_n satisfies the equation $m_n = \frac{x_1 - x_0}{2^n(x_n - x_{n+1})}$.

On the other hand, (3.5) guarantees that[1] $x_n - x_{n+1} < \frac{x_1 - x_0}{3^{n-1}}$, so we get $m_n > \frac{3^{n-1}}{2^n} \to \infty$, and therefore $\operatorname{lip} f(x_0) = \infty$.

Definition 3.6 (acceptable on)**.** Suppose that g is defined on $[0,1]$ and $\{x_n\}$ is acceptable for $x_0 \in (0,1)$. Then we say that g is *acceptable on* $[x_0, x_1]$ if there is a constant c such that $g = f_{\{x_n\}} + c$ on $[x_0, x_1]$.

Definition 3.7 (parallelogram $P_{L,\varepsilon}$ and q_P)**.** Suppose that $0 < \varepsilon < 1$ and L is a finite, closed line segment in $\mathbb{R}^2$ with positive slope m. Let A and B denote the endpoints of L and define $P = P_{L,\varepsilon}$ to be the closed parallelogram with L as one of its diagonals and the boundary of P made up of the line segments with A and B as endpoints and slopes $(1+\varepsilon)m$ and $(1-\varepsilon)m$. We call L the *main diagonal* of P. The situation is schematically represented in Figure 1. We also define $q_P = (1 + 3\varepsilon)m$.

Definition 3.8 (direct descendant)**.** Suppose that $0 < \delta, \varepsilon < 1$, and L is a finite, closed line segment in $\mathbb{R}^2$ with positive slope, moreover $P = P_{L,\varepsilon}$ and $Q = P_{M,\delta}$ are parallelograms with M being a closed line segment sharing an endpoint with L and $Q \subset P$. Then we say that Q *is a direct descendant of* P.

[1] For $n = 1$, the inequality follows from the fact that the sequence of the x_n decreases to x_0.

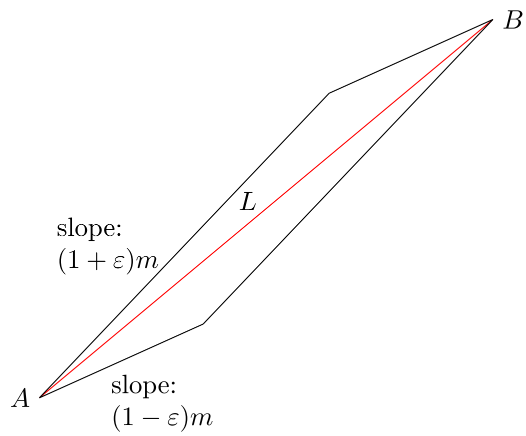

Figure 1: Parallelogram as considered in Definition 3.7

Remark 3.9. Note that if Q is a direct descendant of P, then $\delta \leq \varepsilon$ and $q_Q \leq q_P$.

We next state a simple lemma, which will be useful in the proof of the existence of the function described in Theorem 3.2. In the lemma we use the following notation: Given two points $A, B \in \mathbb{R}^2$, we define $[A, B]$ to be the closed line segment with A and B as the endpoints. We leave the proof of the lemma, which is straightforward, to the reader.

Lemma 3.10. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous with $\frac{f(b)-f(a)}{b-a} = m > 0$ and define the parallelogram P as $P_{L,\varepsilon}$, where $L = [(a, f(a)), (b, f(b))]$ and $\varepsilon \leq 1/2$. Suppose that the graph of f is contained in P. Then for any $x \in (a, b)$ and $r = \min\{x - a, b - x\}$ we have*

$$q_f(x, r) \leq q_P. \tag{3.9}$$

Definition 3.11 (line segment L_I)**.** If f is linear on the interval $I = [a, b]$, then we define $L_I = [(a, f(a)), (b, f(b))]$.

Definition 3.12 (fundamental pair, fundamental envelope)**.** Suppose that f is an increasing homeomorphism of $[0, 1]$ onto itself and $\mathcal{I} = \{I_n\}$ is a countable collection of non-overlapping intervals that are closed and such that $\bigcup I_n = [0, 1]$, f is linear on each $I_n = [a_n, b_n]$ and each $a_n \notin S$. Then we say that $(f, \mathcal{I})$ is a *fundamental pair.*

Finally, given a fundamental pair $(f, \mathcal{I})$ with $\mathcal{I} = \{I_n\}$ and $L_n = L_{I_n}$ and a sequence $\{\varepsilon_n\}$ where $0 < \varepsilon_n \leq 1/2$ for all $n \in \mathbb{N}$, we define

$$P_{f,\mathcal{I},\{\varepsilon_n\}} = \bigcup_{n=1}^{\infty} P_{L_n,\varepsilon_n}$$

and call $P_{f,\mathcal{I},\{\varepsilon_n\}}$ a *fundamental envelope* of $(f,\mathcal{I})$. Thus, $P_{f,\mathcal{I},\{\varepsilon_n\}}$ is a union of non-overlapping, closed parallelograms, which cover the graph of f. Note that we require that $\varepsilon_n \le 1/2$ for all $n \in \mathbb{N}$.

Definition 3.13 (successor)**.** Suppose that $(f,\mathcal{I})$ is a fundamental pair with $\mathcal{I} = \{I_n\} = \{[a_n, b_n]\}$. We say that the fundamental pair $(g,\mathcal{J})$ is a *successor* to $(f,\mathcal{I})$ if there is $m \in \mathbb{N}$ such that $g = f$ on $[0, a_m] \cup [b_m, 1]$ and there are $\beta_1, \beta_2 \in (a_m, b_m)$ such that $[a_m, \beta_1], [\beta_2, b_m] \in \mathcal{J}$ and $\{[a_n, b_n]\}_{n \neq m} \subset \mathcal{J}$.

Lemma 3.14. *Suppose that $(f,\mathcal{I})$ is a fundamental pair with $\mathcal{I} = \{[a_n, b_n]\}$ and fundamental envelope $P = P_{f,\mathcal{I},\{\varepsilon_n\}}$. Let $\alpha > 0$ and suppose that the point $x_0 \in (0,1)\backslash(\bigcup_{k=1}^{\infty}\{a_k, b_k\})$, that is $x_0 \in (a_m, b_m)$ for some $m \in \mathbb{N}$. Then we can find a positive δ with $[x_0 - \delta, x_0 + 2\delta] \subset (a_m, b_m)$, a sequence $x_1 = x_0 + \delta, x_2, x_3, \ldots$ that is acceptable for x_0, and a fundamental pair $(g,\mathcal{J})$ with envelope Q satisfying:*

$$g = f \text{ on } [0, a_m] \cup [b_m, 1], \tag{3.10}$$

$$g \text{ is linear with slope } 1 \text{ on } [x_0 - \delta, x_0] \text{ and on } [x_0 + \delta, x_0 + 2\delta], \tag{3.11}$$

$$g \text{ is acceptable on } [x_0, x_1], \tag{3.12}$$

$$|g - f| < \alpha \text{ on } [0, 1], \tag{3.13}$$

$$\begin{aligned} \mathcal{J} = \{I_n\}_{n \neq m} \cup \{[a_m, x_0 - \delta], [x_0 - \delta, x_0], [x_0 + \delta, x_0 + 2\delta], \\ [x_0 + 2\delta, b_m]\} \cup \{[x_{n+1}, x_n] \colon n \in \mathbb{N}\}, \end{aligned} \tag{3.14}$$

$$Q \subset P, \tag{3.15}$$

recalling that S is the exceptional set from Theorem 3.2 and also appears in Definition 3.12 we also have

$$S \cap (\{x_0 - \delta, x_0 + \delta, x_0 + 2\delta\} \cup \{x_n : n \in \mathbb{N}\}) = \emptyset. \tag{3.16}$$

Remark 3.15. Note that it follows from (3.10) and (3.14) that $(g,\mathcal{J})$ is a successor of $(f,\mathcal{I})$. Note also that δ can be chosen to be arbitrarily small.

Proof of Lemma 3.14. Assume that $f, \mathcal{I}, \{\varepsilon_n\}$, and P are as in the statement of the lemma and let $x_0 \in (a_m, b_m)$. Then $(x_0, f(x_0))$ lies on the segment $L_m = L_{[a_m,b_m]}$ and is contained in the interior of $P_m = P_{L_m,\varepsilon_m}$. It follows that we can choose $\delta > 0$ small enough to ensure that

$$[x_0 - \delta, x_0 + 2\delta] \times [f(x_0) - \delta, f(x_0) + 2\delta] \subset \operatorname{int}(P_m). \tag{3.17}$$

We also require that $(\{x_0 - \delta, x_0 + \delta, x_0 + 2\delta\} \cup \bigcup_n \{x_n : n \in \mathbb{N}\}) \cap S = \emptyset$.

Let K_1 and K_2 be the closed line segments connecting $(a_m, f(a_m))$ with $(x_0 - \delta, f(x_0) - \delta)$ and $(x_0 + 2\delta, f(x_0) + 2\delta)$ with $(b_m, f(b_m))$, respectively.

