Astron. Nachr. / AN **000**, No. 00, 1 – 6 (0000) / **DOI *please set DOI!***

A Planetary System with an Escaping Mars

Á. Süli[1] and **R. Dvorak**[2]

[1] Eötvös Loránd University, Department of Astronomy, Pázmány Péter sétány 1/A, Budapest H-1518, Hungary
[2] Institute for Astronomy, University of Vienna, Türkenschanzstr. 17, A-1180 Wien, Austria

The dates of receipt and acceptance should be inserted later

Key words solar system: general – instabilities – methods: numerical

The chaotic behaviour of the motion of the planets in our Solar System is well established. In this work to model a hypothetical extrasolar planetary system our Solar System was modified in such a way that we replaced the Earth by a more massive planet and let the other planets and all the orbital elements unchanged. The major result of former numerical experiments with a modified Solar System was the appearance of a chaotic window at $\kappa_E \in (4, 6)$, where the dynamical state of the system was highly chaotic and even the body with the smallest mass escaped in some cases. On the contrary for very large values of the mass of the Earth, even greater than that of Jupiter regular dynamical behaviour was observed. In this paper the investigations are extended to the complete Solar System and showed, that this chaotic window does still exist. Tests in different 'Solar Systems' clarified that including only Jupiter and Saturn with their actual masses together with a more 'massive' Earth ($4 < \kappa_E < 6$) perturbs the orbit of Mars so that it can even be ejected from the system. Using the results of the Laplace-Lagrange secular theory we found secular resonances acting between the motions of the nodes of Mars, Jupiter and Saturn. These secular resonances give rise to strong chaos, which is the cause of the appearance of the instability window.

1 Introduction

In a previous work we studied in detail how our planets would evolve when their masses would be different from those which they have nowadays. The primary goal was there to study the stability of the systems with regard to their masses without changing the actual distribution of the orbital elements of the planets. Such systems may serve as models for exoplanetary systems with a comparable distribution of the semimajor axes, relatively small eccentricities and two dominating masses like Jupiter and Saturn. Additionally it can be shown with this kind of research how terrestrial like planets would dynamically evolve in such systems. In a first experiment (Dvorak & Süli 2002 = paper I) the masses of the three terrestrial planets Venus, Earth and Mars were uniformly enlarged and took as a dynamical model the truncated Solar System from the planets Venus to Saturn (=**Ve2Sa**). It turned out that the system remained in a dynamically stable state for enlargement factors up to more than 200. We then started to enlarge the masses of the inner planets separately (Dvorak et al. 2005 = paper II and Süli et el. 2005 = paper III) again in the model **Ve2Sa** and found especially interesting results for the case when we enlarged the mass of the Earth by multiplying it with the mass factor κ_E. It turned out that the new system under consideration is stable (with quasiperiodic motions of the planets) up to quite a large mass of the Earth. In the experiments the systems were stable for 20 million years when the mass of the Earth was up to about 1.6-fold mass of Jupiter. But we found a surprising exception: around $\kappa_E = 5$ in all our computations the modified Solar System is in a highly chaotic state with Mars suffering from large eccentricities and even from escapes. In continuation of this work the interesting dynamical behaviour is now studied in the complete Solar System including also Mercury and also the outer two ice giants Uranus and Neptune.

2 Methods of investigation

As has been shown in many different studies the use of long term integration of the motions in the planetary system gives reliable results up to at least several hundred millions of years for a qualitative study of the orbits (e.g. Ito & Tanikawa 2002). This means that we have a good knowledge of the semimajor axis, the eccentricities and the inclinations of the orbits of the planets involved. On a long term scale the motions of the planets are chaotic which was shown by different authors (e.g. Laskar 1988, Laskar 1996; Murray & Holman 1999; Lecar et al. 2001). Additionally in a work by Laskar (1994) he found that in a very far future (some 10^9 years) by several slight modifications of the initial conditions in the Solar System – where the semimajor axes were kept constant – Mercury could get eccentricities close to 1. Already in the abstract he claims that "The chaotic diffusion of Mercury is so large that its eccentricity can potentially reach values very close to 1, and ejection of this planet out of the Solar system resulting from close encounter with Venus is possible in less than 3.5 Gyr." Our modification is somewhat more drasticall but it is not our purpose to simulate our Solar System as it is but to investigate it as a special model for extrasolar planetary systems.

For our new investigation we used a program already used and tested in many other applications of orbit dynamics, namely the Lie-integration (e.g. Dvorak et al. 2003; Asghari et al. 2004). The method is based on the integration of differential equations with Lie-series and uses the property of recurrence formulae for the Lie-terms. This method has an automatic step size control which makes its results reliable also for eccentric orbits, whilst no additional computations are necessary to accomplish (in contrary to symplectic methods). The details of the method are described in the appendix.

