

Free field representation for the $O(3)$ nonlinear sigma model and bootstrap fusion

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The possibility of the application of the free field representation developed by Lukyanov for massive integrable models is investigated in the context of the $O(3)$ sigma model. We use the bootstrap fusion procedure to construct a free field representation for the $O(3)$ Zamolodchikov-Faddeev algebra and to write down a representation for the solutions of the form-factor equations which is similar to the ones obtained previously for the sine-Gordon and $SU(2)$ Thirring models. We discuss also the possibility of developing further this representation for the $O(3)$ model and comment on the extension to other integrable field theories.

I. INTRODUCTION

Two-dimensional integrable field theories are currently a prime area of research in the context of quantum field theory. There is a huge number of models in two spacetime dimensions which are exactly solvable. A lot of different approaches were developed to deal with these theories.

The first breakthrough in the analysis of integrable models with massive spectra was the famous paper by Zamolodchikov and Zamolodchikov [1], in which they calculated the exact S-matrices of some interesting models using bootstrap ideas. The scattering theory in these models is drastically simplified due to the existence of an infinite number of conserved currents. The many-particle S-matrix is factorized into a product of incoherent two-particle scattering amplitudes, hence the name Factorized Scattering Theory (FST). However one must realize that in these models the correspondence between the conjectured exact S-matrices and the quantum field theory itself is very weak: it is based on some information about the spectrum and symmetries of the theory. There is the so-called CDD ambiguity which plagues the uniqueness of the solution for the S-matrix. In most cases one chooses a "minimal solution" of the equations of the FST bootstrap, which means an appropriate fixing of this ambiguity.

Since the S-matrix is so simple, it doesn't really contain all information about the theory: it is an on-shell quantity and for the calculation of correlation functions one must go off-shell. However with the help of the two-particle S-matrix one can write down equations for specific matrix elements of local operators. These are the so-called form-factor equations and were used to calculate various off-shell quantities. With the knowledge of the form-factors of local fields one can reconstruct all the quantities associated to these local operators, such as e.g. correlation functions. This program was started long ago [2], but the major step forward was taken by Smirnov, who in a series of papers calculated the form-factors of some important operators in the sine-Gordon, $SU(2)$ Thirring and $O(3)$ nonlinear sigma models [3–6].

Then a question arises: what is the structure behind the models which makes it possible to solve the form-factor equations which is essentially equivalent to solving a very complicated Riemann-Hilbert problem? This line of research was taken up by Bernard, Leclair and Smirnov who analysed the nonlocal symmetries of these models. They found that the relevant structure is nothing but quantum affine algebras and their representation theory [7]. Smirnov cleared up the connection of the Zamolodchikov-Fateev operators, which play a very important role (we discuss them in the sequel) and the quantum affine symmetry algebra. It turned out [8] that the ZF operators are just vertex operators for the representations of the quantum affine algebra. The form-factor axioms proved to be related to the deformed Knizhnik-Zamolodchikov equations which are the quantum analogues of the usual Knizhnik-Zamolodchikov equations for the affine Kac-Moody algebras [9]. Quantum affine algebras were given a vertex operator representation in [10]. This and other conformal field theory analogies suggest that a bosonization technique based on free fields can be useful in calculating form-factors in the integrable model. This route was taken up by Lukyanov [11,12], who constructed a free field representation for the sine-Gordon and $SU(2)$ Thirring models and used it to give an integral representation for the form-factors. In this paper we use mainly the results of his work [12].

The question we investigate is the following: how can we apply the bosonization technique to the $O(3)$ sigma model? Since there is a simple relationship (which we call the bootstrap fusion) between the $SU(2)$ Thirring model S-matrix and the S-matrix of the $O(3)$ sigma model, this seems to be rather trivial. However, it was already noticed by Smirnov, that the corresponding limit of the $SU(2)$ Thirring model form-factors is singular and therefore one must be careful when trying to define this procedure. Here we intend to make this fusion procedure on the level of the

free field representation. The other problem with this fusion is that the form-factors obtained does not satisfy one of the form-factor axioms, namely the one which gives the position and residues of the kinematical poles. We identify the source of this problem and solve it in the same way as Smirnov did, i.e. by introducing some multiplier functions which do not destroy the validity of the other form-factor axioms but correct for this problem. The main result is that we obtain a procedure for calculating generating functions of form-factors in the O(3) sigma model.