Note that K_1 and K_2 are contained in $\text{int}(P_m)\cup\{(a_m,f(a_m)),(b_m,f(b_m))\}$, and therefore we can choose ε_0 small enough so that $P_{K_i,\varepsilon_0}\subset P_m$ for $i=1,2$. Now let $x_1=x_0+\delta$, choose $x_2,x_3,\dots$ so that $\{x_n\}$ is acceptable for x and define

(3.18)

$$g(x)=\begin{cases} f(x) & x\in[0,a_m]\cup[b_m,1],\\ f(a_m)+\frac{f(x_0)-\delta-f(a_m)}{x_0-\delta-a_m}(x-a_m) & x\in[a_m,x_0-\delta],\\ f(x_0)-x_0+x & x\in[x_0-\delta,x_0],\\ f(x_0)+f_{\{x_n\}}(x) & x\in[x_0,x_0+\delta],\\ f(x_0)-x_0+x & x\in[x_0+\delta,x_0+2\delta],\\ f(x_0)+2\delta+\frac{f(b_m)-f(x_0)-2\delta}{b_m-x_0-2\delta}(x-x_0-2\delta) & x\in[x_0+2\delta,b_m]. \end{cases}$$

Then (3.10) to (3.12) hold trivially and if $\delta>0$ is chosen small enough, we have (3.13) as well. Let $\mathcal{J}$ be defined by (3.14). It remains to choose our envelope Q such that $Q\subset P$. For each $k\in\mathbb{N}$, we let the set M_k be defined as $M_k=[(x_{k+1},g(x_{k+1})),(x_k,g(x_k))]$ and in case $k=0$, we define $M_0=[(x_0-\delta,f(x_0-\delta)),(x_0,f(x_0))]$ and for $k=\infty$, we set

$$M_\infty=[(x_0+\delta,f(x_0+\delta)),(x_0+2\delta,f(x_0+2\delta))].$$

Note that for any ε such that $0<\varepsilon<1$ and any $k\in\mathbb{N}\cup\{0,\infty\}$ we have

$$P_{M_k,\varepsilon}\subset[x_0,x_1]\times[g(x_0),g(x_1)]\subset\text{int}(P_m)\subset P.$$

Setting

$$Q=\bigcup_{i=1,\,i\neq m}^{\infty}P_{L_i,\varepsilon_i}\cup\Big(\bigcup_{i=1}^{2}P_{K_i,\varepsilon_0}\Big)\cup\Big(\bigcup_{i=0}^{\infty}P_{M_i,1/2}\Big)\cup P_{M_\infty,1/2},$$

we see that Q is an envelope for g and $Q\subset P$, as desired. □

Proof of Theorem 3.2. We begin by setting $f_0(x)=x$ on the interval $[0,1]$ and $\mathcal{I}=\{[0,1]\}$ and letting $P_0=P_{f_0,\mathcal{I},1/2}$ be the envelope associated with f_0. Note that $(f_0,\mathcal{I})$ is a fundamental pair. Now using Lemma 3.14 and Remark 3.15 inductively and recalling that $S=\{s_1,s_2,\dots\}$, it is easy to see that for each $n\in\mathbb{N}$ we can choose $\delta_n>0$ (assume $\delta_n\searrow 0$) and a fundamental

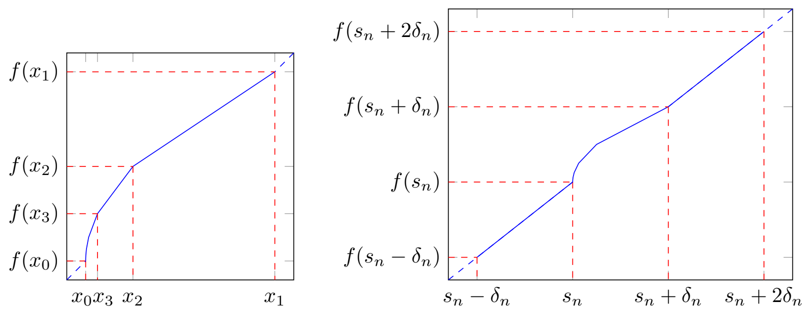

Figure 2: Parts of f_{x_0} and f_n, respectively

pair $(f_n, \mathcal{I}_n)$ with envelope P_n such that

$$f_n \text{ is acceptable on } [s_n, s_n+\delta_n], \tag{3.19}$$

$$f_n \text{ is linear with slope 1 on } [s_n-\delta_n, s_n] \text{ and on } [s_n+\delta_n, s_n+2\delta_n], \tag{3.20}$$

$$\{s_n-\delta_n, s_n+\delta_n, s_n+2\delta_n\} \cap S = \emptyset, \tag{3.21}$$

$$|f_n - f_{n-1}| < 2^{-n} \text{ on } [0,1], \tag{3.22}$$

$$P_n \subset P_{n-1}, \tag{3.23}$$

$$(f_n, \mathcal{I}_n) \text{ is a successor of } (f_{n-1}, \mathcal{I}_{n-1}). \tag{3.24}$$

We also require that for each $n \in \mathbb{N}$ we have

$$[s_n-\delta_n, s_n+2\delta_n] \subset (a_m, b_m) \text{ where } [a_m, b_m] \in \mathcal{I}_{n-1}, \tag{3.25}$$

$$\text{and} \quad [a_m, s_n-\delta_n], [s_n+2\delta_n, b_m] \in \mathcal{I}_n. \tag{3.26}$$

Using (3.22), we can define f as the pointwise limit of the sequence $\{f_n\}$ on $[0,1]$. Clearly, f is continuous and increasing on $[0,1]$. We extend f to the whole real line by defining $f(x) = x$ outside of $[0,1]$. It remains to show that $l_f^\infty = S$ and that $\operatorname{Lip} f(x) < \infty$ for $x \notin \overline{S}$.

Thus, suppose that $x \notin \overline{S}$. If $x \notin [0,1]$, then clearly $\operatorname{Lip} f(x) = 1 < \infty$. Similarly, if $x = 0$ or $x = 1$, then $\operatorname{Lip} f(x) < \infty$ since f is linear on $(-\infty, 0]$ and on $[1, \infty)$ and the graph of f restricted to $[0,1]$ is contained in P_0. Thus we may assume that $x \in (0,1)$. Now suppose that for some $n \in \mathbb{N}$ the point $(x, f_n(x))$ is a vertex of one of the parallelograms that make up P_n. In this case, since $x \notin S$ and $x \in (0,1)$, it follows that $(x, f_n(x))$ and hence $(x, f(x))$ is the shared vertex of two adjoining parallelograms in P_n. Since the graph of f restricted to $[0,1]$ is contained in P_n, it follows again that $\operatorname{Lip} f(x) < \infty$. Thus, we may further assume that for each $n \in \mathbb{N}$ the point $(x, f_n(x))$ is not a vertex of any parallelogram of P_n. Since $x \notin \overline{S}$, it follows from the construction of f, that f is linear on an open interval containing x and hence, once again, we have $\operatorname{Lip} f(x) < \infty$.

To finish the proof we need to show that $l_f^\infty = S$. First of all, consider $s_n \in S$. From (3.23) it follows that the graph of $f|_{[0,1]}$ is contained in P_n and therefore by (3.19) and Remark 3.5 it clearly follows that $\operatorname{lip} f(s_n) = \infty$.

Now suppose that $x \notin S$. Since we have already established earlier that $\operatorname{Lip} f(x) < \infty$ if $x \notin (0,1)$, we may assume that $x \in (0,1)$. Similarly, we may also assume that for each $n \in \mathbb{N}$ the point $(x, f_n(x))$ is not a vertex of any parallelogram of P_n, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define $J_n = (s_n - \delta_n, s_n + 2\delta_n)$ and we consider two cases depending on whether or not x is contained in finitely or infinitely many of these intervals.

Suppose, first of all, that x is contained in only finitely many of these intervals J_n, and choose N to be the largest n such that $x \in J_n$.

Recalling that for any n, P_n is a countable union of non-overlapping parallelograms $\{Q_{n,j}\}$, we let Q_n be the parallelogram in this collection containing the point $(x, f(x))$. For each $n \in \mathbb{N}$ we also define the line segment L_n by $L_n = [(a_n, f_n(a_n)), (b_n, f_n(b_n))]$ and set $\varepsilon_n > 0$ so that $Q_n = P_{L_n, \varepsilon_n}$. Then $x \in (a_n, b_n)$ for all $n \in \mathbb{N}$.

We note that for each $n \geq N$, the parallelogram Q_{n+1} is a direct descendant of Q_n. This fact follows from (3.23), (3.25) and (3.26). Since the intervals $[a_n, b_n]$ are nested, we can let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ so we have $a \leq x \leq b$. If $a < x < b$, then the graph of f is linear on $[a,b]$ and it follows, as above, that $\operatorname{Lip} f(x) < \infty$, so trivially $\operatorname{lip} f(x) < \infty$. Now suppose that either $x = a$ or $x = b$. For each $n \in \mathbb{N}$ define $r_n = \min\{x - a_n, b_n - x\}$. Then, by Remark 3.9 and Lemma 3.10, we have $q_f(x, r_n) \leq q_{Q_N}$ for all $n \geq N$. Since $r_n \to 0$, it follows that $\operatorname{lip} f(x) < \infty$ in this case as well.

Finally, we assume that x is contained in infinitely many of the intervals J_n. In particular, suppose that the point x is contained in J_n. If $x \in (s_n - \delta_n, s_n) \cup (s_n + \delta_n, s_n + 2\delta_n)$, then, from (3.20) and the fact that the graph of f is contained in P_n, we get $q_f(x,r) \leq 5/2$, as computed in (3.9), where $r = \min\{|x - s_n + \delta_n|, |x - s_n|, |x - s_n - \delta_n|, |x - s_n - 2\delta_n|\}$, so $r \leq 1/2 \cdot \delta_n$. On the other hand, suppose that $x \in (s_n, s_n + \delta_n)$. Using (3.19), (3.20), and that the graph of $f|_{[0,1]}$ is contained in P_n, we get

$$f(s_n + 2\delta_n) - f(s_n + \delta_n) = f(s_n + \delta_n) - f(s_n) = f(s_n) - f(s_n - \delta_n) = \delta_n,$$

and now it follows easily from this and the fact that f is increasing that $q_f(x, \delta_n) \leq 2$.