During the long term integrations we checked the evolution of the action like elements, which – in case of a chaotic orbit – show quite irregular behaviour: small to moderate jumps in the eccentricities and inclinations and also in the semimajor axis can be found. Finally in many cases we found a 'quasiescape'[1] of the planet with a small mass, i. e. Mars. What we were interested in is to investigate more in detail the dynamical behaviour of a 'Solar-like' planetary system (with all terrestrial planets and the ice giants) when we increase the mass of the Earth with a factor κ_E between 4 and 6.

3 The dynamical models

First we have undertaken computations in different dynamical models inside this 'chaotic window': in the truncated models **Ve2Ju**, **Ve2Ma** and **Ea2Ma** the motion of Mars did not show any signs of chaos when we enlarged the mass of the Earth via κ_E. These carefull examinations showed that the inclusion of Mercury did not significantly modified the dynamical evolution of the inner planets, although it's mass is still comparable to the those of the other three terrestrial planets. It therefore seems clear that the couple of Jupiter-Saturn in the model **Ve2Sa** is – together with a more massive Earth – responsible for the escapes of Mars.

The effect of Uranus and Neptune on the dynamics of the planets and on the size and location of the chaotic window (if it is still exist) is of high concern. In order to study the dynamics of these systems several numerical integrations were performed in the chaotic window. To present the main features of the results the specific value of $\kappa_E = 4.7$ was selected (chosen just as one out of several others in this window). For this mass factor the evolution of the action like elements will be shown and discussed in details and also a comparison with the former results is given.

3.1 The 'truncated' planetary system Ve2Sa

In Fig. 1 (upper panel) we can see a kind of irregular variations in the semimajor axis of Mars; they go together with large values of the eccentricities (middle panel). During these phases the inclination (lower panel) is always relatively small. On the contrary when the inclination is large the variations

[1] with orbital eccentricities $e > 0.91$

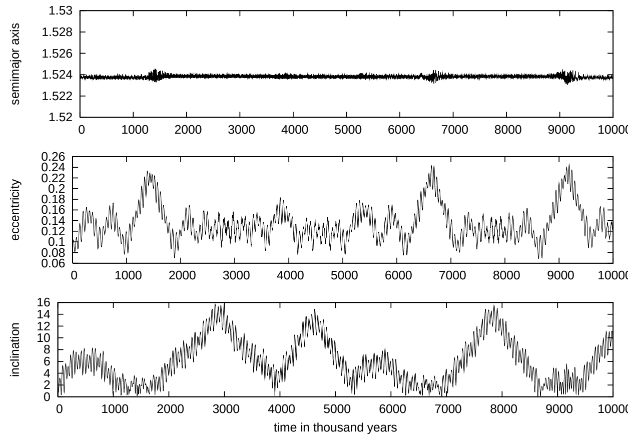


Fig. 1 Orbital evolution of Mars in the dynamical model **Ve2Sa** with $\kappa_E = 4.7$; the larger variations in the semimajor axis (upper plot) coincides with larger eccentricities (middle graph) and relative small inclinations (lower graph).

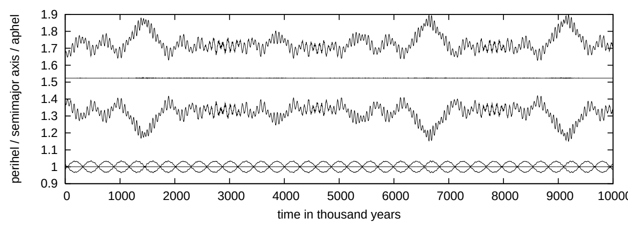


Fig. 2 The time evolution of the semimajor axes the perihelia and aphelia distances of Mars and Earth in the dynamical model **Ve2Sa** with $\kappa_E = 4.7$ for 10 million years.

in the eccentricity and the value itself is relatively small. This is a consequence of the fact that Delaunay element $H = \sqrt{a(1-e^2)}\cos i$ changes slowly with time. In Fig. 2 the semimajor axis, the perihelion and aphelion distances of Mars together with the respective orbital elements of the Earth are plotted. As one can see the two orbits are still far from intersection, nevertheless whenever Mars comes close to the Earth these kind of 'punches' act as larger perturbations on its semimajor axis (see upper panel of Fig. 1).

