The Kohn mode for trapped Bose gases within the dielectric formalism

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The presence of undamped harmonic center of mass oscillations of a weakly interacting Bose gas in a harmonic trap is demonstrated within the dielectric formalism for a previously introduced finite temperature approximation including exchange. The consistency of the approximation with the Kohn theorem is thereby demonstrated. The Kohn modes are found explicitly, generalizing an earlier zero-temperature result found in the literature. It is shown how the Kohn mode disappears from the single-particle spectrum, while remaining in the density oscillation spectrum, when the temperature increases from below to above the condensation temperature.

I. INTRODUCTION

In 1961 *W. Kohn* [1] first noticed that the electron-electron interaction does not change the cyclotron resonance frequency in a bulk three dimensional gas. *Brey et al.* [2] and *Dobson* [3] expanded the proof as a *generalized Kohn theorem* for parabolic quantum wells. Independent of the interactions the bare harmonic-oscillator frequency is preserved in the excitation spectra. These oscillations describe the center of mass movements along the main axes of the trap. They are used for calibration in the experiments with trapped Bose gases [4].

Many different model approximations were derived since the discovery of Bose condensation in trapped dilute Bose gases. For them the satisfaction of the Kohn theorem provides an important consistency check. That the Kohn theorem is satisfied within the zero temperature Bogoliubov equations for a trapped dilute Bose gas was proven by *Fetter and Rokhsar* [5]. The Kohn theorem was demonstrated by *Griffin and coworkers* for various approximations leading to hydrodynamic equations [6]. *Stoof and coworkers* checked the presence of the Kohn modes in an approximate variational approach to the kinetic theory also in the collisionless regime [7]. A finite temperature theory based on a linear response formalism was given by *Minguzzi and Tosi* [8] and the validity of the Kohn theorem in this approach was mentioned in their paper, but without proof.

The purpose of the present paper is to examine the validity of the Kohn theorem in a specific finite temperature approximation within the dielectric formalism. It was introduced and studied in a previous paper [9]. It was shown there that its results agree with those results which can also be calculated in the *Minguzzi and Tosi* approach [8], but that it gives additional results, in particular for the 1-particle Green's functions.

The starting point is the closed set of eigenvalue equations for the single particle Green's functions and the density-density correlation functions. We first identify the eigenmodes for the center of mass movement. Then we show that they are eigenfunctions of the density autocorrelation function at the bare trap frequency.

We furthermore prove the validity of the Kohn theorem in the single particle spectra in the Bose condensed system and show how the Kohn modes disappear from the single-particle spectrum, but remain in the density autocorrelation spectrum if the number of particles in the condensate goes to zero.

II. EIGENVALUE EQUATIONS FOR THE DENSITY AND SINGLE PARTICLE AUTOCORRELATION FUNCTIONS

In this section we consider the eigenvalue equations for the single particle excitations and the density autocorrelation function above and below the critical temperature for Bose condensation T_c. The eigenfrequencies ω_k and the eigenmodes $\delta n_{\mathrm{tot},k}(\boldsymbol{r})$ of the density correlation function $\chi(\boldsymbol{r}, \tau; \boldsymbol{r}', \tau') = -\langle T_\tau [\tilde{n}(\boldsymbol{r}, \tau) \tilde{n}(\boldsymbol{r}', \tau')]\rangle$, where $\tilde{n}(\boldsymbol{r}) = \hat{n}(\boldsymbol{r}) - \langle \hat{n}(\boldsymbol{r})\rangle$, are given by

$$\int d^3 \boldsymbol{r}' \chi^{-1}(\boldsymbol{r}, \boldsymbol{r}', \omega_k) \delta n_{\mathrm{tot},k}(\boldsymbol{r}') = 0 \,. \qquad (1)$$

It is useful to express χ in terms of its interaction-irreducible part $\tilde{\chi}$ via

$$\chi(\boldsymbol{r}, \boldsymbol{r}', \omega)$$
$$= \tilde{\chi}(\boldsymbol{r}, \boldsymbol{r}', \omega) + \frac{g}{\hbar} \int d^3 \boldsymbol{r}_1 \, \tilde{\chi}(\boldsymbol{r}, \boldsymbol{r}_1, \omega) \chi(\boldsymbol{r}_1, \boldsymbol{r}', \omega) \,. \quad (2)$$

Here and in the following the *tilde*-sign denotes *interaction-irreducible* quantities, which are given by diagrams which cannot be split into two by cutting a single interaction line.

Then Eq. (1) takes the form

$$\delta n_{\text{tot},k}(\boldsymbol{r}) = \frac{g}{\hbar} \int d^3 \boldsymbol{r}' \tilde{\chi}(\boldsymbol{r}, \boldsymbol{r}', \omega_k) \delta n_{\text{tot},k}(\boldsymbol{r}') \,. \qquad (3)$$