Thus, if $x \in J_n$, then $q_f(x,r) \leq 5/2$ for some $r \leq \delta_n$. Therefore, if x is contained in infinitely many intervals J_n, we get $\operatorname{lip} f(x) < \infty$, and we are done with the proof of Theorem 3.2. $\square$

A simple modification of the construction of f in Theorem 3.2 yields the following result.

Corollary 3.16. *Given a countable set S contained in an open interval (a,b), there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $l_f^\infty = S$, $\operatorname{Lip} f(x) < \infty$ for $x \notin \overline{S}$ and f satisfies:*

$$0 \leq f(x) \leq \min\{x-a, b-x\} \text{ for all } x \in (a,b), \tag{3.27}$$

$$f(x) = 0 \text{ for all } x \notin (a,b). \tag{3.28}$$

The goal in this subsection is the proof of the following result.

Proposition 3.17. *Let $E \subset \mathbb{R}$ be a nowhere dense, perfect set. Then there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $l_f^\infty = E$ and f is constant on open intervals contiguous to E and therefore* $\operatorname{Lip} f(x) = 0$ *for all $x \notin E$.*

In the proof of Proposition 3.17, we will make use of the following simple lemma, whose proof we leave to the reader.

Lemma 3.18. *Let $k > 1$ and $f\colon \mathbb{R} \to \mathbb{R}$. Then*

$$\operatorname{lip} f(x) = +\infty \iff \lim_{n\to\infty} q_f(x, k^{-n}) = \infty.$$

Definition 3.19 (strongly intersecting)**.** An interval J is said to *strongly intersect* a set F if $F \cap \operatorname{int}(J) \neq \emptyset$.

Definition 3.20 (dyadic interval)**.** Let $k \in \mathbb{N}$. We say the interval J is a *dyadic interval of scale* 4^{-k} if $J = [\frac{j}{4^k}, \frac{j+1}{4^k}]$ for some $j \in \mathbb{Z}$.

The construction of f in Proposition 3.17 uses the following lemma. We leave the details of its proof to the reader.

Lemma 3.21. *Let E be a nowhere dense, bounded, perfect set. Denoting $m := \min E$ and $M := \max E$, we write $[m,M] \setminus E = \bigcup_{j=1}^{\infty} I_j$, where the intervals I_j are pairwise disjoint, open, and satisfy $|I_j| \geq |I_{j+1}|$. Then there is a sequence of integers $0 =: h_0 < h_1 < h_2 < \cdots$ such that if J is a dyadic interval of scale 4^{-k} strongly intersecting E, then there are at least two indices i such that $h_{k-1} < i \leq h_k$ and $I_i \subset J$.*

Proof of Proposition 3.17. Assume that E is nowhere dense and perfect. We also assume without loss of generality that E is bounded and define $m := \min E$ and $M := \max E$. We then choose a collection of open intervals

$\{I_j\}_{j=1}^{\infty} = \{(a_j, b_j)\}_{j=1}^{\infty}$ and a sequence of indices $\{h_j\}$ as in Lemma 3.21. Further, we define $I_{-1} = (-\infty, m)$ and $I_0 = (M, \infty)$ and let $b_0 = m$ and $a_0 = M$. After defining f on $\mathbb{R}\backslash(m, M)$ by setting

$$f(x) = \begin{cases} 0 & \text{if } x \leq m, \\ 1 & \text{if } x \geq M, \end{cases}$$

we next proceed by induction to define f on each $\overline{I_j} = [a_j, b_j]$. First, we define f equal to $1/2$ on $\overline{I_1}$. Now suppose that we have defined f on $\overline{I_1}, \overline{I_2}, \ldots, \overline{I_{i-1}}$ and assume that $h_{k-1} < i \leq h_k$. Let

$$c_i = \max_{0 \leq j \leq i-1,\, b_j < a_i} b_j \quad \text{and} \quad d_i = \min_{0 \leq j \leq i-1,\, a_j > b_i} a_j.$$

In order to define f on $\overline{I_i} = [a_i, b_i]$, we consider two cases. First, assume that I_i is contained in a dyadic interval J of scale 4^{-k} and there is exactly one other interval I_j with $1 \leq j \leq i-1$ such that $I_j \subset J$. If $|f(c_i) - f(d_i)| < 2^{-k}$, then we define f on $[a_i, b_i]$ as $f(c_i) + 2^{-k+1}$. In all other cases, we assign the mean value of $f(c_i)$ and $f(d_i)$ to f on $[a_i, b_i]$.

Our next task is to extend the definition of f to the whole real line. We begin with a pair of definitions and a couple of simple lemmas.

Definition 3.22 (adjacent at level)**.** Suppose that $-1 \leq i, j \leq n$ and $b_i < a_j$. We say that $I_i = (a_i, b_i)$ and $I_j = (a_j, b_j)$ are *adjacent at level* n if none of the first n intervals $I_1, I_2, \ldots, I_n$ are located between I_i and I_j:

$$[b_i, a_j] \cap I_l = \emptyset \text{ for } l = 1, 2, \ldots, n. \tag{3.29}$$

Definition 3.23. For each $k \in \mathbb{N}$ we define $\mathcal{J}_k$ to be the collection of dyadic intervals J of scale 4^{-k} such that $E \cap \operatorname{int}(J) \neq \emptyset$.

Lemma 3.24. *If $I_i = (a_i, b_i)$ and $I_j = (a_j, b_j)$ are adjacent at level $n \geq h_k$, then $[b_i, a_j]$ intersects at most 8 distinct intervals from $\mathcal{J}_{k+1}$.*

Proof. If the intervals I_i and I_j are adjacent at level n where $n \geq h_k$, then it follows that $[b_i, a_j]$ does not contain any intervals from $\mathcal{J}_k$.

Therefore, $[b_i, a_j]$ intersects at most 2 distinct intervals from $\mathcal{J}_k$, and the lemma easily follows. □

Lemma 3.25. *For each $j \in \mathbb{N} \cup \{-1, 0\}$ let y_j be the value assigned to f on I_j so that $f(x) = y_j$ for all $x \in I_j$. Suppose that $I_i = (a_i, b_i)$ and $I_j = (a_j, b_j)$ with $b_i < a_j$ are adjacent at level $n \geq h_k$. Let $\lambda = \min\{y_i, y_j\}$ and $\Lambda = \max\{y_i, y_j\}$. Then for any $l \geq n$ such that $I_l \subset [b_i, a_j]$ we have*

$$\lambda \leq y_l \leq \Lambda + 16 \cdot 2^{-k}. \tag{3.30}$$

Proof. Assume that I_i and I_j as well as l are as in the lemma. Then I_l has intervals I_s and I_t that are adjacent at level $l-1$ lying on its left and right, respectively. Assume for the moment that $h_k < l \leq h_{k+1}$. Then either $y_l = \frac{y_s+y_t}{2}$ or

$$y_l = y_s + 2^{-k}. \tag{3.31}$$

But by Lemma 3.24, equation (3.31) is applied at most 8 times for values of l such that $n < l \leq h_{k+1}$ and $I_l \subset [b_i, a_j]$. It follows that $\lambda \leq y_l \leq \Lambda + 8 \cdot 2^{-k}$ if $n < l \leq h_{k+1}$ and $I_l \subset [b_i, a_j]$. Applying this same argument inductively on each interval $h_i < l \leq h_{i+1}$ for $i \in \{k+1, k+2, \dots\}$, we get (3.30). □

We now resume the proof of Proposition 3.17; our next objective is to show that f can be continuously extended to the whole real line. To that end, we pick an arbitrary point $x \in E$ and aim to show that the oscillation of f at x is 0. For each $i \in \mathbb{N}$ we define

$$s_i = \max_{-1 \leq j \leq i,\, b_j \leq x} b_j \quad \text{and} \quad t_i = \min_{0 \leq j \leq i,\, a_j \geq x} a_j.$$

We also define

$$m_j = \inf_{t \in [s_j, t_j] \setminus E} f(t) \quad \text{and} \quad M_j = \sup_{t \in [s_j, t_j] \setminus E} f(t).$$

Note that the nowhere denseness of E implies that

$$\bigcap_{j=1}^{\infty} [s_j, t_j] = \{x\}. \tag{3.32}$$

Therefore, in order to show that we may extend f continuously at x, it suffices to prove that $M_j - m_j \to 0$ as $j \to \infty$.

To that end, we let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ so large that

$$17 \cdot 2^{-k} < \varepsilon. \tag{3.33}$$

We next define an increasing sequence of integers $1 = j_1 < j_2 < j_3 < \cdots$ inductively as follows: For each $i \geq 1$ we let j_{i+1} be the smallest integer $j > j_i$ such that $[s_j, t_j] \neq [s_{j_i}, t_{j_i}]$. Now, using (3.32), choose i so that $t_{j_i} - s_{j_i} < 4^{-k}$ and $j_i > h_k$. Suppose that $|f(t_{j_i}) - f(s_{j_i})| \geq 2^{-k}$. In that case, f is defined on $I_{j_{i+1}}$ as $\frac{f(s_{j_i}) + f(t_{j_i})}{2}$, implying $|f(t_{j_{i+1}}) - f(s_{j_{i+1}})| = \frac{1}{2}|f(t_{j_i}) - f(s_{j_i})|$.