In Fig. 3 we see the results of a continuation of the computations shown in Fig. 2 up to the moment when the eccentricities of the orbit of Mars reaches values of ≈ 0.8 after 54 Myrs. From Fig. 2 and 3 it is clear that the semimajor axis of the Earth is constantly 1.0 AU and its aphelion distance reaches periodically 1.05 AU, therefore when the eccentricity of Mars is higher than 0.31 its orbit may cross that of the Earth. Throughout the integration this limit is approached for several short time interval. Although the distance be-

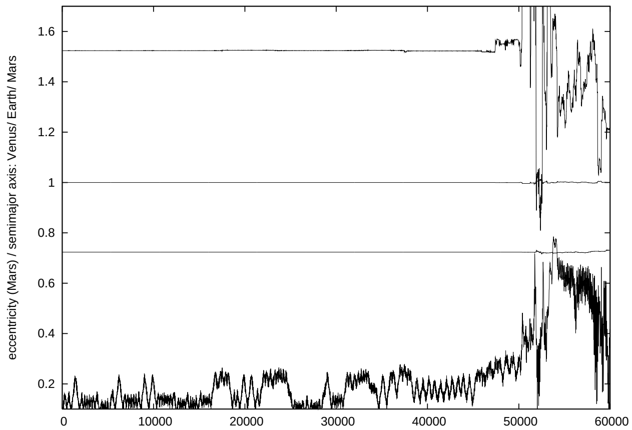


Fig. 3 Orbital Evolution of the semimajor axes for the three terrestrial planets (upper lines) and the eccentricity of Mars for 60 million years in the model **Ve2Sa** ($\kappa_E = 4.7$).

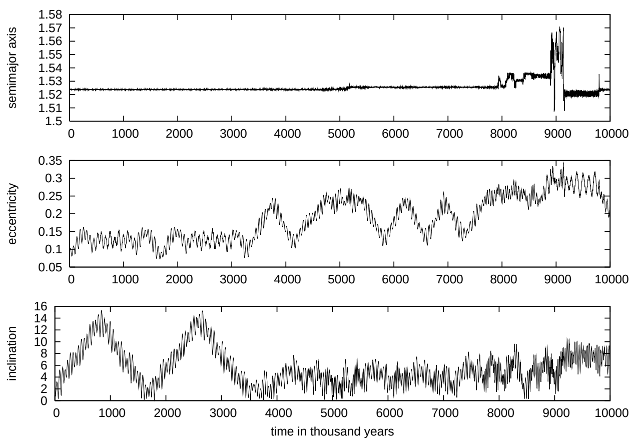


Fig. 4 Evolution of the the action like variables of Mars for 10 million years in the model **Me2Ne**; semimajor axis (upper graph), eccentricity (middle graph) and inclination (lower graph) ($\kappa_E = 4.7$).

tween the two orbits (≈ 0.1 AU) during these periods are still an order of magnitude bigger than the Hill-sphere of Earth (≈ 0.01 AU) the Earth can strongly perturb the motion of Mars.

After 45 million years the eccentricity of Mars begins to grow secularly, reaches values bigger than 0.31. This is followed by a cascade mechanism caused by the mutual orbital crossings: the eccentricity of Mars suffers from very big jumps with amplitude as high as 0.75: the orbit of Mars crosses the orbits of the Earth and Venus too and it is only a matter of time that the highly chaotic orbit leads to a subsequent escape of Mars.

3.2 The complete system Me2Ne

In Fig. 4 (top graph) one can observe a small increase (jump) in the semimajor axis after almost 4 million years (because of the difference in the y-scale compared to Fig. 1 it is not well visible but it is present) together with an increase in the

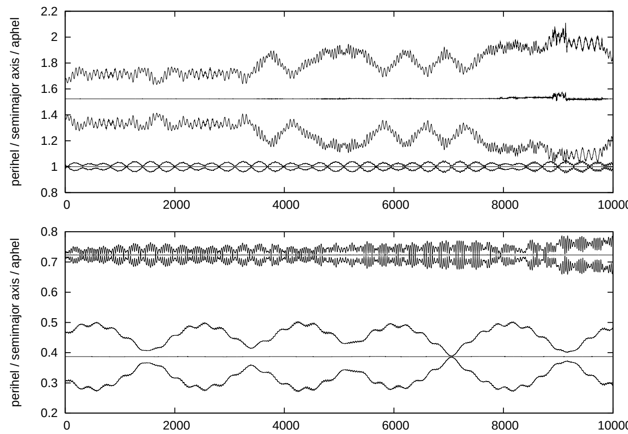


Fig. 5 Orbital evolution of the semimajor axis and the perihelia and aphelia distances for the Earth and Mars (upper panel) for 10 million years in the model **Me2Ne**; the same elements are plotted for Mercury and Venus (lower panel) ($\kappa_E = 4.7$).

eccentricity. From the same moment on the inclination (bottom graph) stays in a mode of only small irregular changes up to the end of integration of 10 million years. Large irregular variations superimposed on a high mean value of $e \approx 0.26$ after 7.8 million years lead to strong variations in the semimajor axis ($1.51 < a < 1.57$) of Mars as it can be seen in Fig. 4 (upper graph). In Fig. 5 (upper graph) this dynamical behaviour is well explained when we see that the aphelion distance of the Earth and the perihelion distance of Mars almost equals after nine millions of years. This means that Mars may enter into the Hill-sphere of the Earth. In the same Fig. 5 (lower graph) one can see that also Venus is suffering from an increase in the eccentricity. It is well visible that Mercury does not play any important role in the dynamical evolution during the first ten million years (lower graph in Fig. 5). In both models discussed the motion of the outer planets did not show any visible different behaviour compared to the actual Solar System.