Besides we consider the matrix of Green's functions

$$G_{\alpha,\beta}(\boldsymbol{r}, \tau; \boldsymbol{r}', \tau') = -\langle T_\tau \left[\hat{\varphi}_\alpha(\boldsymbol{r}, \tau) \hat{\varphi}_\beta^\dagger(\boldsymbol{r}', \tau') \right] \rangle \qquad (4)$$

with field operators in Matsubara representation $\hat{\varphi}_1(\boldsymbol{r}, \tau) \equiv \hat{\varphi}(\boldsymbol{r}, \tau)$ and $\hat{\varphi}_2(\boldsymbol{r}, \tau) \equiv \hat{\varphi}^\dagger(\boldsymbol{r}, \tau)$. The corresponding eigenfrequencies ω_k' and eigenfunctions $\varphi_{\alpha,k}$ are determined by

$$\int d^3 \boldsymbol{r}' G_{\alpha\beta}^{-1}(\boldsymbol{r}, \boldsymbol{r}', \omega_k') \varphi_{k,\beta}(\boldsymbol{r}') = 0 \,. \qquad (5)$$

Of course we need to make approximations for $\tilde{\chi}$ and $G_{\alpha\beta}$ and describe now the model we previously introduced within the framework of the dielectric formalism [9,10]. The approximations will be further detailed in the subsections II A and II B and the appendix where they will be used to derive the eigenvalue equations for the different excitation spectra. At this point it is sufficient to fix the approximations for the thermal density $n_{\text{T}}(\boldsymbol{r})$. The external trap potential $U(\boldsymbol{r})$ is

$$U(\boldsymbol{r}) = \frac{1}{2} m \sum_{i=1,2,3} \Omega_i^2 x_i^2 \,, \qquad (6)$$

where Ω_i are the main trap frequencies.
The effective interaction potential $v(\boldsymbol{r}, \boldsymbol{r}')$ has the form $v(\boldsymbol{r}, \boldsymbol{r}') = g\delta(\boldsymbol{r} - \boldsymbol{r}')$ with the constant interaction strength g. It is taken to contain already the complete series of ladder-diagrams describing repeated two-particle scattering, i.e. it is the 2-particle T-matrix.
The stationary thermal density $n_{\text{T}}(\boldsymbol{r})$ is calculated using as auxiliary quantity the stationary Hartree-Fock Greek's function G^{HF}

$$\left(\hbar\omega - \hat{H}^{HF}(\boldsymbol{r}) \right) G^{HF}(\boldsymbol{r}, \boldsymbol{r}', \omega) = \hbar\delta(\boldsymbol{r} - \boldsymbol{r}') \,. \qquad (7)$$

The Hartree-Fock Hamiltonian is

$$\hat{H}^{HF}(\boldsymbol{r}) = -\frac{\hbar^2}{2m} \Delta + U(\boldsymbol{r}) - \mu + 2g n_{\text{tot}}(\boldsymbol{r}) \,, \qquad (8)$$

with the chemical potential μ and the total density $n_{\text{tot}}(\boldsymbol{r}) = n_{\text{T}}(\boldsymbol{r}) + n_{\text{c}}(\boldsymbol{r})$, where n_{c} is the condensate density and n_{T} is

$$n_{\text{T}}(\boldsymbol{r}) = \frac{k_B T}{\hbar} \lim_{\eta \to 0} \sum_n e^{i\omega_n \eta} G^{HF}(\boldsymbol{r}, \boldsymbol{r}, \omega_n)$$
$$= \sum_k n(\varepsilon_k) |\varphi_k^{HF}(\boldsymbol{r})|^2 \,. \qquad (9)$$

ω_n are the Matsubara frequencies $2n\pi k_B T/\hbar$ and $n(\varepsilon_k)$ is the Bose factor to the eigenenergy ε_k for the eigenfunction $\varphi_k^{HF}(\boldsymbol{r})$ of the Hartree-Fock Hamiltonian. The latter will be taken as real, in the following, without restriction of generality because the Hamiltonian (8) is real. Within the model [9] all internal lines of diagrams are calculated with G^{HF}.

A. The uncondensed Bose gas

We first consider the uncondensed Bose gas. Here, the lowest order contribution to $\tilde{\chi}$ is the bubble diagram $\tilde{\chi}^0$ containing two Hartree-Fock Green's functions

$$\tilde{\chi}^0(\boldsymbol{r}, \boldsymbol{r}', \omega) =$$
$$-\frac{1}{\beta\hbar} \sum_n G^{HF}(\boldsymbol{r}, \boldsymbol{r}', i\omega_n) G^{HF}(\boldsymbol{r}', \boldsymbol{r}, i\omega_n - \omega) \qquad (10)$$
$$= \sum_{k,l} \varphi_k^{HF}(\boldsymbol{r}) \varphi_l^{HF}(\boldsymbol{r}) \frac{(n(\varepsilon_k) - n(\varepsilon_l))}{\omega + (\varepsilon_k - \varepsilon_l)/\hbar} \varphi_k^{HF}(\boldsymbol{r}') \varphi_l^{HF}(\boldsymbol{r}')$$
$$\overset{l \leftrightarrow k}{=} \frac{1}{2} \sum_{k,l} \varphi_k^{HF}(\boldsymbol{r}) \varphi_l^{HF}(\boldsymbol{r}) \left(n(\varepsilon_k) - n(\varepsilon_l) \right)$$
$$\times \left[\frac{1}{\omega + (\varepsilon_k - \varepsilon_l)/\hbar} \right.$$
$$\left. + \frac{1}{-\omega + (\varepsilon_k - \varepsilon_l)/\hbar} \right] \varphi_k^{HF}(\boldsymbol{r}') \varphi_l^{HF}(\boldsymbol{r}') \,. \qquad (11)$$

