TBA, NLO Lüscher correction, and double wrapping in twisted AdS/CFT

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Abstract

The ground-state energy of integrably-twisted theories is analyzed in finite volume. We derive the leading and next-to-leading order (NLO) Lüscher-type corrections for large volumes of the vacuum energy for integrable theories with twisted boundary conditions and twisted S-matrix. We then derive the twisted thermodynamic Bethe ansatz (TBA) equations to describe exactly the ground state, from which we obtain an untwisted Y-system. The two approaches are compared by expanding the TBA equations to NLO, and exact agreement is found. We give explicit results for the $O(4)$ model and for the three-parameter family of γ-deformed (non-supersymmetric) planar AdS/CFT model, where the ground-state energy can be nontrivial and can acquire finite-size corrections. The NLO corrections, which correspond to double-wrapping diagrams, are explicitly evaluated for the latter model at six loops.

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1 Introduction

The AdS/CFT correspondence in the planar limit can be described by a two-dimensional integrable quantum field theory. The finite-volume energy levels of this integrable theory correspond on one side to the string energies in the curved $AdS_5 \times S^5$ background, while to the anomalous dimensions of gauge-invariant single-trace operators on the other side. Integrability provides tools to solve the finite-volume spectral problem exactly. (For recent reviews with many references, see [1, 2].)

For large volume, L, (long operators of size L), the asymptotic Bethe ansatz [3, 4] determines the spectrum including all polynomial corrections in L^{-1}. In the weak-coupling limit, this result is exact up to L loops; but over L loops, wrapping diagrams start to contribute [5]. In the integrable quantum field theory, they show up as exponentially small vacuum polarization effects: virtual particles circling around the space-time modifies the energy levels [6]. These effects have a systematic expansion which counts how many times virtual particles encircle the space-time cylinder (or diagrams wrap around). The leading-order (LO) Lüscher correction corresponds to a single circle or wrapping. Together with the asymptotic Bethe ansatz, they provide an exact result up to $2L$ loops. The next-to-leading (NLO) Lüscher correction corresponds to two circles and double wrapping. Including their contribution describes the energy levels/anomalous dimensions exactly up to $3L$ loops.

For an exact description, valid for any number of loops, one has to sum up all virtual processes. For the ground state, this is done by the thermodynamic Bethe ansatz (TBA), which evaluates the saddle point of the partition function for large Euclidean times in the mirror (space-time rotated) description [7, 8, 9, 10, 11, 12, 13, 14]. The TBA provides coupled integral equations for infinitely-many unknown functions, whose solutions determine the exact ground-state energy and satisfy the so-called Y-system relations, which is characteristic for the model and are the same for all the excited states [15]. What is different for the excited states is the analytical structure of these Y-functions [16, 17, 18]. Using additional inputs, such as discontinuity relations [19, 20] and analytical structure, the Y-system can be turned into integral equations for excited states [21, 22], which provide the solution of the finite-volume spectral problem. An ultimate solution would be to replace the infinite Y-system with a finite T-Q system (see attempts [23, 24, 25, 26, 22] in this direction), which would lead to nonlinear integral equations (NLIE) for only finitely-many unknowns.

In the present paper, we would like to analyze the ground state of the three-parameter family of γ-deformed planar AdS/CFT theories [27, 28, 29, 30, 31], for which we refer as γ-deformed theory from now on. Contrary to the undeformed or β-deformed theories, in the most general case, no supersymmetry is preserved, so the ground state is indeed nontrivial and affected by wrapping corrections. The planar gauge theory is nevertheless ultraviolet finite and scale-invariant [32]. This is an ideal laboratory to test ideas directly on the ground state, which actually contains all information about the theory.

The γ-deformation can be implemented in several distinct ways: in [33] it was described as an operatorial twisted boundary condition (the twist depends on the particle number); in [34, 35] as a (c-number) twisted boundary condition and a twisted scattering matrix; finally in [36] the authors showed that the untwisted Y-system with twisted asymptotic conditions is consistent with the LO Lüscher (single wrapping) correction as calculated on the gauge-theory side. In this paper, based on our previous work [35], we choose twisted boundary condition and twisted S-matrix.

We begin by analyzing in Sec 2 the effect of a twisted boundary condition on the ground state in general. We derive exact expressions for the LO and NLO Lüscher corrections valid for any integrable theory with a twisted boundary condition. The LO correction contains information about the spectrum of the (mirror) theory, while the NLO contains the logarithmic derivative of the scattering matrix. We show that a Drinfeld-Reshetikhin type twist [37] of the scattering matrix does not affect the ground-state energy. We then demonstrate the effect of the twist in the TBA equations in general. These equations provides the exact description of the ground state for any finite size. By expanding the result for large sizes, we must recover the LO and NLO Lüscher corrections. This is explicitly elaborated in the examples that follow.

As a warm up in a simpler case, we analyze in Sec. 3 the $O(4)$ model with twisted boundary condi-

tions. After calculating the LO and NLO Lüscher corrections, we derive the so-called raw (canonical) TBA equations, which contain the twist as chemical potentials. Interestingly, the twist does not show up in the simplified TBA equations except in the asymptotic behavior of the Y-functions. As a consequence, the Y-system is the same as the untwisted one. We solve the simplified TBA equations at NLO and compare with the NLO Lüscher correction. We find complete agreement.

We turn in Sec. 4 to the γ-deformed AdS/CFT model. We calculate first the LO Lüscher correction. In calculating the NLO correction, we determine the determinant of the two-particle S-matrix $S^{Q_1 Q_2}$ in all the $su(2)_L \otimes su(2)_R$ sectors for the generic Q_1 and Q_2 bound-state case. We then derive the raw TBA equations from first principles by evaluating exactly the chemical potentials originating from the twisted boundary condition. (For the untwisted case, the TBA equations were formulated in [10, 11, 12, 13, 14].) The twist disappears from the simplified equations, just as it does in the O(4) case. (See [38] for a general argument on this.) The twist nevertheless reappears in the asymptotic boundary conditions for the Y-functions. Since the simplified equations are not twisted, neither is the Y-system, as was anticipated by the authors of [36, 39]. Our derivation confirms their assumption. We then expand the TBA equations to NLO and compare with the result of the NLO Lüscher correction. We find complete agreement again.

We evaluate in Sec. 5 the weak-coupling expansion of the NLO Lüscher correction, which corresponds to double-wrapping diagrams. We explicitly compute this correction for $L = 3$, thereby obtaining the anomalous dimension of the operator $\mathrm{Tr}Z^3$ in the twisted gauge theory up to six loops.

Finally, Sec. 6 contains our conclusion and outlook.

2 Finite-size corrections of the vacuum energy

In this section we analyze the finite-size corrections for the ground state with a twisted boundary condition. We consider an integrable $(1 + 1)$-dimensional quantum field theory that possesses just one multiplet of particles with the same dispersion relation. The particles are labeled by α, and their interaction is described by the two-particle scattering matrix $S^{\delta\gamma}_{\alpha\beta}(p_1, p_2)$, which does not admit any bound states.[1] We are interested in the ground-state energy of a system of size L with a c-number twisted boundary condition in terms of the scattering data. The twisted boundary condition is defined by means of a conserved charge J, which commutes with the scattering matrix $[J, S] = 0$. The twists are implemented by introducing a so-called defect line on the circle. It has the effect that, whenever a particle of type α crosses the defect line from the left to the right, it picks up the transmission phase $e^{i\gamma J_\alpha}$, where γ is the twist angle supposed to be real. If the particle moves oppositely, then it picks up the inverse phase $e^{-i\gamma J_\alpha}$. This ensures that if we formulate the Bethe-Yang equation by moving one particle around the circle and scattering with all the other particles and with the defect line in both directions, then we obtain equivalent equations.

In deriving the finite-size energy of the vacuum with the defect line, $E_0^d(L)$, we analyze the twisted Euclidean torus partition function from two different perspectives, see Figure 1.
By compactifying the time-like direction with period R and taking the $R \to \infty$ limit, the ground-state energy of the twisted system can be extracted from the twisted partition function as

$$\lim_{R \to \infty} Z^d(L, R) = \lim_{R \to \infty} Tr \left(e^{-H^d(L)R} \right) = e^{-E_0^d(L)R} + \dots . \tag{2.1}$$

In the alternative description in which the role of Euclidean time, $\tilde{x} = -it$, and space, x, are exchanged, the defect will be localized at a constant imaginary time $\tilde{t} = -ix$ of the mirror model. It acts as an operator of the periodic Hilbert space of the mirror model defined by the configurations on a fixed-$\tilde{t}$ slice. The action of this operator can be calculated from the transmission phase [40]. In the present case, the operator is simply $e^{i\gamma J}$, and we can evaluate the twisted partition function alternatively as

$$Z^d(L, R) = \mathrm{Tr}(e^{-\tilde{H}(R)L} e^{i\gamma J}), \tag{2.2}$$

[1] With a view to later applying this formalism to AdS/CFT, we do not assume relativistic invariance; hence, the two-particle S-matrix need not be a function of the difference of the particles' momenta.
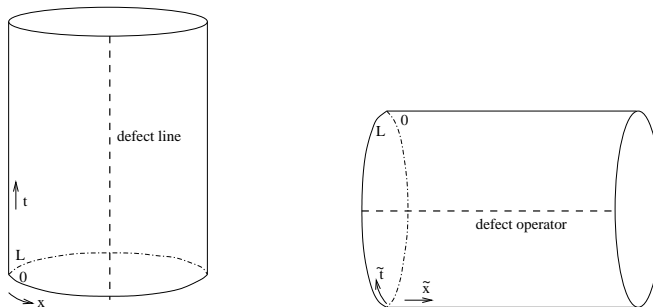
Figure 1: Two possible locations of a defect. On the left it is located in space, and it introduces a twisted boundary condition. On the right it is located in (Euclidean) time, and it acts as an operator on the periodic Hilbert space.

where we use a tilde ˜ to help distinguish quantities in the mirror model. In the first subsection, we suppose that the volume L is large and expand the partition function at leading and next-to-leading orders. In this way, we derive the LO and NLO Lüscher-type corrections for the ground state energy of the twisted system. Then, in the second subsection, we comment on how one can evaluate the partition function in the saddle-point approximation to obtain the twisted thermodynamic Bethe ansatz (TBA) equations.

2.1 Large-volume expansion

In this subsection, we evaluate the twisted partition function at LO and NLO for large volumes (i.e., L is large, and $R \to \infty$). This means that we keep the first two nontrivial terms in the expansion of the twisted partition function

$$\lim_{R \to \infty} \text{Tr}(e^{-\tilde{H}(R)L} e^{i\gamma J}) = 1 + \sum_{k,\alpha} e^{i\gamma J_\alpha - \tilde{\epsilon}(\tilde{p}_k)L} + \sideset{}{'}\sum_{k,l,(\alpha,\beta)} e^{i\gamma J_{(\alpha,\beta)} - (\tilde{\epsilon}(\tilde{p}_k) + \tilde{\epsilon}(\tilde{p}_l))L} + \dots, \qquad (2.3)$$

where k, l are the labels of the allowed mirror momenta $\tilde{p}$; α is the color index of the one-particle and (α, β) is that of the two-particle state. The sum $\sum'$ is taken over the distinct two-particle states. J is the conserved charge such that J_α denotes its eigenvalue on the one particle, while $J_{(\alpha,\beta)}$ is its eigenvalue on the two-particle state. Finally, $\tilde{\epsilon}(\tilde{p})$ denotes the energy of the mirror particle. Clearly, the defect does not affect the energy levels, but nevertheless modifies the twisted partition function. Calculations based on the expansion of the partition function for large volumes can be found for boundary entropies in [41], while for the boundary ground state energy in [42].

2.1.1 Leading-order calculation

In evaluating the twisted partition function at LO, we analyze the one-particle contributions. In a finite but large volume, R, the momentum is quantized as

$$e^{i\tilde{p}_k R} = 1 \qquad \rightarrow \qquad \frac{R}{2\pi} \tilde{p}_k = k \in \mathbb{Z}, \qquad (2.4)$$

which is independent of the color index $\alpha = 1, \dots, N$. In the $R \to \infty$ limit, the allowed momenta become dense, and the summation can be turned into integration. The change from the discrete label k to the continuous momentum variable $\tilde{p}$ is dictated by the Bethe-Yang equation above as

$$\sum_k \to R \int \frac{d\tilde{p}}{2\pi}. \qquad (2.5)$$

3

Taking the logarithm of the twisted partition function, the ground-state energy can be obtained

$$E_0^d(L) = -\lim_{R\to\infty} R^{-1} \log\left[\mathrm{Tr}(e^{-\tilde{H}(R)L} e^{i\gamma J})\right] . \tag{2.6}$$

Expanding the log as $\log(1 + x) = x + O\left(x^2\right)$ and keeping the first term, we obtain

$$E_0^d(L) = E_0^{(1)}(L) + O(e^{-2\tilde{\epsilon}(0)L}), \qquad E_0^{(1)}(L) = -\mathrm{Tr}(e^{i\gamma J}) \int \frac{d\tilde{p}}{2\pi} e^{-\tilde{\epsilon}(\tilde{p})L} , \tag{2.7}$$

where the color summation gives $\sum_\alpha e^{i\gamma J_\alpha} = \mathrm{Tr}(e^{i\gamma J})$, which is basically the character of the particles' representation. The physical meaning of this formula is clear: The finite-volume vacuum contains virtual particles, and they modify the vacuum energy by virtual processes. The leading volume-dependent process is when a particle and anti-particle pair appears from the vacuum, and then the particle travels around the world and annihilates with the anti-particle on the other side. Clearly, in so doing, it crosses the defect line and picks up the phase which, when summed up for the multiplet, results in the character.

2.1.2 Next-to-leading order calculation

At the NLO energy correction, we have to expand the logarithm of the partition function (2.6) to second order: $\log(1 + x) = x - \frac{x^2}{2} + O\left(x^3\right)$. This will include the square of the one-particle term and the two-particle term. The former, however, contains a factor R^2 which would lead to a divergence in the $R \to \infty$ limit, and has to be canceled against a similar part of the two-particle term. We evaluate now the two-particle contribution and see the needed cancellation. From the remaining terms, we obtain the NLO energy correction.

In calculating the two-particle term, we must first determine the allowed momenta. In very large volume R, the momentum quantization conditions are given by the Bethe-Yang (or, in other terminology, the asymptotic Bethe ansatz) equations. As the scattering mixes the color indices, we begin by diagonalizing the two-particle S-matrix:

$$e^{iR\tilde{p}_k} S_\mu^\nu(\tilde{p}_k, \tilde{p}_l)\psi_\nu = \psi_\mu \to e^{iR\tilde{p}_k} e^{i\delta_\mu(\tilde{p}_k, \tilde{p}_l)} = 1 . \tag{2.8}$$

The two-particle S-matrix has N^2 eigenvalues, and we denote their phases by $\delta_\mu(\tilde{p}_k, \tilde{p}_l)$ for $\mu = 1, \ldots, N^2$. Unitarity implies $\delta_\mu(\tilde{p}_k, \tilde{p}_l) = -\delta_\mu(\tilde{p}_l, \tilde{p}_k)$ mod 2π. We assume that the particles are fermionic: $S(\tilde{p}, \tilde{p}) = -\mathbb{I}$, thus $\delta_\mu(\tilde{p}, \tilde{p}) = \pi$. Taking the logarithm of the equations (2.8) for a given eigenvalue, we arrive at the Bethe-Yang equations

$$\begin{aligned}
\frac{R}{2\pi}\tilde{p}_k + \frac{1}{2\pi}\delta_\mu(\tilde{p}_k, \tilde{p}_l) &= k , \\
\frac{R}{2\pi}\tilde{p}_l - \frac{1}{2\pi}\delta_\mu(\tilde{p}_k, \tilde{p}_l) &= l .
\end{aligned} \tag{2.9}$$

The fermionic nature of the particles excludes $k = l$; and in summing up over two-particle states, $k > l$ is understood. In changing to momentum integration, it is better to reorganize the sum as $\sum_{k>l} f(k, l) = \frac{1}{2}\sum_{k,l} f(k, l) - \frac{1}{2}\sum_k f(k, k)$, since the summand $f(k, l) = e^{i\gamma J - (\tilde{\epsilon}(\tilde{p}_k) + \tilde{\epsilon}(\tilde{p}_l))L}$ is symmetric. The diagonal part, $-\frac{1}{2}\sum_k f(k, k)$, has the one-particle quantization rule (2.4); thus, changing to integration as in (2.5) the contribution to the energy turns out to be:

$$E_0^{(2,1)}(L) = \frac{1}{2}\mathrm{Tr}(e^{i\gamma J})^2 \int \frac{d\tilde{p}}{2\pi} e^{-2\tilde{\epsilon}(\tilde{p})L} , \tag{2.10}$$

where we used that $\sum_{(\alpha,\beta)} e^{i\gamma J_{(\alpha,\beta)}} = \sum_\mu e^{i\gamma J_\mu} = \mathrm{Tr}(e^{i\gamma J})^2$.

4

We now transform $\frac{1}{2}\sum_{k,l} f(k,l)$ into a double integral. To this end, we compute the Jacobian for the change of variables $(k,l) \to (\tilde{p}_k, \tilde{p}_l)$:

$$\begin{vmatrix} \frac{\partial k}{\partial \tilde{p}_k} & \frac{\partial k}{\partial \tilde{p}_l} \\ \frac{\partial l}{\partial \tilde{p}_k} & \frac{\partial l}{\partial \tilde{p}_l} \end{vmatrix} = \frac{1}{(2\pi)^2} \begin{vmatrix} R + \delta_{\mu,k} & \delta_{\mu,l} \\ -\delta_{\mu,k} & R - \delta_{\mu,l} \end{vmatrix} = \frac{1}{(2\pi)^2} \left[R^2 + R(\delta_{\mu,k} - \delta_{\mu,l}) \right] , \qquad (2.11)$$

where $\delta_{\mu,k} = \partial_{\tilde{p}_k}\delta_\mu(\tilde{p}_k, \tilde{p}_l)$ and $\delta_{\mu,l} = \partial_{\tilde{p}_l}\delta_\mu(\tilde{p}_k, \tilde{p}_l)$. As already mentioned, the terms which contribute to the ground-state energy have to be proportional to R. Indeed the dangerous R^2 term

$$\frac{R^2}{2}\mathrm{Tr}(e^{i\gamma J})^2 \int \frac{d\tilde{p}_1}{2\pi} \int \frac{d\tilde{p}_2}{2\pi} e^{-(\tilde{\epsilon}(\tilde{p}_1)+\tilde{\epsilon}(\tilde{p}_2))L} \qquad (2.12)$$

will cancel against the $-\frac{x^2}{2}$ term of the expansion of the logarithm of the one-particle contribution. The second term of the Jacobi determinant (2.11) is proportional to the volume R, and contributes to the ground-state energy as

$$E_0^{(2,2)}(L) = - \int \frac{d\tilde{p}_1}{2\pi} e^{-\tilde{\epsilon}(\tilde{p}_1)L} \int \frac{d\tilde{p}_2}{2\pi} e^{-\tilde{\epsilon}(\tilde{p}_2)L} \sum_\mu e^{i\gamma J_\mu} \partial_{\tilde{p}_1}\delta_\mu(\tilde{p}_1, \tilde{p}_2) , \qquad (2.13)$$

where we have used that $\delta_\mu(\tilde{p}_1, \tilde{p}_2)$ is antisymmetric in its arguments; and that, as the twist commutes with the scattering matrix $[e^{i\gamma J}, S] = 0$, both can be diagonalized in the same basis. We note that

$$\sum_\mu e^{i\gamma J_\mu} \partial_{\tilde{p}_1}\delta_\mu(\tilde{p}_1, \tilde{p}_2) = -i\partial_{\tilde{p}_1}\mathrm{Tr}(e^{i\gamma J}\log[S(\tilde{p}_1, \tilde{p}_2)]) . \qquad (2.14)$$

In particular, this implies that if the S-matrix is twisted (à la Drinfeld-Reshetikhin [37]) with another conserved charge $\tilde{S} = FSF$, such that $[e^{i\gamma J}, F] = 0$, then the finite-size correction is the same as in the undeformed case:

$$\begin{aligned} \partial_{\tilde{p}_1}\mathrm{Tr}(e^{i\gamma J}\log(\tilde{S})) &= \partial_{\tilde{p}_1}\sum_\alpha e^{i\gamma J_\alpha}\mathrm{Tr}_\alpha \log(F_\alpha S_\alpha F_\alpha) = \partial_{\tilde{p}_1}\sum_\alpha e^{i\gamma J_\alpha} \log\det(F_\alpha S_\alpha F_\alpha) \\ &= \partial_{\tilde{p}_1}\sum_\alpha e^{i\gamma J_\alpha} \log\det S_\alpha = \partial_{\tilde{p}_1}\mathrm{Tr}(e^{i\gamma J}\log(S)) , \end{aligned} \qquad (2.15)$$

where we have denoted by F_α (S_α) the matrix F (S) in the subspace where J has eigenvalue J_α, respectively; and we have used the fact that $\det(F_\alpha S_\alpha F_\alpha) = \det S_\alpha \det F_\alpha^2$.