The paper is organized as follows. Section II gives a brief review of the Zamolodchikov-Faddeev algebra and an introduction to the free field representation. Section III defines the notion of the bootstrap fusion and discusses the representation of the O(3) ZF-algebra obtained in this way. In Section IV we write down the form-factor axioms and give the idea of the solution. Then we discuss the regularization of the free field construction. Section V contains the result of evaluation of the traces and the discussion of the problem of the integration. We then give an explicit example in section VI, where we show the calculation of the two-particle form-factor of the O(3) current and compare the result to the one found in the literature. Section VII is reserved for the discussion. The paper contains an appendix in which we show a direct method of proving the ZF-algebra commutation relations.

II. REVIEW OF THE FREE FIELD REPRESENTATION

We first study the SU(2) Thirring model described by the Lagrangian density

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - gj_i^\mu j_{i\mu}. \quad (2.1)$$

The field ψ describes an isospin doublet of fermions and the current j_i^μ is the SU(2) current

$$j_i^\mu = \frac{1}{2}\bar{\psi}\gamma^\mu\tau_i\psi, \quad (2.2)$$

where the τ_i , $i = 1, 2, 3$ are the Pauli-matrices. The spectrum of the model consists of a massive isospin doublet of kinks and a free scalar. We will be concerned only with the kink sector. We introduce the rapidity variable describing the energy and momentum of the on-shell kink states by the definition $p^0 = m \cosh \beta$, $p^1 = m \sinh \beta$. The scattering processes of the kinks are described by the following S-matrix:

$$S_{i,j}^{k,l} = S_0(\beta) \frac{1}{\beta - \pi i} (\beta \delta_i^k \delta_j^l - \pi i \delta_i^l \delta_j^k), \quad (2.3)$$

where $i, j, k, l = +, -$ denote the isotopic indices and

$$S_0(\beta) = \frac{\Gamma(\frac{1}{2} + \frac{\beta}{2\pi i})\Gamma(-\frac{\beta}{2\pi i})}{\Gamma(\frac{1}{2} - \frac{\beta}{2\pi i})\Gamma(\frac{\beta}{2\pi i})}. \quad (2.4)$$

This S-matrix was obtained using the bootstrap procedure and the properties of the conjectured spectrum of the model. These are purely algebraic reasonings and the satisfactory connection to the field theory defined by the Lagrangian (2.1) is still missing as we discussed in the introduction. One should also keep in mind however that there are many cases in which profound arguments can be presented for the correctness of the bootstrap approach by comparing with another known approaches such as perturbation theory, $1/N$ expansion and lattice simulation (see for example the discussion in [13]). This gives the encouragement for the hope that one day we really can establish the correspondence between the different descriptions.

The charge conjugation matrix for the doublet kinks is $C = i\sigma_2$ which satisfies the unusual property $C^t = -C$. As it is known in two dimensions more general statistics are possible than just bosonic or fermionic ones. These new ones correspond to representations of the braid group instead of the representations of the symmetric group. Actually the kinks describe spin-1/4 particles. The S-matrix (and the model) has a global SU(2) symmetry under which the kinks transform as the fundamental representation of the group.

The Hilbert space of an integrable model is defined by the representation of the formal Zamolodchikov-Faddeev (ZF) algebra, which is defined by the relations

$$Z_i(\beta_1)Z_j(\beta_2) = S_{i,j}^{k,l}(\beta_{12})Z_l(\beta_2)Z_k(\beta_1), \quad \beta_{12} = \beta_1 - \beta_2. \quad (2.5)$$

The commutation relations of the Z-operators reflect the scattering of the particles. From now on we will refer to them as the scattering relations. The space of states furnishes a representation of the ZF algebra. The Hilbert space

structure is fixed by giving the relations for the adjoint operators in that representation (we denote it by π_A and use the letter A for $\pi_A(Z)$)

$$\begin{aligned} A^{i\dagger}(\beta_1)A^{j\dagger}(\beta_2) &= A^{l\dagger}(\beta_2)A^{k\dagger}(\beta_1)S_{k,l}^{i,j}(\beta_{12}) \\ A_i(\beta_1)A^{j\dagger}(\beta_2) &= A^{k\dagger}(\beta_2)S_{i,k}^{j,l}(\beta_{12})A_l(\beta_1) + 2\pi\delta_i^j\delta(\beta_{12}). \end{aligned} \quad (2.6)$$

However in the sequel we will be concerned with another type of representation of the ZF algebra in which we do not have the conjugate operators. Here we would like to stress that the representation π_A is nothing else but the space of asymptotic particle states (in and out) of the model as can be found e.g. in [12].