Here we first performed the Matsubara sum and then symmetrized the obtained expression with respect to $k \leftrightarrow l$.
$\tilde{\chi}$ is represented, within the model, by summing up the multiple particle-hole scattering processes including exchange with the result, in symbolic notation

$$\tilde{\chi}(\boldsymbol{r}, \boldsymbol{r}', \omega) =$$
$$\int d^3 \boldsymbol{r}_1 \left[1 - \frac{g}{\hbar} \tilde{\chi}^0(\omega) \right]^{-1} (\boldsymbol{r}, \boldsymbol{r}_1) \tilde{\chi}^0(\boldsymbol{r}_1, \boldsymbol{r}', \omega) \,. \qquad (12)$$

Inserting these diagrams $\tilde{\chi}$ in Eq. (3), multiplying from the left with the integral operator $\left[1 - (g/\hbar)\tilde{\chi}^0(\omega) \right]$ and combining similar terms, the density correlation spectra are given by

$$\delta n_{\text{tot},k}(\boldsymbol{r}) = 2\frac{g}{\hbar} \int d^3 \boldsymbol{r}' \tilde{\chi}^0(\boldsymbol{r}, \boldsymbol{r}', \omega_k) \delta n_{\text{tot},k}(\boldsymbol{r}') \,. \qquad (13)$$

Above T_c the single particle Green's function in our model is just the Hartree-Fock approximation. Of course the corresponding eigenfrequencies $\omega_k' = \varepsilon_k/\hbar$ given by

$$\int d^3 \boldsymbol{r}' \{G^{HF}\}^{-1}(\boldsymbol{r}, \boldsymbol{r}', \omega_k') \varphi_k^{HF}(\boldsymbol{r}') = \varphi_k^{HF}(\boldsymbol{r}) \qquad (14)$$

differ from those of the density correlation spectra ω_k.

B. The Bose condensed gas

In the presence of the condensate the two kind of excitations (1) and (5) share the same frequencies. This fact is a consequence of the underlying spontaneous symmetry breaking and can be conveniently formulated by using the dielectric formalism. This formalism expresses all the relevant quantities in terms of proper irreducible quantities (whose graphs remain connected even if one cuts an

arbitrary, single interaction line or propagator). For the Bose condensed gas we have to include interaction processes with the condensate atoms, involving many further diagrams within our approximation. In order to avoid a somewhat lengthy detour we refer the interested reader to the appendix, where we briefly summarize the approximations and the derivation of the following equations. These approximations have been presented in detail in reference [10]. The complete set of equations for the eigenfunctions within our model are (Eq.(A12))

$$\left(\tilde{H}_0(\boldsymbol{r}) + gn_{\mathrm{c}}(\boldsymbol{r})\right)\varphi_{1,k}(\boldsymbol{r}) + g\Phi_0^2(\boldsymbol{r})\varphi_{2,k}(\boldsymbol{r}) = \\ \hbar\omega_k\varphi_{1,k}(\boldsymbol{r}) - 2g\Phi_0(\boldsymbol{r})\delta n_{\mathrm{T},k}(\boldsymbol{r}) , \quad (15)$$

$$\left(\tilde{H}_0(\boldsymbol{r}) + gn_{\mathrm{c}}(\boldsymbol{r})\right)\varphi_{2,k}(\boldsymbol{r}) + g\Phi_0^{*2}(\boldsymbol{r})\varphi_{1,k}(\boldsymbol{r}) = \\ -\hbar\omega_k\varphi_{2,k}(\boldsymbol{r}) - 2g\Phi_0^*(\boldsymbol{r})\delta n_{\mathrm{T},k}(\boldsymbol{r}) , \quad (16)$$

where we defined

$$\delta n_{\mathrm{c},k}(\boldsymbol{r}) = \Phi_0^*(\boldsymbol{r})\varphi_{1,k}(\boldsymbol{r}) + \Phi_0(\boldsymbol{r})\varphi_{2,k}(\boldsymbol{r}) \quad (17)$$

and

$$\delta n_{\mathrm{T},k}(\boldsymbol{r}) = \delta n_{\mathrm{tot},k}(\boldsymbol{r}) - \delta n_{\mathrm{c},k}(\boldsymbol{r}) . \quad (18)$$

Due to the presence of the condensate we have quasiparticle excitations with two-component vector fields $\varphi_{k,\alpha}(\boldsymbol{r})$ ($\alpha = 1, 2$) as eigenfunctions.

The condensate wave function $\Phi_0(\boldsymbol{r})$ is calculated from the finite-temperature generalization of the Gross-Pitaevskii-equation

$$\tilde{H}_0(\boldsymbol{r})\Phi_0(\boldsymbol{r}) = 0 , \quad (19)$$

with the Hamiltonian $\tilde{H}_0(\boldsymbol{r})$

$$\tilde{H}_0(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\Delta + U(\boldsymbol{r}) - \mu + g|\Phi_0(\boldsymbol{r})|^2 + 2gn_{\mathrm{T}}(\boldsymbol{r}) . \quad (20)$$

We allow complex solutions of Eq. (20), i.e. our considerations apply also to the case where vortices are present. The total density is given by the sum $n_{\mathrm{T}} + n_{\mathrm{c}}$ with (9) and $n_{\mathrm{c}}(\boldsymbol{r}) = |\Phi_0(\boldsymbol{r})|^2$.