We conclude that the LO and NLO corrections to the finite-volume vacuum energy in the twisted theory come only from the twisted boundary condition, and are given by

$$\begin{aligned} E_0^d(L) &= E_0^{(1)}(L) + E_0^{(2,1)}(L) + E_0^{(2,2)}(L) \\ &= -\mathrm{Tr}_1(e^{i\gamma J})\int \frac{d\tilde{p}}{2\pi} e^{-\tilde{\epsilon}(\tilde{p})L} + \frac{1}{2}\mathrm{Tr}_1(e^{i\gamma J})^2 \int \frac{d\tilde{p}}{2\pi} e^{-2\tilde{\epsilon}(\tilde{p})L} \\ &\quad + \int \frac{d\tilde{p}_1}{2\pi} e^{-\tilde{\epsilon}(\tilde{p}_1)L} \int \frac{d\tilde{p}_2}{2\pi} e^{-\tilde{\epsilon}(\tilde{p}_2)L} i\partial_{\tilde{p}_1}\mathrm{Tr}_2(e^{i\gamma J}\log[S(\tilde{p}_1, \tilde{p}_2)]) , \end{aligned} \qquad (2.16)$$

where the omitted terms are of order of $O(e^{-3\tilde{\epsilon}(0)L})$, and $\mathrm{Tr}_i()$ for $i = 1, 2$ means that the trace is taken over the one- or two-particle states, respectively. This derivation is an alternative formulation of the virial expansion of the partition function in statistical physics. (See also the result for the $O(n)$ case [43].)

Our result (2.16) can also be used to make the connection between the scattering description and other descriptions of the theory. Indeed, given an integral equation for the ground-state energy, we can extract from it the S-matrix by expanding for large volume to NLO.

2.2 Twisted TBA

We have so far supposed that the physical volume L is large, and we have calculated the LO and NLO energy corrections. If the volume is not large and we are interested in the exact description of the vacuum, we have to evaluate the contributions of multiparticle states. This, in the untwisted case, is done by the TBA; and we shall now see how the derivations are modified in the presence of the twist.

The first step in calculating the partition function is the determination of the momentum quantization of multiparticle states. This is done by solving the Bethe-Yang equations by means of the asymptotic Bethe ansatz (BA). Here, in addition to the physical momentum-carrying particles, one has to introduce so-called magnonic particles that take care of the non-diagonal nature of the scattering. They are useful objects, since in terms of them the scattering can be regarded as diagonal. One then analyzes the various "diagonal" scattering matrices and looks for bound states: i.e., complex string-like solutions of the asymptotic BA equations. The scattering matrices of the bound states are determined from the scattering matrices of their constituents. Let us label the particles (momentum-carrying, magnonic and their bound states) by a multilabel n; and their scattering matrices by $S_{nm}(u_1^n, u_2^m)$, where $u_i^{n_i}$ is some generalized rapidity of a particle of type n_i. Greek indices such as α will denote magnons only. The asymptotic BA equations for large particle numbers (thermodynamic limit) takes the generic form

$$-1 = e^{i\tilde{p}_n(u_k^n)R} \prod_m \prod_l S_{nm}(u_k^n, u_l^m) \,, \tag{2.17}$$

where the mirror momentum vanishes for magnons $\tilde{p}_\alpha(u^\alpha) = 0$, and $S_{nn}(u_k^n, u_k^n) = -1$. We note that not only the momentum, but also the energy vanishes for magnons, $\tilde{\epsilon}_\alpha = 0$. Thus, the magnonic equations can be inverted, without changing their physical meaning. We have to choose such equations which give rise to positive particle densities in the thermodynamic limit. In this limit, the partition function is dominated by finite-density configurations. The density of the particles (holes) of type n can be introduced as $\rho_n = \frac{\Delta N_n}{R\Delta\tilde{p}}$, $(\bar{\rho}_n = \frac{\Delta\bar{N}_n}{R\Delta\tilde{p}})$, where $\Delta N_n(\Delta\bar{N}_n)$ denotes the number of particles (holes) in the interval $(\tilde{p}, \tilde{p} + \Delta\tilde{p})$, respectively. In terms of these densities, the energy of the configuration is

$$\tilde{E}[\rho] = R \sum_n \int d\tilde{p}\, \rho_n(\tilde{p})\, \tilde{\epsilon}_n(\tilde{p}) = R \sum_n \int du\, \rho_n(u)\, \tilde{\epsilon}_n(u) \,, \tag{2.18}$$

while the entropy is

$$S[\rho, \bar{\rho}] = R \sum_n \int du \left[(\rho_n + \bar{\rho}_n) \log(\rho_n + \bar{\rho}_n) - \rho_n \log \rho_n - \bar{\rho}_n \log \bar{\rho}_n \right] \,. \tag{2.19}$$

The particle and the hole densities are not independent, and the derivative of the logarithm of the asymptotic BA (2.17) connects them as

$$\rho_n + \bar{\rho}_n - \frac{1}{2\pi}\partial_u \tilde{p}_n = \int du' \sum_m K_{nm}(u, u')\rho_m(u') =: K_{nm} \star \rho_m \,, \tag{2.20}$$

where $K_{nm}(u, u') = \frac{1}{2\pi i}\partial_u \log S_{nm}(u, u')$. If we had inverted any of the asymptotic BA equations, then we would have obtained the sign-changed kernel here. By choosing the proper signs of the kernels for the magnons, we can ensure the positivity of all the densities. If we had started instead with the Drinfeld-Reshetikhin-twisted S-matrix, then S_{nm} in (2.17) would be replaced by $\tilde{S}_{nm}$, which differs from S_{nm} by constant phases; and these phases would disappear from the kernel K_{nm}. Consequently, the TBA equations are independent of twists of the S-matrix, as is the Lüscher correction (2.15).

We have seen that the twist does not change the energy levels of the periodic mirror system, but nevertheless modifies the partition function. Since the twist commutes with the scattering matrix, the particles of the asymptotic BA equations which diagonalize the multiparticle scatterings will have diagonal twist eigenvalues, too. Let us denote the eigenvalue of $i\gamma J$ on a particle with label n by μ_n.

The total contribution of the twist on the multiparticle state is

$$\mu[\rho] = R \sum_n \int du\, \rho_n(u)\, \mu_n\,. \tag{2.21}$$

In terms of these quantities, the partition function can be written as

$$Z^d(L,R) = \text{Tr}(e^{-\tilde{H}(R)L} e^{i\gamma J}) = \int \prod_n d[\rho_n, \bar{\rho}_n] e^{S[\rho,\bar{\rho}] + \mu[\rho] - L\tilde{E}[\rho]}\,. \tag{2.22}$$

Evaluating the integrals in the saddle-point approximation, the minimizing condition for the pseudo-energies $\epsilon_n = \log\frac{\bar{\rho}_n}{\rho_n}$ turns out to be

$$\epsilon_n + \mu_n = \tilde{\epsilon}_n L - \log(1 + e^{-\epsilon_m}) \star K_{mn}\,. \tag{2.23}$$

Once we have calculated the pseudo-energies, the ground-state energy can be extracted from the saddle-point value as

$$E_0^d(L) = -\sum_n \int \frac{du}{2\pi} \partial_u \tilde{p}_n \log(1 + e^{-\epsilon_n})\,. \tag{2.24}$$

Clearly the only difference compared with the untwisted case is the appearance in the TBA equations (2.23) of the chemical potential μ_n, which is proportional to the charge of the particle. (TBA equations with chemical potentials have been studied previously; see e.g. [44].)

As the determination of the magnons and their charges is model dependent, we work out the details in the following for the $O(4)$ model, and then for twisted planar AdS/CFT.

3 Case study: $O(4)$ model

In this section, as a warm-up, we elaborate explicitly the simpler case of the twisted $O(4)$ model, also known as the $su(2)$ principal chiral model. We calculate the LO and NLO Lüscher corrections, derive the twisted TBA equations, and compare the two approaches by expanding the TBA equations up to second order.

The $O(4)$ model is a relativistic theory containing one multiplet of particles with mass m. The dispersion relation $E(p) = \sqrt{m^2 + p^2}$ can be parameterized in terms of the rapidity as

$$E(\theta) = m \cosh \pi \theta\,, \qquad p(\theta) = m \sinh \pi \theta\,. \tag{3.1}$$

The particles transform under the bifundamental representation of $su(2)$. The two-particle S-matrix is the simplest $su(2) \otimes su(2)$ symmetric, unitary and crossing-invariant scattering matrix [45, 46]

$$S(\theta) = \frac{S_0^2(\theta)}{(\theta - i)^2} \hat{S}(\theta) \otimes \hat{S}(\theta)\,, \qquad \hat{S}(\theta) = \theta\,\mathbb{I} - i\,\mathbb{P}\,, \tag{3.2}$$

where $\theta = \theta_1 - \theta_2$, and the scalar factor

$$S_0(\theta) = i\,\frac{\Gamma(\frac{1}{2} - \frac{i\theta}{2})\Gamma(\frac{i\theta}{2})}{\Gamma(\frac{1}{2} + \frac{i\theta}{2})\Gamma(-\frac{i\theta}{2})} \tag{3.3}$$

does not have any poles in the physical strip, showing the absence of physical bound states.

We analyze this theory on a circle of size L with a twisted boundary condition. We twist the theory with independent twist angles $\gamma_\mp$ for the left and right $su(2)$ factors, respectively:

$$e^{i\gamma J} = e^{i\gamma_- J_0 \otimes \mathbb{I} + i\gamma_+ \mathbb{I} \otimes J_0} = e^{i\gamma_- J_0} \otimes e^{i\gamma_+ J_0} = \text{diag}(\dot{q}, \dot{q}^{-1}) \otimes \text{diag}(q, q^{-1})\,, \tag{3.4}$$

where J_0 has eigenvalues ± 1 on the two components of the doublet, and $\dot{q} = e^{i\gamma_-}$, $q = e^{i\gamma_+}$. We could also twist the S-matrix, i.e. change $S \to FSF$, but this would have no effect on the ground-state energy, as explained in (2.15).

3.1 Lüscher corrections

We now proceed to evaluate the Lüscher correction for the vacuum (2.16). As the theory is relativistically invariant, the mirror dispersion relation is $\tilde{\epsilon}(\tilde{p}) = \sqrt{m^2 + \tilde{p}^2}$, which we parameterize in terms of the rapidity as above: $\tilde{p}(\theta) = m \sinh \pi\theta$. In this parameterization, the leading-order result for the ground-state energy is

$$E_0^{(1)}(L) = -[2]_q [2]_{\dot{q}} m \int \frac{d\theta}{2} \cosh \pi\theta \, e^{-mL \cosh \pi\theta} \,, \tag{3.5}$$

where we used that

$$\mathrm{Tr}(e^{i\gamma J}) = \mathrm{Tr}(e^{i\gamma_- J_0}) \mathrm{Tr}(e^{i\gamma_+ J_0}) = (\dot{q} + \dot{q}^{-1})(q + q^{-1}) = [2]_q [2]_{\dot{q}} \,. \tag{3.6}$$

It is useful to introduce the q-numbers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{3-n} + q^{1-n} \,, \tag{3.7}$$

for which $[n]_q \to n$ in the untwisted limit $q \to 1$.

In the second-order correction, we have the term without the scattering matrix

$$E_0^{(2,1)}(L) = \frac{1}{2} [2]_q^2 [2]_{\dot{q}}^2 m \int \frac{d\theta}{2} \cosh \pi\theta \, e^{-2mL \cosh \pi\theta} \,. \tag{3.8}$$

In the other term, we have to diagonalize the two-particle S-matrix

$$S(\theta) = S_0^2(\theta) S_1(\theta) \otimes S_2(\theta) \,, \qquad S_1(\theta) = S_2(\theta) = \frac{1}{\theta - i} \hat{S}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\theta}{\theta - i} & \frac{-i}{\theta - i} & 0 \\ 0 & \frac{-i}{\theta - i} & \frac{\theta}{\theta - i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{3.9}$$

The twist matrix acts on the two-particle states as

$$e^{i\gamma J} = e^{i\gamma_- J_0} \otimes e^{i\gamma_+ J_0} = \dot{A} \otimes A = \mathrm{diag}(\dot{q}^2, 1, 1, \dot{q}^{-2}) \otimes \mathrm{diag}(q^2, 1, 1, q^{-2}) \,, \tag{3.10}$$

and commutes with the scattering matrix. The twist and the S-matrix can be diagonalized in the same basis, where the S-matrix eigenvalues take the form

$$S = S_0^2 \, \Lambda \otimes \Lambda = S_0^2 \, \mathrm{diag}(1, 1, \frac{\theta + i}{\theta - i}, 1) \otimes \mathrm{diag}(1, 1, \frac{\theta + i}{\theta - i}, 1) \,, . \tag{3.11}$$

For the Lüscher correction, we need to calculate $\mathrm{Tr}(e^{i\gamma J}(-i\partial_\theta) \log S)$. As the scattering matrix has the specific tensor product structure (3.9), we can write

$$\begin{aligned} \mathrm{Tr}(e^{i\gamma J} \log S) &= \mathrm{Tr}((\dot{A} \otimes A)(2 \log S_0 \mathbb{I} \otimes \mathbb{I} + \log S_1 \otimes \mathbb{I} + \mathbb{I} \otimes \log S_2)) \\ &= \mathrm{Tr}(\dot{A}) \mathrm{Tr}(A) 2 \log S_0 + \mathrm{Tr}(A) \mathrm{Tr}(\dot{A} \log S_1) + \mathrm{Tr}(\dot{A}) \mathrm{Tr}(A \log S_2) \\ &= 2 \mathrm{Tr}(\dot{A}) \mathrm{Tr}(A) \log S_0 + \mathrm{Tr}(A) \sum_i \dot{A}_i \log \Lambda_i + \mathrm{Tr}(\dot{A}) \sum_i A_i \log \Lambda_i \,. \end{aligned} \tag{3.12}$$

In Fourier space, the logarithmic derivatives take a particularly simple form:

$$\begin{aligned} K_{00}(\theta) &= \frac{1}{2\pi i} \partial_\theta \log S_0^2(\theta) &\to \quad \tilde{K}_{00}(\omega) &= \frac{2t}{t + t^{-1}} \,, \\ K(\theta) &= \frac{1}{2\pi i} \partial_\theta \log \frac{\theta + i}{\theta - i} &\to \quad \tilde{K}(\omega) &= -t^2 \,, \end{aligned} \tag{3.13}$$

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where we have indicated the Fourier transform by tilde, and $t = e^{-\frac{|\omega|}{2}}$. The integrand of the second order Lüscher correction is finally

$$\frac{1}{2\pi}\text{Tr}(e^{i\gamma J}(-i\partial_\theta)\log S) = [2]_q^2[2]_{\dot{q}}^2 K_{00} + \left([2]_q^2 + [2]_{\dot{q}}^2\right) K \,. \tag{3.14}$$

In terms of these quantities, the second part of the Lüscher correction is

$$
\begin{aligned}
E_0^{(2,2)}(L) &= -[2]_q^2[2]_{\dot{q}}^2 \frac{m}{2} \int d\theta_1 \, e^{-mL\cosh\pi\theta_1} \int d\theta_2 \cosh\pi\theta_2 \, e^{-mL\cosh\pi\theta_2} \\
&\quad \times \left\{ K_{00}(\theta_1 - \theta_2) + ([2]_q^{-2} + [2]_{\dot{q}}^{-2})K(\theta_1 - \theta_2) \right\} \,.
\end{aligned} \tag{3.15}
$$

3.2 Twisted TBA

Following the general procedure outlined in section 2.2, in order to formulate the twisted TBA equations, we need to classify the particles: momentum-carrying, magnons and their bound states. We also have to calculate their scattering matrices; and, additionally to the untwisted case, we also must identify the twist charge on all the excitations.

3.2.1 Raw twisted TBA

In order to derive the mirror nested asymptotic BA equations, we start with an N-particle state consisting of down-spin particles only. We label these particles by 0. They scatter on each other as

$$S_{00}(\theta) = S_0(\theta)^2 \,, \tag{3.16}$$

and they have the dispersion relation $\tilde{\epsilon}_0(\tilde{p}) = \tilde{\epsilon}(\tilde{p})$. As the J_0 eigenvalue of the lower component is -1 on both $su(2)$ sides, the chemical potential is $\mu_0 = -i\gamma_- - i\gamma_+$. We can now introduce up-spins in the sea of down-spins. These are the magnons, which do not change the energy and momentum, rather describe the polarization degrees of freedom. We label them by 1 for the right $su(2)$ factor, and by -1 for the left $su(2)$ factor. Let us first focus on the positive (right) part, and denote magnon rapidities by u. The magnons scatter on the massive particles and on themselves as

$$S_{01}(\theta - u) = \frac{\theta - u + \frac{i}{2}}{\theta - u - \frac{i}{2}}\,, \qquad S_{11}(u - u') = \frac{u - u' - i}{u - u' + i}\,, \tag{3.17}$$

respectively. The magnons do not have any energy and momentum $\tilde{\epsilon}_1(u) = \tilde{p}_1(u) = 0$, but they do have chemical potential. Since a magnon swaps a spin from down to up, it changes the charge by 2: $\mu_1 = 2i\gamma_+$. This means that a state with m up-spins and $N - m$ down-spins, which contains N type-0 particles and m type-1 particles, has J_0 charge $-N + 2m$. Inspecting the magnon scattering matrices, we can conclude that a magnon and a massive particle cannot form bound states. In contrast, magnons among themselves can bound. Bound states in the thermodynamic limit consist of strings of any length $M \in \mathbb{N}$:

$$u_j = u + i\frac{M + 1 - 2j}{2}\,, \qquad j = 1, \ldots, M \,. \tag{3.18}$$

We label this string as M. Clearly, the $M = 1$ string is the magnon itself. The scattering of the M-string and the massive particle can be calculated from the bootstrap,

$$S_{0M}(\theta - u) = \prod_{j=1}^{M} S_{01}(\theta - u_j) = \frac{\theta - u + \frac{i}{2}M}{\theta - u - \frac{i}{2}M}\,. \tag{3.19}$$

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As $S_{0M}(\theta - u)S_{M0}(u - \theta) = 1$, we conclude that $S_{0M}(u) = S_{M0}(u)$. Similarly, the magnon-magnon scatterings are given by

$$
\begin{aligned}
S_{MM'}(u - u') &= \prod_{j=1}^{M}\prod_{j'=1}^{M'} S_{11}(u_j - u'_{j'}) \qquad\qquad (3.20) \\
&= \left(\frac{u - u' - \frac{i}{2}|M - M'|}{u - u' + \frac{i}{2}|M - M'|}\right)\left(\frac{u - u' - \frac{i}{2}(|M - M'| + 2)}{u - u' + \frac{i}{2}(|M - M'| + 2)}\right)^2 \\
&\quad \times \ldots \left(\frac{u - u' - \frac{i}{2}(M + M' - 2)}{u - u' + \frac{i}{2}(M + M' - 2)}\right)^2\left(\frac{u - u' - \frac{i}{2}(M + M')}{u - u' + \frac{i}{2}(M + M')}\right).
\end{aligned}
$$

These bound states have no energy and momentum $\tilde{\epsilon}_M(u) = \tilde{p}_M(u) = 0$, while their chemical potential is the sum of their constituents': $\mu_M = 2Mi\gamma_+$.

Similar considerations apply to the left excitations, which are denoted by $-M$. They scatter only on themselves and on the massive particle, such that the scattering is independent of the sign of M. The only difference is in the chemical potential, as the twists are different on the two sides: $\mu_{-M} = 2Mi\gamma_-$.

Summarizing, we have particles for any $M \in \mathbb{Z}$. The only massive excitation that has nontrivial energy and momentum has the label 0; all others are magnons. The scattering kernels in Fourier space have the form

$$
\tilde{K}_{00} = \frac{2t}{(t + t^{-1})}, \qquad \tilde{K}_{0n} = \tilde{K}_{n0} = -t^n, \qquad \tilde{K}_{nm} = \frac{t + t^{-1}}{t - t^{-1}}(t^{n+m} - t^{|n-m|}) - \delta_{nm}, \qquad (3.21)
$$

where $t = e^{-\frac{|\omega|}{2}}$ and $n, m > 0$. For the other values, we have $K_{0n} = K_{0-n}$, $K_{n0} = K_{-n0}$, $K_{-n-m} = K_{nm}$ and $K_{-nm} = K_{n-m} = 0$.

In the general procedure, one has to invert the magnonic equations before introducing the magnon densities. In so doing, one obtains the "raw" (canonical) twisted TBA equations

$$
\epsilon_0 + \mu_0 = L\tilde{\epsilon}_0 - \log(1 + e^{-\epsilon_0}) \star K_{00} + \sum_{M \neq 0} \log(1 + e^{-\epsilon_M}) \star K_{M0}, \qquad (3.22)
$$

$$
\epsilon_M + \mu_M = -\log(1 + e^{-\epsilon_0}) \star K_{0M} + \sum_{M' \neq 0} \log(1 + e^{-\epsilon_{M'}}) \star K_{M'M}, \quad M \neq 0. \qquad (3.23)
$$

These equations for the untwisted ($\mu = 0$) case reduce to those in [47], although in slightly different convention.

3.2.2 Universal TBA and Y-system

Using identities among the kernels, we now bring the TBA equations (3.22), (3.23) to a universal local form. This means that the pseudo-energies can be associated with vertices of a two-dimensional lattice, such that only neighboring sites couple to each other with the following universal kernel

$$
s\,I_{MN} = \delta_{MN} - (K + 1)^{-1}_{MN}, \qquad s(\theta) = \frac{1}{2\cosh \pi\theta}, \qquad (3.24)
$$

where $I_{MN} = \delta_{M+1,N} + \delta_{M-1,N}$ and $(K+1)^{-1}_{MN} \star (K_{NL} + \delta_{NL}) = \delta_{ML}$. We also have $(K_{n1} + \delta_{n1}) \star s = -K_{0n}$, which can be easily seen in Fourier space where $\tilde{s} = \frac{1}{t + t^{-1}}$.