Similarly, if $|f(t_{j_{i+1}}) - f(s_{j_{i+1}})| \geq 2^{-k}$, then

$$|f(t_{j_{i+2}}) - f(s_{j_{i+2}})| = \frac{1}{2}|f(t_{j_{i+1}}) - f(s_{j_{i+1}})|.$$

It follows that we can find $l > i$ such that $|f(t_{j_l}) - f(s_{j_l})| < 2^{-k}$. Applying Lemma 3.25 and inequality (3.33), we get $M_{j_n} - m_{j_n} < \varepsilon$ for all $n \geq l$, as desired.

We have now established that we can continuously extend f to all of $\mathbb{R}$. Moreover, it follows from the construction that f is constant on open intervals contiguous to E so it remains to demonstrate that $\operatorname{lip} f(x) = \infty$ for all $x \in E$.

Assume that $x \in E$. Using the fact that E is perfect, we can choose a sequence of dyadic intervals $\{J_i\}$ such that each $J_i \in \mathcal{J}_i$ and $x \in J_i$. Given an interval J_i, let $I_m = (a_m, b_m)$ and $I_n = (a_n, b_n)$ be two intervals in J_i; actually we want them to be the two intervals chosen first with this membership property. We further assume that $b_m < a_n$. From our rules for defining f on the intervals $\{I_j\}$ it follows that $|f(b_m) - f(a_n)| \geq 2^{-i-1}$. Since $a_n, b_m, x \in J_i$, we see that $q_f(x, 4^{-i}) \geq \frac{2^{-i-2}}{4^{-i}} = 2^{i-2}$. Letting $i \to \infty$ and using Lemma 3.18, we end up with $\operatorname{lip} f(x) = \infty$, and we are done with the proof of Proposition 3.17. □

3.2 Countable union of nowhere dense, perfect sets

Now, we start to look at unions of closed sets. To begin with, we look at countable unions of nowhere dense, perfect sets.

Proposition 3.26. *Suppose $F \subset \mathbb{R}$ is a countable union of perfect, nowhere dense sets. Then there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that f is constant on intervals contiguous to $\overline{F}$ and $l_f^\infty = F$.*

In order to prove Proposition 3.26, we will need the following lemma:

Lemma 3.27. *Suppose that $E \subset F$, where E is closed and F is perfect, nowhere dense, and bounded. Then there exists a collection $\mathcal{I} = \mathcal{I}_{E,F}$ of pairwise disjoint, closed intervals $I = [a_I, b_I]$ satisfying:*

$$F \backslash E \subset \bigcup_{I \in \mathcal{I}} I, \tag{3.34}$$

$$I \cap E = \emptyset \quad \text{for all } I \in \mathcal{I}, \tag{3.35}$$

$$F \cap I \text{ is perfect for all } I \in \mathcal{I}, \tag{3.36}$$

$$\{a_I, b_I\} \subset F \quad \text{for all } I \in \mathcal{I}. \tag{3.37}$$

Moreover, we can choose the intervals so that each closed subinterval of an interval contiguous to E intersects only finitely many elements in $\mathcal{I}$.

Proof. Assume that E and F are as in the statement of the lemma. If $F \setminus E$ is empty, then the result holds by taking the empty collection, so we assume that $F \setminus E \neq \emptyset$. Let $\mathcal{J}$ be the collection of open intervals contiguous to E that intersect F. To prove the lemma, it suffices to show that for each $J \in \mathcal{J}$ we can find a collection $\mathcal{I}_J$ of pairwise disjoint, closed intervals contained in J, which cover $F \cap J$ and such that for each $I \in \mathcal{J}$ the set $F \cap I$ is perfect and the endpoints of I lie in F. Additionally, we have to take care of the moreover-clause in the statement.

Assume that $J = (a, b) \in \mathcal{J}$. Let $c = \inf F \cap J$ and $d = \sup F \cap J$. If $c \neq a$ and $d \neq b$, then we simply take $\mathcal{I}_J = \{[c, d]\}$. Otherwise, suppose that $c \neq a$ and $d = b$. In this case, using the fact that F is perfect and nowhere dense, we can choose a sequence of open intervals $(c_1, d_1), (c_2, d_2), \dots$ such that $c < c_n < d_n < c_{n+1} < d$ for $n = 1, 2, \dots$ such that $d_n \to d$ and such that each (c_n, d_n) is contiguous to F. We then let $\mathcal{I}_J = \{[c, c_1], [d_1, c_2], [d_2, c_3], \dots\}$.

It is easy to check that $\mathcal{I}_J$ has the desired properties in this case. The cases where $c = a$ and $d \neq b$ and where $c = a$ and $d = b$ are handled similarly. □

Before setting out on the proof of Proposition 3.26, we state a few helpful definitions.

Definition 3.28 (Functions Φ_I and g_F)**.** Given a bounded, open interval $I = (a, b)$, we define

$$\Phi_I(x) = \begin{cases} \min\{x - a, b - x\} & \text{if } x \in (a, b), \\ 0 & \text{if } x \notin (a, b). \end{cases}$$

For intervals I of the form $(-\infty, a)$ or (a, ∞), where $a \in \mathbb{R}$, we define

$$\Phi_I(x) = \begin{cases} |x - a| & \text{if } x \in I, \\ 0 & \text{if } x \notin I. \end{cases}$$

Given a bounded, nowhere dense perfect set $F \subset \mathbb{R}$, we also define

$$g_F = \sum_I \Phi_I,$$

where the sum is taken over all bounded intervals I which are contiguous to F.

Proof of Proposition 3.26. First, given any bounded, nowhere dense perfect set E, we use Proposition 3.17 to find and fix a continuous function

$f_E\colon \mathbb{R}\to\mathbb{R}$ such that $l^\infty_{f_E}=E$, the function f_E is constant on intervals contiguous to E, vanishes on $(-\infty,\min(E)]\cup[\max(E),\infty)$, and $0\le f_E\le 1$ on $\mathbb{R}$.

One can readily prove the following useful observation:

Observation 3.29. Let E be bounded, nowhere dense and perfect, and $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function that is constant on all intervals contiguous to E and satisfies $l^\infty_f=E$. Assume $g\colon\mathbb{R}\to\mathbb{R}$ satisfies the equality $g(x)=f(x)$ for all $x\in E$. Then we have $\operatorname{lip} g(x)=\infty$ for all $x\in E$.

Assume that $F=\bigcup_{n=1}^\infty F_n$, where each F_n is nowhere dense and perfect. We shall now construct a continuous function f such that $l^\infty_f=F$. We assume without loss of generality that each F_n is bounded and that the sets F_n are nested and differ: $F_n\subsetneq F_{n+1}$.

Now set $f_1=f_{F_1}$ and $g_1=g_{F_1}$. We will construct f in such a way that $f_1\le f\le f_1+g_1$. Since $g_1=0$ on F_1, Observation 3.29 implies that $\operatorname{lip} f(x)=\infty$ for all $x\in F_1$.

Using Lemma 3.27, for each $n>1$, we let $\mathcal{I}_n=\mathcal{I}_{F_{n-1},F_n}$ be a pairwise disjoint collection of closed intervals $I=[a_I,b_I]$ satisfying equations (3.34) to (3.37) with $E=F_{n-1}$ and $F=F_n$. For each element $I\in\mathcal{I}_n$ we choose $O_I=(c_I,d_I)$ to be the open interval contiguous to F_{n-1} that contains I and define $F_I=I\cap F_n$. Then we choose $0<s_I\le\frac{1}{2^n}$ such that

$$s_I f_{F_I}+g_{F_I}\le\Phi_{O_I}\text{ on }I. \tag{3.38}$$

Having already set $f_1=f_{F_1}$, we define for $n>1$

$$f_n=\sum_{I\in\mathcal{I}_n}s_I f_{F_I}. \tag{3.39}$$

Finally, we set $f=\sum_{n=1}^\infty f_n$. Since each f_n is continuous and $0\le f_n\le\frac{1}{2^n}$ for $n>1$, it follows that f is continuous. Note also that f is constant on each open interval contiguous to $\overline{F}$. It remains to show that $l^\infty_f=F$.

We begin by showing that $\operatorname{lip} f(x)=\infty$ on F. To that end, let $x\in F$. For notational convenience, we define $\tilde f_n=\sum_{j=1}^n f_j$, so $f=\lim_{n\to\infty}\tilde f_n$. From (3.38) it follows that

$$\tilde f_n\le\tilde f_{n+1}\le\tilde f_{n+1}+g_{F_{n+1}}\le\tilde f_n+g_{F_n},$$

and therefore

$$\tilde f_n\le f\le\tilde f_n+g_{F_n} \tag{3.40}$$

for all $n \in \mathbb{N}$. Notice also that for all n and $k \in \mathbb{N}$ we have that f_{n+k} is 0 on F_n and therefore $f = \tilde{f}_n$ on F_n. Since $l^{\infty}_{\tilde{f}_n} = F_n$, it follows from Observation 3.29 that $\operatorname{lip} f(x) = \infty$ for all $x \in F$.

We are left with showing that $\operatorname{lip} f(x) < \infty$ for $x \in \overline{F} \setminus F$. Assume that $x \in \overline{F} \setminus F$. Then for each $n \in \mathbb{N}$ there is an open interval $I_n = (a_n, b_n)$ that is contiguous to F_n and that contains x. Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ so we have $a \le x \le b$. Since $x \in \overline{F}$, we have either $a = x$ or $b = x$. Note that $\tilde{f}_n$ is constant on I_n and from (3.40), we know that $\tilde{f}_n \le f \le \tilde{f}_n + \Phi_{I_n}$ on I_n. It follows that $q_f(x, r_n) \le 2$ where $r_n = \min\{x - a_n, b_n - x\}$.