In the appendix we show (see Eq. (A10)) that in the presence of a Bose condensate Eq. (13) must be rewritten as

$$\delta n_{\mathrm{T},k}(\boldsymbol{r}) = 2\frac{g}{\hbar}\int d^3\boldsymbol{r}'\tilde{\chi}^0(\boldsymbol{r},\boldsymbol{r}',\omega_k)\delta n_{\mathrm{tot},k}(\boldsymbol{r}') , \quad (21)$$

which can be also expressed in terms of $\delta n_{\mathrm{c},k}$

$$\delta n_{\mathrm{T},k}(\boldsymbol{r}) = 2\frac{g}{\hbar}\int d^3\boldsymbol{r}_1\int d^3\boldsymbol{r}_2 \\ \times \left[1 - 2\frac{g}{\hbar}\tilde{\chi}^0(\omega_k)\right]^{-1}(\boldsymbol{r},\boldsymbol{r}_1)\tilde{\chi}^0(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega_k)\delta n_{\mathrm{c},k}(\boldsymbol{r}_2) . \quad (22)$$

With Eq. (17) we see that Eq. (15) and (16) still form a set of homogeneous coupled equations for $\varphi_{\alpha,k}$.

For $n_{\mathrm{c}} = 0$ Eq. (15) agrees with the Hartree-Fock equation. In the low temperature region $T \to 0$ we have $\tilde{\chi}^0 \to 0$ and according to Eq.(21) $\delta n_T \to 0$. Therefore, at $T \approx 0$ Eqs. (15) and (16) approach the usual Hartree-Fock Bogoliubov Popov equations.

Eqs. (15) and (16) agree with the linearization of the time dependent Gross-Pitaevskii equation around its stationary solution $\Phi_0(\boldsymbol{r})$ if the thermal density is assumed to be time-dependent and is also linearized around its stationary value $n_{\mathrm{T}}(\boldsymbol{r})$. Of course we then need additional equations like Eqs. (22) and (17) to fix $\delta n_{\mathrm{T},k}$.

III. THE KOHN THEOREM

The Kohn theorem states that the center of mass oscillations of the total density n_{tot} along the main axes of the trap are unchanged by the interactions and appear at the bare trap frequencies Ω_i as exact eigenvalues.

For the proof we choose arbitrarily one of those directions $\boldsymbol{e_i}$. The center of mass movement is just a displacement of the total density in the trap.

Therefore, the eigenfunctions to the Kohn frequency Ω_i should be given by the infinitesimal displacement operations $\eta\,\partial_i$ acting on the stationary densities ($\eta \ll 1$) $\eta\,\partial_i n(\boldsymbol{r}) = n(\boldsymbol{r} + \eta\,\boldsymbol{e}_i) - n(\boldsymbol{r}) = \delta n_i(\boldsymbol{r})$.

The Kohn mode $\delta n_{\mathrm{tot},i}(\boldsymbol{r})$ should therefore be proportional to $\partial_i n_{\mathrm{tot}}(\boldsymbol{r}) = \partial_i n_{\mathrm{c}}(\boldsymbol{r}) + 2\sum_k n(\varepsilon_k)\varphi_k^{HF}(\boldsymbol{r})\partial_i\varphi_k^{HF}(\boldsymbol{r})$ if we insert the expression for n_{T}.

We prove that the Kohn theorem is satisfied in two steps where we set $\omega = \Omega_i$ in all the expressions. First we show that we get $\delta n_{\mathrm{T},i} = \partial_i n_{\mathrm{T}}$ on the left hand side of Eq. (21) if we insert $\delta n_{\mathrm{tot},i} = \partial_i n_{\mathrm{tot}}$ on the right hand side. Using the result $\delta n_{\mathrm{T},i} = \partial_i n_{\mathrm{T}}$ in Eqs. (15) and (16) we calculate the corresponding $\varphi_{\alpha,k}$ from which we derive $\delta n_{\mathrm{c},i} = \partial_i n_{\mathrm{c}}$. We hereby reobtain $\delta n_{\mathrm{tot},i} = \partial_i n_{\mathrm{tot}}$ proving that $\partial_i n_{\mathrm{tot}}$ and $\varphi_{\alpha,k}$ are eigenfunctions of the equations to the trap frequency Ω_i. In the following the effect of the partial differential operator on the eigenfunctions of $\hat{H}^{HF}$ are evaluated by using the commutator relations

$$\left[\partial_i, \hat{H}^{HF}(\boldsymbol{r})\right] = m\Omega_i^2 x_i + 2g\partial_i n_{\mathrm{tot}}(\boldsymbol{r}) , \quad (23)$$

$$\left[x_i, \hat{H}^{HF}(\boldsymbol{r})\right] = \frac{\hbar^2}{m}\partial_i . \quad (24)$$

We obtain the matrix elements of the coordinate and the differential operators within the basis of the Hartree-Fock eigenfunctions

$$(\varepsilon_k - \varepsilon_l)\langle l|x_i|k\rangle = \frac{\hbar^2}{m}\langle l|\partial_i|k\rangle , \quad (25)$$

$$(\varepsilon_k - \varepsilon_l)\langle l|\partial_i|k\rangle = m\Omega_i^2\langle l|x_i|k\rangle \\ + 2g\langle l|\partial_i n_{\mathrm{tot}}(\boldsymbol{r})|k\rangle , \quad (26)$$

where we used the abbreviation

$$\langle l|\hat{A}|k\rangle = \int d^3\boldsymbol{r}\,\varphi_l^{HF}(\boldsymbol{r})\hat{A}\varphi_k^{HF}(\boldsymbol{r})\,. \tag{27}$$