4 Determination of the secular frequencies

For a possible explanation of the strong chaotic behaviour of the system for special values of κ_E we have applied the first order secular theory of Laplace-Lagrange. It can be used when the eccentricities and inclinations can be regarded as small quantities, the orbits are not crossing and no mean motion commensurabilities are present. It is also important that the masses involved are small with respect to the primary body, which is for sure the case even with a 6-fold masses in the case of the Earth (still much smaller than the gas giants). With the orbital elements:

$$\begin{pmatrix} h \\ k \end{pmatrix} = e \cdot \begin{matrix} \sin\varpi \\ \cos\varpi \end{matrix}, \qquad \begin{pmatrix} p \\ q \end{pmatrix} = i \cdot \begin{matrix} \sin\Omega \\ \cos\Omega \end{matrix} \tag{1}$$

the Laplace-Lagrange solution reads:

$$\begin{pmatrix} h_s \\ k_s \end{pmatrix} = \sum_{j=1}^{n} M_s^{(j)} \begin{matrix} \sin \\ \cos \end{matrix} (g_j t + \beta_j) , \qquad (2)$$

$$\begin{pmatrix} p_s \\ q_s \end{pmatrix} = \sum_{j=1}^{n} L_s^{(j)} \begin{matrix} \sin \\ \cos \end{matrix} (f_j t + \gamma_j) , \qquad (3)$$

where N is the number of bodies ($N = 5$ for $\mathbf{Ve2Sa}$ and $N = 8$ for $\mathbf{Me2Ne}$), $M_s^{(j)}, L_s^{(j)}$ are the amplitudes, g_j, f_j are the secular frequencies, and β_j, γ_j are the phases.

To determine the first order solution of the dynamical model as a function of the mass factor κ_E, we have computed the $g_j(\kappa_E)$, $f_j(\kappa_E)$, $M_s^{(j)}(\kappa_E)$, $L_s^{(j)}(\kappa_E)$ and $\beta_j(\kappa_E)$, $\gamma_j(\kappa_E)$ functions for $\kappa_E \in [4, 6]$ with the aid of the MAPLE algebra manipulation package. The comparison of the results from our determination up to the first order with Bretagnon 1974, 1982 and Kneževic 1986 showed satisfactory agreement for the model $\mathbf{Me2Ne}$ (with $\kappa_E = 1$).

The orbital elements of the sth planet are described by Eq. (2) and Eq. (3), which are the sum of harmonic oscillations. Using these formulae it can be calculated that the planets' eccentricities and inclinations are varying between given limits with quasiperiodic oscillations. Due to the positive g_j secular angular velocities the apsidal lines of the planets are rotating in the same direction as the planets, whereas the nodes accordingly to the negative f_j secular angular velocities (see the first line of Table 1) are are rotating in the opposite direction. Upon these mean rotations quasiperiodic variations are superimposed. Both the apsidal and nodal motions can be approximated by average angular velocities, which are to a first approximation equal with the frequencies of those harmonious terms which are multiplied by the largest amplitudes:

$$e_s \cdot \begin{matrix} \sin \varpi_s \\ \cos \varpi_s \end{matrix} \approx M_s^{(J)} \begin{matrix} \sin \\ \cos \end{matrix} (g_J t + \beta_J) , \qquad (4)$$

$$i_s \cdot \begin{matrix} \sin \Omega_s \\ \cos \Omega_s \end{matrix} \approx L_s^{(K)} \begin{matrix} \sin \\ \cos \end{matrix} (f_K t + \gamma_K) , \qquad (5)$$

where $M_s^{(J)} = \mathbf{max}_j |M_s^{(j)}|$, $L_s^{(K)} = \mathbf{max}_j |L_s^{(j)}|$ and the average angular velocities of the sth planet are given by g_J and f_K. In this manner the secular frequencies can be associated with each planet. We note that this assignment is not unambiguous.