Since $r_n \to 0$, we get $\operatorname{lip} f(x) \le 2$, and we are done with the proof. □

3.3 Meager F_σ sets

In this section we improve Proposition 3.26 by removing the requirement that the sets in the union be perfect. More precisely, we prove the following:

Proposition 3.30. *Let $F \subset \mathbb{R}$ be a meager F_σ set, that is it is the countable union of closed, nowhere dense sets. Then there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $l^{\infty}_f = F$ and such that* $\operatorname{Lip} f$ *is finite on $\mathbb{R} \setminus \overline{F}$.*

In order to accomplish our goal of proving Proposition 3.30, we will need some preliminary results that allow us to write a given F_σ set specifically tailored to our use. This is the content of the next section.

3.3.1 Auxiliary results

Lemma 3.31. *Let F be an F_σ set that is not countable nor closed. Then there are countably many perfect, nowhere dense sets $\{P_n\}_{n\in\mathbb{N}}$, and a countable set C such that*

$$F = \bigcup_{n\in\mathbb{N}} P_n \cup \operatorname{int}(F) \cup C,$$

$$C \cap \operatorname{int}(F) = C \cap P_n = P_n \cap \operatorname{int}(F) = \emptyset \quad \text{for all } n \in \mathbb{N}. \tag{3.41}$$

Proof. As F is an F_σ set, there are countably many closed sets F_n such that $F_n \subset F_{n+1}$ and

$$F = \bigcup_{n\in\mathbb{N}} F_n.$$

We let $O = \operatorname{int}(F)$ be the interior of F. The sets $F_n \setminus O = F_n \cap (\mathbb{R} \setminus O)$ are closed. By the Cantor–Bendixson theorem, there are perfect sets P_n and countable sets C_n such that

$$F_n \setminus O = P_n \cup C_n.$$

This means the following for F:

$$F = \bigcup_{n\in\mathbb{N}} (F_n \setminus O) \cup O = \bigcup_{n\in\mathbb{N}} (P_n \cup C_n) \cup O.$$

We denote by C the countable set $\bigcup_{n\in\mathbb{N}} C_n \setminus (\bigcup_{n\in\mathbb{N}} P_n)$. The equalities in (3.41) now follow from the definitions of the sets in question. □

In order to prove Proposition 3.30 and Theorem 3.1, we need a better version of Lemma 3.31:

Lemma 3.32. *Let F be an F_σ set that is not countable nor closed. Then there are countably many perfect, nowhere dense sets $\{P_n\}_{n\in\mathbb{N}}$ and a countable set D such that*

$$F = \bigcup_{n\in\mathbb{N}} P_n \cup \operatorname{int}(F) \cup D,$$

$$D \cap \operatorname{int}(F) = D \cap \overline{\bigcup_{n\in\mathbb{N}} P_n} = \bigcup_{n\in\mathbb{N}} P_n \cap \operatorname{int}(F) = \emptyset, \tag{3.42}$$

and the sets P_n are nested for all $n \in \mathbb{N}$, that is $P_n \subset P_{n+1}$.

Proof. We assume that the representation of F is already as in Lemma 3.31. Let $C_1 = C \cap (\overline{\bigcup_{n=1}^{\infty} P_n})$. We only look at the case where C_1 is infinite and write $C_1 = \{c_1, c_2, \dots\}$. For each $c_n \in C_1$ we will construct a perfect, nowhere dense set P'_n with the following properties:

$$P_n \subset P'_n \subset F, \tag{3.43}$$

$$P'_n \subset C_1 \cup \bigcup_{n=1}^{\infty} P_k \subset \overline{\bigcup_{k=1}^{\infty} P_k}, \tag{3.44}$$

$$c_n \in P'_n. \tag{3.45}$$

Defining $D = C \backslash C_1$ and replacing each P_n with $\bigcup_{k=1}^{n} P'_k$, it is clear that Lemma 3.32 will follow from Lemma 3.31 and (3.43) to (3.45).

We now proceed with the construction of P'_n. Using that $c_n \in \overline{\bigcup_{k=1}^{\infty} P_k}$ and that each P_k is perfect and nowhere dense, we find a subsequence $\{P_{k_i}\}$ of $\{P_k\}$ and intervals $I_j = [a_j, b_j]$, such that

$$I_j \cap I_i = \emptyset \text{ if } j \neq i, \tag{3.46}$$

$$I_j \cap P_{k_j} \text{ is perfect,} \tag{3.47}$$

$$a_j \to c_n. \tag{3.48}$$

Defining $P'_n = P_n \cup (\bigcup_{j=1}^{\infty} (I_j \cap P_{k_j})) \cup \{c_n\}$, it is straightforward to check that P'_n is perfect, nowhere dense, and that (3.43) to (3.45) all hold, completing the proof. □

3.3.2 Proof of Proposition 3.30

Proof of Proposition 3.30. Assume that F is the countable union of closed, nowhere dense sets. Then, by Lemma 3.32, we have

$$F = (\bigcup_{n=1}^{\infty} P_n) \cup D,$$

where each P_n is perfect and nowhere dense, D is countable and the intersection $\overline{\bigcup_{n=1}^{\infty} P_n} \cap D$ is empty. Let $P = \overline{\bigcup_{n=1}^{\infty} P_n}$. Then we can write $\mathbb{R}\backslash P$ as the countable union of pairwise disjoint open intervals: $\mathbb{R}\backslash P = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Using Corollary 3.16 for each $n \in \mathbb{N}$ we find a continuous $h_n \colon \mathbb{R} \to \mathbb{R}$ such that

$$l_{h_n}^{\infty} = D \cap (a_n, b_n), \tag{3.49}$$

$$0 \le h_n(x) \le \min\{x - a_n, b_n - x\} \text{ for all } x \in (a_n, b_n), \tag{3.50}$$

$$\operatorname{Lip} h_n \text{ is finite on } (a_n, b_n)\backslash\overline{D}, \tag{3.51}$$

and

$$h_n(x) = 0 \text{ for all } x \notin (a_n, b_n). \tag{3.52}$$

We let $h = \sum_{n=1}^{\infty} h_n$ and note that it follows from (3.49) to (3.52) that h is continuous on $\mathbb{R}$, $l_h^{\infty} = D$, and $\operatorname{Lip} h$ is finite on $\mathbb{R}\backslash\overline{D}$. Now, using Proposition 3.26, we choose a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying $l_g^{\infty} = \bigcup_{n=1}^{\infty} P_n$ and that $\operatorname{Lip} g$ is finite on $\mathbb{R}\backslash\overline{\bigcup_{n=1}^{\infty} P_n}$ and therefore finite on $\mathbb{R}\backslash\overline{F}$.

We claim that $f = g + h$ has the desired properties.

First note that since $\mathbb{R}\backslash\overline{F} \subset \mathbb{R}\backslash\overline{D}$, it follows that $\operatorname{Lip} h$ is finite on $\mathbb{R}\backslash\overline{F}$ and thus, both $\operatorname{Lip} h$ and $\operatorname{Lip} g$ are finite on $\mathbb{R}\backslash\overline{F}$. Therefore, $\operatorname{Lip} f$ is finite on $\mathbb{R}\backslash\overline{F}$, as required.

To conclude the proof we need to show that $l_f^{\infty} = F$. Since g is constant on each (a_n, b_n), and each h_n is constant on $\mathbb{R}\backslash(a_n, b_n)$, it follows from (3.49) that $l_f^{\infty} \cap (\mathbb{R}\backslash P) = D$. It remains to verify that $l_f^{\infty} \cap P = \bigcup_{n=1}^{\infty} P_n$. This follows easily from the fact that $l_g^{\infty} = \bigcup_{n=1}^{\infty} P_n$ and that $\operatorname{Lip} h(x) < \infty$ on $\mathbb{R}\backslash(\bigcup_{n=1}^{\infty}(a_n, b_n))$; consequently we have finished the proof. □

3.4 Union of closed sets

The proof of Theorem 3.1 now follows rather easily from Proposition 3.30 and the following two lemmas.

Lemma 3.33. *Given any open interval* (a,b) *and* $h>0$ *there exists a continuous function* $f\colon \mathbb{R}\to\mathbb{R}$ *such that*

$$f = 0 \text{ on } \mathbb{R}\backslash(a,b), \tag{3.53}$$

$$0 \le f(x) \le h\cdot \min\{x-a, b-x\} \text{ for } x\in(a,b), \tag{3.54}$$

$$\operatorname{lip} f(x) = \infty \text{ for all } x\in(a,b). \tag{3.55}$$

Note that if f is as in the lemma, then we also have that $\operatorname{Lip} f$ is finite on $\mathbb{R}\backslash(a,b)$.