In the first step we have to insert $\delta n_{\text{tot},k} = \partial_i n_{\text{tot}}$ in the expression (11) for the right hand side of Eq. (21) with $\omega = \Omega_i$

$$\delta n_{\text{T},i}(\boldsymbol{r}) = g\sum_{k,l}\varphi_k^{HF}(\boldsymbol{r})\varphi_l^{HF}(\boldsymbol{r})\,(n(\varepsilon_k) - n(\varepsilon_l))$$
$$\times \left[\frac{\langle k|\delta n_{\text{tot},i}|l\rangle}{\hbar\Omega_i + (\varepsilon_k - \varepsilon_l)} + \frac{\langle k|\delta n_{\text{tot},i}|l\rangle}{-\hbar\Omega_i + (\varepsilon_k - \varepsilon_l)}\right]\,. \tag{28}$$

We further concentrate on the coefficients corresponding to $\varphi_k^{HF}(\boldsymbol{r})\varphi_l^{HF}(\boldsymbol{r})\,(n(\varepsilon_k) - n(\varepsilon_l))$

$$g\left[\frac{\langle l|\delta n_{\text{tot},i}|k\rangle}{\hbar\Omega_i + (\varepsilon_k - \varepsilon_l)} + \frac{\langle l|\delta n_{\text{tot},i}|k\rangle}{-\hbar\Omega_i + (\varepsilon_k - \varepsilon_l)}\right]$$
$$= 2g\left[\frac{(\varepsilon_k - \varepsilon_l)\langle l|\delta n_{\text{tot},i}|k\rangle}{(\varepsilon_k - \varepsilon_l)^2 - (\hbar\Omega_i)^2}\right] \tag{29}$$

$$\stackrel{(26)}{=} \left[\frac{(\varepsilon_k - \varepsilon_l)^2\langle l|\partial_i|k\rangle - (\varepsilon_k - \varepsilon_l)m\Omega_i^2\langle l|x_i|k\rangle}{(\varepsilon_k - \varepsilon_l)^2 - (\hbar\Omega_i)^2}\right] \tag{30}$$

$$\stackrel{(25)}{=} \left[\frac{(\varepsilon_k - \varepsilon_l)^2\langle l|\partial_i|k\rangle - (\hbar\Omega_i)^2\langle l|\partial_i|k\rangle}{(\varepsilon_k - \varepsilon_l)^2 - (\hbar\Omega_i)^2}\right] \tag{31}$$

$$= \langle l|\partial_i|k\rangle\,. \tag{32}$$

We obtain the result

$$\delta n_{\text{T},i}(\boldsymbol{r})$$
$$= \sum_{k,l}\langle l|\partial_i|k\rangle\varphi_k^{HF}(\boldsymbol{r})\varphi_l^{HF}(\boldsymbol{r})\,(n(\varepsilon_k) - n(\varepsilon_l)) \tag{33}$$

$$= 2\sum_{k,l}n(\varepsilon_k)\varphi_k^{HF}(\boldsymbol{r})\varphi_l^{HF}(\boldsymbol{r})\langle l|\partial_i|k\rangle\,, \tag{34}$$

where in the last step we used the antisymmetry of $\langle l|\partial_i|k\rangle$ under exchange of k and l. The expression (34) for $\delta n_{\text{T},i}$ is exactly the partial differential of n_{T} with respect to x_i. Therefore, we get by comparing the left and the right hand side of Eq. (21) $\delta n_{\text{T},i} = \partial_i n_{\text{T}}$.
In the second step we have to insert $\delta n_{\text{T},i} = \partial_i n_{\text{T}}$ in Eqs. (15) and (16). But then it is straight forward to see that Eqs. (15) and (16) are solved for $\omega = \Omega_i$ and $\delta n_{\text{T},i} = \partial_i n_{\text{T}}$ by setting

$$\varphi_{1,i}(\boldsymbol{r}) = (\partial_i - x_i m\Omega_i/\hbar)\Phi_0(\boldsymbol{r})\,, \tag{35}$$
$$\varphi_{2,i}(\boldsymbol{r}) = (\partial_i + x_i m\Omega_i/\hbar)\Phi_0^*(\boldsymbol{r})\,. \tag{36}$$

Inserting these expressions in Eq. (17) we get $\delta n_{\text{c}}(\boldsymbol{r}) = \partial_i n_{\text{c}}(\boldsymbol{r})$ and hence $\delta n_{\text{tot}}(\boldsymbol{r}) = \delta n_{\text{c}}(\boldsymbol{r}) + \delta n_{\text{T}}(\boldsymbol{r})$ $= \partial_i n_{\text{tot}}(\boldsymbol{r})$, completing the proof. Therefore the expressions $\partial_i n_{\text{tot}}(\boldsymbol{r})$ and $\varphi_{\alpha,i}(\boldsymbol{r})$ are eigenfunctions with the main trap frequencies Ω_i as eigenfrequencies.
Fetter and Rokhsar [5] proved that (35), (36) are the Kohn modes of the Bogoliubov equations at $T = 0$. Our preceding considerations generalize their proof to finite T. In the whole calculation we have chosen to fix the