In Fig. 6 we compare the frequencies f_2 and f_3 assigned to the planets Earth and Mars, respectively, in the models **Ve2Sa** and **Me2Ne**, which show only a small shift along the κ_E axis: the minimum distance is 0.1567 arcsec/year for $\kappa_E \approx 5.00$ in the model **Ve2Sa** and 0.1538 arcsec/year for $\kappa_E \approx 5.20$ in the model **Me2Ne**

A study of Table 1 shows that the largest amplitude are, in the solution for Mars, $L_4^{(2)}$ and $L_4^{(3)}$, for Jupiter $L_5^{(2)}$ and $L_5^{(3)}$ and for Saturn $L_6^{(2)}, L_6^{(3)}$. Accordingly the f_2 and f_3 frequencies, can be associated with Mars, Jupiter and Saturn. The orbital plane of Mars therefore on the average rotates together with those of Jupiter and Saturn, giving rise to chaotic behaviour. The equality of two apsidal or nodal rates is referred to in Solar System as a secular resonance. In this case we have three secular resonances: $\dot{\Omega}_M \approx \dot{\Omega}_J$, $\dot{\Omega}_M \approx \dot{\Omega}_S$ and $\dot{\Omega}_J \approx \dot{\Omega}_S$. We suspect that these secular resonances are the main source of the observed chaos, and produce the chaotic window.

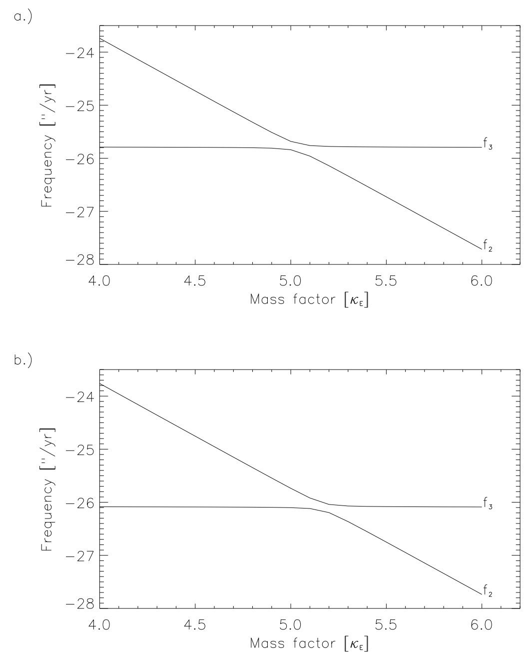


Fig. 6 Secular frequencies f_j in the model **Ve2Sa** (a) and **Me2Ne** (b) with respect to the mass factor κ.

5 Discussion

In modelling extrasolar planetary systems we took our Solar System as starting point for several models, where we increased the masses of the planets involved until to the moment of instability. In former papers it turned out that only a significantly larger mass of the Earth (the other planets' masses were left unchanged) could lead to a decay of these modified systems. In the article we focus on a surprising "chaotic window" in the dynamical evolution of our planetary system when we increase the mass of the Earth by the massfactor $4 < \kappa_E < 6$. Former results of paper III have unveiled this interesting dynamical behaviour for a truncated Solar system model ($\mathbf{Ve2Sa}$). We used a heuristic way to find out the reason for this unexpected strong chaotic behaviour: we numerically integrated different dynamical models. It turned out that only the couple Jupiter-Saturn together with the Earth (with a larger mass) is the cause for the subsequent escape of Mars. In the comparison of the two models **Ve2Sa** with **Me2Ne** we see that there is in principle no difference for the state of chaoticity of the orbits: in both models Mars suffers sooner or later from close approaches with the Earth because of the large values of the eccentricity (Fig. 7). This is true for the whole interval $4 \leq \kappa_E \leq 6$ which we tested with a

Table 1 The f_j secular frequencies, and the $L_s^{(j)}$ amplitudes of the model for $\kappa_E = 5.2$ determined by the Laplace-Lagrange theory. f_j in arcsec/yr.