Let us introduce the Y-functions:

$$
Y_0 = e^{-\epsilon_0}, \qquad Y_M = e^{\epsilon_M}, \quad M \neq 0. \qquad (3.25)
$$

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We take the equations (3.23) for Y_M, act with the operator $\delta_{MN} - s\, I_{MN} = (K+1)^{-1}_{MN}$ from the right, and use the kernel identity $K_{0N} \star (K+1)^{-1}_{NM} = -s\, \delta_{M,1}$. Since the chemical potentials are annihilated by the discrete Laplacian

$$\mu_M \star (sI_{MN} - \delta_{MN}) = \frac{1}{2}(\mu_{N-1} + \mu_{N+1}) - \mu_N = 0\,, \tag{3.26}$$

they completely disappear from the equations, and we arrive at

$$\log Y_M \;=\; I_{MM'} \log(1 + Y_{M'}) \star s\,, \quad M \neq 0\,. \tag{3.27}$$

Finally, we take the equations for $M = \pm 1$ and convolute them with the kernel s. We combine these equations with the massive equation (3.22). Using the magic property of the kernel $K_{00} = -2s \star K_{01}$, and exploiting that $\mu_0 + \frac{1}{2}(\mu_1 + \mu_{-1}) = 0$, we obtain the equation for the massive node

$$\log Y_0 + mL \cosh \pi\theta = (\log(1 + Y_1) + \log(1 + Y_{-1})) \star s\,. \tag{3.28}$$

Thus, the twists completely disappear from the "simplified" equations (3.27), (3.28). Nevertheless, they enter in the asymptotics of the Y-functions as

$$\lim_{M \to \infty} \frac{1}{M} \log Y_{\pm M} = -2i\gamma_\pm\,, \tag{3.29}$$

since the kernels in (3.23) vanish in this limit. After all, it should not come as a surprise that the Y-system is not twisted,

$$Y_M^+ Y_M^- = (1 + Y_{M-1})(1 + Y_{M+1})\,, \qquad Y^\pm(\theta) = Y(\theta \pm \frac{i}{2})\,. \tag{3.30}$$

The ground-state energy contains the contribution of the only massive node,

$$E_0(L) = -\frac{m}{2} \int d\theta \cosh \pi\theta \, \log(1 + Y_0)\,. \tag{3.31}$$

3.2.3 Asymptotic expansion

We now make a LO and NLO asymptotic expansion of the simplified TBA equations (3.27), (3.28) for $L \to \infty$.

At leading order, Y_0 is exponentially small and the other Y functions are constant. Let us expand the Y-functions as

$$Y_M = \mathcal{Y}_M(1 + y_M) + \ldots\,, \tag{3.32}$$

and determine all functions iteratively. The Y-system at leading order will be split into two independent constant Y-systems. The solutions with the correct initial and asymptotic behaviors will determine the exponentially small leading-order $\mathcal{Y}_0$ in terms of $\mathcal{Y}_{\pm 1}$. Then, in calculating the NLO y_M functions, we can proceed independently for the two parts. Again, the initial condition is provided by $\mathcal{Y}_0$, which appears as a multiplicative factor; while uniqueness is provided by the vanishing asymptotics $\lim_{M \to \infty} y_M = 0$. The $y_{\pm 1}$ obtained in this way will determine the NLO correction y_0, which is needed for the energy correction.

Let us now carry out these calculations. Using the fact that $s \star f = \frac{1}{2}f$ if f is constant, we see from (3.28) that

$$\log \mathcal{Y}_0 = -mL \cosh \pi\theta + \frac{1}{2}\log(1 + \mathcal{Y}_1) + \frac{1}{2}\log(1 + \mathcal{Y}_{-1})\,, \tag{3.33}$$

where the LO constant Y-functions satisfy the relations

$$(\mathcal{Y}_M)^2 = (1 + \mathcal{Y}_{M-1})(1 + \mathcal{Y}_{M+1})\,, \quad M \neq 0\,, \tag{3.34}$$

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as follows from (3.27). The solution with the correct asymptotics (3.29) is [2]

$$\mathcal{Y}_M = [M]_q[M+2]_q\,, \qquad \mathcal{Y}_{-M} = [M]_{\dot q}[M+2]_{\dot q}\,. \tag{3.35}$$

Clearly, the twist dependence reenters through the asymptotic solution. This means that at leading non-vanishing order

$$Y_0 \approx \mathcal{Y}_0 = \sqrt{(1+\mathcal{Y}_1)(1+\mathcal{Y}_{-1})}e^{-mL\cosh\pi\theta} = [2]_q[2]_{\dot q}e^{-mL\cosh\pi\theta}\,, \tag{3.36}$$

which, when substituted back into the energy formula (3.31), reproduces the leading-order Lüscher correction (3.5). Actually, expanding the log in the energy formula (3.31) to second order $\log(1+Y_0) = Y_0 - \frac{1}{2}Y_0^2$ reproduces also $E_0^{(2,1)}$ in (3.8). Thus, we need to expand the Y-functions to NLO to obtain the remaining $E_0^{(2,2)}$ in (3.15).

We see from (3.28) and (3.32) that the massive node has the NLO expansion

$$Y_0 = \mathcal{Y}_0\left(1 + s\star\left(\frac{\mathcal{Y}_1}{1+\mathcal{Y}_1}y_1 + \frac{\mathcal{Y}_{-1}}{1+\mathcal{Y}_{-1}}y_{-1}\right)\right) + \dots\,. \tag{3.37}$$

We need to calculate $y_{\pm 1}$. We expand the TBA equations (3.27), keeping only the linear terms in y,

$$y_k = s\star\left(\frac{\mathcal{Y}_{k+1}}{1+\mathcal{Y}_{k+1}}y_{k+1} + \frac{\mathcal{Y}_{k-1}}{1+\mathcal{Y}_{k-1}}y_{k-1}\right)\,, \quad k \neq 0\,. \tag{3.38}$$

We solve this equation by Fourier transform

$$(t+t^{-1})\tilde y_k = \frac{[k+1]_q[k+3]_q}{[k+2]_q^2}\tilde y_{k+1} + \frac{[k-1]_q[k+1]_q}{[k]_q^2}\tilde y_{k-1}\,, \tag{3.39}$$

where we have also used the result (3.35) and the identity $1 + [k-1]_q[k+1]_q = [k]_q^2$. Being a second-order difference equation, the generic solution contains two parameters. These parameters can be fixed by demanding that $\lim_{k\to\infty}\tilde y_k = 0$ and $\tilde{\mathcal{Y}}_0 = \lim_{k\to 0}\mathcal{Y}_k\tilde y_k$. The result is

$$\tilde y_k = t^k\frac{[k+1]_q}{[2]_q[k]_q[k+2]_q}([k+2]_q - [k]_q t^2)\tilde{\mathcal{Y}}_0\,, \qquad \tilde y_{-k} = \tilde y_k(q\to\dot q)\,, \tag{3.40}$$

which is just the deformed version of the $O(4)$ solution [47]. Thus, for the needed $y_{\pm 1}$, we have

$$\tilde y_1 = \left(t^1 - \frac{t^3}{[3]_q}\right)\tilde{\mathcal{Y}}_0\,, \qquad \tilde y_{-1} = \left(t^1 - \frac{t^3}{[3]_{\dot q}}\right)\tilde{\mathcal{Y}}_0\,. \tag{3.41}$$

Performing inverse Fourier transform,

$$y_1 = -\left(K_{01} - \frac{K_{03}}{[3]_q}\right)\star\mathcal{Y}_0\,, \qquad y_{-1} = -\left(K_{01} - \frac{K_{03}}{[3]_{\dot q}}\right)\star\mathcal{Y}_0\,. \tag{3.42}$$

Substituting back into (3.37), we obtain

$$Y_0 = [2]_q[2]_{\dot q}e^{-mL\cosh\pi\theta}\left(1 + s\star\left[(K_{03} - [3]_q K_{01})[2]_q^{-2} + (K_{03} - [3]_{\dot q}K_{01})[2]_{\dot q}^{-2}\right]\star[2]_q[2]_{\dot q}e^{-mL\cosh\pi\theta}\right)\,. \tag{3.43}$$

Comparing the double-convolution term with $E_0^{(2,2)}$ in (3.15) in Fourier space, we obtain complete agreement.

---

[2]The twists $\gamma_\pm$ have small positive imaginary parts in order to suppress large-M magnonic contributions to the partition function (2.22).

4 Twisted AdS/CFT

In this section, we apply the previous methodology to the twisted AdS/CFT model. After defining the model by its scattering matrix, dispersion relation and twist matrix, we derive the LO and NLO Lüscher corrections. As the model has infinitely many massive bound states $Q \in \mathbb{N}$, in the NLO Lüscher correction we have a sum of the form $\sum_{Q_1,Q_2=1}^{\infty}$. We first elaborate the summand $Q_1 = Q_2 = 1$ in detail, and we then treat the general case, which entails detailed knowledge of all scattering matrices $S^{Q_1 Q_2}$. We next derive the twisted TBA equations by evaluating the charges of the magnons and their bound states in the thermodynamic limit of the mirror asymptotic BA. The twist, just as in the $O(4)$ model, disappears from the universal equations, which lead to the untwisted Y-system. We expand the TBA equations to NLO and compare to the Lüscher correction, and again find perfect agreement.

The AdS/CFT integrable model has an $su(2|2) \otimes su(2|2)$ symmetry. The elementary particle transforms under the bifundamental representation of $su(2|2)$. For one copy of $su(2|2)$, Latin indices $a = 1, 2$ label the bosonic, while Greek indices $\alpha = 3, 4$ label the fermionic components of the four-dimensional representation. We will introduce twist in the bosonic subspace by the generator L_0, which has nonvanishing diagonal matrix elements: $(L_0)_1^1 = 1$ and $(L_0)_2^2 = -1$.

The symmetry completely determines the left/right scattering matrix, which has the nonvanishing amplitudes

$$S_{aa}^{aa} = S_{ab}^{ab} + S_{ab}^{ba} = a_1 = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \sqrt{\frac{x_2^+}{x_2^-}} \sqrt{\frac{x_1^-}{x_1^+}}, \qquad S_{ab}^{ab} - S_{ab}^{ba} = a_2, \tag{4.1}$$

$$S_{\alpha\alpha}^{\alpha\alpha} = S_{\alpha\beta}^{\alpha\beta} + S_{\alpha\beta}^{\beta\alpha} = a_3 = -1, \qquad S_{\alpha\beta}^{\alpha\beta} - S_{\alpha\beta}^{\beta\alpha} = a_4, \tag{4.2}$$

$$S_{ab}^{\alpha\beta} = -\frac{1}{2} \epsilon_{ab} \epsilon^{\alpha\beta} a_5, \qquad S_{\alpha\beta}^{ab} = -\frac{1}{2} \epsilon_{\alpha\beta} \epsilon^{ab} a_6, \tag{4.3}$$

$$S_{a\alpha}^{a\alpha} = a_7, \qquad S_{a\alpha}^{\alpha a} = a_8, \qquad S_{\alpha a}^{a\alpha} = a_9, \qquad S_{\alpha a}^{\alpha a} = a_{10}, \tag{4.4}$$

where $a, b \in \{1, 2\}$ with $a \neq b$; $\alpha, \beta \in \{3, 4\}$ with $\alpha \neq \beta$; and the various coefficients can be extracted from [48].[3] For $Q_1 = Q_2 = 1$ we shall need explicitly only a_1, since – as a consequence of some identities among the various coefficients – we shall be able to express the Lüscher corrections purely in terms of it. The scattering matrix depends independently on the momenta of the particles p_1 and p_2 via

$$\frac{x^+}{x^-} = e^{ip}, \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}, \tag{4.5}$$

where $g = \sqrt{\lambda}/(2\pi)$ and $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. The full scattering matrix has the form

$$\mathcal{S}_{11}(p_1, p_2) = S_{sl(2)}^{11}(p_1, p_2) \left[S_{su(2|2)}^{11}(x_1^\pm, x_2^\pm) \otimes S_{su(2|2)}^{11}(x_1^\pm, x_2^\pm) \right]^{-1}, \tag{4.6}$$

where $S_{sl(2)}^{11}(p_1, p_2)$ is the scalar factor

$$S_{sl(2)}^{11}(u, u') = \frac{u - u' + \frac{i}{g}}{u - u' - \frac{i}{g}} \Sigma_{11}^{-2}, \qquad \Sigma_{11} = \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \sigma, \tag{4.7}$$

with σ being the dressing factor. We remark that $\mathcal{S}_{11}$ denotes actually the inverse of the AFZ S-matrix [48], since we are using the relativistic convention $1 = e^{ipL} \prod_j S(p, p_j)$, as in Section 2, instead of $e^{ipL} = \prod_j S(p, p_j)$.

The dispersion relation can be easily expressed in terms of $x^\pm$ as

$$E = -\frac{ig}{2} \left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-} \right). \tag{4.8}$$

---

[3]Indeed, $a_1, \ldots, a_{10}$ are given by the coefficients of the ten terms in Eq. (8.7) in [48], respectively.

In analogy with the $O(4)$ model, we introduce different twists for the two $su(2|2)$ factors, which we label by $\alpha = \pm$,

$$e^{i\gamma J} = e^{i\gamma_- L_0} \otimes e^{i\gamma_+ L_0} = \mathrm{diag}(\dot{q}, \dot{q}^{-1}, 1, 1) \otimes \mathrm{diag}(q, q^{-1}, 1, 1) \,, \tag{4.9}$$

where again $q = e^{i\gamma_+}$, $\dot{q} = e^{i\gamma_-}$; and $\gamma_\pm$ are related to the deformation parameters γ_i used in [29, 30] by $\gamma_\pm = (\gamma_3 \pm \gamma_2)\frac{L}{2}$.

The scattering matrix has poles, which signal the existence of bound states. These states transform under the $4Q$-dimensional totally *symmetric* representation of $su(2|2)$ for any $Q \in \mathbb{N}$. The dispersion relation of the bound states can be obtained from (4.8) by changing the shortening condition to

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2iQ}{g} \,. \tag{4.10}$$

The matrix part of the scattering matrix can be fixed [49] from the Yangian symmetry [50], while the scalar factor can be determined [51] from the bootstrap principle.

The mirror model has the analytically-continued scattering matrix: $x^\pm(p) \to x^\pm(\tilde{p})$, where $\tilde{p} = -iE$. Since the physical domains of p and $\tilde{p}$ are different, the bound states are different, too. The mirror bound states transform under the $4Q$-dimensional totally *antisymmetric* representation of $su(2|2)$, and the twist charge acts as

$$e^{i\gamma_+ L_0} = \mathrm{diag}(\mathbb{I}_{Q-1}, \mathbb{I}_{Q+1}, q\mathbb{I}_Q, q^{-1}\mathbb{I}_Q) \,. \tag{4.11}$$

The scattering matrix of the antisymmetric bound states are related to those of the symmetric ones by changing the labels $1 \leftrightarrow 3$, $2 \leftrightarrow 4$ and simultaneously flipping $x^\pm \leftrightarrow x^\mp$ inside the matrix part. Combining this with the previously mentioned notational differences, we can use the following scattering matrices to calculate the Lüscher correction:

$$\mathcal{S} = S_{sl(2)}^{Q_1 Q_2}(S_{su(2|2)}^{Q_1 Q_2} \otimes S_{su(2|2)}^{Q_1 Q_2}) \,, \tag{4.12}$$

where

$$S_{sl(2)}^{Q_1 Q_2}(u_1, u_2) = \prod_{j_1=1}^{Q_1} \prod_{j_2=1}^{Q_2} S_{11}(u_{j_1}^1, u_{j_2}^2) \,, \quad u_{j_n}^n = u_n + (Q_n + 1 - 2j_n)\frac{i}{g} \,. \tag{4.13}$$

and $S_{su(2|2)}^{Q_1 Q_2}$ denotes the symmetric-symmetric bound state scattering matrix in the conventions of [49].

4.1 Lüscher corrections

The derivation of Section 2 is not general enough to describe the AdS/CFT problem. We have to incorporate two new features: the existence of fermions, and of multiple species of particles that are labeled by the charge Q. The fermionic nature can be taken into account by changing the trace to the supertrace. This is equivalent to imposing antiperiodic boundary conditions on the fermions, which can be implemented by an $e^{i\pi F}$ twist, where F is the fermion number operator:

$$\mathrm{STr}_Q(e^{i\gamma J}) = \mathrm{Tr}_Q((-1)^F e^{i\gamma J}) = \mathrm{Tr}_Q(e^{i(\pi F + \gamma J)}) = \mathrm{STr}_Q(e^{i\gamma_- L_0})\mathrm{STr}_Q(e^{i\gamma_+ L_0}) = ([2]_q - 2)([2]_{\dot{q}} - 2)Q^2 \,. \tag{4.14}$$

Clearly, the supertrace vanishes in the untwisted $q \to 1$ limit. The generalization of the derivation of Section 2 will contain the scattering matrices $\mathcal{S}^{Q_1 Q_2}$. They arise from two-particle states with charges Q_1 and Q_2. As the species are different, we should not constrain the summation on the quantization numbers $\sum_{k<l}$, and must keep all $\sum_{k,l}$, as they label distinct two-particle states. One can verify that the dangerous R^2 terms from the determinant cancel against the cross terms coming from the square of the one-particle contribution. Otherwise the derivation goes along the same lines as before. As a

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final result, we obtain the LO and NLO Lüscher correction as follows:

$$E_0^{(1)} = -\sum_Q \mathrm{STr}_Q(e^{i\gamma J}) \int \frac{d\tilde{p}}{2\pi} e^{-\tilde{\epsilon}_Q(\tilde{p})L}, \tag{4.15}$$

$$E_0^{(2,1)} = \frac{1}{2}\sum_Q \mathrm{STr}_Q(e^{i\gamma J})^2 \int \frac{d\tilde{p}}{2\pi} e^{-2\tilde{\epsilon}_Q(\tilde{p})L}, \tag{4.16}$$

$$E_0^{(2,2)} = \sum_{Q_1,Q_2=1}^{\infty} \int \frac{d\tilde{p}_1}{2\pi} e^{-L\tilde{\epsilon}_{Q_1}(\tilde{p}_1)} \int \frac{d\tilde{p}_2}{2\pi} e^{-L\tilde{\epsilon}_{Q_2}(\tilde{p}_2)} i\partial_{\tilde{p}_1} \mathrm{STr}_{Q_1Q_2}(e^{i\gamma J} \log \mathcal{S}^{Q_1Q_2}(\tilde{p}_1,\tilde{p}_2)), \tag{4.17}$$

cf. Eqs. (2.7), (2.10) and (2.13), (2.14), respectively. Here and below it is understood that $\tilde{p}_i = \tilde{p}_{Q_i}$.

In evaluating these expressions, we note that the mirror dispersion relation is defined via $\tilde{p} = -iE$ and $\tilde{\epsilon} = -ip$. This dispersion relation can be then encoded into

$$e^{\tilde{\epsilon}_Q(\tilde{p})} = \frac{x^-}{x^+}, \qquad \frac{2\tilde{p}}{g} = x^- - \frac{1}{x^-} - x^+ + \frac{1}{x^+}, \tag{4.18}$$

where the shortening condition (4.10) is satisfied.