These relations can be converted to a new particle basis in which the fundamental operators of the SU(2) Thirring model are usual spin-1/2 fermions. To achieve this we redefine the S-matrix (2.3) of the model as follows

$$\hat{S}_{i,j}^{k,l}(\beta) = (-1)^{(i+l)}S_{i,j}^{k,l}(\beta) \quad (2.7)$$

where $(-1)^i = +1$ for $i = -$ and -1 for $i = +$. This also means a redefinition of the phases of the ZF operators. We will drop the hat from now on. This will cause no confusion since we will only use the new basis of auxiliary spin-1/2 particles. The net effect of the phases is that the coproduct rule for the SU(2) symmetry changes: we get an SU(2)₋₁ quantum symmetry as in [12]. We also give the formula for the S-matrix elements in this basis in detail:

$$\begin{aligned} S_{++}^{++}(\beta) &= S_{--}^{--}(\beta) = S_0(\beta), \\ S_{+-}^{+-}(\beta) &= S_{-+}^{-+}(\beta) = S_0(\beta)\frac{\beta}{i\pi - \beta}, \\ S_{-+}^{+-}(\beta) &= S_{+-}^{-+}(\beta) = S_0(\beta)\frac{i\pi}{i\pi - \beta}. \end{aligned} \quad (2.8)$$

The charge conjugation matrix is given by

$$C_{ij} = \delta_{i+j,0}. \quad (2.9)$$

Now we turn to the free field representation. Lukyanov [12] introduces a free field $\phi(\beta)$ with the following properties:

$$\begin{aligned} [\phi(\beta_1), \phi(\beta_2)] &= \ln S_0(\beta_2 - \beta_1), \\ \langle 0|\phi(\beta_1)\phi(\beta_2)|0\rangle &= -\ln g(\beta_2 - \beta_1), \end{aligned} \quad (2.10)$$

where S_0 is the function defined in eqn. (2.4) and we give the formula for the function g below (2.15). The consistency of the two relations above requires

$$S_0(\beta) = \frac{g(-\beta)}{g(\beta)}. \quad (2.11)$$

The field ϕ is represented (after a proper ultraviolet cut-off procedure) on a Fock space. We will build up operators which represent the ZF algebra on that space.

The field

$$\bar{\phi}(\beta) = \phi(\beta + i\frac{\pi}{2}) + \phi(\beta - i\frac{\pi}{2}), \quad (2.12)$$

which satisfies the commutation relations

$$[\bar{\phi}(\beta_1), \bar{\phi}(\beta_2)] = \ln \frac{\beta_2 - \beta_1 - i\pi}{\beta_2 - \beta_1 + i\pi}, \quad (2.13)$$

and similar ones with $\phi(\beta)$, plays an important role as well. The corresponding two-point functions can also be written down as

$$\begin{aligned} \langle 0|\bar{\phi}(\beta_1)\phi(\beta_2)|0\rangle &= \ln w(\beta_2 - \beta_1), \\ \langle 0|\bar{\phi}(\beta_1)\bar{\phi}(\beta_2)|0\rangle &= -\ln \bar{g}(\beta_2 - \beta_1), \end{aligned} \quad (2.14)$$

where we defined the following functions

$$\begin{aligned}
g(\beta) &= k^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{i\beta}{2\pi})}{\Gamma(\frac{i\beta}{2\pi})}, \\
w(\beta) &= k^{-1} \frac{2\pi}{i(\beta + i\frac{\pi}{2})}, \\
\bar{g}(\beta) &= -k^2 \frac{\beta(\beta + i\pi)}{4\pi^2},
\end{aligned} \tag{2.15}$$

where k is a normalization constant. The field $\bar{\phi}$ is a much more convenient object than the field ϕ since it satisfies much simpler relations and its two-point function is closely analogous to the two-point function of a free conformal scalar field.

Now we are ready to define the following vertex operators a la Lukyanov

$$\begin{aligned}
V(\beta) &= \exp(i\phi(\beta)) = (g(0))^{\frac{1}{2}} : \exp(i\phi(\beta)) :, \\
\bar{V}(\beta) &= \exp(-i\bar{\phi}(\beta)) = (\bar{g}(0))^{\frac{1}{2}} : \exp(-i\bar{\phi}(\beta)) :,
\end{aligned} \tag{2.16}$$

where $:$ denotes an appropriate normal ordering [12]. These operators need to be regularized because the functions g and \bar{g} have simple zeros at $\beta = 0$. Using the analogy with conformal theory we define the regularized values of the functions g, \bar{g} at $\beta = 0$ to be

$$\begin{aligned}
g_{reg}(0) &= \lim_{\beta \rightarrow 0} \frac{g(\beta)}{\beta} =: \rho^2, \\
\bar{g}_{reg}(0) &= \lim_{\beta \rightarrow 0} \frac{\bar{g}(\beta)}{\beta} =: \bar{\rho}^2.
\end{aligned} \tag{2.17}$$