f_j	-48.769679	-26.194843	-26.041015	-9.7343131	-7.0784088	-2.8810179	-0.67270058	0.0
$L_s^{(j)}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
Mercury	$2.879{\cdot}10^{-3}$	$1.993{\cdot}10^{-3}$	$2.444{\cdot}10^{-3}$	$1.616{\cdot}10^{-1}$	$6.545{\cdot}10^{-2}$	$1.640{\cdot}10^{-3}$	$1.600{\cdot}10^{-4}$	$2.796{\cdot}10^{-2}$
Venus	$-5.241{\cdot}10^{-2}$	$-7.126{\cdot}10^{-3}$	$-9.535{\cdot}10^{-3}$	$-2.747{\cdot}10^{-3}$	$2.190{\cdot}10^{-2}$	$1.259{\cdot}10^{-3}$	$1.520{\cdot}10^{-4}$	$2.796{\cdot}10^{-2}$
Earth	$7.427{\cdot}10^{-3}$	$-3.862{\cdot}10^{-3}$	$-5.503{\cdot}10^{-3}$	$-2.975{\cdot}10^{-3}$	$2.026{\cdot}10^{-2}$	$1.225{\cdot}10^{-3}$	$1.510{\cdot}10^{-4}$	$2.796{\cdot}10^{-2}$
Mars	$-2.277{\cdot}10^{-3}$	$3.867{\cdot}10^{-1}$	$3.723{\cdot}10^{-1}$	$-1.876{\cdot}10^{-3}$	$1.141{\cdot}10^{-2}$	$1.051{\cdot}10^{-3}$	$1.465{\cdot}10^{-4}$	$2.796{\cdot}10^{-2}$
Jupiter	$-4.370{\cdot}10^{-6}$	$-1.875{\cdot}10^{-3}$	$4.059{\cdot}10^{-3}$	$9.975{\cdot}10^{-6}$	$-1.132{\cdot}10^{-4}$	$7.469{\cdot}10^{-4}$	$1.376{\cdot}10^{-4}$	$2.796{\cdot}10^{-2}$
Saturn	$2.056{\cdot}10^{-6}$	$4.589{\cdot}10^{-3}$	$-1.017{\cdot}10^{-2}$	$1.823{\cdot}10^{-5}$	$-1.652{\cdot}10^{-4}$	$6.113{\cdot}10^{-4}$	$1.327{\cdot}10^{-4}$	$2.796{\cdot}10^{-2}$
Uranus	$-6.524{\cdot}10^{-9}$	$-1.946{\cdot}10^{-4}$	$4.374{\cdot}10^{-4}$	$-4.657{\cdot}10^{-6}$	$7.325{\cdot}10^{-5}$	$-1.390{\cdot}10^{-2}$	$-1.315{\cdot}10^{-4}$	$2.796{\cdot}10^{-2}$
Neptune	$7.922{\cdot}10^{-10}$	$-2.170{\cdot}10^{-5}$	$4.889{\cdot}10^{-5}$	$-4.275{\cdot}10^{-7}$	$4.827{\cdot}10^{-6}$	$1.642{\cdot}10^{-3}$	$-1.388{\cdot}10^{-3}$	$2.796{\cdot}10^{-2}$

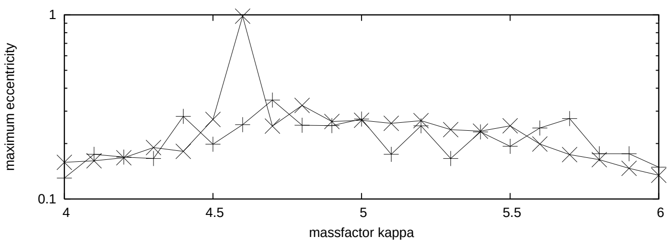


Fig. 7 Comparison of the maximum value of the eccentricity of Mars in the models **Ve2Sa** ($\times$) and **Me2Ne** (+) for 10 million years with respect to the mass factor κ_E.

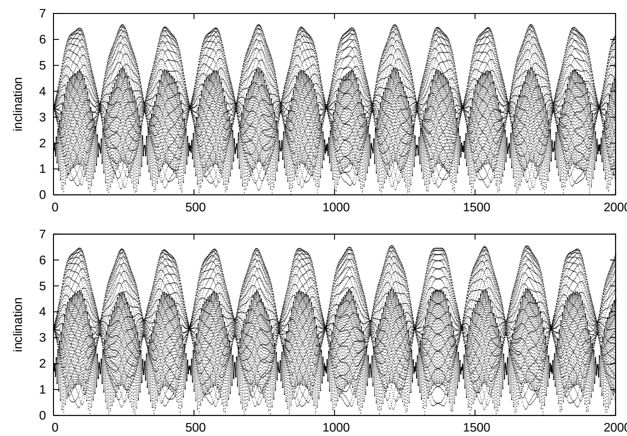


Fig. 8 Comparison of the inclinations of the planets Venus and Mars in the models **Ve2Sa** (upper graph) and **Me2Ne** (lower graph) for the Earth with a mass of Saturn for 2 million years.

step of $\Delta\kappa_E = 0.1$ for both models. On the contrary, with larger values of $\kappa_E < 540 \approx 1.8 M_{Jupiter}$ a very regular dynamical behaviour was observed for all planets. This regular dynamical evolution can be seen in the quasiperiodic behaviour of the inclinations of the planets (Fig. 8). In the upper panel one can see the inclinations of Venus (max(i)=$6.°5$) and Mars (max(i)= $5°$) for the model **Ve2Sa**, in the lower panel the same quantities in the model **Me2Ne**. It is evident that qualitatively both plots agree quite well. The first quantitative differences appear after 1 million years in the inclination of Venus. The same overall behaviour can be observed for the eccentricities (not shown). All these orbits are stable. But why do we have strong chaos in this window of κ which does not appear for larger values up to a mass factor which correspond to an Earth comparable to Jupiter? Using the results of the Laplace-Lagrange secular theory we found secular resonances acting between the motions of the nodes of Mars, Jupiter and Saturn. These secular resonances give rise to strong chaos, which is the primary cause of the appearance of the chaotic window, and eventually the escape of Mars. The properties of the dynamics of the model in the chaotic window must be further analysed by higher order secular theory. The final answer to this problem is highly interesting for future research on the dynamics of extrasolar planetary systems especially when we will have evidence via observations – primarely from space missions like KEPLER, DARWIN and TPF– that some of them are also hosting terrestrial planets.