The leading Lüscher correction for the vacuum (4.15) receives contributions from each particle

$$E_0^{(1)} = -([2]_q - 2)([2]_{\dot{q}} - 2)\sum_Q Q^2 \int \frac{d\tilde{p}}{2\pi} e^{-\tilde{\epsilon}_Q(\tilde{p})L}. \tag{4.19}$$

The simple part of the NLO correction (4.16) is also straightforward to compute

$$E_0^{(2,1)} = \frac{1}{2}([2]_q - 2)^2([2]_{\dot{q}} - 2)^2 \sum_Q Q^4 \int \frac{d\tilde{p}}{2\pi} e^{-2\tilde{\epsilon}_Q(\tilde{p})L}. \tag{4.20}$$

In order to calculate the $E_0^{(2,2)}$-part of the NLO correction, we need the supertrace of the logarithmic derivative of the mirror S-matrix (4.12):

$$\mathrm{STr}\left(e^{i\gamma_- L_0} \otimes e^{i\gamma_+ L_0} \log \mathcal{S}\right). \tag{4.21}$$

We now diagonalize the twist matrix and the scattering matrix on the same basis,

$$e^{i\gamma_- L_0} \otimes e^{i\gamma_+ L_0} = \dot{A} \otimes A = \mathrm{diag}(\dot{A}_1, \ldots, \dot{A}_n) \otimes \mathrm{diag}(A_1, \ldots, A_n),$$
$$\mathcal{S} = \Lambda \otimes \Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_n) \otimes \mathrm{diag}(\Lambda_1, \ldots, \Lambda_n), \tag{4.22}$$

where Λ_i are the eigenvalues of $S_{su(2|2)}^{Q_1Q_2}$, and $n = 16Q_1Q_2$. Calculation similar to the one in the $O(4)$ model gives

$$\mathrm{STr}(e^{i\gamma J} \log \mathcal{S}) = \mathrm{STr}\left(\dot{A} \otimes A(\log S_{sl(2)}^{Q_1Q_2} \mathbb{I} \otimes \mathbb{I} + \log S_{su(2|2)}^{Q_1Q_2} \otimes \mathbb{I} + \mathbb{I} \otimes \log S_{su(2|2)}^{Q_1Q_2})\right) \tag{4.23}$$

$$= \mathrm{STr}(\dot{A}) \mathrm{STr}(A) \log S_{sl(2)}^{Q_1Q_2} + \sum_i (-1)^{F_i}\left(\mathrm{STr}(A)\dot{A}_i + \mathrm{STr}(\dot{A})A_i\right) \log \Lambda_i.$$

Using the derivative of this expression, we can express the NLO Lüscher correction (4.17) in the following form:

$$E_0^{(2,2)} = \sum_{Q_1,Q_2=1}^{\infty} Q_1 Q_2 \int \frac{d\tilde{p}_1}{2\pi} e^{-L\tilde{\epsilon}_{Q_1}(\tilde{p}_1)} \int \frac{d\tilde{p}_2}{2\pi} e^{-L\tilde{\epsilon}_{Q_2}(\tilde{p}_2)} i\partial_{\tilde{p}_1} \times$$

$$\left[Q_1 Q_2 (2 - [2]_q)^2 (2 - [2]_{\dot{q}})^2 \log S_{sl(2)}^{Q_1Q_2}(\tilde{p}_1,\tilde{p}_2) \right.$$

$$\left. + \sum_i (-1)^{F_i}\left(\dot{A}_i(2 - [2]_q)^2 + A_i(2 - [2]_{\dot{q}})^2\right) \log \Lambda_i^{Q_1Q_2}(\tilde{p}_1,\tilde{p}_2) \right]. \tag{4.24}$$

4.1.1 NLO Lüscher correction: the case $Q_1 = Q_2 = 1$

To warm up, let us evaluate the NLO Lüscher correction for the simplest $Q_1 = Q_2 = 1$ case. We focus on the matrix part in (4.24). Performing the calculation explicitly, we obtain

$$(2 - [2]_{\dot{q}})^2 i\partial_{\tilde{p}_1}\left\{([3]_q - 1)\log a_1 + \log\left[a_1 a_3^3\left((a_1 + 2a_2)(a_3 + 2a_4) - 4a_7 a_8\right)\right]\right.$$
$$\left. - 2[2]_q\log(a_5 a_6 - a_{10}a_9)\right\} + (q \leftrightarrow \dot{q}),\tag{4.25}$$

where $(-1)^F = (1, 1, -1, -1)$. Using the explicit expressions for the coefficients found in [48], we observe the following identities

$$a_5 a_6 - a_{10}a_9 = a_1,\qquad (a_1 + 2a_2)(a_3 + 2a_4) - 4a_7 a_8 = -a_1.\tag{4.26}$$

Substituting these identities into (4.25), we obtain a very simple expression for the matrix part of the NLO Lüscher correction for $Q_1 = Q_2 = 1$ in terms of only a_1,

$$(2 - [2]_q)^2(2 - [2]_{\dot{q}})^2\left(\frac{[2]_{\dot{q}}}{2 - [2]_{\dot{q}}} + \frac{[2]_q}{2 - [2]_q}\right)\int\frac{d\tilde{p}_1}{2\pi}e^{-L\tilde{\epsilon}_1(\tilde{p}_1)}\int\frac{d\tilde{p}_2}{2\pi}e^{-L\tilde{\epsilon}_1(\tilde{p}_2)}i\partial_{\tilde{p}_1}\log a_1(\tilde{p}_1, \tilde{p}_2).\tag{4.27}$$

4.1.2 NLO Lüscher correction: the general case (Q_1, Q_2)

Although the above approach can also be used for the cases $(Q_1, Q_2) = (1, 2), (2, 2)$ for which the explicit S-matrices are available [52], it is impractical for higher-dimensional cases. Clearly, a more powerful approach is needed to treat the general case. Observe from (4.24) that the NLO Lüscher correction involves the quantity $\sum_i(-1)^{F_i}A_i\log\Lambda_i^{Q_1 Q_2}$, and a similar quantity with A_i replaced by $\dot{A}_i$. We exploit the fact that the $su(2|2)$ part of the γ_+ twist $e^{i\gamma J} = \mathbb{I}\otimes e^{i\gamma_+ L_0}$ involves nontrivially only the $su(2)_R$ factor in $su(2)_L \otimes su(2)_R \subset su(2|2)$, as is evident from (4.11). Since $su(2)_L \otimes su(2)_R$ is the symmetry of the scattering matrix, we can perform an expansion in the left (s_L) and right (s_R) spins:

$$\sum_i(-1)^{F_i}A_i\log\Lambda_i^{Q_1, Q_2} = \sum_{(s_L, s_R)}\text{STr}[(\mathbb{I}\otimes e^{i\gamma_+ L_0})\log S^{Q_1 Q_2}(s_L, s_R)]$$
$$= \sum_{(s_L, s_R)}(-1)^{2s_R}(2s_L + 1)[2s_R + 1]_q\log\det S^{Q_1 Q_2}(s_L, s_R),\tag{4.28}$$

where $S^{Q_1 Q_2}(s_L, s_R)$ is the 2-particle S-matrix in the sector with left and right $su(2)$ spins s_L and s_R[4], and we calculated the traces as $\text{Tr}_{s_L}(\mathbb{I}) = 2s_L + 1$ and $\text{STr}_{s_R}(e^{i\gamma_+ L_0}) = (-1)^{2s_R}[2s_R + 1]_q$, respectively. The sum is over all the possible values of s_L and s_R for the given values of Q_1 and Q_2.

The problem of computing the NLO Lüscher correction therefore reduces to the determination of $\det S^{Q_1 Q_2}(s_L, s_R)$ for general values of Q_1 and Q_2, and for all the possible corresponding values of s_L and s_R. This is a formidable technical challenge, since individual S-matrix elements – and particularly the eigenvalues – are not known explicitly enough in general, and those that are known explicitly enough [52, 59] generally have very complicated expressions. Nevertheless, it turns out that – remarkably – these determinants have simple compact expressions, which are constructed from a small number of elementary building blocks.

We propose that, with both particles in *symmetric* representations and $Q_1, Q_2 > 1$, the determinants are given by the expressions in Table 1. In order to save writing, we have introduced the following notation

$$U_0 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-},\quad U_1 = \sqrt{\frac{x_1^+}{x_1^-}},\quad U_2 = \sqrt{\frac{x_2^-}{x_2^+}},\quad U_3 = \frac{x_1^+ x_2^+ - 1}{x_1^- x_2^- - 1},\quad S_Q = \frac{u_1 - u_2 - \frac{iQ}{g}}{u_1 - u_2 + \frac{iQ}{g}},\tag{4.29}$$

---

[4]In other words, $\det S(s_L, s_R) = \prod_i\Lambda_i(s_L, s_R)$, where $\Lambda_i(s_L, s_R)$ are the eigenvalues of the 2-particle S-matrix corresponding to eigenstates which are also $su(2)_L \otimes su(2)_R$ highest-weight states with given values of s_L and s_R. For further details, see appendix A. As usual, the spins s_L, s_R are non-negative integers or half-odd integers.

s_R	$\det S^{Q_1Q_2}(s_L, s_R)$	s_L
1	$U_0U_1U_2 S_{2s_L+2}S_{2s_L+4}\ldots S_{Q_1+Q_2-2}$	$\frac{1}{2}|Q_{12}|, \frac{1}{2}(|Q_{12}|+2),\ldots,\frac{1}{2}(Q_1+Q_2-2)$
$\frac{1}{2}$	$U_0U_1U_2$	$\frac{1}{2}(Q_1+Q_2-1)$
$\frac{1}{2}$	$(U_0U_1U_2)^4 S^2_{2s_L+1}S^4_{2s_L+3}\ldots S^4_{Q_1+Q_2-2}$	$\frac{1}{2}(|Q_{12}|+1),\ldots,\frac{1}{2}(Q_1+Q_2-3)$
$\frac{1}{2}$	$(U_0U_1U_2)^2 \left(\frac{U_2U_3}{U_1}\right)^{\delta_{21}} S^2_{|Q_{12}|+2}S^2_{|Q_{12}|+4}\ldots S^2_{Q_1+Q_2-2}$	$\frac{1}{2}(|Q_{12}|-1)\geq 0$
0	1	$\frac{1}{2}(Q_1+Q_2)$
0	$(U_0U_1U_2)^3 S_{2s_L}$	$\frac{1}{2}(Q_1+Q_2-2)$
0	$(U_0U_1U_2)^5 S_{2s_L}S^4_{2s_L+2}$	$\frac{1}{2}(Q_1+Q_2-4)$
0	$(U_0U_1U_2)^5 S_{2s_L}S^4_{2s_L+2}S^5_{2s_L+4}\ldots S^5_{Q_1+Q_2-2}$	$\frac{1}{2}(|Q_{12}|+2),\ldots,\frac{1}{2}(Q_1+Q_2-6)$
0	$(U_0U_1U_2)^4 \left(\frac{U_2U_3}{U_1}\right)^{\delta_{21}} S^3_{|Q_{12}|+2}S^4_{|Q_{12}|+4}\ldots S^4_{Q_1+Q_2-2}$	$\frac{1}{2}|Q_{12}|\neq 0$
0	$U_0U_1U_2 \left(\frac{U_2U_3}{U_1}\right)^{\delta_{21}} S_{|Q_{12}|+2}S_{|Q_{12}|+4}\ldots S_{Q_1+Q_2-2}$	$\frac{1}{2}(|Q_{12}|-2)\geq 0$
0	$(U_0U_1U_2)^3 S^2_2 S^3_4 S^3_6 \ldots S^3_{Q_1+Q_2-2}$	$0=Q_{12}$

Table 1: The values of $\det S^{Q_1Q_2}(s_L,s_R)$ for $Q_1,Q_2 > 1$ and for all possible s_R and s_L, where $\delta_{21} = \frac{Q_{21}}{|Q_{21}|} = \pm 1$, and $Q_{ij} = Q_i - Q_j$. See (4.29) for further notations.

where $x_j^\pm$ are the parameters of the Q_j bound-state representation, and $u_j \pm \frac{iQ_j}{g} = x_j^\pm + \frac{1}{x_j^\pm}$. Note that there are only three possible values of the right-spin, namely $s_R = 0, \frac{1}{2}, 1$, as $2s_R$ counts the number of fermions in the basis of the Hilbert space. If at least one of either Q_1 or Q_2 is 1, then the corresponding results are collected in Table 2. A brief account of how these results were obtained is presented in Appendix A.

Substituting the results from Tables 1 and 2 into (4.28), and carefully simplifying the resulting expression, we obtain

$$
\begin{aligned}
\sum_i (-1)^{F_i} A_i \log \Lambda_i^{Q_1Q_2} = \quad & [3]_q\big(Q_1Q_2 \log U_0U_1U_2 + \mathcal{K}^{Q_1Q_2}\big) \\
& -[2]_q\big((4Q_1Q_2 - Q_1 - Q_2)\log U_0 + 2Q_2(2Q_1-1)\log U_1 \\
& \qquad +2Q_1(2Q_2-1)\log U_2 + (Q_2-Q_1)\log U_3 + 4\mathcal{K}^{Q_1Q_2}\big) \\
& +[1]_q\big((5Q_1Q_2 - 2Q_1 - 2Q_2)\log U_0 + Q_2(5Q_1-4)\log U_1 \\
& \qquad +Q_1(5Q_2-4)\log U_2 + 2(Q_2-Q_1)\log U_3 + 5\mathcal{K}^{Q_1Q_2}\big),
\end{aligned}
\tag{4.30}
$$

where we have defined

$$
\mathcal{K}^{Q_1Q_2} = \sum_{j=0}^{Q_1-1} (Q_2 - Q_1 + 2j + 1) \sum_{k=1}^{Q_1-j-1} \log S_{Q_2-Q_1+2j+2k}.
\tag{4.31}
$$

In deriving the result (4.30), we have made use of the fact that $\log S_Q$ is an antisymmetric function of Q (i.e., $\log S_{-Q} = -\log S_Q$, up to an irrelevant additive constant), which in particular implies that $\mathcal{K}^{Q_1Q_2} = \mathcal{K}^{Q_2Q_1}$. We emphasize that (4.30) holds for any $Q_1, Q_2 \in \mathbb{N}$. An analogous result can be derived for $\sum_i (-1)^{F_i} \dot{A}_i \log \Lambda_i^{Q_1Q_2}$ by replacing $q \to \dot{q}$ in (4.30).

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s_R	$\det S^{Q_1 Q_2}(s_L, s_R)$	s_L
1	$U_0 U_1 U_2$	$\frac{1}{2}(Q_1 + Q_2 - 2)$
$\frac{1}{2}$	$U_0 U_1 U_2$	$\frac{1}{2}(Q_1 + Q_2 - 1)$
$\frac{1}{2}$	$(U_0 U_1 U_2)^2 \left(\frac{U_2 U_3}{U_1}\right)^{\delta_{21}}$	$\frac{1}{2}(|Q_{12}| - 1) \geq 0$
0	1	$\frac{1}{2}(Q_1 + Q_2)$
0	$(U_0 U_1 U_2)^2 \left(\frac{U_2 U_3}{U_1}\right)^{\delta_{21}}$	$\frac{1}{2}|Q_{12}| \neq 0$
0	$U_0 U_1 U_2 \left(\frac{U_2 U_3}{U_1}\right)^{\delta_{21}}$	$\frac{1}{2}(|Q_{12}| - 2) \geq 0$
0	$U_0 U_1 U_2$	$0 = Q_{12} \quad (Q_1 = Q_2 = 1)$

Table 2: The values of $\det S^{Q_1 Q_2}(s_L, s_R)$ for all possible s_R and s_L if either Q_1 or Q_2 is 1.

Thus, in order to calculate the Lüscher correction, we have to plug (4.30) into the formula (4.24):

$$
\begin{aligned}
E_0^{(2,2)} &= \sum_{Q_1, Q_2=1}^{\infty} Q_1 Q_2 \int_{-\infty}^{\infty} \frac{d\tilde{p}_1}{2\pi} e^{-L\tilde{\epsilon}_{Q_1}(\tilde{p}_1)} \int_{-\infty}^{\infty} \frac{d\tilde{p}_2}{2\pi} e^{-L\tilde{\epsilon}_{Q_2}(\tilde{p}_2)} \\
&\times \ i\partial_{\tilde{p}_1} \bigg\{ (2 - [2]_{\dot{q}})^2 \Big[[3]_q \left(Q_1 Q_2 \log U_0 U_1 U_2 + \mathcal{K}^{Q_1 Q_2} \right) \\
&\qquad\qquad - [2]_q \Big((4Q_1 Q_2 - Q_1 - Q_2) \log U_0 + 2Q_2(2Q_1 - 1) \log U_1 \\
&\qquad\qquad\qquad + 2Q_1(2Q_2 - 1) \log U_2 + (Q_2 - Q_1) \log U_3 + 4\mathcal{K}^{Q_1 Q_2} \Big) \\
&\qquad\qquad + [1]_q \Big((5Q_1 Q_2 - 2Q_1 - 2Q_2) \log U_0 + Q_2(5Q_1 - 4) \log U_1 \\
&\qquad\qquad\qquad + Q_1(5Q_2 - 4) \log U_2 + 2(Q_2 - Q_1) \log U_3 + 5\mathcal{K}^{Q_1 Q_2} \Big) \Big] \\
&\quad + (q \leftrightarrow \dot{q}) \\
&\quad + Q_1 Q_2 (2 - [2]_q)^2 (2 - [2]_{\dot{q}})^2 \log S_{sl(2)}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) \bigg\}.
\end{aligned}
\tag{4.32}
$$

We shall compare this result to the TBA output in Section 4.4.4.

4.2 Twisted TBA equations

In [19, 20, 13], the authors derived the TBA equations for the AdS/CFT model with the most general chemical potentials. Hence, the TBA equations for the γ-deformed theories correspond to some special cases. However, since we must determine precisely the charges/chemical potentials of the various excitations in terms of the deformation parameters, we now briefly sketch the derivation.

In order to derive the TBA equations, we have to recall the various types of excitations (both massive and magnonic) and their scattering matrices; and we must calculate their twist charges. We label the fundamental massive particle as $Q = 1$, corresponding to the $(3\dot{3})$ label of the fundamental representation. The S-matrix of this kind of particles is in fact given by (4.7) and they can form

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bound states for any Q with string-like complex roots defined like in (4.13). We label such a massive composite particle by Q and the scattering matrix of such particles is (4.13). Since the twist charge acts trivially in the $(3, 4)$ subspace, the massive particles are not charged: $\mu_Q = 0$.

We now focus on the magnonic excitations. They encode the color $su(2|2)$ structure of the scattering, and come in independent left and right copies. We first consider the right part. We label a magnon, which introduces label 2 in the sea of massive 3-particles, by y. It scatters trivially on itself, but nontrivially on the massive particles

$$ S_{1y}(u, y) = \frac{x^-(u) - y}{x^+(u) - y} \sqrt{\frac{x^+(u)}{x^-(u)}}, \qquad S_{Qy}(u, y) = \prod_{j=1}^{Q} S_{1y}(u_j, y). \tag{4.33} $$

The twist charge of the y particles is $\mu_y = -i\gamma_+$.

We can also introduce the label 1 in the sea of 2-particles. These particles are labeled by w. They scatter nontrivially only on the y particles and on themselves:

$$ S_{yw}(y, w) = S_{-1}(v(y) - w), \qquad S_{ww}(w, w') = S_2(w - w'), \tag{4.34} $$

where $v(y) = y + y^{-1}$, and $S_n(u)$ is defined as in (4.29), namely

$$ S_n(u) = \frac{u - \frac{in}{g}}{u + \frac{in}{g}}. \tag{4.35} $$

The twist charge of these particles is $\mu_w = 2i\gamma_+$.

As the scattering matrix $S_{yw}(y, w)$ has a difference form in the variable $v(y) = y + y^{-1}$, we might use the parameter v instead of y. The inverse of the relation, however, is not unique. Defining $y_-(v) = \frac{1}{2}(v - i\sqrt{4 - v^2})$ with the branch cuts running from $\pm\infty$ to ±2, we can describe any y with $\Im m(y) < 0$ for $v \in [-2, 2]$. Clearly $y_+(v) = y_-(v)^{-1}$ describes the other $\Im m(y) > 0$ case; and in the scattering matrices S_{1y} which depend on y, and not on v, we have to specify which root is taken. As a consequence, we have two types of y particles $y|\delta$ with $\delta = \pm$; and the scattering matrices split as $S_{1y}(u, y) \to S_{1y|\delta}(u, v) := S_{1y}(u, y_\delta(v))$. Clearly, the $y|\delta$ magnons scatter on the momentum bound states as $S_{Q\,y|\delta}(u, v) = \prod_j S_{1\,y|\delta}(u_j, v)$.

Let us now focus on the magnonic bound states. Detailed investigation showed [53] that v and w particles can form bound states for any positive integer M. It consist of $2M$ v-particles $v_{\pm(M+2-2j)} = v \pm (M + 2 - 2j)\frac{i}{g}$ for $j = 1, \ldots, M$ with $y_j = (y_{-j}^{-1})^*$, and M w-particles with synchronized parameters $w_j = v + (M + 1 - 2j)\frac{i}{g}$ for $j = 1, \ldots, M$. The scattering matrix of the $v|M$ particle with all other particles is simply the product of the scatterings of each of its individual constituents

$$ S_{v|M\,i}(v, q) = \prod_{j=1}^{M+1} S_{y|-\,i}(v_{M+2-2j}, q) \prod_{j}^{M} S_{wi}(w_j, q) \prod_{j=1}^{M-1} S_{y|+\,i}(v_{M-2j}, q). \tag{4.36} $$

The twist charge of the bound state simply sums up to $\mu_{v|M} = 2M(-i\gamma_+) + M2i\gamma_+ = 0$.

The w-type particles can form bound states among themselves: an N-string of w-particles can be formed as $w_j = w + (N + 1 - 2j)\frac{i}{g}$. The scattering of the N-string with any other particle is

$$ S_{w|N\,i}(w, q) = \prod_{j=1}^{N} S_{w\,i}(w_j, q), \tag{4.37} $$

while the twist charge is $\mu_{w|N} = 2Ni\gamma_+$.

We summarize the various scattering matrices and chemical potentials in Table 3.

Once we know the chemical potentials, we can calculate the kernels

$$ K_{jj'}(u, u') = \frac{1}{2\pi i} \partial_u \log S_{jj'}(u, u'), \tag{4.38} $$

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	Q_2	$v\|M_2$	$w\|N_2$	$y\|\delta_2$	μ
Q_1	$S_{Q_1 Q_2}$	$S_{Q_1 v\|M_2}$	1	$S_{Q_1 y\|\delta_2}$	0
$v\|M_1$	$S_{v\|M_1 Q_2}$	$S_{v\|M_1 v\|M_2}$	1	$S_{v\|M_1 y\|\delta_2}$	0
$w\|N_1$	1	1	$S_{w\|N_1 w\|N_2}$	$S_{w\|N_1 y\|\delta_2}$	$2N_1 i\gamma$
$y\|\delta_1$	$S_{y\|\delta_1 Q_2}$	$S_{y\|\delta_1 v\|M_2}$	$S_{y\|\delta_1 w\|N_2}$	1	$-i\gamma$

Table 3: Scattering matrices of the various particles and their chemical potentials for any of the two $su(2|2)$ wings.

and write the TBA equations one by one. To ensure positive particle densities, we have to invert the equations for $v|M$ and for $y|-$. The equation for the massive nodes then read as

$$\epsilon_{Q_2} = L\tilde{\epsilon}_{Q_2} - \log(1+e^{-\epsilon_{Q_1}}) \star K_{Q_1 Q_2} + \sum_{\alpha=\pm} \log(1+e^{-\epsilon_{v|M}^{\alpha}}) \star K_{v|M\,Q_2} - \delta \log(1+e^{-\epsilon_{y|\delta}^{\alpha}}) \star K_{y|\delta\,Q_2} \,. \quad (4.39)$$

Note that for particles of type $v|M$ and $y|\delta$, we must include contributions of the two $su(2|2)$ copies, which we denote by $\alpha = \pm$. The remaining equations are valid for the two $su(2|2)$ factors separately, so we omit the α index:

$$\begin{aligned}
\epsilon_{v|M} &= -\log(1+e^{-\epsilon_{Q_2}}) \star K_{Q_2\,v|M} + \log(1+e^{-\epsilon_{v|M'}}) \star K_{v|M'\,v|M} - \delta\log(1+e^{-\epsilon_{y|\delta}}) \star K_{y|\delta\,v|M}\,, \\
\epsilon_{w|N} &= -\mu_{w|N} - \log(1+e^{-\epsilon_{w|N'}}) \star K_{w|N'\,w|N} - \delta\log(1+e^{-\epsilon_{y|\delta}}) \star K_{y|\delta\,w|N}\,, \quad (4.40) \\
\epsilon_{y|\delta} &= i\pi - \mu_{y|\delta} - \log(1+e^{-\epsilon_{Q_2}}) \star K_{Q_2\,y|\delta} + \log(1+e^{-\epsilon_{v|M}}) \star K_{v|M\,y|\delta} - \log(1+e^{-\epsilon_{w|N}}) \star K_{w|N\,y|\delta}\,.
\end{aligned}$$

Once these equations are solved, the ground-state energy can be obtained as

$$E_0(L) = -\sum_{Q_2=1}^{\infty} \int \frac{du}{2\pi} \partial_u \tilde{p}_{Q_2} \log(1+e^{-\epsilon_{Q_2}})\,. \quad (4.41)$$

In [19, 20] the authors analyzed the TBA equations with generic chemical potentials, and formulated the requirement under which the Y-system remains unchanged. Our chemical potentials, which correspond to γ-deformations, satisfy their requirement.