normalization of the density fluctuations $\delta n_{\text{tot},i} = \partial_i n_{\text{tot}}$. By performing a simple partial integration we get the normalization of the Kohn eigenfunctions $\varphi_{1,i}(\boldsymbol{r}), \varphi_{2,i}(\boldsymbol{r})$ according to $\int d^3\boldsymbol{r}(|\varphi_{1,i}(\boldsymbol{r})|^2 - |\varphi_{2,i}(\boldsymbol{r})|^2) = 2m\Omega_i N_{\text{c}}/\hbar$, where N_{c} is the total number of atoms in the condensate. With $N_{\text{c}} \to 0$ the Kohn modes are seen to disappear from the single-particle spectrum.
For $N_{\text{c}} = 0$ (i.e. $\Phi_0 = 0 = n_c$) Eq. (15) agrees with the usual Hartree-Fock equation where the Kohn mode is no longer present in the single particle spectra. But the Kohn modes survive in the density correlation spectra given by Eq. (21) for $n_{\text{tot}} = n_{\text{T}}$. In this case the proof of the Kohn theorem is already finished after the first step of our proof.

IV. DISCUSSION AND SUMMARY

The fundamental difference between a spatially homogeneous and a trapped Bose gas is the absence of the conservation las of momentum in the latter, which plays an important role in the theory of homogeneous systems. Certainly any approximation made in homogeneous systems should satisfy this fundamental law. In the special case of harmonic trapping potentials the momentum conservation law is replaced by Kohn's theorem, which states that there are three exact special modes where the center of mass oscillates harmonically with the three main trap frequencies. While this is easily proven for the exact Hamiltonian, the question whether any given approximation still satisfies this theorem is, in general, a rather nontrivial one. In the case of Bose condensed gases the question acquires an additional aspect, because of the coincidence of single particle and density oscillation spectra in Bose condensates. The Kohn mode, which is a density oscillation mode, must then have a single particle counterpart, which somehow disappears from the spectrum if the temperature is raised, so that the condensate disappears and the single particle spectrum and the density oscillation spectrum become decoupled.
In the present paper we have examined these questions for a specific approximation which includes direct interaction and exchange and is formulated within the dielectric formalism, which guarantees form the onset the coincidence of single particle and density oscillation spectra in the Bose condensed regime. By deriving a closed set of equations (15)-(18) for the single particle modes and the density modes generalizing the usual Bogoliubov-deGennes equations in Popov approximation, and solving them for the special coupled modes, corresponding for the densities just to a translation, we could not only verify the Kohn theorem for the closed set of equations, but obtain explicit expressions for the single particle modes (35), (36). These expressions for the latter are beautifully simple. They generalize the corresponding zero-temperature result obtained by *Fetter and Rokhsar*

by replacing the zero-temperature solution of the Gross-Pitaevskii equation by its finite counterpart of our specific model within the dielectric formalism. In this form our results (35), (36) show explicitly how the single particle component of the Kohn mode vanishes while keeping the density oscillations component unchanged, if the number of particles in the condensate is sent to zero.

A simpler version of the model treated here leaves out all exchange processes. This is then the simplest model of an interacting Bose gas which one can set up within the dielectric formalism. It was analyzed in detail for a homogeneous Bose gas in [11], but one can also analyze it for the trapped gas along the lines of [9] extending work in [12]. Using the same procedure as employed in the present paper one can readily show that the Kohn theorem is respected also by this simpler model.

APPENDIX:

In the first part of the appendix we introduce the necessary quantities given by our model [9,10] including exchange to describe the density autocorrelation function for $T < T_c$. Then we show how Eqs. (15),(16), which we have used here as a convenient starting point, are obtained. We first note that the contributions to $\tilde{\chi}$ above T_c given in Eq.(12) are not only *interaction line irreducible* but also *propagator line irreducible*. We denote these so-called *regular* contributions by $\tilde{\chi}^{(r)}$. Below T_c we get contributions to $\tilde{\chi}$ in addition to the regular ones, namely [9]

$$\tilde{\chi}(\boldsymbol{r},\boldsymbol{r}',\omega) = \tilde{\chi}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega) + \int d^3\boldsymbol{r}_1 \int d^3\boldsymbol{r}_2 \tilde{\Lambda}_\alpha^{(r)}(\boldsymbol{r},\boldsymbol{r}_1,\omega)$$
$$\times \tilde{G}_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)\tilde{\Lambda}_\beta^{(r)*}(\boldsymbol{r}_2,\boldsymbol{r}',\omega). \quad (A1)$$

The interaction line irreducible Green's functions $\tilde{G}_{\alpha\beta}$ are related to the exact Green's functions $G_{\alpha\beta}$ by the Dyson-type equation

$$G_{\alpha,\beta}(\boldsymbol{r},\boldsymbol{r}',\omega) = \tilde{G}_{\alpha,\beta}(\boldsymbol{r},\boldsymbol{r}',\omega)$$
$$+\frac{g}{\hbar}\int d^3\boldsymbol{r}_1 \int d^3\boldsymbol{r}_2 \int d^3\boldsymbol{r}_3 \tilde{G}_{\alpha,\gamma}(\boldsymbol{r},\boldsymbol{r}_1,\omega)$$
$$\times\tilde{\Lambda}_\gamma^*(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)\Lambda_\delta(\boldsymbol{r}_2,\boldsymbol{r}_3,\omega)G_{\delta,\beta}(\boldsymbol{r}_3,\boldsymbol{r}',\omega), \quad (A2)$$

where the vertex functions Λ_α are related to their regular parts by

$$\Lambda_\alpha(\boldsymbol{r},\boldsymbol{r}',\omega) = \tilde{\Lambda}_\alpha(\boldsymbol{r},\boldsymbol{r}',\omega)$$
$$+\frac{g}{\hbar}\int d^3\boldsymbol{r}_1 \tilde{\chi}^{(r)}(\boldsymbol{r},\boldsymbol{r}_1,\omega)\Lambda_\alpha(\boldsymbol{r}_1,\boldsymbol{r}',\omega). \quad (A3)$$