Proof of Lemma 3.33. We start with a definition: If f is linear on $[a,b]$ with $f(a)=c$ and $f(b)=d$, then we define $f^{[a,b]}\colon [a,b]\to\mathbb{R}$ so that $f^{[a,b]}$ is linear on each of the intervals $[a,a+\frac{b-a}{3}],[a+\frac{b-a}{3},a+\frac{2(b-a)}{3}]$ and $[a+\frac{2(b-a)}{3},b]$ and so that

$$f^{[a,b]}(a) = f^{[a,b]}\left(a+\frac{2(b-a)}{3}\right) = c \text{ and } f^{[a,b]}\left(a+\frac{b-a}{3}\right) = f^{[a,b]}(b) = d.$$

We next define an auxiliary function g on $[0,1]$ as follows:

We begin by setting $g_0(x)=x$ on $[0,1]$. For each $n\in\mathbb{N}$ we define

$$I_{n,j} = \left[\frac{j}{3^n},\frac{j+1}{3^n}\right] \text{ for } j\in\{0,1,\ldots,3^n-1\}.$$

Next we define a sequence of functions $\{g_n\}$ recursively on $[0,1]$ so that

$$g_n|_{I_{2n-1,j}} = g_{n-1}^{I_{2n-1,j}} \text{ for all } n\in\mathbb{N} \text{ and } j\in\{0,1,\ldots,3^{2n-1}-1\}.$$

Then since each g_n is continuous on $[0,1]$ and $\|g_n-g_{n-1}\|\le 2\cdot 3^{-n}$ for each $n\in\mathbb{N}$, it follows that $g=\lim_{n\to\infty} g_n$ is continuous on $[0,1]$. It is also easy to verify that $q_g(x,3^{-2n})\ge 3^n/2$ for each $x\in(0,1)$ and for each $n\in\mathbb{N}$. It follows from Lemma 3.18 that $\operatorname{lip} g=\infty$ on $(0,1)$. Finally, given an interval $[a,b]$, we define

$$f(x) = \begin{cases} h\Phi_{(a,b)}(x)\frac{1+g(\frac{x-a}{b-a})}{2} & \text{if } x\in(a,b),\\ 0 & \text{if } x\notin(a,b).\end{cases}$$

It is easy to verify that f has the desired properties. □

Lemma 3.34. *Given an open set* $O\subset\mathbb{R}$*, there exists a continuous function* $f\colon\mathbb{R}\to\mathbb{R}$ *such that* $l_f^\infty = O$ *and* $\operatorname{Lip} f(x)<\infty$ *for all* $x\in\mathbb{R}\backslash O$.

Proof. Let O be equal to $\bigcup_n (a_n, b_n)$, where the intervals (a_n, b_n) are pairwise disjoint. For each n we construct a continuous function $f_n \colon \mathbb{R} \to \mathbb{R}$ satisfying (3.53) to (3.55) with a and b replaced with a_n and b_n, f replaced with f_n and h replaced by $1/2$. Letting $f = \sum_n f_n$ it is easy to verify that $l_f^\infty = O$ and $\operatorname{Lip} f(x) \leq 1$ for all $x \in \mathbb{R}\backslash O$. □

Proof of Theorem 3.1. Let F be a countable union of closed sets and define $O = \operatorname{int}(F)$ and $E = F\backslash O$. It follows that E is a countable union of nowhere dense closed sets, so by Proposition 3.30 there exists a continuous function g such that $l_g^\infty = E$ and $\operatorname{Lip} g$ is finite on $\mathbb{R}\backslash\overline{E}$. Moreover, using Lemma 3.34, we can find a continuous function h such that $l_h^\infty = O$ and $\operatorname{Lip} h$ is finite on $\mathbb{R}\backslash O$. Let $f = g + h$. Since $\operatorname{lip} g = \infty$ on E and $\operatorname{Lip} h < \infty$ on $E \subset \mathbb{R}\backslash O$, we see that $\operatorname{lip} f = \infty$ on E. Similarly, since $\operatorname{Lip} g < \infty$ on $O \subset \mathbb{R}\backslash\overline{E}$ and $\operatorname{lip} h = \infty$ on O, we have $\operatorname{lip} f = \infty$ on O. Finally, since $\operatorname{lip} g < \infty$ on $\mathbb{R}\backslash F \subset \mathbb{R}\backslash E$ and $\operatorname{Lip} h < \infty$ on $\mathbb{R}\backslash F \subset \mathbb{R}\backslash O$, we have $\operatorname{lip} f < \infty$ on $\mathbb{R}\backslash F$, finishing the proof. □

3.5 The G_δ case

In this section, we prove the following result:

Theorem 3.35. *For every G_δ set $E \subset \mathbb{R}$, there exists a continuous function f such that $L_f^\infty = l_f^\infty = E$.*

We begin with a few definitions:

First of all, for each interval I and each $n \in \mathbb{N}$ we define the function $\Phi_{I,n}$ by $\Phi_{I,n}(x) = \Phi_I(x)/5^n$, where Φ_I is as previously defined in Definition 3.28. Let $I = (a, b)$ be an interval and $n \in \mathbb{N}$. A basic building block in the construction of the function f advertised in Theorem 3.35 will be the function $\Psi_{I,n}(x)$ that we now proceed to define.

We begin by defining a countable set of points

$$B_{I,n} = \{x_k\}_{k\in\mathbb{Z}} \tag{3.56}$$

contained in I. First set $x_0 = \frac{a+b}{2}$ and $x_1 = \frac{a+b}{2} + \frac{b-a}{2\cdot 5^{2n}}$. For $k \in \mathbb{N}$, let $x_{2k+1} = x_{2k-1} + \frac{b-a}{5^{2n}}(\frac{5^{2n}-1}{5^{2n}+1})^k$ and $x_{2k} = \frac{x_{2k-1}+x_{2k+1}}{2}$ and set $x_{-k} = a + b - x_k$. A bit of calculation should now convince the reader that $x_k < x_{k+1}$ for all $k \in \mathbb{Z}$, $\lim_{k\to\infty} x_k = b$ and $\lim_{k\to\infty} x_{-k} = a$.

Now define $\Psi = \Psi_{I,n} \colon \mathbb{R} \to \mathbb{R}$ to be the unique continuous function with the following properties: $\Psi(x) = 0$ for all $x \notin I$, $\Psi(x_{2k+1}) = 0$ for all $k \in \mathbb{Z}$, the function Ψ is linear with slope 5^n on each interval $[x_{2k+1}, x_{2k+2}]$ and

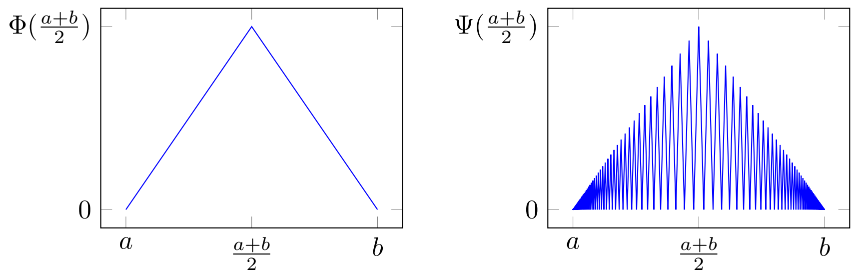

Figure 3: $\Phi_{I,n}$ and $\Psi_{I,n}$

linear with slope -5^n on each interval $[x_{2k}, x_{2k+1}]$. Note that the graph of $\Psi|_I$ consists of countably many straight line segments of slope $\pm 5^n$.

One can also verify that

$$0 \le \Psi_{I,n}(x) \le \Phi_{I,n}(x) \text{ for all } x \in \mathbb{R}. \tag{3.57}$$

Additionally, we have

$$\Psi_{I,n}(x_{2k}) = \Phi_{I,n}(x_{2k}) \text{ for all } k \in \mathbb{Z}. \tag{3.58}$$

For future reference we record the following useful observations:

$$\max_j (x_{j+1} - x_j) = x_1 - x_0 = \frac{b-a}{2 \cdot 5^{2n}}, \tag{3.59}$$

$$\frac{12}{13} \le \frac{5^{2n}-1}{5^{2n}+1} \le \frac{x_{j+2} - x_{j+1}}{x_{j+1} - x_j} \le \frac{5^{2n}+1}{5^{2n}-1} \le \frac{13}{12}, \tag{3.60}$$

$$\left| \frac{\Psi(x_{2j+2})}{\Psi(x_{2j})} \right| = \begin{cases} \frac{5^{2n}-1}{5^{2n}+1} & j \ge 0, \\ \frac{5^{2n}+1}{5^{2n}-1} & j < 0. \end{cases} \tag{3.61}$$

Note that from (3.57), (3.58), and the definition of $\Phi_{I,n}$ we get

$$\Psi_{I,n}(x) \le \Psi_{I,n}(x_{2j}) + \frac{|x - x_{2j}|}{5^n} \text{ for all } x \in I. \tag{3.62}$$

Given a collection of open sets $\{G_1, G_2, \ldots\}$ for each $n \in \mathbb{N}$ we let $\mathcal{I}_n$ be the unique collection of pairwise disjoint open intervals whose union is G_n so $G_n = \bigcup_{I \in \mathcal{I}_n} I$. Given such a collection, by using (3.56) and the definition after it we define:

$$X_n = \bigcup_{I \in \mathcal{I}_n} B_{I,n}. \tag{3.63}$$

The following simple lemma is easily proved by induction.