4.3 Universal TBA equations and Y-system

The TBA equations can usually be brought into a local form. As already remarked, this means that the pseudo-energies can be drawn on a two-dimensional lattice, such that only neighboring sites couple to each other with the universal kernel

$$s\,I_{MN} = \delta_{MN} - (K+1)_{MN}^{-1}\,, \qquad s(u) = \frac{g}{4\cosh\frac{g\pi u}{2}}\,, \quad (4.42)$$

where $I_{MN} = \delta_{M+1,N} + \delta_{M-1,N}$ and $(K+1)_{MN}^{-1} \star (K_{NL}+\delta_{NL}) = \delta_{ML}$. To simplify the notation, we introduce the following Y-functions[5]

$$Y_Q = e^{-\epsilon_Q}\,, \qquad Y_{v|M} = e^{\epsilon_{v|M}}\,, \qquad Y_{w|N} = e^{\epsilon_{w|N}}\,, \qquad Y_{\delta} = -e^{\epsilon_{y|\delta}}\,. \quad (4.43)$$

Clearly, we have two copies for the magnonic Y-functions: $Y_{v|M}^{\alpha}, Y_{w|N}^{\alpha}, Y_{\delta}^{\alpha}$ where $\alpha = \pm$ refers to the two $su(2|2)$ copies. Acting with the operator (4.42) on the TBA equations (4.39), (4.40), and using kernel identities such as $(K+1)_{MN}^{-1} \star K_N = s\,\delta_{M,1}$ as well as the special properties of the chemical

---

[5]To compare with [13, 14], we note that $Y_{w|N} = Y_{N|w}^{AF}$, $Y_{v|M} = Y_{M|vw}^{AF}$ and $K_{vx}^{Q_1 Q_2} = K_{vwx}^{Q_1 Q_2\,AF}$. Also, $K_n(u) = \frac{1}{2\pi i}\frac{d}{du}\log S_n(u)$, where $S_n(u)$ is defined in (4.35); its Fourier transform is $\tilde{K}_n = \text{sign}(n)t^{|n|}$, $t = e^{-\frac{|\omega|}{g}}$.

potentials $\mu_{w|N-1} - 2\mu_{w|N} + \mu_{w|N+1} = 0$ and $\mu_{w|1} = -2\mu_y$, we arrive at their simplified form [14]. For later purposes, we write the simplified equations for $v|M$ and $w|N$ magnons, and a useful combination (hybrid) of the un-simplified equations for Q and y particles [16]

$$
\begin{aligned}
\log Y_{Q_2} &= -L\tilde{\epsilon}_{Q_2} + \log(1 + Y_{Q_1}) \star \left(K_{sl(2)}^{Q_1 Q_2} + 2s \star K_{vx}^{Q_1 - 1, Q_2} \right) \\
&\quad + \sum_{\alpha=\pm} \Bigg[\log\left(1 + Y_{v|1}^\alpha\right) \star s \,\hat{\star}\, K_{yQ_2} + \log(1 + Y_{v|Q_2 - 1}^\alpha) \star s - \log \frac{1 - Y_-^\alpha}{1 - Y_+^\alpha} \hat{\star} s \star K_{vx}^{1Q_2} \\
&\quad + \frac{1}{2} \log \frac{1 - \frac{1}{Y_-^\alpha}}{1 - \frac{1}{Y_+^\alpha}} \hat{\star} K_{Q_2} + \frac{1}{2} \log(1 - \frac{1}{Y_-^\alpha})(1 - \frac{1}{Y_+^\alpha}) \hat{\star} K_{yQ_2} \Bigg],
\end{aligned}
\tag{4.44}
$$

$$
\log Y_-^\alpha Y_+^\alpha = -\log(1 + Y_{Q_2}) \star K_{Q_2} + 2\log(1 + Y_{Q_2}) \star K_{xv}^{Q_2 1} \star s + 2\log \frac{1 + Y_{v|1}^\alpha}{1 + Y_{w|1}^\alpha} \star s,
\tag{4.45}
$$

$$
\log \frac{Y_+^\alpha}{Y_-^\alpha} = \log(1 + Y_{Q_2}) \star K_{Q_2 y},
\tag{4.46}
$$

$$
\log Y_{v|M}^\alpha = -\log(1 + Y_{M+1}) \star s + I_{MN} \log(1 + Y_{v|N}^\alpha) \star s + \delta_{M1} \log \frac{1 - Y_-^\alpha}{1 - Y_+^\alpha} \hat{\star} s,
\tag{4.47}
$$

$$
\log Y_{w|M}^\alpha = I_{MN} \log(1 + Y_{w|N}^\alpha) \star s + \delta_{M1} \log \frac{1 - \frac{1}{Y_-^\alpha}}{1 - \frac{1}{Y_+^\alpha}} \hat{\star} s,
\tag{4.48}
$$

where in the convolution $\hat{\star}$ we integrate over the interval $[-2, 2]$ only. To conform with part of the literature, we have renamed some kernels $K_{vx}^{MQ} = K_{v|M\,Q}$, $K_{xv}^{QM} = K_{Q\,v|M}$, $K_{yQ} = K_{y|-Q} + K_{y|+Q}$, $K_{Qy} = K_{Q\,y|-} - K_{Q\,y|+}$. The ground-state energy is given by summing the contributions of the massive nodes only:

$$
E_0(L) = -\sum_{Q=1}^\infty \int \frac{du}{2\pi} \partial_u \tilde{p}_Q \log(1 + Y_Q).
\tag{4.49}
$$

Evidently, as in the case of the $O(4)$ model, the chemical potentials and so the twists completely disappear from the simplified equations: They show up only in the asymptotics of the $Y_{w|N}$ functions, as $\lim_{N \to \infty} \log Y_{w|N} = -\mu_{w|N}$. It follows that the Y-system relations are not modified by the twists, as was supposed in [36]. Equations (4.48)-(4.49) together with the asymptotic prescription give the complete solution for the finite-size energy of the twisted AdS/CFT model for any coupling g. We now check this solution against LO and NLO Lüscher corrections.

4.4 Asymptotic expansion

We now expand the simplified TBA equations to leading and next-to-leading order. We expand any Y-functions as

$$
Y = \mathcal{Y}(1 + y + \dots).
\tag{4.50}
$$

We solve iteratively these equations similarly to the $O(4)$ case: At leading order, all the massive nodes Y_Q are exponentially small, which splits the Y-system into two independent subsystems which have constant asymptotic solutions. These constant values then determine the LO exponentially small expressions for $\mathcal{Y}_Q$. At NLO, we obtain linear integral equations for the y corrections of the two subsystems, whose initial values are provided by the asymptotic $\mathcal{Y}_Q$ functions. The solution of the linearized equations determine the NLO correction for the massive nodes y_Q, which provides the NLO energy correction.

4.4.1 Leading-order expansion

At LO, the massive $\mathcal{Y}_Q$ functions are exponentially small, and we can neglect the convolutions involving all $\log(1 + Y_Q)$. The magnonic $\mathcal{Y}_\pm^\alpha$, $\mathcal{Y}_{v|M}^\alpha$, $\mathcal{Y}_{w|N}^\alpha$ functions are constants. From (4.46), we see that

$\mathcal{Y}_+^\alpha = \mathcal{Y}_-^\alpha$. It then follows from (4.47) and (4.48) that the equations for $\mathcal{Y}_{v|M}^\alpha$ and $\mathcal{Y}_{w|N}^\alpha$ are the same as those for one of the wings of the $O(4)$ model (3.27). From the asymptotic behavior, we see that the solution for $v|M$ is the same as in the undeformed model, while the solution for $w|N$ is that of the deformed model:

$$\mathcal{Y}_{v|M}^\alpha = M(M+2)\,, \qquad \mathcal{Y}_{w|N}^+ = [N]_q[N+2]_q\,, \qquad \mathcal{Y}_{w|N}^- = [N]_{\dot{q}}[N+2]_{\dot{q}}\,. \tag{4.51}$$

Since $1 \star s = \frac{1}{2}$, the equations (4.45) for $\mathcal{Y}_\pm^\alpha$ can be solved as

$$\mathcal{Y}_+^\alpha = \mathcal{Y}_-^\alpha = \sqrt{\frac{1 + \mathcal{Y}_{v|1}^\alpha}{1 + \mathcal{Y}_{w|1}^\alpha}} = \frac{2}{[2]_\alpha}\,, \tag{4.52}$$

where we have further streamlined the notation by defining

$$[n]_+ = [n]_q\,, \qquad [n]_- = [n]_{\dot{q}}\,. \tag{4.53}$$

The sign in (4.52) can be fixed by the last equation in (4.40), and is consistent with the vanishing of the ground-state energy (4.56) in the undeformed ($q, \dot{q} \to 1$) limit. We now use that $1 \hat{\star} K_{yQ} = 1$ (see (6.12) in [13]) to write

$$\log \mathcal{Y}_Q = -L\,\tilde{\epsilon}_Q + \frac{1}{2} \sum_{\alpha=\pm} \left[\log\left(1 + \mathcal{Y}_{v|1}^\alpha\right) + \log(1 + \mathcal{Y}_{v|Q-1}^\alpha) + \log\left(1 - \frac{1}{\mathcal{Y}_-^\alpha}\right)\left(1 - \frac{1}{\mathcal{Y}_+^\alpha}\right) \right]\,. \tag{4.54}$$

Using the asymptotic solution (4.51), (4.52), we obtain the leading-order result for Y_Q

$$\mathcal{Y}_Q = (2 - [2]_q)(2 - [2]_{\dot{q}})Q^2 e^{-L\tilde{\epsilon}_Q(\tilde{p})}\,. \tag{4.55}$$

Substituting back into the energy formula (4.49), the LO correction reads as

$$E_0(L) \simeq E_0^{(1)}(L) = -\sum_{Q=1}^\infty \int \frac{d\tilde{p}}{2\pi} \mathcal{Y}_Q = -(2 - [2]_q)(2 - [2]_{\dot{q}}) \sum_{Q=1}^\infty Q^2 \int \frac{d\tilde{p}}{2\pi} e^{-L\tilde{\epsilon}_Q(\tilde{p})}\,, \tag{4.56}$$

which agrees with the result (4.19) that we obtained from the Lüscher calculation.

4.4.2   NLO expansion

Expanding the energy formula (4.49) to NLO, we obtain

$$E_0(L) = -\sum_{Q=1}^\infty \int \frac{d\tilde{p}}{2\pi} \log(1 + Y_Q) \simeq -\sum_{Q=1}^\infty \int \frac{d\tilde{p}}{2\pi} \mathcal{Y}_Q(1 + y_Q) + \sum_{Q=1}^\infty \int \frac{d\tilde{p}}{2\pi} \frac{1}{2} \mathcal{Y}_Q^2\,, \tag{4.57}$$

The quadratic term nicely reproduces our previous result (4.20) for $E_0^{(2,1)}$, since using again (4.55) gives

$$E_0^{(2,1)}(L) = \sum_{Q=1}^\infty \int \frac{d\tilde{p}}{2\pi} \frac{1}{2} \mathcal{Y}_Q^2 = \frac{1}{2}(2 - [2]_q)^2(2 - [2]_{\dot{q}})^2 \sum_{Q=1}^\infty Q^4 \int \frac{d\tilde{p}}{2\pi} e^{-2L\tilde{\epsilon}_Q(\tilde{p})}\,. \tag{4.58}$$

In order to evaluate

$$E_0^{(2,2)}(L) = -\sum_{Q=1}^\infty \int \frac{d\tilde{p}}{2\pi} \mathcal{Y}_Q y_Q\,, \tag{4.59}$$

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we must first calculate y_Q. This will be given by the solution of the following linearized set of TBA equations: [6]

$$y_{Q_2} = \mathcal{Y}_{Q_1} \star \left(K_{sl(2)}^{Q_1 Q_2} + 2s \star K_{vx}^{Q_1-1,Q_2} \right) + \sum_{\alpha=\pm} \left[A_{v|1}^{\alpha} y_{v|1}^{\alpha} \star s \, \hat{\star} K_{yQ_2} + A_{v|Q_2-1}^{\alpha} y_{v|Q_2-1}^{\alpha} \star s \right.$$

$$\left. - \frac{y_-^{\alpha} - y_+^{\alpha}}{1 - \frac{1}{\mathcal{Y}_+^{\alpha}}} \hat{\star} s \star K_{vx}^{1Q_2} + \frac{y_-^{\alpha} - y_+^{\alpha}}{2(\mathcal{Y}_+^{\alpha} - 1)} \hat{\star} K_{Q_2} + \frac{y_-^{\alpha} + y_+^{\alpha}}{2(\mathcal{Y}_+^{\alpha} - 1)} \hat{\star} K_{yQ_2} \right], \tag{4.60}$$

$$y_+^{\alpha} + y_-^{\alpha} = 2 \left(A_{v|1}^{\alpha} y_{v|1}^{\alpha} - A_{w|1}^{\alpha} y_{w|1}^{\alpha} \right) \star s - \mathcal{Y}_{Q_2} \star K_{Q_2} + 2\mathcal{Y}_{Q_2} \star K_{xv}^{Q_2 1} \star s, \tag{4.61}$$

$$y_+^{\alpha} - y_-^{\alpha} = \mathcal{Y}_{Q_2} \star K_{Q_2 y}, \tag{4.62}$$

$$y_{v|M}^{\alpha} = \left(A_{v|M-1}^{\alpha} y_{v|M-1}^{\alpha} + A_{v|M+1}^{\alpha} y_{v|M+1}^{\alpha} \right) \star s - \mathcal{Y}_{M+1} \star s + \delta_{M1} \frac{y_-^{\alpha} - y_+^{\alpha}}{1 - \frac{1}{\mathcal{Y}_+^{\alpha}}} \hat{\star} s, \tag{4.63}$$

$$y_{w|N}^{\alpha} = \left(A_{w|N-1}^{\alpha} y_{w|N-1}^{\alpha} + A_{w|N+1}^{\alpha} y_{w|N+1}^{\alpha} \right) \star s + \delta_{N1} \frac{y_+^{\alpha} - y_-^{\alpha}}{1 - \mathcal{Y}_+^{\alpha}} \hat{\star} s, \tag{4.64}$$

where

$$A_{v|M}^{\alpha} = \frac{\mathcal{Y}_{v|M}^{\alpha}}{1 + \mathcal{Y}_{v|M}^{\alpha}} = \frac{M(M+2)}{(M+1)^2}, \qquad A_{w|N}^{\alpha} = \frac{\mathcal{Y}_{w|N}^{\alpha}}{1 + \mathcal{Y}_{w|N}^{\alpha}} = \frac{[N]_{\alpha}[N+2]_{\alpha}}{[N+1]_{\alpha}^2}. \tag{4.65}$$

We start with the equation (4.64) for $y_{w|N}^{\alpha}$. The difference between $\alpha = +$ and $\alpha = -$ is only in the asymptotics (4.51), (4.52). Since one equation can be obtained from the other by interchanging $q \leftrightarrow \dot{q}$, we do not write out explicitly the α index. Replacing $y_+ - y_-$ in (4.64) with the contributions from the massive nodes (4.62), and using the explicit form of the asymptotic solution, we obtain an equation similar to the one for the $O(4)$ case:

$$y_{w|N} = \left(\frac{[N-1][N+1]}{[N]^2} y_{w|N-1} + \frac{[N+1][N+3]}{[N+2]^2} y_{w|N+1} \right) \star s + \delta_{N1} c_1 \star s, \tag{4.66}$$

where

$$c_1 = \frac{[2]}{[2]-2} \mathcal{Y}_Q \star \hat{K}_{Qy}, \qquad \hat{K}_{Qy}(u,v) = K_{Qy}(u,v) \left(\Theta(v+2) - \Theta(v-2) \right), \tag{4.67}$$

and $\Theta(v)$ is the standard unit step function. We solve the difference equation in Fourier space. We use that $\tilde{s} = (2 \cosh \frac{\omega}{g})^{-1} = (t + t^{-1})^{-1}$ where $t \equiv e^{-\frac{|\omega|}{g}}$. The solution which decreases for large N (to respect the asymptotics of $Y_{w|N}$) and is compatible with the $\delta_{N,1}$ term is

$$\tilde{y}_{w|N} = \frac{\tilde{c}_1 t}{[2]} \left(\frac{[N+1]}{[N]} t^{N-1} - \frac{[N+1]}{[N+2]} t^{N+1} \right). \tag{4.68}$$

We now analyze the equation (4.63) for $y_{v|M}^{\alpha}$. This difference equation is not the same as for the undeformed $O(4)$ model, as it has inhomogeneous terms,

$$y_{v|M} = \left(\frac{(M-1)(M+1)}{M^2} y_{v|M-1} + \frac{(M+1)(M+3)}{(M+2)^2} y_{v|M+1} \right) \star s - \mathcal{Y}_{M+1} \star s + \delta_{M1} c_2 \star s, \tag{4.69}$$

where

$$c_2 = \frac{2}{[2]-2} \mathcal{Y}_Q \star \hat{K}_{Qy}. \tag{4.70}$$

Taking the Fourier transform, we obtain the difference equation

$$(t + t^{-1}) \tilde{y}_{v|M} = \frac{(M-1)(M+1)}{M^2} \tilde{y}_{v|M-1} + \frac{(M+1)(M+3)}{(M+2)^2} \tilde{y}_{v|M+1} - \tilde{\mathcal{Y}}_{M+1} + \delta_{M1} \tilde{c}_2. \tag{4.71}$$

---

[6]We note that in [17] there is an erroneous term in Eq. (2.7): $-Y_Q^0 \star s$ should be instead $-Y_Q^0 \star K_Q$, as in (4.61).

The general solution with two arbitrary parameters A_1 and A_2 reads as

$$
\begin{aligned}
\tilde{y}_{v|M} &= \left(\frac{M+1}{M} t^{M-1} - \frac{M+1}{M+2} t^{M+1} \right) \left(A_1 - \sum_{k=1}^{M} \frac{\tilde{\mathcal{Y}}_{k+1} t^{-k-2} \left(t^{-2}k - k - 2 \right)}{\left(t^{-2} - 1 \right)^3 (k+1)} \right) \\
&+ \left(\frac{M+1}{M} t^{1-M} - \frac{M+1}{M+2} t^{-1-M} \right) \left(A_2 - \sum_{k=1}^{M} \frac{\tilde{\mathcal{Y}}_{k+1} t^{k-2} \left(t^{-2}(k+2) - k \right)}{\left(t^{-2} - 1 \right)^3 (k+1)} \right) . \quad (4.72)
\end{aligned}
$$

The parameters can be fixed from $\lim_{M \to \infty} \tilde{y}_{v|M} = 0$ and from the $M = 1$ term as

$$
A_1 = t^2 \left(\frac{t^{-1}}{2} \tilde{c}_2 - A_2 \right), \qquad A_2 = \sum_{k=1}^{\infty} \frac{\tilde{\mathcal{Y}}_{k+1} t^{k-2} \left(t^{-2}(k+2) - k \right)}{\left(t^{-2} - 1 \right)^3 (k+1)} . \quad (4.73)
$$

The NLO hybrid equation for y_{Q_2} is (4.60); we plug into it the equations (4.61) and (4.62), and obtain

$$
\begin{aligned}
y_{Q_2} &= \mathcal{Y}_{Q_1} \star \left(K_{sl(2)}^{Q_1 Q_2} + 2s \star K_{vx}^{Q_1 - 1, Q_2} \right) + \sum_{\alpha = 1, 2} \left[\frac{A_{v|1}^{\alpha}}{1 - \frac{1}{\mathcal{Y}_+^{\alpha}}} y_{v|1}^{\alpha} \star s \hat{\star} K_{yQ_2} - \frac{A_{w|1}^{\alpha}}{\mathcal{Y}_+^{\alpha} - 1} y_{w|1}^{\alpha} \star s \hat{\star} K_{yQ_2} \right. \\
&+ \frac{\mathcal{Y}_{Q_1} \star K_{xv}^{Q_1 1}}{\mathcal{Y}_+^{\alpha} - 1} \star s \hat{\star} K_{yQ_2} + \frac{\mathcal{Y}_{Q_1} \star K_{Q_1 y}}{1 - \frac{1}{\mathcal{Y}_{\pm}^{\alpha}}} \hat{\star} s \star K_{vx}^{1 Q_2} - \frac{\mathcal{Y}_{Q_1} \star K_{Q_1 y}}{2(\mathcal{Y}_{\pm}^{\alpha} - 1)} \hat{\star} K_{Q_2} - \frac{\mathcal{Y}_{Q_1} \star K_{Q_1}}{2(\mathcal{Y}_+^{\alpha} - 1)} \hat{\star} K_{yQ_2} \\
&\left. + A_{v|Q_2 - 1}^{\alpha} y_{v|Q_2 - 1}^{\alpha} \star s \right] , \quad (4.74)
\end{aligned}
$$

Since $y_{v|1}$ and $y_{w|1}$ can be expressed in terms of $\mathcal{Y}_Q$, we see that the solution for y_{Q_2} has the general form

$$
y_{Q_2} = \mathcal{Y}_{Q_1} \star K_{sl(2)}^{Q_1 Q_2} + \mathcal{Y}_{Q_1} \star \mathcal{M}^{Q_1 Q_2} . \quad (4.75)
$$

Consider the first term $\mathcal{Y}_{Q_1} \star K_{sl(2)}^{Q_1 Q_2}$. It is easy to see that its contribution to the integrand in the energy formula (4.59)

$$
\mathcal{Y}_{Q_2} y_{Q_2} = \mathcal{Y}_{Q_2} \left(\mathcal{Y}_{Q_1} \star K_{sl(2)}^{Q_1 Q_2} \right), \quad (4.76)
$$

with $\mathcal{Y}_{Q_2}$ given by (4.55), matches with the "scalar part" of the integrand of the Lüscher correction $E_0^{(2,2)}$ in (4.24). We now proceed to analyze the remaining contribution in (4.75), and show that it gives the "matrix part" of the integrand of the Lüscher correction.