The regular vertex functions $\tilde{\Lambda}_\alpha^{(r)}$ express the coupling of the single-particle Green's functions $\tilde{G}_{\alpha\beta}$ to the density autocorrelation function $\tilde{\chi}$. The coupling comes from excitation processes out of and relaxation processes into the condensate.

In the model introduced in [9,10] the vertex functions $\tilde{\Lambda}_\alpha^{(r)}$ contain the lowest order contributions given by $\Phi_{\alpha,0}^*(\boldsymbol{r})$ and a series of higher order diagrams corresponding to the diagrams summed up in Eq. (12)

$$\tilde{\Lambda}_\alpha^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega) = \Phi_{\alpha,0}^*(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}') + \frac{g}{\hbar}\int d^3\boldsymbol{r}_1 \tilde{\Lambda}_\alpha^{(r)}(\boldsymbol{r},\boldsymbol{r}_1,\omega)$$
$$\times\tilde{\chi}^0(\boldsymbol{r}_1,\boldsymbol{r}',\omega). \quad (A4)$$

If we allow Φ_0 to be complex we have to use Φ_0 for the excitation processes due to the corresponding annihilation of a condensate atom and Φ_0^* in the opposite case. In order to simplify the notations we write $\Phi_{\alpha,0}$ ($\alpha = 1,2$) with $\Phi_{1,0} = \Phi_0$ and $\Phi_{2,0} = \Phi_0^*$.

$\tilde{G}_{\alpha\beta}$ is in general expressed by the regular part $\tilde{\Sigma}_{\alpha\beta}^{(r)}$ of the self-energies in the form

$$\left[\frac{1}{\hbar}\tilde{G}(\omega)\right]^{-1}(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}')$$
$$\times\begin{pmatrix} \hbar\omega + \frac{\nabla^2\hbar^2}{2m} + \mu - U(\boldsymbol{r}) & 0 \\ 0 & -\hbar\omega + \frac{\nabla^2\hbar^2}{2m} + \mu - U(\boldsymbol{r}) \end{pmatrix}$$
$$-\tilde{\Sigma}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega). \quad (A5)$$

The approximation of the model for $\tilde{\Sigma}^{(r)}$ is defined by

$$\tilde{\Sigma}_{\alpha\beta}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega) = \delta(\boldsymbol{r}-\boldsymbol{r}')(gn_c(\boldsymbol{r}') + 2gn_T(\boldsymbol{r}'))\delta_{\alpha\beta},$$
$$+\frac{g^2}{\hbar}\Phi_{\alpha,0}(\boldsymbol{r})\tilde{\chi}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega)\Phi_{\beta,0}^*(\boldsymbol{r}'). \quad (A6)$$

Like in Eqs. (12),(A4) we add to the lowest order regular self-energy diagrams $gn_c(\boldsymbol{r}') + 2gn_T(\boldsymbol{r}')$ appearing in the diagonal of $\tilde{\Sigma}^{(r)}$ the sum over multiple particle-hole scattering processes including exchange with additional factors Φ_0^*, Φ_0 due to excitation processes out of and the absorption processes into the condensate.

Below T_c the eigenfunctions $\varphi_1(\boldsymbol{r})$ and $\varphi_2(\boldsymbol{r})$ can be expressed as a linear functional of the corresponding eigenfunction $\delta n_{\mathrm{tot},k}(\boldsymbol{r})$ of the density autocorrelation function spectra (*see* [12])

$$\varphi_{\alpha,k}(\boldsymbol{r}) = \int d^3\boldsymbol{r}_1 \int d^3\boldsymbol{r}_2 \sum_\beta \tilde{G}_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}_1,\omega_k)$$
$$\times\tilde{\Lambda}^{(r)*}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega_k)\delta n_{\mathrm{tot},k}(\boldsymbol{r}_2). \quad (A7)$$

Without any approximations for the regular quantities $\tilde{\Sigma}_{\alpha\beta}^{(r)}$, $\tilde{\chi}^{(r)}$ and $\tilde{\Lambda}_\alpha^{(r)}$ the functions $\varphi_\alpha(\boldsymbol{r})$ would fulfill Eq. (5) if $\delta n_{\mathrm{tot},k}(\boldsymbol{r})$ fulfills Eq. (3).