Lemma 3.36. *Let E be a G_δ set with empty interior. Then we can find open sets $G_1, G_2, \ldots$ such that*

$$E = \bigcap_{n=1}^{\infty} G_n, \tag{3.64}$$

$$G_{n+1} \subset G_n \text{ for all } n \in \mathbb{N}, \tag{3.65}$$

$$\sup_{I \in \mathcal{I}_1} |I| \leq 1, \tag{3.66}$$

$$\text{and each } I \in \mathcal{I}_{n+1} \text{ contains at most one point of } X_n. \tag{3.67}$$

Proof of Theorem 3.35. We first consider the case where E is a G_δ set with empty interior. Assume that the sets $G_1, G_2, \ldots$ have been chosen as in Lemma 3.36. For each $n \in \mathbb{N}$ define

$$f_n(x) = \sum_{I \in \mathcal{I}_n} \Psi_{I,n}(x) \tag{3.68}$$

and let

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

It follows from (3.59) and (3.65) to (3.68), and the definition of $\Psi_{I,n}$ that f is continuous, so we are left to show that $\operatorname{lip} f(x) = \infty$ for all $x \in E$ and $\operatorname{Lip} f(x) < \infty$ for all $x \notin E$.

First we show that $\operatorname{Lip} f(x) < \infty$ for all $x \notin E$. Assume that $x \notin E$. Then we can choose $K \in \mathbb{N}$ such that $x \notin G_n$ for all $n > K$, and we write $f = S_K + T_K$, where $S_K = \sum_{n=1}^{K} f_n$ and $T_K = \sum_{n=K+1}^{\infty} f_n$. Since each f_n is 5^n-Lipschitz, it follows that S_K is Lipschitz and therefore $\operatorname{Lip} S_K(x) < \infty$. Thus, to establish that $\operatorname{Lip} f(x) < \infty$ it suffices to prove the inequality $\operatorname{Lip} T_K(x) < \infty$. Now for each $n \in \mathbb{N}$ we define

$$h_n = \sum_{I \in \mathcal{I}_n} \Phi_{I,n},$$

so we have, by the fact that $\Phi_{I,n}$ bounds $\Psi_{I,n}$ from above as detailed in (3.57), that $0 \leq f_n(t) \leq h_n(t)$ for all $t \in \mathbb{R}$ and therefore

$$0 \leq T_K(t) \leq \sum_{n=K+1}^{\infty} h_n(t) =: R_K(t) \text{ for all } t \in \mathbb{R}. \tag{3.69}$$

That h_n is 5^{-n}-Lipschitz implies that R_K is Lipschitz. It therefore follows from (3.69) and the fact that $0 = T_K(x) = R_K(x)$ that $\operatorname{Lip} T_K(x) < \infty$ as desired.

It remains to show that $\operatorname{lip} f(x) = \infty$ for all $x \in E$. Let $x \in E$ and assume without loss of generality that $x = 0$. For each $n \in \mathbb{N}$ we choose $I_n \in \mathcal{I}_n$ such that $0 \in I_n$ and we define $\Psi_n = \Psi_{I_n,n}$. We next write $f = g + h$ where

$$g = \sum_{n=1}^{\infty} \Psi_n \text{ and } h = \sum_{n=1}^{\infty} \sum_{I \in \mathcal{I}_n,\, I \neq I_n} \Psi_{I,n}.$$

Using the same argument as above, one can show that $\operatorname{Lip} h(0) < \infty$, and therefore it suffices to prove that $\operatorname{lip} g(0) = \infty$.

For each $n \in \mathbb{N}$ choose x_j^n, $x_{j+1}^n \in B_{I_n,n} =: B_n$ so that $0 \in [x_j^n, x_{j+1}^n]$, choose J_n to be the larger of the two intervals $[x_j^n, 0]$ and $[0, x_{j+1}^n]$ and define $r_n = |J_n|$. Note that

$$r_n \geq \frac{1}{2}(x_{j+1}^n - x_j^n) \tag{3.70}$$

and $(\operatorname{int}(J_n)) \cap B_n = \emptyset$. Using these facts along with (3.59), (3.60) and (3.67) we get

$$2 \cdot 5^{2n} r_{n+1} \leq |I_{n+1}| \leq 5 r_n. \tag{3.71}$$

We also note that Ψ_n is linear on J_n with slope $\pm 5^n$. Define m_n by the equality $m_n = \max\{\Psi_n(x_j^n), \Psi_n(x_{j+1}^n)\}$ and note that from (3.70) we get $m_n \leq 2 \cdot 5^n r_n$. It follows from (3.62) that

$$\Psi_n(t) \leq 2 \cdot 5^n r_n + \frac{1}{5^n}(|t| + r_n) \text{ for all } t. \tag{3.72}$$

Let $g_k = \sum_{j=1}^{k} \Psi_j$. Since each Ψ_j is 5^j-Lipschitz, it follows that g_k is $\frac{5^{k+1}}{4}$-Lipschitz. Therefore, since $g_n = \Psi_n + g_{n-1}$ and $|\Psi_n'| = 5^n$ on J_n, we get

$$|g_n(t) - g_n(s)| \geq \frac{3}{4} \cdot 5^n\, |t - s| \text{ for all } s, t \in J_n. \tag{3.73}$$

Now we define $[a_n, b_n] = J_n$ and $q(r) = q_g(0, r)$ and further note that $q(r_n) \geq \frac{|g(b_n) - g(a_n)|}{r_n}$.

We need to show that $\lim_{r \searrow 0} q(r) = \infty$. We begin by showing that $q(r_n) \to \infty$ as $n \to \infty$. From (3.71) we get

$$\sup_{t \in \mathbb{R}} \Psi_n(t) \leq \frac{1}{2 \cdot 5^n} |I_n| \leq \frac{1}{2 \cdot 5^{n-1}} r_{n-1}, \tag{3.74}$$

and it follows that

$$\sum_{k > n} \Psi_k(t) \leq \frac{1}{5^n} r_n \text{ for all } t \in \mathbb{R}. \tag{3.75}$$

But from (3.73) we know that $|g_n(b_n) - g_n(a_n)| \geq \frac{3}{4} \cdot 5^n r_n$, so it follows that $|g(b_n) - g(a_n)| \geq \frac{1}{2} \cdot 5^n r_n$, and therefore $q(r_n) \geq \frac{1}{2} \cdot 5^n$. This implies that $q(r_n)$ converges to ∞.

We now consider the interval $[r_{n+1}, r_n]$. We need to show that $q(r)$ remains large on the entire interval and not just at the endpoints, r_{n+1} and r_n. Note that for any $r > 0$ and $s \geq 1$ we have $q(sr) \geq q(r)/s$.

We assume without loss of generality that $a_n = 0$ so $J_n = [0, r_n]$. Then from (3.73) we have

$$|g_n(t) - g_n(0)| \geq \frac{3}{4} \cdot 5^n t \text{ for all } t \in [0, r_n]. \tag{3.76}$$

Now, using (3.72), we get

$$\Psi_{n+1}(t) \leq \frac{5^n}{2} t \text{ for all } t \geq 25 r_{n+1}. \tag{3.77}$$

Finally, using (3.75), (3.76), and (3.77), we get $|g(t) - g(0)| \geq \frac{1}{5} \cdot 5^n t$ on $[25 r_{n+1}, r_n]$ and therefore $q(r) \geq 5^{n-1}$ on $[25 r_{n+1}, r_n]$. It follows that $\lim_{r \searrow 0} q(r) = \infty$ so $\operatorname{lip} g(0) = \infty$, as desired.

To complete the proof of Theorem 3.35, we need to remove the assumption that E has empty interior. Thus we now assume that E is a G_δ set with non-empty interior. Let $O = \operatorname{int}(E)$ and $E' = E \setminus O$. Then E' is a G_δ set and using what we have proved so far we can find a function h such that $\operatorname{lip} h(x) = \infty$ on E' and $\operatorname{Lip} h(x) < \infty$ on $\mathbb{R} \setminus E'$.

In Lemma 3.34, we constructed a continuous function g satisfying the equality $\operatorname{lip} g(x) = \infty$ on O and the inequality $\operatorname{Lip} g(x) < \infty$ on the complement of O. Setting $f = g + h$ we have the result we need. □

4 Typical Results

In this section we consider continuous functions defined on $\mathbb{R}^d$, where d is a positive integer. Our goal here is to show that the typical continuous function has vanishing lip at points of a residual set of full measure. We start with some auxiliary results.

Notation 4.1 (balls $B_\infty(x, r)$ and $U_\infty(x, r)$ and Lebesgue measure $|E|$)**.** Let $d \in \mathbb{N}$, choose $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $r > 0$. We denote by $B_\infty(x, r)$ and $U_\infty(x, r)$ the closed and open balls with respect to the maximum norm on $\mathbb{R}^d$, centered at x and with radius r, respectively. We denote the d-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^d$ by $|E|$.

Theorem 4.2. *The typical function* $f \in C([0,1]^d)$ *satisfies* $\operatorname{lip} f(x) = 0$ *at points of a residual set of full measure in* $[0,1]^d$.

Proof. We aim to find a residual set of functions in $C([0,1]^d)$ such that each member f satisfies $\operatorname{lip} f(x) = 0$ at all points of a set of full measure (depending on f) in $[0,1]^d$; the residuality part then follows automatically, as for any continuous function f, the set of points x with $\operatorname{lip} f(x) = 0$ is of the type G_δ by a simple multidimensional generalization of Lemma 2.4 (and, of course, full measure implies density).