4.4.3 NLO TBA correction: the case $Q_1 = Q_2 = 1$

To warm up, let us evaluate the NLO correction for the $Q_1 = Q_2 = 1$ case; thus, we calculate $\mathcal{M}^{11}$. In so doing, we can freely put $\mathcal{Y}_{Q_2} = 0$ for $Q_2 > 1$. The corresponding solutions read as

$$
\begin{aligned}
y_{w|1} &= \frac{[2]}{[2] - 2} \mathcal{Y}_1 \star K_{1y} \hat{\star} \left(K_1 - \frac{1}{[3]} K_3 \right) , \quad (4.77) \\
y_{v|1} &= \frac{2}{[2] - 2} \mathcal{Y}_1 \star K_{1y} \hat{\star} \left(K_1 - \frac{1}{3} K_3 \right) , \\
y_+ - y_- &= \mathcal{Y}_1 \star K_{1y} , \\
y_+ + y_- &= \frac{2}{[2] - 2} \mathcal{Y}_1 \star K_{1y} \hat{\star} \left(\left(\frac{3}{2} - \frac{[3]}{[2]} \right) K_1 - \left(\frac{1}{2} - \frac{1}{[2]} \right) K_3 \right) \star s - \mathcal{Y}_1 \star K_1 + 2 \mathcal{Y}_1 \star K_{xv}^{11} \star s .
\end{aligned}
$$

It is convenient to substitute these solutions directly into (4.60), i.e.,

$$
y_1 = \mathcal{Y}_1 \star K_{sl(2)}^{11} + \sum_{\alpha = \pm} \left[A_{v|1} y_{v|1}^{\alpha} \star s \hat{\star} K_{y1} - \frac{y_-^{\alpha} - y_+^{\alpha}}{1 - \frac{1}{\mathcal{Y}_+^{\alpha}}} \hat{\star} s \star K_{vx}^{11} + \frac{y_-^{\alpha} - y_+^{\alpha}}{2(\mathcal{Y}_+^{\alpha} - 1)} \hat{\star} K_1 + \frac{y_-^{\alpha} + y_+^{\alpha}}{2(\mathcal{Y}_+^{\alpha} - 1)} \hat{\star} K_{y1} \right] . \quad (4.78)
$$

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Using the explicit form of the asymptotic solutions, one can see that the terms involving the convolution with K_3 completely cancel. Exploiting further that $K_{1y}\hat{\star}K_1 = K_{xv}^{11}$ (which can be shown using relations from Sec. 6 in [13]), we arrive at

$$y_1 = \mathcal{Y}_1 \star K_{sl(2)}^{11} + \sum_{\alpha=\pm} \left[\frac{[2]_\alpha}{2([2]_\alpha - 2)} \left(\mathcal{Y}_1 \star K_{1y}\hat{\star}K_1 + \mathcal{Y}_1 \star K_1\hat{\star}K_{y1} - 2\mathcal{Y}_1 \star K_{xv}^{11} \star s\hat{\star}K_{y1} \right) \right.$$
$$\left. + \frac{[3]_\alpha - 3}{([2]_\alpha - 2)^2} \mathcal{Y}_1 \star K_{xv}^{11} \star s\,\hat{\star}K_{y1} - \frac{2}{[2]_\alpha - 2}\mathcal{Y}_1 \star K_{1y}\hat{\star}s \star K_{vx}^{11} \right]. \quad (4.79)$$

This expression further simplifies to

$$y_1 = \mathcal{Y}_1 \star K_{sl(2)}^{11} + \sum_{\alpha=\pm} \left[\frac{[2]_\alpha}{2([2]_\alpha - 2)} \mathcal{Y}_1 \star (K_{1y}\hat{\star}K_1 + K_1\hat{\star}K_{y1}) + \frac{2}{[2]_\alpha - 2}\mathcal{Y}_1 \star \left(K_{xv}^{11} \star s\hat{\star}K_{y1} - K_{1y}\hat{\star}s \star K_{vx}^{11} \right) \right]. \quad (4.80)$$

In the second term, using $K_{1y}\hat{\star}K_1 = K_{xv}^{11}$ and $K_{vx}^{11} = K_1\hat{\star}K_{y1}$, we can write

$$K_{xv}^{11} \star s\hat{\star}K_{y1} - K_{1y}\hat{\star}s \star K_{vx}^{11} = K_{1y}\hat{\star}(K_1 \star s - s \star K_1)\hat{\star}K_{y1} = 0, \quad (4.81)$$

as both s and K_1 depend on the differences of their arguments, and therefore their convolution is commutative. In the previous term in (4.80), we can obtain

$$K_{1y}\hat{\star}K_1 + K_1\hat{\star}K_{y1} = \frac{1}{2\pi i}\partial_{u_1} \log \left(\frac{x_1^- - x_2^+}{x_1^- - 1/x_2^+} \frac{x_1^+ - 1/x_2^-}{x_1^+ - x_2^-} \frac{u_1 - u_2 - 2i/g}{u_1 - u_2 + 2i/g} \right)$$
$$= \frac{1}{\pi i}\partial_{u_1} \log \left(\frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \sqrt{\frac{x_1^+}{x_1^-}} \right) = -\frac{1}{\pi i}\partial_{u_1} \log a_1(u_1, u_2), \quad (4.82)$$

where we have used identities from Sec. 6 in [13] and Eq. (3.7) in [14], and we have recalled the definition in (4.1) of a_1. The final expression for the $Q = Q' = 1$ contribution to the energy (4.59) is therefore given by

$$y_1 = \mathcal{Y}_1 \star K_{sl(2)}^{11} + \mathcal{Y}_1 \star \mathcal{M}^{11}, \qquad \mathcal{M}^{11} = \frac{1}{2\pi i}\partial_{u_1} \log a_1(u_1, u_2) \sum_\alpha \frac{[2]_\alpha}{2 - [2]_\alpha}, \quad (4.83)$$

which completely reproduces the result (4.27) obtained directly from the Lüscher correction.

4.4.4 NLO TBA correction for any Q_1, Q_2

We now consider the general case. Let us recall the result (4.74) for y_{Q_2}

$$y_{Q_2} = \mathcal{Y}_{Q_1} \star \left(K_{sl(2)}^{Q_1 Q_2} + 2s \star K_{vx}^{Q_1-1,Q_2} \right) + \sum_{\alpha=1,2} \left[\frac{A_{v|1}^\alpha}{1 - \frac{1}{\mathcal{Y}_+^\alpha}}y_{v|1}^\alpha \star s\hat{\star}K_{yQ_2} - \frac{A_{w|1}^\alpha}{\mathcal{Y}_+^\alpha - 1}y_{w|1}^\alpha \star s\hat{\star}K_{yQ_2} \right.$$

$$+ \frac{\mathcal{Y}_{Q_1} \star K_{xv}^{Q_1 1}}{\mathcal{Y}_+^\alpha - 1} \star s\hat{\star}K_{yQ_2} + \frac{\mathcal{Y}_{Q_1} \star K_{Q_1 y}}{1 - \frac{1}{\mathcal{Y}_\pm^\alpha}}\hat{\star}s \star K_{vx}^{1Q_2} - \frac{\mathcal{Y}_{Q_1} \star K_{Q_1 y}}{2(\mathcal{Y}_\pm^\alpha - 1)}\hat{\star}K_{Q_2} - \frac{\mathcal{Y}_{Q_1} \star K_{Q_1}}{2(\mathcal{Y}_+^\alpha - 1)}\hat{\star}K_{yQ_2}$$

$$\left. + A_{v|Q_2-1}^\alpha y_{v|Q_2-1}^\alpha \star s \right], \quad (4.84)$$

and analyze it term by term. Since we have already checked in Section (4.4.2) the matching of the first term with the scalar part of the Lüscher result, we start by considering the second term of (4.84), which can be rewritten as

$$2\mathcal{Y}_{Q_1} \star s \star K_{vx}^{Q_1-1,Q_2} = 2\mathcal{Y}_{Q_1} \star K_{Q_1-1} \star s\hat{\star}K_{yQ_2} + 2\mathcal{Y}_{Q_1} \star \sum_{j=0}^{Q_1-2} K_{Q_2-Q_1+2j+1} \star s, \quad (4.85)$$

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where we used the property $s \star K_Q = K_Q \star s$, valid for any Q. Now we consider the terms in the square brackets of (4.84), again suppressing the index α. Using the solution (4.72) for $M = 1$ and taking its inverse Fourier transform, we can express the first term as

$$
\frac{A_{v|1}}{1 - \frac{1}{\mathcal{Y}_+}} y_{v|1} \star s \hat{\star} K_{yQ_2} \quad = \quad \frac{3}{2 - [2]} \left\{ \frac{\mathcal{Y}_{Q_1} \star K_{Q_1 y}}{2 - [2]} \hat{\star} \left(\frac{K_3}{3} - K_1 \right) \right.
$$
$$
\left. + \quad \frac{1}{3} \frac{\mathcal{Y}_{Q_1}}{Q_1} \star [(Q_1 - 1)K_{Q_1+1} - (Q_1 + 1)K_{Q_1-1}] \right\} \star s \hat{\star} K_{yQ_2} , \quad (4.86)
$$

where the term in the second line can be rewritten, by using the identity $(K_{n+1} + K_{n-1} + n\delta_{n,\pm 1}\delta)\star s = K_n$, as

$$
-\frac{2}{2 - [2]} \mathcal{Y}_{Q_1} \star K_{Q_1-1} \star s \hat{\star} K_{yQ_2} + \frac{Q_1 - 1}{Q_1(2 - [2])} \mathcal{Y}_{Q_1} \star K_{Q_1} \hat{\star} K_{yQ_2} .
$$

The K_3 contribution in the first line of (4.86) cancels, as we have already seen in the $Q_1 = Q_2 = 1$ case, with the successive term in (4.84)

$$
-\frac{A_{w|1}}{\mathcal{Y}_+ - 1} y_{w|1} \star s \hat{\star} K_{yQ_2} = -\frac{1}{(2 - [2])^2} \mathcal{Y}_{Q_1} \star K_{Q_1 y} \hat{\star} (K_3 - [3]K_1) \star s \hat{\star} K_{yQ_2} \quad , \quad (4.87)
$$

while the terms with K_1 give

$$
\frac{[3] - 3}{(2 - [2])^2} \mathcal{Y}_{Q_1} \star K_{Q_1 y} \hat{\star} K_1 \star s \hat{\star} K_{yQ_2} . \quad (4.88)
$$

Summing this contribution to the first two terms in the second line of (4.84), we obtain

$$
\frac{2}{2 - [2]} \left(\mathcal{Y}_{Q_1} \star K_{Q_1 y} \hat{\star} s \star K_1 \hat{\star} K_{yQ_2} - \mathcal{Y}_{Q_1} \star K_{Q_1 y} \hat{\star} K_1 \star s \hat{\star} K_{yQ_2} \right)
$$
$$
+\frac{[2]}{2 - [2]} \mathcal{Y}_{Q_1} \star K_{Q_1-1} \star s \hat{\star} K_{yQ_2} + \frac{2}{2 - [2]} \mathcal{Y}_{Q_1} \star K_{Q_1 y} \hat{\star} s \star K_{Q_2-1} , \quad (4.89)
$$

where we used the identities $K_{vx}^{1Q} = K_1 \hat{\star} K_{yQ} + K_{Q-1}$ and $K_{xv}^{Q1} = K_{Qy} \hat{\star} K_1 + K_{Q-1}$. [7] As already noticed for the case $Q_1 = Q_2 = 1$, the first line in the expression above vanishes because $K_1 \star s = s \star K_1$. The successive two terms in the second line of (4.84) give

$$
-\frac{[2]}{2\pi(2 - [2])} \mathcal{Y}_{Q_1} \star i\partial_{u_1} \log a_1^{Q_1 Q_2}(u_1, u_2) , \quad (4.90)
$$

where we used the identity (4.82) generalized for any Q_1, Q_2,

$$
K_{Q_1 y} \hat{\star} K_{Q_2} + K_{Q_1} \hat{\star} K_{yQ_2} = \frac{1}{\pi i} \partial_{u_1} \log \left(\frac{x_1^{-Q_1} - x_2^{+Q_2}}{x_1^{+Q_1} - x_2^{-Q_2}} \sqrt{\frac{x_1^{+Q_1}}{x_1^{-Q_1}}} \right) = -\frac{1}{\pi i} \partial_{u_1} \log a_1^{Q_1 Q_2}(u_1, u_2) . \quad (4.91)
$$

Let us turn to the last and most complicated term. Using the inverse Fourier transform of (4.72) for $M = Q_2 - 1$, we can write it as follows

$$
A_{v|Q_2-1} y_{v|Q_2-1} \star s \quad = \quad \frac{\mathcal{Y}_{Q_1}}{Q_1 Q_2} \star \sum_{k=0}^{Q_1-1} k(k - Q_1) \left[(Q_2 + 1)K_{Q_2-Q_1+2k-1} - (Q_2 - 1)K_{Q_2-Q_1+2k+1} \right] \star s
$$
$$
+ \quad \frac{\mathcal{Y}_{Q_1} \star K_{Q_1 y}}{Q_2(2 - [2])} \hat{\star} \left[(Q_2 - 1)K_{Q_2+1} - (Q_2 + 1)K_{Q_2-1} \right] \star s , \quad (4.92)
$$

---

[7] The latter identity is reported in footnote 4 of [16]; the former can be derived analogously using equations (6.19) and (6.39) in [13]. The same equations, together with (6.14), can also be used to obtain (4.85).

where the second line can be expressed as

$$-\frac{2}{2-[2]}\mathcal{Y}_{Q_1}\star K_{Q_1y}\hat{\star}s\star K_{Q_2-1}+\frac{Q_2-1}{Q_2(2-[2])}\mathcal{Y}_{Q_1}\star K_{Q_1y}\hat{\star}K_{Q_2}\,. \tag{4.93}$$

Now, taking into account that in summing over α the first term in (4.92) gets a factor 2 and the other terms get similar coefficients with $q\to\dot{q}$, we can sum all the contributions above to get the solution for y_{Q_2} for generic values of Q_1,Q_2:

$$\begin{aligned}
y_{Q_2} &= \mathcal{Y}_{Q_1}\star\left\{K_{sl(2)}^{Q_1Q_2}+2\sum_{j=0}^{Q_1-2}K_{Q_2-Q_1+2j+1}\star s-\frac{\partial_{u_1}\log a_1^{Q_1Q_2}(u_1,u_2)}{\pi i}\right.\\
&\quad -\sum_{\alpha=\pm}\left[\frac{\partial_{u_1}\log a_2^{Q_1Q_2}(u_1,u_2)}{2\pi iQ_2(2-[2]_\alpha)}+\frac{\partial_{u_1}\log a_2^{Q_2Q_1}(u_2,u_1)^\star}{2\pi iQ_1(2-[2]_\alpha)}\right]\\
&\quad \left.+\frac{2}{Q_1Q_2}\sum_{k=0}^{Q_1-1}k(k-Q_1)\left[(Q_2+1)K_{Q_2-Q_1+2k-1}-(Q_2-1)K_{Q_2-Q_1+2k+1}\right]\star s\right\},\tag{4.94}
\end{aligned}$$

where we used the following identity

$$K_{Q_1y}\hat{\star}K_{Q_2}=\frac{1}{2\pi i}\partial_{u_1}\log\left(\frac{x_1^{-Q_1}-x_2^{+Q_2}}{x_1^{+Q_1}-x_2^{-Q_2}}\frac{x_1^{+Q_1}-1/x_2^{+Q_2}}{x_1^{-Q_1}-1/x_2^{-Q_2}}\right)\equiv\frac{1}{2\pi i}\partial_{u_1}\log a_2^{Q_1Q_2}(u_1,u_2)\,,\tag{4.95}$$

its hermitian conjugate (recall that $x(u)^*=1/x(u^*)$ in the mirror kinematics) [8]

$$K_{Q_1}\hat{\star}K_{yQ_2}=\frac{1}{2\pi i}\partial_{u_1}\log\left(\frac{x_1^{-Q_1}-x_2^{+Q_2}}{x_1^{+Q_1}-x_2^{-Q_2}}\frac{x_1^{-Q_1}x_2^{-Q_2}-1}{x_1^{+Q_1}x_2^{+Q_2}-1}\frac{x_1^{+Q_1}}{x_1^{-Q_1}}\frac{x_2^{-Q_2}}{x_2^{+Q_2}}\right)=\frac{1}{2\pi i}\partial_{u_1}\log a_2^{Q_2Q_1}(u_2,u_1)^*\,,\tag{4.96}$$

and $a_2^{Q_1Q_2}(u_1,u_2)\,a_2^{Q_2Q_1}(u_2,u_1)^*=\left[a_1^{Q_1Q_2}(u_1,u_2)\right]^{-2}$. Moreover, we can write the sum of the two convolutions involving the universal kernel $s(u)$ in (4.94) as

$$\begin{aligned}
&\frac{1}{Q_1Q_2}\sum_{k=0}^{Q_1-1}k(k-Q_1)\left[(Q_2+1)K_{Q_2-Q_1+2k-1}-(Q_2-1)K_{Q_2-Q_1+2k+1}\right]\star s\\
&+\sum_{j=0}^{Q_1-2}K_{Q_2-Q_1+2j+1}\star s=\frac{1}{2\pi iQ_1Q_2}\partial_{u_1}\mathcal{K}^{Q_1Q_2}\,,\tag{4.97}
\end{aligned}$$

where we used the definition (4.31) of $\mathcal{K}^{Q_1Q_2}$. Remarkably, despite the long computation, the final expression for y_{Q_2} is quite simple and reads

$$\begin{aligned}
y_{Q_2} &= \mathcal{Y}_{Q_1}\star\frac{1}{2\pi i}\partial_{u_1}\left\{\log S_{sl(2)}^{Q_1Q_2}+\frac{2}{Q_1Q_2}\mathcal{K}^{Q_1Q_2}-2\log a_1^{Q_1Q_2}(u_1,u_2)\right.\\
&\quad \left.-\sum_{\alpha=\pm}\frac{1}{(2-[2]_\alpha)}\left[\frac{1}{Q_2}\log a_2^{Q_1Q_2}(u_1,u_2)+\frac{1}{Q_1}\log a_2^{Q_2Q_1}(u_2,u_1)^\star\right]\right\}.\tag{4.98}
\end{aligned}$$

---

[8]Actually, identities (4.82), (4.91), and (4.96) are valid up to vanishing derivatives $\partial_{u_1}\log\sqrt{\frac{x_2^-}{x_2^+}}$, $\partial_{u_1}\log\sqrt{\frac{x_2^{-Q_2}}{x_2^{+Q_2}}}$ and

$\partial_{u_1}\log\frac{x_2^{+Q_2}}{x_2^{-Q_2}}$, respectively.

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Substituting this result, together with the result (4.55) for $\mathcal{Y}_{Q_1}$, into the formula (4.59) for the energy correction, we obtain

$$
\begin{aligned}
E_0^{(2,2)} &= \sum_{Q_1,Q_2=1}^{\infty} Q_1 Q_2 \int_{-\infty}^{\infty} \frac{d\tilde{p}_1}{2\pi} e^{-L\tilde{\epsilon}_{Q_1}(\tilde{p}_1)} \int_{-\infty}^{\infty} \frac{d\tilde{p}_2}{2\pi} e^{-L\tilde{\epsilon}_{Q_2}(\tilde{p}_2)} \\
&\quad \times\ i\partial_{\tilde{p}_1} \Bigg\{ (2-[2]_{\dot{q}})^2 \Big[[3]_q \left(-Q_1 Q_2 \log a_1^{Q_1 Q_2} + \mathcal{K}^{Q_1 Q_2} \right) \\
&\qquad\qquad\qquad - [2]_q \left(-4 Q_1 Q_2 \log a_1^{Q_1 Q_2} - Q_1 \log a_2^{Q_1 Q_2} - Q_2 \log a_2^{Q_2 Q_1 *} + 4\mathcal{K}^{Q_1 Q_2} \right) \\
&\qquad\qquad\qquad + [1]_q \left(-5 Q_1 Q_2 \log a_1^{Q_1 Q_2} - 2 Q_1 \log a_2^{Q_1 Q_2} - 2 Q_2 \log a_2^{Q_2 Q_1 *} + 5\mathcal{K}^{Q_1 Q_2} \right) \Big] \\
&\qquad + (q \leftrightarrow \dot{q}) \\
&\qquad + Q_1 Q_2 (2-[2]_q)^2 (2-[2]_{\dot{q}})^2 \log S_{sl(2)}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) \Bigg\}.
\end{aligned}
\tag{4.99}
$$

Finally, through the following identifications

$$
a_1^{Q_1 Q_2} = (U_0 U_1 U_2)^{-1}, \quad a_2^{Q_1 Q_2} = U_0 U_2^2 U_3, \quad a_2^{Q_2 Q_1 *} = U_0 U_1^2 U_3^{-1}, \tag{4.100}
$$

we find full agreement with the result (4.32) from the Lüscher computation.