The vertex operators defined in (2.16) obey the following very important relations

$$\begin{aligned}
V(\beta_1)V(\beta_2) &= \rho^2 g(\beta_2 - \beta_1) : V(\beta_1)V(\beta_2) :, \\
\bar{V}(\beta_1)V(\beta_2) &= \rho \bar{\rho} w(\beta_2 - \beta_1) : V(\beta_1)V(\beta_2) :, \\
\bar{V}(\beta_1)\bar{V}(\beta_2) &= \bar{\rho}^2 \bar{g}(\beta_2 - \beta_1) : \bar{V}(\beta_1)\bar{V}(\beta_2) :,
\end{aligned} \tag{2.18}$$

Eqn. (2.18) plays a key role in proving the ZF scattering relations.

Then in analogy with the Coulomb gas representation of rational conformal field theory one introduces a screening charge χ by the definition

$$\langle u|\chi|v \rangle = \eta^{-1} \langle u| \frac{1}{2\pi} \int_C d\gamma \bar{V}(\gamma) |v \rangle, \tag{2.19}$$

where C is a contour specified in the following manner: assuming that all matrix elements $\langle u|\bar{V}(\gamma)|v \rangle$ are meromorphic functions decreasing at the infinity faster than γ^{-1} , the contour goes from $\Re\gamma = -\infty$ to $\Re\gamma = +\infty$ and lies above all singularities whose positions depend on $|u \rangle$ and below all singularities depending on $|v \rangle$. (η is a normalization parameter which is chosen to be $(-i\pi)^{1/2}$. This normalization guarantees the correct value for the residue of the operator product (see proposition (ii) below).

Having defined these objects, one can write down the following operators

$$\begin{aligned}
Z_+(\beta) &= V(\beta), \\
Z_-(\beta) &= i(\chi V(\beta) + V(\beta)\chi).
\end{aligned} \tag{2.20}$$

Then Lukyanov proves the following propositions:

(i) These operators satisfy the SU(2) Thirring model ZF relations.

(ii) The singular part of the operator product $Z_i(\beta_2)Z_j(\beta_1)$ considered as a function of the complex variable β_2 for real β_1 in the upper half plane $\Im\beta_2 \geq 0$ contains only one simple pole at $\beta_2 = \beta_1 + i\pi$ with residue $-iC_{ij}$.

(i) and (ii) can be proven by technics similar to those used in conformal field theory. Here we see that the operator χ plays the role of the step operator of the quantum group. It can be compared with the work of Gomez and Sierra [14] on the Coulomb gas representation of rational conformal field theories where they showed that the positive step operators of the quantum group can be identified with a contour creation operator defined by the help of the screening charges. Here we see a very close structure in the definition of the operator χ and in the formulae for the bosonization of the ZF operators.

III. BOOTSTRAP FUSION AND THE ZF ALGEBRA OF THE NONLINEAR SIGMA MODEL

Here we recall some basic facts about the nonlinear sigma model. It is defined to be the dynamics of a scalar field taking values on the surface of the two-dimensional sphere S^2 :

$$\mathcal{L} = \frac{1}{g^2} \int d^2x \partial_\mu n^a \partial_\mu n^a, \quad n^a n^a = 1, \quad a = 1, 2, 3. \quad (3.1)$$

The spectrum of the model consists of a massive isospin triplet of scalars. The model displays spontaneous mass generation. It is asymptotically free and the coupling g can be traded for the true parameter of the model, Λ , which is the scale characterizing the running coupling in the \overline{MS} scheme. This is the well-known phenomenon of dimensional transmutation. The ratio of the particle mass m and the scale Λ was calculated by Hasenfratz et al. [15,16] using Bethe ansatz techniques and the conjectured exact S-matrix obtained in [1]. This calculation also provides a way of fixing the CDD ambiguity by comparison with the m/Λ ratio obtained from perturbation theory.