First, for each $n \in \mathbb{N}$ set

$$\alpha_n := \sqrt[d]{1 - 2^{-n}}, \quad \beta_n := \frac{1 - \alpha_n}{n}, \tag{4.1}$$

and define $\mathcal{C}_n$ as the set of all finite sequences $(C_i)_{i=1}^k$ (where k can be any natural number) of pairwise disjoint closed cubes in $[0,1]^d$ (i.e. closed balls in the maximum norm on $\mathbb{R}^d$) such that for each $i \in \{1, \ldots, k\}$ the side length $l(C_i)$ of C_i is less than $1/n$, and $\left|\bigcup_{i=1}^k C_i\right| > 1 - 2^{-n}$. Finally, for each $n \in \mathbb{N}$ we define

$$A_n := \big\{ f \in C([0,1]^d) : \exists \{C_i\}_{i=1}^k \in \mathcal{C}_n \quad \exists \{a_i\}_{i=1}^k \subset \mathbb{R} \\ \forall i \in \{1, \ldots, k\}, \quad \forall x \in C_i : |f(x) - a_i| < \beta_n l(C_i) \big\}.$$

The proof will be finished when we prove the following two claims:

(a) The intersection $A := \bigcap_{n=1}^\infty A_n$ is residual in $C([0,1]^d)$; in fact, A_n is open and dense for each n.

(b) If $f \in A$, then $\operatorname{lip} f(x) = 0$ for almost every $x \in [0,1]^d$.

To prove claim (a), we first observe that A_n is open for any n; to that end, fix n and $f \in A_n$. Take $\{C_i\}_{i=1}^k \in \mathcal{C}_n$ and $\{a_i\}_{i=1}^k \subset \mathbb{R}$ witnessing the fact that $f \in A_n$. Thus we have for each $i \in \{1, \ldots, k\}$ and each $x \in C_i$ that

$$|f(x) - a_i| < \beta_n l(C_i).$$

But the finitely many cubes C_i are compact, so there exists $\gamma > 0$ (depending only on $\{C_i\}_{i=1}^k$) such that

$$|f(x) - a_i| < \beta_n l(C_i) - \gamma$$

for each $x \in C_i$. Now, if $g \in C([0,1]^d)$ is such that $\|f - g\|_\infty < \gamma$, then for each i and each $x \in C_i$ we have

$$|g(x) - a_i| \leq |g(x) - f(x)| + |f(x) - a_i| < \gamma + \beta_n l(C_i) - \gamma = \beta_n l(C_i).$$

Thus, $g \in A_n$, and A_n is open.

To prove the density of A_n in $C([0,1]^d)$, let there be given an arbitrary function $f \in C([0,1]^d)$ and $\varepsilon > 0$; we want to find a function $g \in A_n$ such that $\|f-g\|_\infty < \varepsilon$. From the uniform continuity of f we obtain a $\delta > 0$ such that for each $x \in [0,1]^d$ and each $y \in B_\infty(x,\delta) \cap [0,1]^d$ we have $|f(x)-f(y)| < \varepsilon/2$. Next, let us find $\{x_i\}_{i=1}^k \subset [0,1]^d$ and $\{r_i\}_{i=1}^k \subset (0,\delta)$ such that $\{B_\infty(x_i,r_i)\}_{i=1}^k \in \mathcal{C}_n$ (clearly we can do that; recall that B_∞ denotes the closed ball in the maximum norm), and take a number $\gamma \in (0,1)$ so close to 1 that also $\{B_\infty(x_i,\gamma r_i)\}_{i=1}^k \in \mathcal{C}_n$. Define

$$\tilde{g}(x) = \begin{cases} f(x_i) & \text{if } x \in B_\infty(x_i,\gamma r_i), \\ f(x) & \text{if } x \in [0,1]^d \setminus \bigcup_{i=1}^k U_\infty(x_i,r_i). \end{cases}$$

Hence $\tilde{g}$ is clearly continuous on the closed subspace

$$\bigcup_{i=1}^k B_\infty(x_i,\gamma r_i) \cup \left([0,1]^d \setminus \bigcup_{i=1}^k U_\infty(x_i,r_i) \right),$$

and we can use Tietze's Theorem to continuously extend $\tilde{g}$ to the whole $[0,1]^d$. We denote the extension by $\tilde{g}$ as well.

However, the statement of Tietze's Theorem gives us no control on the distance between $\tilde{g}$ and f, and so we need to perform a simple truncation procedure to ensure g will indeed be close to f. We define $g\colon [0,1]^d \to \mathbb{R}$ as

$$g(x) = \begin{cases} \min\left\{\max\left\{\tilde{g}(x), f(x_i) - \frac{\varepsilon}{2}\right\}, f(x_i) + \frac{\varepsilon}{2}\right\} & \text{if } x \in B_\infty(x_i,r_i), \\ \tilde{g}(x) & \text{otherwise.} \end{cases}$$

To see that g is continuous, take any $i \in \{1,\dots,k\}$ and observe that for each $x \in \partial B_\infty(x_i,r_i)$, we have $\tilde{g}(x) = f(x) \in (f(x_i)-\varepsilon/2, f(x_i)+\varepsilon/2)$. Therefore, the truncation can only change the function f in the interior of the cubes, so the continuity is preserved.

To see that $\|f-g\|_\infty < \varepsilon$, take any $x \in [0,1]^d$. As g coincides with f outside $\bigcup_{i=1}^k B_\infty(x_i,r_i)$, we can assume that $x \in B_\infty(x_i,r_i)$ for some i. But then $f(x) \in (f(x_i)-\varepsilon/2, f(x_i)+\varepsilon/2)$, and $g(x) \in [f(x_i)-\varepsilon/2, f(x_i)+\varepsilon/2]$, whence $|f(x)-g(x)| < \varepsilon$. This shows that A_n is dense, and the proof of claim (a) is complete.

To prove claim (b), take any function $f \in \bigcap_{n=1}^\infty A_n$. Our goal is to prove that for almost every $x \in [0,1]^d$ we have $\operatorname{lip} f(x) = 0$. Since $f \in A_n$ for every $n \in \mathbb{N}$, for each n we can choose sequences of cubes $\{C_i^{(n)}\}_{i=1}^{k_n}$ in $\mathcal{C}_n$ and of points $\{a_i^{(n)}\}_{i=1}^{k_n}$ such that for all $i \in \{1,\dots,k_n\}$ and all $x \in C_i^{(n)}$ we have

$$\left| f(x) - a_i^{(n)} \right| < \beta_n l\left(C_i^{(n)}\right). \tag{4.2}$$

For a positive number α and a cube C, we denote by αC the cube that has the same centre as C and satisfies $l(\alpha C) = \alpha \cdot l(C)$. Recalling the definition in (4.1), we set

$$Z = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \bigcup_{i=1}^{k_n} \alpha_n C_i^{(n)}.$$

First we observe that $|Z| = 1$, that is the set Z has full measure in $[0,1]^d$. Indeed, for a fixed n we have $\left|\alpha_n C_i^{(n)}\right| = \alpha_n^d \left|C_i^{(n)}\right|$. As $\alpha_n < 1$, the cubes $\alpha_n C_i^{(n)}$ are pairwise disjoint. Therefore,

$$\begin{aligned} \left|\bigcup_{i=1}^{k_n} \alpha_n C_i^{(n)}\right| &= \alpha_n^d \left|\bigcup_{i=1}^{k_n} C_i^{(n)}\right| > \alpha_n^d \left(1 - 2^{-n}\right) \\ &= \left(1 - 2^{-n}\right)^2 > 1 - 2 \cdot 2^{-n}. \end{aligned}$$

Hence, for a fixed $m \in \mathbb{N}$ we obtain that

$$\left|\bigcap_{n=m}^{\infty} \bigcup_{i=1}^{k_n} \alpha_n C_i^{(n)}\right| > 1 - \sum_{n=m}^{\infty} 2 \cdot 2^{-n} = 1 - 2^{2-m},$$

and the right-hand side tends to 1 as $m \to \infty$, which implies the desired conclusion $|Z| = 1$.

Finally, we take an arbitrary $x \in Z$; we want to prove that $\operatorname{lip} f(x) = 0$. Since $x \in Z$ means that there exists an $m \in \mathbb{N}$ such that for each $n \geq m$ there is an index $i_n \in \{1, \ldots, k_n\}$ such that $x \in \alpha_n C_{i_n}^{(n)} \subset C_{i_n}^{(n)}$. For any such $n \geq m$, we have from (4.2) that for each $y \in C_{i_n}^{(n)}$,

$$\left|f(y) - a_{i_n}^{(n)}\right| < \beta_n l\left(C_{i_n}^{(n)}\right).$$

In particular, since $x \in \alpha_n C_{i_n}^{(n)}$, we have this for each $y \in B_\infty(x, r_n)$ where

$$r_n = (1 - \alpha_n) \cdot l\left(C_{i_n}^{(n)}\right),$$

and it follows that

$$\sup_{y \in B_\infty(x, r_n)} \frac{|f(y) - f(x)|}{r_n} \leq \frac{2\beta_n l\left(C_{i_n}^{(n)}\right)}{r_n} = \frac{2\beta_n}{1 - \alpha_n} = \frac{2}{n}.$$

As

$$r_n < l\left(C_{i_n}^{(n)}\right) < \frac{1}{n},$$

r_n tends to 0. It follows that

$$\liminf_{n \to \infty} \sup_{y \in B_\infty(x, r_n)} \frac{|f(y) - f(x)|}{r_n} = 0,$$

which concludes the proof. $\square$

Remark 4.3. In [3], the first author studied micro tangent sets of functions defined on the interval $[0,1]$. Theorem 5 in his paper leads to an alternative proof of Theorem 4.2 for $d = 1$.

Remark 4.4. One might wonder if it is even true that the typical function has a vanishing lip everywhere. However, Lemma 1.1 in [1] implies that if lip of a function vanishes outside a countable set, then the function is differentiable almost everywhere. Thus, for the typical function f, there is an uncountable set where $\operatorname{lip} f(x)$ differs from 0.

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