5 Weak-coupling expansion

In this section we calculate the weak-coupling expansion of the ground-state energy of the twisted AdS/CFT model. In order to perform the weak-coupling expansion, we use the parameterization

$$
x^{\pm}(\tilde{p}) = \frac{(\tilde{p} - iQ)}{2g} \left(\sqrt{1 + \frac{4g^2}{Q^2 + \tilde{p}^2}} \mp 1 \right), \tag{5.1}
$$

which follows from (4.10) and (4.18). At leading order in g, and so at weak coupling, we have

$$
x^- = \frac{\tilde{p} - iQ}{g} + O(g), \qquad x^+ = \frac{g}{\tilde{p} + iQ} + O(g^3). \tag{5.2}
$$

5.1 LO contribution, single wrapping

The LO correction can be calculated from (4.56) by using the expansion of the exponential term appearing in $\mathcal{Y}_Q$:

$$
e^{-L\tilde{\epsilon}_Q(\tilde{p})} = \sum_{j=0}^{\infty} c_j \frac{g^{2(L+j)}}{(\tilde{p}^2 + Q^2)^{L+j}}. \tag{5.3}
$$

In particular $c_0 = 1$, while the higher-order terms can be easily generated with Mathematica. Using the fact that $\frac{1}{n!} f^{(n)}(z) = \oint \frac{dw}{2\pi i} \frac{f(w)}{(w-z)^{n+1}}$, we perform the integral in (4.56) by residues

$$
\int_{-\infty}^{\infty} \frac{d\tilde{p}}{2\pi} \frac{1}{(\tilde{p}^2 + Q^2)^k} = \binom{2k-2}{k-1} (2Q)^{1-2k}. \tag{5.4}
$$

The summation over Q gives rise to a series of ζ-functions:

$$
E_0^{(1)}(L) = -(2-[2]_q)(2-[2]_{\dot{q}}) \sum_{j=0}^{\infty} c_j 2^{1-2(L+j)} \binom{2(L+j)-2}{L+j-1} \zeta_{2(L+j)-3} g^{2(L+j)}. \tag{5.5}
$$

This result is exact up to g^{4L} where the NLO Lüscher correction starts to play a role. We evaluate the leading g^{4L}-order contribution of the NLO Lüscher correction in the next subsection.

5.2 NLO contribution, double wrapping

The simplest term of the NLO correction comes from (4.58) and contains $\mathcal{Y}_Q^2$. Its contribution at order g^{4L} can be calculated using eq. (5.4) to be

$$E_0^{(2,1)}(L) = (2 - [2]_q)^2 (2 - [2]_{\dot{q}})^2 2^{-4L} \binom{4L-2}{2L-1} \zeta_{4L-5} g^{4L}. \tag{5.6}$$

The most complicated term is $E_0^{(2,2)}(L)$. We have to evaluate (4.59) based on the solution given in (4.98). The twist dependence comes in two distinct ways as:

$$E_0^{(2,2)}(L) = (2 - [2]_q)^2 (2 - [2]_{\dot{q}})^2 \left[A(L) + B(L) \left(\frac{1}{[2]_q - 2} + \frac{1}{[2]_{\dot{q}} - 2} \right) \right] g^{4L}. \tag{5.7}$$

We first calculate $B(L)$ for any value of L. The weak-coupling expansion of the functions $a_2^{Q_1 Q_2}$ and $a_2^{Q_2 Q_1 *}$ are given by

$$\partial_{\tilde{p}_1} \log a_2^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) = O(g^2), \qquad \partial_{\tilde{p}_1} \log a_2^{Q_2 Q_1}(\tilde{p}_2, \tilde{p}_1)^* = \frac{2iQ_1}{\tilde{p}_1^2 + Q_1^2} + O(g^2). \tag{5.8}$$

We substitute these results into (4.98) and then into (4.59), we perform the integrals as in (5.4), and sum up the independent terms to obtain:

$$B(L) = -2^{1-4L} \binom{2L-2}{L-1} \binom{2L}{L} \zeta_{2L-1} \zeta_{2L-3}. \tag{5.9}$$

This gives the complete answer for the given $(2 - [2]_q)(2 - [2]_{\dot{q}})(4 - [2]_q - [2]_{\dot{q}})$ dependence of the double-wrapping correction at leading nonvanishing order for any L.

We now proceed to calculate $A(L)$. It acquires contributions from the first line of (4.98), which we denote by $A_{sl(2)}$, $A_{\mathcal{K}}$ and A_1, respectively,

$$A(L) = A_{sl(2)}(L) + A_{\mathcal{K}}(L) + A_1(L), \tag{5.10}$$

where

$$A_{sl(2)}(L) = \sum_{Q_1, Q_2} Q_1^2 Q_2^2 \int \frac{d\tilde{p}_1}{2\pi} e^{-L\epsilon_{Q_1}(\tilde{p}_1)} \int \frac{d\tilde{p}_2}{2\pi} e^{-L\epsilon_{Q_2}(\tilde{p}_2)} i\partial_{\tilde{p}_1} \log S_{sl(2)}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2), \tag{5.11}$$

$$A_{\mathcal{K}}(L) = 2 \sum_{Q_1, Q_2} Q_1 Q_2 \int \frac{d\tilde{p}_1}{2\pi} e^{-L\epsilon_{Q_1}(\tilde{p}_1)} \int \frac{d\tilde{p}_2}{2\pi} e^{-L\epsilon_{Q_2}(\tilde{p}_2)} i\partial_{\tilde{p}_1} \mathcal{K}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2), \tag{5.12}$$

$$A_1(L) = -2 \sum_{Q_1, Q_2} Q_1^2 Q_2^2 \int \frac{d\tilde{p}_1}{2\pi} e^{-L\epsilon_{Q_1}(\tilde{p}_1)} \int \frac{d\tilde{p}_2}{2\pi} e^{-L\epsilon_{Q_2}(\tilde{p}_2)} i\partial_{\tilde{p}_1} \log a_1^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2). \tag{5.13}$$

In order to compute A_1, we expand $a_1^{Q_1 Q_2}$ to leading order in g:

$$\partial_{\tilde{p}_1} \log a_1^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) = -\frac{iQ_1}{\tilde{p}_1^2 + Q_1^2} + O(g^2). \tag{5.14}$$

Substituting the result back into (5.13) gives

$$A_1(L) = -2^{1-4L} \binom{2L-2}{L-1} \binom{2L}{L} \zeta_{2L-2} \zeta_{2L-3}. \tag{5.15}$$

Observe that the transcendentality of $A(L)$ and $B(L)$ are different. It seems the deformation $2 - [2]$ carries transcendentality 1. A similar effect was observed already in [54, 38].

To calculate $A_{sl(2)}$, we have to expand the logarithm of the dressing factor $\log S_{sl(2)}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2)$ in the mirror-mirror kinematics. According to [13], it has the structure $\log S_{sl(2)}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) = -\log S_{su(2)}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) - 2\log \Sigma^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2)$. Hence, we can write

$$\frac{1}{2\pi i}\partial_{\tilde{p}_1} \log S_{sl(2)}^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) = -K_{Q_1 Q_2} - \frac{1}{\pi i}\partial_{\tilde{p}_1} \log \Sigma^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2). \tag{5.16}$$

Explicitly performing the weak-coupling expansion of (6.14) in [51], we obtain (see (B.4))

$$i\partial_{\tilde{p}_1} \log \Sigma^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) = -\frac{1}{2}\left[\psi(1 - \frac{i}{2}(\tilde{p}_1 + iQ_1)) - \psi(1 + \frac{1}{2}(i(\tilde{p}_1 - \tilde{p}_2) + Q_1 + Q_2)) + c.c\right], \tag{5.17}$$

where $\psi(x) = \partial_x(\log \Gamma(x))$ is the polygamma function. The $su(2)$ scalar factor results in

$$K_{su(2)}^{Q_1 Q_2} = K_{Q_1 Q_2} = -\frac{1}{4\pi}\left[\psi(\frac{1}{2}(i(\tilde{p}_2 - \tilde{p}_1) - Q_1 + Q_2)) + \psi(1 + \frac{1}{2}(i(\tilde{p}_2 - \tilde{p}_1) - Q_1 + Q_2)) \right. \tag{5.18}$$

$$\left. -\psi(\frac{1}{2}(i(\tilde{p}_2 - \tilde{p}_1) + Q_1 + Q_2)) - \psi(1 + \frac{1}{2}(i(\tilde{p}_2 - \tilde{p}_1) + Q_1 + Q_2)) + c.c\right].$$

Finally,

$$i\partial_{\tilde{p}_1}\mathcal{K}^{Q_1 Q_2} = -\frac{1}{8}\left[4(Q_1 - 1)Q_2 + ((Q_1 - Q_2)^2 + (\tilde{p}_1 - \tilde{p}_2)^2) \times \right. \tag{5.19}$$

$$\left. (\psi(1 + \frac{1}{2}(i(\tilde{p}_2 - \tilde{p}_1) - Q_1 + Q_2)) - \psi(\frac{1}{2}(i(\tilde{p}_2 - \tilde{p}_1) + Q_1 + Q_2))) + c.c.\right].$$

Denoting the contributions to $A_{sl(2)}$ by A_Σ and $A_{su(2)}$, we have that

$$A_{sl(2)}(L) = A_\Sigma(L) + A_{su(2)}(L), \tag{5.20}$$

where

$$A_\Sigma(L) = -2\sum_{Q_1, Q_2} Q_1^2 Q_2^2 \int \frac{d\tilde{p}_1}{2\pi}e^{-L\epsilon_{Q_1}(\tilde{p}_1)}\int \frac{d\tilde{p}_2}{2\pi}e^{-L\epsilon_{Q_2}(\tilde{p}_2)}i\partial_{\tilde{p}_1}\log \Sigma^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2), \tag{5.21}$$

$$A_{su(2)}(L) = \sum_{Q_1, Q_2} Q_1^2 Q_2^2 \int \frac{d\tilde{p}_1}{2\pi}e^{-L\epsilon_{Q_1}(\tilde{p}_1)}\int \frac{d\tilde{p}_2}{2\pi}e^{-L\epsilon_{Q_2}(\tilde{p}_2)}2\pi K_{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2). \tag{5.22}$$

Using methods explained in Appendix B, we evaluated the integrals by residues. To demonstrate the structure of the result, we write out explicitly A_Σ (see (B.10) and (B.13)):

$$A_\Sigma(L) = -2^{2-2L}\left(\begin{array}{c}2L-2\\L-1\end{array}\right)\zeta_{2L-3}\sum_{Q_1}\sum_{j=0}^{L-1}\left(\begin{array}{c}L+j-1\\j\end{array}\right)\frac{2^{-2L+1}}{(L-1-j)!}(-Q_1)^{2-L-j}\psi^{(L-j-1)}(Q_1 + 1)$$

$$-\sum_{Q_1, Q_2}\sum_{j_1, j_2=0}^{L-1}\left(\begin{array}{c}L+j_1-1\\j_1\end{array}\right)\frac{2^{-2L+2}}{(L-1-j_1)!}(-Q_1)^{2-L-j_1}$$

$$\times\left(\begin{array}{c}L+j_2-1\\j_2\end{array}\right)\frac{2^{-2L+1}}{(L-1-j_2)!}(-Q_2)^{2-L-j_2}\psi^{(2L-j_1-j_2-2)}(Q_1 + Q_2 + 1). \tag{5.23}$$

These terms can be expressed in terms of multiple zeta values (MZV) by rewriting[9]

$$\psi^{(n)}(Q + 1) = (-1)^{n+1}n!(\zeta(n+1) - \sum_{j=1}^{Q}j^{-n-1}), \tag{5.24}$$

---

[9]For $n = 0$, one has to replace $\zeta(1)$ with γ_E.

and performing the sums explicitly. We will, however, not pursue this calculation further here as we did not find an explicit answer for generic L. The integrals can be evaluated similarly for $A_{su(2)}$ and $A_{\mathcal{K}}$ with a similar structural final result, although some care must be taken to the $Q_1 - Q_2$ dependent term for $Q_1 = Q_2$. In the next subsection, we present explicit results for the smallest nontrivial length: $L = 3$.

5.3 Specific calculations for $L = 3$

The LO wrapping correction (5.5) for $L = 2$ is divergent, as we have for $j = 0$ the term $\zeta_{2L-3} = \zeta_1$. Similar observations were made in [55, 38]. We therefore focus now on the first nontrivial case, namely $L = 3$. The LO correction for this case goes as follows:

$$E_0^{(1)}(3) = -(2 - [2]_q)(2 - [2]_{\dot{q}}) \left(\frac{3}{16}\zeta_3 g^6 - \frac{15}{16}\zeta_5 g^8 + \frac{945}{256}\zeta_7 g^{10} - \frac{3465}{256}\zeta_9 g^{12} + \dots \right). \tag{5.25}$$

The simple double-wrapping contribution (5.6) at leading order is

$$E_0^{(2,1)}(3) = (2 - [2]_q)^2(2 - [2]_{\dot{q}})^2 \frac{63}{1024}\zeta_7 g^{12}. \tag{5.26}$$

In calculating the term $E_0^{(2,2)}(3)$, we recall from (5.7) that

$$E_0^{(2,2)}(3) = (2 - [2]_q)^2(2 - [2]_{\dot{q}})^2 \left[A(3) + B(3) \left(\frac{1}{[2]_q - 2} + \frac{1}{[2]_{\dot{q}} - 2} \right) \right] g^{12}. \tag{5.27}$$

From (5.9), we have

$$B(3) = -\frac{15}{256}\zeta_3\zeta_5. \tag{5.28}$$

We calculated the contributions to $A(3)$ one by one. The simplest is

$$A_1(3) = -\frac{15}{256}\zeta_3\zeta_4, \tag{5.29}$$

as follows from (5.15). In the more complicated terms, we calculated the integrals by residues as explained in Appendix B. Then, in summing up the expressions, we employed the following strategies:

- We performed the sums analytically by replacing the polygamma functions with harmonic sums using (5.24), and then rearranging all the sums into MZVs. These MZVs could then be expressed in terms of elementary ones, which contained only products of simple zetas with transcendentality less than or equal to 7.

- Alternatively, for terms involving polygamma functions depending on $Q_1 + Q_2$, we replaced the polygamma functions with their integral representations

$$\psi^{(n)}(z) = \int_0^\infty \left(\delta_{n,0} \frac{e^{-t}}{t} - (-1)^n \frac{t^n e^{-tz}}{1 - e^{-t}} \right) dt, \tag{5.30}$$

and performed the summations $\sum_{Q_1,Q_2=1}^\infty$ explicitly. The remaining integral over t could be evaluated numerically with very high precision (100 digits), and the result could be expressed in terms of products of zeta functions (and the Euler constant γ_E) with the help of the online MZV calculator, EZ-Face. [10]

- Finally, for polygamma functions depending on $Q_1 - Q_2$, we evaluated the sums numerically as $\sum_{Q_1,Q_2=1}^\infty = 2\sum_{Q_1}^\infty \sum_{Q_2=1}^{Q_1-1} + \sum_{(Q_1=Q_2)=1}^\infty$, and again expressed the result in terms of zeta functions using EZ-Face.

---

[10]EZ-Face is documented in [56], and can be accessed at `http://oldweb.cecm.sfu.ca/projects/EZFace/index.html`

We found the following results:

$$A_\Sigma(3) = \frac{81}{1024}\zeta_3\zeta_4 + \frac{21}{512}\zeta_2\zeta_5 - \frac{441}{2048}\zeta_7,$$

$$A_{su(2)}(3) = -\frac{9}{512}\zeta_3\zeta_4 + \frac{315}{4096}\zeta_7,$$

$$A_\mathcal{K}(3) = -\frac{9}{256}\zeta_3^2 - \frac{3}{1024}\zeta_3\zeta_4 - \frac{21}{512}\zeta_2\zeta_5 + \frac{63}{512}\zeta_7. \tag{5.31}$$

By summing up, we obtain the total A contribution

$$A(3) = A_\Sigma(3) + A_{su(2)}(3) + A_\mathcal{K}(3) + A_1(3) = -\frac{9}{256}\zeta_3^2 - \frac{63}{4096}\zeta_7. \tag{5.32}$$

Thus, the total anomalous dimension is

$$\begin{aligned}
E_0(3) &= E_0^{(1)}(3) + E_0^{(2,1)}(3) + E_0^{(2,2)}(3) + \ldots \\
&= -(2 - [2]_q)(2 - [2]_{\dot{q}})\left(\frac{3}{16}\zeta_3 g^6 - \frac{15}{16}\zeta_5 g^8 + \frac{945}{256}\zeta_7 g^{10} - \frac{3465}{256}\zeta_9 g^{12} + \ldots\right) \\
&\quad -(2 - [2]_q)(2 - [2]_{\dot{q}})\left([2]_q + [2]_{\dot{q}} - 4\right)\frac{15}{256}\zeta_3\zeta_5 g^{12} + \ldots \\
&\quad +(2 - [2]_q)^2(2 - [2]_{\dot{q}})^2\left(-\frac{9}{256}\zeta_3^2 + \frac{189}{4096}\zeta_7\right)g^{12} + \ldots, \tag{5.33}
\end{aligned}$$

where we recall that $2 - [2]_q = 4\sin(\frac{\gamma_+}{2})^2$ and $2 - [2]_{\dot{q}} = 4\sin(\frac{\gamma_-}{2})^2$ in terms of the deformation parameters $\gamma_\pm = (\gamma_3 \pm \gamma_2)\frac{3}{2}$, as in our case $L = 3$.

The result (5.33) is indeed the total anomalous dimension, since the vacuum energy does not receive any contributions from the asymptotic Bethe ansatz. Remarkably, even though at intermediate stages of the computation there appear terms involving even zeta functions and Euler's constant γ_E, all such terms finally cancel.

6   Conclusion

We have computed the NLO finite-volume correction to the vacuum energy in twisted AdS/CFT by two apparently independent approaches: Lüscher (4.32) and TBA (4.99). The fact that both approaches yield identical results provides a strong consistency check on the AdS/CFT S-matrices and TBA equations that have been developed in the literature, as well as on the final result. This result is expressed in terms of a double infinite sum of contributions from the infinitely-many types of massive mirror bound states. Our computations check the complete (both horizontal and vertical parts of the) Y-system, and go beyond the five-loop calculations presented in [17, 57, 58], which checked at the single wrapping order only the vertical part.

Our result is valid for any value of the coupling constant. However, by making a weak-coupling expansion, we have obtained a prediction (5.33) for the anomalous dimension of the operator $\text{Tr}Z^3$ in the twisted gauge theory up to six loops. It should be possible to check this prediction directly in perturbation theory by taking into account both single-wrapping and double-wrapping diagrams. To our knowledge, this is the first complete computation of double wrapping in the literature. It may be interesting to investigate also the strong-coupling limit.

The key results needed for the NLO Lüscher computation were the determinants of the (untwisted) AdS/CFT S-matrices in all the $su(2)_L \otimes su(2)_R$ sectors, presented in Tables 1 and 2. The simplicity of these results suggests that they may have some group-theoretical formulation. In particular, it should be possible to find a general proof, presumably based on $su(2|2)$ Yangian symmetry.

It would be interesting to extend our analysis of finite-size corrections in twisted AdS/CFT, which has so far been restricted to the ground state, to excited states beyond the LO result of [38]. It would

also be interesting to understand the origin of the divergence of the LO and NLO results for $L = 2$, which was already noticed in similar contexts in [55, 38]. Finally, one can now begin to contemplate triple and higher wrapping.

Acknowledgments

A   Determinants of S-matrices in the $su(2)_L \otimes su(2)_R$ sectors

We describe here how we obtained the results in Tables 1 and 2 for $\det S^{(Q_1,Q_2)}(s_L, s_R)$, the determinants of the AdS/CFT S-matrices in the $su(2)_L \otimes su(2)_R$ sectors, which enter into the NLO Lüscher computation. Our straightforward approach was to explicitly compute these determinants for small values of Q_1 and Q_2 (up to 8), and then infer the general pattern.