Acknowledgements. We thank the Wissenschaftlich-technisches Abkommen Österreich-Ungarn Project A12-2004. Most of the numerical integrations were accomplished on the NIIDP (National Information Infrastructure Development Program) supercomputer in Hungary. This study was supported by the International Space Science Institute (ISSI) and benefit from the ISSI team 'Evolution of Habitable Planets'.

References

Asghari N. et al.: 2004, A&A, 426, 353
Bretagnon, P.: 1974, A&A, 30, 341
Bretagnon, P.: 1982, A&A, 114, 278
Delva, M.: 1984, Celestial Mechanics 34, 145
Delva, M.: 1985, A&A Supp. 60, 277
Dvorak, R., Müller, P., Kallrath, J.: 1993, A&A, 274, 627
Dvorak R., Süli, A., Freistetter F.: 2005, in eds. Z. Knežević IAU Colloq. 197, Dynamics of populations of planetary systems, p. 63
Dvorak R., Pilat-Lohinger E., Funk B., Freistetter F.: 2003, A&A, 398, L1
Dvorak R., Süli, A.: 2002, Celest. Mech. & Dyn. Astron., 83, 77
Gröbner, W., 1967, Die Lie-Reihen und ihre Anwendungen (Berlin, VEB, Deutscher Verlag der Wissenschaften)
Hanslmeier A., Dvorak R.: 1984, A&A, 132, 203
Ito T., Tanikawa K.: 2002, MNRAS, 336, 483
Kneževič Z.: 1986, Celest. Mech., 38, 123

Laskar J.: 1988, A&A, 198, 341
Laskar J.: 1994, A&A, 287, L9
Laskar J.: 1996, Celest. Mech. & Dyn. Astron, 64, 115
Lecar M., Franklin F.A., Holman M.J., Murray N.W.: 2001, Annu. Rev. A & A, 39, 581
Lichtenegger H., 1984, Celest. Mech., 34, 357
Murray N., Holman M.: 1999, Sci. 283, 1877
Süli À., Dvorak, R. & Freistetter F.: 2005, MNRAS 363, 1, 241

A The Lie-integration

Because our integration method is not one which is used by many colleagues we shortly explain how the LIE-integration works. This method, based on an idea by Gröbner 1967, has been used by our group since 1984 (e.g. Hanslmeier & Dvorak 1984; Lichtenegger 1984; Delva 1984,1985; Dvorak et al. 1993). It turned out to be a precise and fast tool, which can be also used when the orbits of a system suffer from close encounters. This is done by computing for every step the optimal step size for the desired precision of the integration.

Let D denote a linear differential operator; the point $z = (z_1, z_2, \ldots, z_n)$ lies in the n-dimensional z-space; the functions $\theta_i(z)$ are holomorphic within a certain domain G, e.g. they can be expanded in converging power series. Let the function $f(z)$ be holomorphic in the same region as $\theta_i(z)$. Then D can be applied to $f(z)$:

$$Df = \theta_1(z)\frac{\partial f}{\partial z_1} + \theta_2(z)\frac{\partial f}{\partial z_2} + \ldots + \theta_n(z)\frac{\partial f}{\partial z_n} \tag{A1}$$

If we proceed applying D to f we get

$$\begin{aligned} D^2 f &= D(Df) \\ &\vdots \\ D^n f &= D(D^{n-1} f) \end{aligned}$$

The **Lie-series** will be defined in the following way;

$$L(z,t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu f(z) = f(z) + tDf(z) + \frac{t^2}{2!} D^2 f(z) + \ldots$$

Because we can write the Taylor-expansion of the exponential function

$$e^{tD} f = 1 + tD^1 + \frac{t^2}{2!} D^2 + \frac{t^3}{3!} D^3 + \ldots \tag{A2}$$

$L(z,t)$ can be written in the symbolic form

$$L(z,t) = e^{tD} f(z) \tag{A3}$$

The convergence proof of $L(z,t)$ is given in detail in Gröbner 1967. The most useful property of Lie-series is the Vertauschungssatz:

Theorem 1. *Let $F(z)$ be a holomorphic function in the neighbourhood of $(z_1, z_2, \ldots, z_n)$ where the corresponding power series expansion converges at the point $(Z_1, Z_2, \ldots, Z_n)$; then we have:*

$$F(Z) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu F(Z) \tag{A4}$$

or

$$F(e^{tD}) z = e^{tD} F(z) \tag{A5}$$

Making use of it we can demonstrate how Lie-series solve differential equations. Let us give the system of differential equations:

$$\frac{dz_i}{dt} = \theta_i(z) \tag{A6}$$

with $(z_1, z_2, \ldots, z_n)$. We postulate that the solution of (A6) can be written as

$$z_i = e^{tD} \xi_i \tag{A7}$$

where ξ_i are the initial conditions $z_i(t = 0)$ and D is the Lie-operator as defined in (A1). In order to prove (A7) we differentiate it with respect to time t and make use of the Vertauschungssatz:

$$\frac{dz_i}{dt} = De^{tD}\xi_i = e^{tD} D\xi_i. \tag{A8}$$

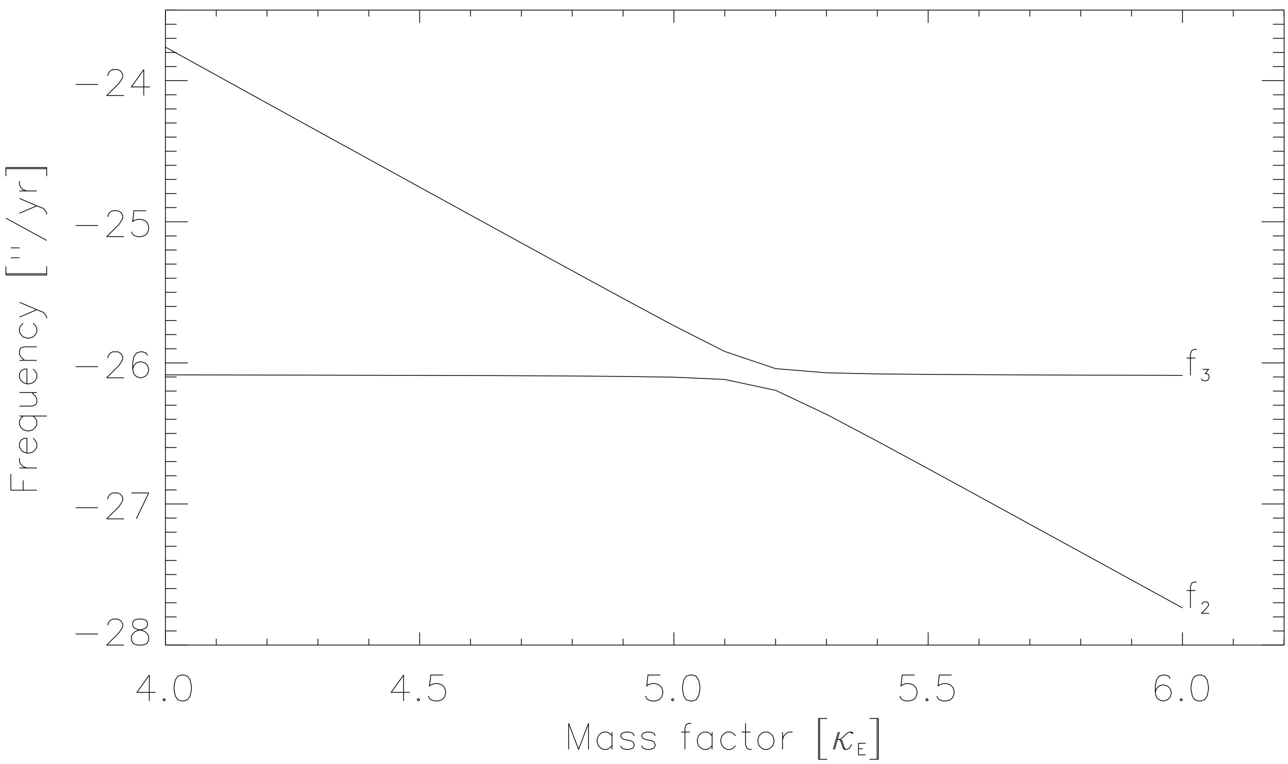
Because of

$$D\xi_i = \theta_i(\xi_i) \tag{A9}$$

we obtain – again by using the Vertauschungssatz – the following result which turns out to be the original differential equation (A6):

$$\frac{dz_i}{dt} = e^{tD}\theta_i(\xi_i) = \theta_i(e^{tD}\xi_i) = \theta_i(z_i) \tag{A10}$$

b.)



−24
−25
−26
−27
−28
Frequency ["/yr]
4.0
4.5
5.0
5.5
6.0
Mass factor [κ_E]
f_3
f_2