We insert Eq. (A1) in Eq. (3) and take into account (A7) to get

$$\delta n_{\mathrm{tot},k}(\boldsymbol{r}) = \int d^3\boldsymbol{r}' \left[\frac{g}{\hbar}\tilde{\chi}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega_k)\delta n_{\mathrm{tot},k}(\boldsymbol{r}')\right.$$
$$\left.+\tilde{\Lambda}_\alpha^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega_k)\varphi_{\alpha,k}(\boldsymbol{r}')\right] \quad (A8)$$

5

Defining $\delta n_{\mathrm{c},k}(\boldsymbol{r}) = \Phi_0^*(\boldsymbol{r})\varphi_{1,k}(\boldsymbol{r}) + \Phi_0(\boldsymbol{r})\varphi_{2,k}(\boldsymbol{r})$ and separating out the the lowest order diagram $\Phi_{\alpha,0}^*$ of $\tilde{\Lambda}_\alpha^{(r)}$ we derive an equation for $\delta n_{\mathrm{T},k} = \delta n_{\mathrm{tot},k} - \delta n_{\mathrm{c},k}$

$$\delta n_{\mathrm{T},k}(\boldsymbol{r}) = \frac{g}{\hbar} \int d^3\boldsymbol{r}'\, \tilde{\chi}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega_k)$$
$$\times (\delta n_{\mathrm{c},k}(\boldsymbol{r}') + \delta n_{\mathrm{tot},k}(\boldsymbol{r}'))\ . \qquad \text{(A9)}$$

Eliminating $\delta n_{\mathrm{c},k}$ by using $\delta n_{\mathrm{c},k} = \delta n_{\mathrm{tot},k} - \delta n_{\mathrm{T},k}$ this equation is solved by

$$\delta n_{\mathrm{T},k}(\boldsymbol{r}) = 2\frac{g}{\hbar} \int d^3\boldsymbol{r}'\, \tilde{\chi}^0(\boldsymbol{r},\boldsymbol{r}',\omega_k)\delta n_{\mathrm{tot},k}(\boldsymbol{r}')\,, \qquad \text{(A10)}$$

as can be verified by using the result (12) for $\tilde{\chi}^{(r)}$.

Acting with the operator $\left[\frac{1}{\hbar}\tilde{G}(\omega_k)\right]_{\alpha\beta}^{-1}(\boldsymbol{r},\boldsymbol{r}')$ on $\varphi_{\beta,k}(\boldsymbol{r}')$ and using (A1)-(A7) we obtain

$$(\pm\hbar\omega_k - \tilde{H}_0(\boldsymbol{r}))\varphi_{\frac{1}{2},k}(\boldsymbol{r})$$
$$= \frac{g^2}{\hbar} \int d^3\boldsymbol{r}'\, \Phi_{\frac{1}{2},0}(\boldsymbol{r})\tilde{\chi}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega_k)\delta n_{\mathrm{c},k}(\boldsymbol{r}')$$
$$+ g\Phi_{\frac{1}{2},0}(\boldsymbol{r}) \int d^3\boldsymbol{r}' \left(\delta(\boldsymbol{r}-\boldsymbol{r}') + \frac{g}{\hbar}\tilde{\chi}^{(r)}(\boldsymbol{r},\boldsymbol{r}',\omega_k)\right)$$
$$\times \delta n_{\mathrm{tot},k}(\boldsymbol{r}')\,, \qquad \text{(A11)}$$

where $\tilde{H}_0$ is given by (20). Eqs. (A11) can be simplified with the help of Eq. (A9) and we can directly give the results

$$(\pm\hbar\omega_k - \tilde{H}_0(\boldsymbol{r}))\varphi_{\frac{1}{2},k}(\boldsymbol{r}) =$$
$$g\Phi_{\frac{1}{2},0}(\boldsymbol{r})\left(\delta n_{\mathrm{tot},k}(\boldsymbol{r}) + \delta n_{\mathrm{T},k}(\boldsymbol{r})\right)\ . \qquad \text{(A12)}$$

Eqs. (A12) agree with Eqs. (15) and (16) using the definition $\delta n_{\mathrm{c},k} = \Phi_0^*\varphi_{1,k} + \Phi_0\varphi_{2,k}$.

[1] W. Kohn, Phys. Rev. **123**, 1242 (1961).

[2] L. Brey, N. F. Johnson, and B. I. Halperin, Phys. Rev. B **40**, 10647 (1989).

[3] J. F. Dobson, Phys. Rev. Lett. **73**, 2244 (1994)

[4] see W. Ketterle, D. S. Durfee, and D. M. Stamper-Kurn, 67, and E. A. Cornell, J. R. Ensher, and C. E. Wieman, 15, in "Proceedings of the International School of Physics - Enrico Fermi", ed. by M.Inguscio, S.Stringari and C. Wieman (IOS Press, 1999).

[5] D. Rokhsar and A. L. Fetter, Phys. Rev. A **57**, 1191 (1998).

[6] E. Zaremba, T. Nikuni, and A. Griffin, J. Low Temp. Phys. **116**, 277 (1999); M. Imamović-Tomasović and A. Griffin, preprint, cond-mat/0010107.

[7] M. J. Bijlsma and H. T. C. Stoof, Phys. Rev. A **60**, 3973 (1999).

[8] A. Minguzzi and M. P. Tosi, J. Phys. Condens. Matter **9**, 10211 (1997).

[9] J. Reidl, A. Csordás, R. Graham, and P. Szépfalusy, Phys. Rev. A **61**, 43606 (2000).

[10] M. Fliesser, J. Reidl, P. Szépfalusy, and R. Graham, preprint, cond-mat/0006475.

[11] P. Szépfalusy and I. Kondor, Ann. Phys. (N.Y.) **82**, 1 (1974).

[12] Gy. Bene, P. Szépfalusy, Phys. Rev. A **58**, R3391 (1998).