For the cases $(Q_1, Q_2) = (1, 1), (1, 2), (2, 2)$, we used the explicit S-matrices from [52] to directly compute the eigenvalues. For the cases $(Q_1, Q_2) = (1, Q)$, we used results from [59]: from Eq. (56) there, it follows that (up to the overall factors),

$$
\begin{aligned}
\det S^{(1,Q)}(\frac{Q-1}{2}, 1) &= a_9^9\,, \\
\det S^{(1,Q)}(\frac{Q}{2}, \frac{1}{2}) &= \frac{1}{Q} \det \begin{pmatrix} a_5^5 & a_5^6 \\ a_6^5 & a_6^6 \end{pmatrix}, \\
\det S^{(1,Q)}(\frac{Q-2}{2}, \frac{1}{2}) &= Q \det \begin{pmatrix} a_7^7 & a_7^8 \\ a_8^7 & a_8^8 \end{pmatrix}, \\
\det S^{(1,Q)}(\frac{Q+1}{2}, 0) &= a_1^1 = 1\,, \\
\det S^{(1,Q)}(\frac{Q-1}{2}, 0) &= \frac{Q+1}{Q-1} \det \begin{pmatrix} a_2^2 & a_2^3 & a_2^4 \\ a_3^2 & a_3^3 & a_3^4 \\ a_4^2 & a_4^3 & a_4^4 \end{pmatrix}, \\
\det S^{(1,Q)}(\frac{Q-3}{2}, 0) &= \frac{Q-1}{2} a_{10}^{10}\,.
\end{aligned}
\tag{A.1}
$$

One can verify using the explicit values of a_i^j that [11]

$$
\begin{aligned}
a_9^9 &= \frac{1}{Q} \det \begin{pmatrix} a_5^5 & a_5^6 \\ a_6^5 & a_6^6 \end{pmatrix} = U_0 U_1 U_2\,, \\
Q \det \begin{pmatrix} a_7^7 & a_7^8 \\ a_8^7 & a_8^8 \end{pmatrix} &= \frac{Q+1}{Q-1} \det \begin{pmatrix} a_2^2 & a_2^3 & a_2^4 \\ a_3^2 & a_3^3 & a_3^4 \\ a_4^2 & a_4^3 & a_4^4 \end{pmatrix} = U_0^2 U_1 U_2^3 U_3\,, \\
\frac{Q-1}{2} a_{10}^{10} &= U_0 U_2^2 U_3\,,
\end{aligned}
\tag{A.2}
$$

---

[11] We note a couple of typos in appendix B of [59]: a_3^2 should not have the factor $x^+ z^+$ in the denominator; and a_3^3 is missing an overall minus sign.

where the notation is defined in (4.29). It is then easy to see that the expressions in Table 2 are consistent with the results (A.1), (A.2).

For general values of (Q_1, Q_2), we made use of the formalism developed in [49]. As an example, let us consider the case $(Q_1, Q_2) = (2, 3)$. Since the state of a single Q-particle (the $4Q$-dimensional totally symmetric representation of $su(2|2)$) has the $su(2)_L \otimes su(2)_R$ decomposition

$$V^{\frac{Q}{2}} \times V^0 + V^{\frac{Q-1}{2}} \times V^{\frac{1}{2}} + V^{\frac{Q-2}{2}} \times V^0 \,, \tag{A.3}$$

the decomposition of the corresponding 2-particle states can be obtained from the tensor product

$$\left(V^1 \times V^0 + V^{\frac{1}{2}} \times V^{\frac{1}{2}} + V^0 \times V^0 \right) \otimes \left(V^{\frac{3}{2}} \times V^0 + V^1 \times V^{\frac{1}{2}} + V^{\frac{1}{2}} \times V^0 \right) \,, \tag{A.4}$$

where in this appendix we denote by $\times$ the tensor product of the $su(2)_L$ and $su(2)_R$ representations. For concreteness, let us focus on the computation of $\det S^{(2,3)}(1, \frac{1}{2})$. The tensor product in (A.4) can be decomposed, by the Clebsch-Gordan theorem, into a sum of irreducible representations of $su(2)_L \otimes su(2)_R$. In this decomposition, there appear four representations with $(s_L, s_R) = (1, \frac{1}{2})$, which are the relevant ones for computing this determinant. These four representations come from the following channels:

$$
\begin{aligned}
1: && \left(V^1 \times V^0 \right) \otimes \left(V^1 \times V^{\frac{1}{2}} \right) \\
2: && \left(V^{\frac{1}{2}} \times V^{\frac{1}{2}} \right) \otimes \left(V^{\frac{1}{2}} \times V^0 \right) \\
3: && \left(V^0 \times V^0 \right) \otimes \left(V^1 \times V^{\frac{1}{2}} \right) \\
4: && \left(V^{\frac{1}{2}} \times V^{\frac{1}{2}} \right) \otimes \left(V^{\frac{3}{2}} \times V^0 \right) \,.
\end{aligned}
\tag{A.5}
$$

The corresponding highest-weight states $|\psi_I^{(Q_1, Q_2)}(s_L, s_R)\rangle$ with $s_L = m_L = 1$ and $s_R = m_R = \frac{1}{2}$ are given (up to an overall normalization factor) by

$$
\begin{aligned}
|\psi_1^{(2,3)}(1, \tfrac{1}{2})\rangle & \propto & |0,1\rangle_2^{\mathrm{II}} - |1,0\rangle_2^{\mathrm{II}} \,, \\
|\psi_2^{(2,3)}(1, \tfrac{1}{2})\rangle & \propto & |0,1\rangle_3^{\mathrm{II}} \,, \\
|\psi_3^{(2,3)}(1, \tfrac{1}{2})\rangle & \propto & |1,0\rangle_4^{\mathrm{II}} \,, \\
|\psi_4^{(2,3)}(1, \tfrac{1}{2})\rangle & \propto & |0,1\rangle_1^{\mathrm{II}} - |1,0\rangle_1^{\mathrm{II}} \,,
\end{aligned}
\tag{A.6}
$$

respectively, where the states $|k, l\rangle_i^{\mathrm{II}}$ are defined in [49]. It is convenient to introduce a basis $|e_i\rangle$ of these so-called type-II states with $N \equiv k + l = 1$:

$$
\begin{aligned}
|e_1\rangle &=& |0,1\rangle_1^{\mathrm{II}} \,, \\
|e_2\rangle &=& |0,1\rangle_2^{\mathrm{II}} \,, \\
|e_3\rangle &=& |0,1\rangle_3^{\mathrm{II}} \,, \\
|e_4\rangle &=& |1,0\rangle_1^{\mathrm{II}} \,, \\
|e_5\rangle &=& |1,0\rangle_2^{\mathrm{II}} \,, \\
|e_6\rangle &=& |1,0\rangle_4^{\mathrm{II}} \,.
\end{aligned}
\tag{A.7}
$$

Although these states are orthogonal, they are not normalized.[12] Indeed, defining

$$n_i \equiv \langle e_i | e_i \rangle \,, \tag{A.8}$$

---

[12]We are grateful to M. de Leeuw for pointing this out to us.

it readily follows from the definitions of the states [49] that here $n_i = (2, 2, 1, 6, 2, 2)$. An orthonormal basis $|\tilde{e}_i\rangle$ is therefore given by

$$|\tilde{e}_i\rangle \equiv \frac{1}{\sqrt{n_i}} |e_i\rangle , \qquad \langle \tilde{e}_i | \tilde{e}_j \rangle = \delta_{ij} . \tag{A.9}$$

The S-matrix acts as

$$S|e_i\rangle = \sum_j |e_j\rangle U_{ji} . \tag{A.10}$$

Numerical values for the coefficients U_{ji} can be computed using formulas in [49], for given numerical values of momenta p_1, p_2, coupling constant g, and representations Q_1, Q_2.[13] Hence, we can obtain the corresponding coefficients $\tilde{U}_{ji}$ in the normalized basis

$$\begin{aligned} \tilde{U}_{ji} &\equiv \langle \tilde{e}_j | S | \tilde{e}_i \rangle \\ &= \sqrt{\frac{n_j}{n_i}} U_{ji} = \sum_{k,l} M_{jk} U_{kl} M_{li}^{-1} , \end{aligned} \tag{A.11}$$

where we have introduced the diagonal matrix $M_{ij} \equiv \sqrt{n_i} \delta_{ij}$. A useful check is that the matrix $\tilde{U}_{ji}$ (unlike U_{ji}) is unitary.

We use (A.6) to express the highest-weight states $|\psi_I^{(Q_1,Q_2)}(s_L, s_R)\rangle$ in terms of the normalized basis

$$|\psi_I^{(Q_1,Q_2)}(s_L, s_R)\rangle = \sum_i |\tilde{e}_i\rangle c_{iI} , \qquad c_{iI} \equiv \langle \tilde{e}_i | \psi_I^{(Q_1,Q_2)}(s_L, s_R) \rangle , \tag{A.12}$$

where the states themselves are normalized,

$$\langle \psi_I^{(Q_1,Q_2)}(s_L, s_R) | \psi_J^{(Q_1,Q_2)}(s_L, s_R) \rangle = \delta_{IJ} . \tag{A.13}$$

We can finally construct the S-matrix in the (s_L, s_R) sector,

$$S_{IJ}^{(Q_1,Q_2)}(s_L, s_R) \equiv \langle \psi_I^{(Q_1,Q_2)}(s_L, s_R) | S | \psi_J^{(Q_1,Q_2)}(s_L, s_R) \rangle = \sum_{i,j} c_{iI}^* \tilde{U}_{ij} c_{jJ} . \tag{A.14}$$

Another useful check is that the matrix $S_{IJ}^{(Q_1,Q_2)}(s_L, s_R)$ is also unitary. Computing numerically the determinant of this matrix [14]

$$\det S^{(Q_1,Q_2)}(s_L, s_R) \equiv \det \left(S_{IJ}^{(Q_1,Q_2)}(s_L, s_R) \right) , \tag{A.15}$$

we find for the case in question (namely, $(Q_1, Q_2) = (2, 3)$ and $(s_L, s_R) = (1, \frac{1}{2})$) that the result coincides with $(U_0 U_1 U_2)^4 S_3^2$, in agreement with Table 1. Other cases (Q_1, Q_2) and other sectors (s_L, s_R) can be treated in a similar way. Note that sectors with $s_R = 1, \frac{1}{2}, 0$ are constructed with states of type I, II, III, respectively. After some effort to accumulate results for sufficiently many cases, the general pattern summarized in Tables 1 and 2 became evident.

Before closing this section, it may be worthwhile to frame the problem that we have addressed here in a more general context. Consider an S-matrix (solution of the Yang-Baxter equation) that is invariant under a group G, which here is $su(2)_L \otimes su(2)_R$. As is well known (see e.g. [60, 61]),

---

[13] We note that version 1 in the arXiv of [49] contains a number of typos, most of which are corrected in the journal. However, some typos remain in the latter. In particular, in (5.14): $\bar{Q}_{ij} = b_i d_j - b_j d_i$. Also, in A^{-1} in (5.17): in the (2,2) element of the big matrix, c_1^- should be instead c_1^+ ; and in the (2,1) matrix element, the sign in front of $[M + (l_1 - l_2)/2]$ should be plus instead of minus. Finally, in (A.8), the formulas for $b_1, \ldots, b_4$ should have sign plus instead of minus; and the formulas for d_2 and d_3 should not have i in the denominator. We are grateful to G. Arutyunov and M. de Leeuw for correspondence on these points.

[14] We use the convention that the determinant of a number (i.e., a 1×1 matrix) is the number itself.

the S-matrix is a matrix S_{ab} defined in the tensor product of two vector spaces $V_a \otimes V_b$ in which representations Π_a and Π_b of G act, [15]

$$[\Pi_a(g) \otimes \Pi_b(g), S_{ab}] = 0, \qquad g \in G. \tag{A.16}$$

The representation space decomposes into a sum of irreducible representations of G parameterized by highest weights Λ_k, which here are (s_L, s_R),

$$V_a \otimes V_b = \sum_k V(\Lambda_k). \tag{A.17}$$

Since the S-matrix is G-invariant (A.16), it has the corresponding spectral resolution

$$S_{ab} = \sum_k \rho_k P_{\Lambda_k}, \tag{A.18}$$

where P_{Λ_k} is a projector onto the irreducible subspace $V(\Lambda_k)$.

In the seminal work [60] on the construction of rational S-matrices, it was essential to assume that the Clebsch-Gordan series (A.17) is multiplicity free (i.e., a given irreducible representation appears at most once), in which case ρ_k in (A.18) is a scalar. For AdS/CFT, the decomposition (A.17) is unfortunately *not* multiplicity free: the Clebsch-Gordan series contains multiple irreducible representations, as we have seen in the example (A.5). Hence, ρ_k becomes an $r \times r$ matrix, where r is the multiplicity of the corresponding irreducible representation with highest weight Λ_k. In the AdS/CFT case, ρ_k is the matrix that we have defined in (A.14). The problem of explicitly determining this matrix can be quite complicated even for rational S-matrices, see e.g. [61]. In the present work, we have restricted to the problem of computing its determinant.

B Details of the weak-coupling expansion

B.1 Weak coupling expansion of the dressing phase

The dressing phase in the mirror-mirror kinematics is given by [51]

$$
\begin{aligned}
-i \log \Sigma_{Q_1 Q_2}(y_1, y_2) &= \Phi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-) - \Phi(y_1^-, y_2^+) + \Phi(y_1^-, y_2^-) \\
&\quad + \frac{1}{2} \left[-\Psi(y_1^+, y_2^+) + \Psi(y_1^+, y_2^-) - \Psi(y_1^-, y_2^+) + \Psi(y_1^-, y_2^-) \right] \\
&\quad - \frac{1}{2} \left[-\Psi(y_2^+, y_1^+) + \Psi(y_2^+, y_1^-) - \Psi(y_2^-, y_1^+) + \Psi(y_2^-, y_1^-) \right] \\
&\quad + \frac{1}{i} \log \left[\frac{i^{Q_1} \Gamma(Q_2 - \frac{i}{2} g(y_1^+ + \frac{1}{y_1^+} - y_2^+ - \frac{1}{y_2^+}))}{i^{Q_2} \Gamma(Q_1 + \frac{i}{2} g(y_1^+ + \frac{1}{y_1^+} - y_2^+ - \frac{1}{y_2^+}))} \left(\frac{1 - \frac{1}{y_1^+ y_2^-}}{1 - \frac{1}{y_1^- y_2^+}} \right) \sqrt{\frac{y_1^+ y_2^-}{y_1^- y_2^+}} \right],
\end{aligned}
\tag{B.1}
$$

where

$$\Psi(x_1, x_2) = i \oint_{C_1} \frac{dw_2}{2\pi i} \frac{1}{w_2 - x_2} \log \frac{\Gamma(1 + i\frac{g}{2}(x_1 + x_1^{-1} - w_2 - w_2^{-1}))}{\Gamma(1 - i\frac{g}{2}(x_1 + x_1^{-1} - w_2 - w_2^{-1}))}, \tag{B.2}$$

and for $\Phi(x_1, x_2)$ we just note that it starts in any kinematics at least with g^2. We calculate the O(1) expansion of the phase (B.2). Using the property $\Psi(x_1, x_2) = \Psi(x_1, 0) - \Psi(x_1, x_2^{-1})$, being valid if $|x_2| \neq 1$, and that for $|x_2| > 1$ it starts at g^2, it is easy to see that we need to calculate $\Psi(x_1, x_2) \equiv \Psi(x_1, 0)$ for $|x_2| < 1$, i.e. for x_2^+. Since we are interested in the derivative of the expanded

---

[15]The representations Π_a and Π_b need not be irreducible representations of G. Indeed, in the AdS/CFT case, they are sums of irreducible representations, as in (A.3).

functions with respect to the first argument ∂_1, we need to expand $-\frac{1}{2}(\Psi(y_1^+, 0) + \Psi(y_1^-, 0))$ only. Rescaling the integration variable w_2 by g and evaluating the leading residue for small g, we obtain

$$\Psi(y_1^+, 0) = i \log \frac{\Gamma(1 + \frac{i}{2}(\tilde{p}_1 + iQ_1))}{\Gamma(1 - \frac{i}{2}(\tilde{p}_1 + iQ_1))} + \dots , \qquad \Psi(y_1^-, 0) = i \log \frac{\Gamma(1 + \frac{i}{2}(\tilde{p}_1 - iQ_1))}{\Gamma(1 - \frac{i}{2}(\tilde{p}_1 - iQ_1))} + \dots \qquad (B.3)$$

The logarithmic derivative of the whole dressing phase is then

$$-\frac{1}{\pi i} \partial_{\tilde{p}_1} \log \Sigma^{Q_1 Q_2}(\tilde{p}_1, \tilde{p}_2) = \frac{1}{2\pi} \left[-\psi(1 - \frac{i}{2}(\tilde{p}_1 + iQ_1)) + \psi(1 + \frac{1}{2}(i(\tilde{p}_1 - \tilde{p}_2) + Q_1 + Q_2)) + c.c \right], \quad (B.4)$$

where c.c. denotes complex conjugate, and we used that $\psi(-\frac{i}{2}(\tilde{p} - iQ)) + c.c = \psi(1 - \frac{i}{2}(\tilde{p} + iQ)) + c.c$ for integer Q.

B.2   Performing the integrals by residues

We demonstrate here how we performed the integrals by evaluating A_Σ (5.21). In view of the result (B.4), we start by evaluating the term with $\psi(1 - \frac{i}{2}(\tilde{p}_1 + iQ_1)) + c.c.$. Its contribution factorizes for the indices $1, 2$ into a product of two factors. The more complicated factor is

$$\sum_{Q_1} Q_1^2 \int_{-\infty}^{\infty} \frac{d\tilde{p}_1}{2\pi} \frac{1}{(\tilde{p}_1^2 + Q_1^2)^L} \left[\psi(1 - \frac{i}{2}(\tilde{p}_1 + iQ_1)) + \psi(1 + \frac{i}{2}(\tilde{p}_1 - iQ_1)) \right]. \qquad (B.5)$$

Let us analyze the pole structure of the integrand. Additionally to the two "kinematical" poles at $\tilde{p} = \pm iQ$, the polygamma function has poles for $\psi(-n)$ if $n \geq 0$. These poles are located on the lower half plane (LHP) for the first and on the upper half plane (UHP) for second polygamma function:

$$\frac{1}{2}(Q_1 + 2 \mp i\tilde{p}_1) = -n \quad \longrightarrow \quad \tilde{p}_1 = \mp i(2(n+1) + Q_1). \qquad (B.6)$$

We now use the trick in [62] of exploiting the reality of the integrand to rewrite the integral as

$$2 \Re e \sum_{Q_1} Q_1^2 \int_{-\infty}^{\infty} \frac{d\tilde{p}_1}{2\pi} \frac{1}{(\tilde{p}_1^2 + Q_1^2)^L} \left[\psi(1 - \frac{i}{2}(\tilde{p}_1 + iQ_1)) \right], \qquad (B.7)$$

and close the contour on the UHP. In so doing, we have to pick up the residue at $\tilde{p}_1 = iQ_1$ only:

$$2i \sum_{Q_1} Q_1^2 \frac{\partial_{\tilde{p}_1}^{L-1}}{(L-1)!} \frac{\psi(1 - \frac{i}{2}(\tilde{p}_1 + iQ_1))}{(\tilde{p}_1 + iQ_1)^L} \Big|_{\tilde{p}_1 = iQ_1}$$

$$= -\sum_{Q_1} \sum_{j=0}^{L-1} \binom{L+j-1}{j} \frac{2^{-2L+2}}{(L-1-j)!} (-Q_1)^{2-L-j} \psi^{(L-j-1)}(Q_1 + 1). \qquad (B.8)$$

We now note that the Q_2-dependent terms give

$$\sum_{Q_2=1}^{\infty} Q_2^2 \int_{-\infty}^{\infty} \frac{d\tilde{p}_2}{2\pi} \frac{1}{(\tilde{p}_2^2 + Q_2^2)^L} = \sum_{Q_2=1}^{\infty} \binom{2L-2}{L-1} Q_2^{3-2L} 2^{1-2L} = 2^{1-2L} \binom{2L-2}{L-1} \zeta_{2L-3}. \qquad (B.9)$$

Hence, the factorizing contribution to A_Σ, which we denote by $A_\Sigma^{(1)}$, is given by

$$A_\Sigma^{(1)} = -2^{1-2L} \binom{2L-2}{L-1} \zeta_{2L-3} \sum_{Q_1} \sum_{j=0}^{L-1} \binom{L+j-1}{j} \frac{2^{-2L+2}}{(L-1-j)!} (-Q_1)^{2-L-j} \psi^{(L-j-1)}(Q_1 + 1). \tag*{(B.10)}$$

Let us concentrate now on the nonfactorizing contributions, which we denote by $A_\Sigma^{(2)}$. Using again the reality trick, we can write

$$A_\Sigma^{(2)} = -2 \sum_{Q_1, Q_2} Q_1^2 Q_2^2 \int \frac{d\tilde{p}_1}{2\pi} \int \frac{d\tilde{p}_2}{2\pi} \frac{1}{(\tilde{p}_2^2 + Q_2^2)^L (\tilde{p}_1^2 + Q_1^2)^L} \psi\left(1 + \frac{1}{2}(Q_1 + Q_2 - i(\tilde{p}_2 - \tilde{p}_1))\right),$$
(B.11)

and close the $\tilde{p}_2$ integration contour on the UHP. By picking up the only residue at $\tilde{p}_2 = iQ_2$, the result is

$$-2i \sum_{Q_2} Q_2^2 \frac{\partial_{\tilde{p}_2}^{L-1}}{(L-1)!} \frac{\psi(1 + \frac{1}{2}(Q_1 + Q_2 - i(\tilde{p}_2 - \tilde{p}_1)))}{(\tilde{p}_2 + iQ_2)^L}\Big|_{\tilde{p}_2 = iQ_2}$$

$$= \sum_{Q_2} \sum_{j=0}^{L-1} \binom{L+j-1}{j} \frac{2^{-2L+2}}{(L-1-j)!} (-Q_2)^{2-L-j} \psi^{(L-j-1)}(Q_2 + 1 + \frac{1}{2}(Q_1 + i\tilde{p}_1)).$$
(B.12)

The next integral we close on the lower half plane and pick up the residue at $-iQ_1$:

$$A_\Sigma^{(2)} = - \sum_{Q_1, Q_2} \sum_{j_1, j_2 = 0}^{L-1} \binom{L + j_1 - 1}{j_1} \frac{2^{-2L+2}}{(L-1-j_1)!} (-Q_1)^{2-L-j_1}$$

$$\times \binom{L + j_2 - 1}{j_2} \frac{2^{-2L+1}}{(L-1-j_2)!} (-Q_2)^{2-L-j_2} \psi^{(2L-j_1-j_2-2)}(Q_1 + Q_2 + 1). \quad \text{(B.13)}$$

Adding the two terms $A_\Sigma = A_\Sigma^{(1)} + A_\Sigma^{(2)}$ gives the result we presented in (5.23).

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