We now make use of the idea which we call the bootstrap fusion which was used by Smirnov to calculate form-factors of the O(3) nonlinear sigma model from those of the SU(2) Thirring model. It is similar to the description of the bound states in the bootstrap approach but in this case the limit is singular (there are no actual bound states in the theory). We define

$$\tilde{Z}_I(\beta) = \lim_{\epsilon \rightarrow 0} A C_I^{ij} Z_i(\beta + \pi i/2 + i\epsilon) Z_j(\beta - \pi i/2 - i\epsilon), \quad (3.2)$$

where A is a normal ordering constant and C_I^{ij} are the Clebsh-Gordan coefficients for the adjoint representation of SU(2) in the product of two fundamental representations. (The capital letters denote adjoint indices, taking the values $I = +, 0, -$). Due to proposition (ii) above the limit is well-defined since the singularity drops out. In this way we obtain particle operators transforming in the adjoint representation of SU(2) or, the same, in the fundamental representation of O(3). The nice property, used by Smirnov, is the following: if we now calculate what kind of scattering relations are satisfied by the \tilde{Z} 's what we find is the O(3) sigma model S-matrix. The calculation proceeds by simple commutations of the Z 's and the result is

$$\tilde{Z}_I(\beta_1) \tilde{Z}_J(\beta_2) = \tilde{S}_{I,J}^{K,L}(\beta_{12}) \tilde{Z}_L(\beta_2) \tilde{Z}_K(\beta_1), \quad (3.3)$$

where \tilde{S} denotes the S-matrix of the nonlinear sigma model which is known to be

$$\tilde{S}_{I,J}^{K,L}(\beta) = \frac{1}{(\beta + \pi i)(\beta - 2\pi i)} \times [\beta(\beta - \pi i) \delta_I^K \delta_J^L - 2\pi i(\beta - \pi i) \delta_I^L \delta_J^K + 2\pi i \beta \delta_{I,-J} \delta^{K,-L}]. \quad (3.4)$$

We use the spin-1/2 basis so the C_I^{ij} are Clebsh-Gordan coefficients of the quantum group SU(2) $_{-1}$ for the spin-1 representation in the product of two spin-1/2 representation. Written out explicitly $\tilde{Z}_I(\beta)$ have the form

$$\begin{aligned} \tilde{Z}_+(\beta) &= \lim_{\epsilon \rightarrow 0} A Z_+(\beta + \pi i/2 + i\epsilon) Z_+(\beta - \pi i/2 - i\epsilon), \\ \tilde{Z}_0(\beta) &= \lim_{\epsilon \rightarrow 0} A \frac{1}{\sqrt{2}} (Z_+(\beta + \pi i/2 + i\epsilon) Z_-(\beta - \pi i/2 - i\epsilon) \\ &\quad - Z_-(\beta + \pi i/2 + i\epsilon) Z_+(\beta - \pi i/2 - i\epsilon)), \\ \tilde{Z}_-(\beta) &= \lim_{\epsilon \rightarrow 0} A Z_-(\beta + \pi i/2 + i\epsilon) Z_-(\beta - \pi i/2 - i\epsilon). \end{aligned} \quad (3.5)$$

We define the new vertex operator

$$\begin{aligned} U(\beta) &= \lim_{\epsilon \rightarrow 0} : V(\beta + i\pi/2 + i\epsilon) V(\beta - i\pi/2 - i\epsilon) : \\ &= \lim_{\epsilon \rightarrow 0} A V(\beta + i\pi/2 + i\epsilon) V(\beta - i\pi/2 - i\epsilon). \end{aligned} \quad (3.6)$$

From this equation the value of A can be read off. However this is just a normalization constant in the pure quadratic scattering relations and therefore can be dropped. One can prove that the operator U can be expressed explicitly in terms of the free fields as

$$U(\beta) = \exp(i\bar{\phi}(\beta)), \quad (3.7)$$

and satisfies the following relations

$$\begin{aligned} U(\beta_1)U(\beta_2) &= \bar{g}(\beta_{12}) : U(\beta_1)U(\beta_2) :, \\ U(\beta_1)\bar{V}(\beta_2) &= \bar{g}^{-1}(\beta_{12}) : U(\beta_1)\bar{V}(\beta_2) :. \end{aligned} \quad (3.8)$$

Calculating the operator products and rearranging them we arrive at the following free field representation of the O(3) sigma model Zamolodchikov-Faddeev operators

$$\begin{aligned} \tilde{Z}_+(\beta) &= U(\beta), \\ \tilde{Z}_0(\beta) &= \frac{i}{\sqrt{2}}(\chi U(\beta) - U(\beta)\chi), \\ \tilde{Z}_-(\beta) &= \frac{1}{2}(\chi\chi U(\beta) - 2\chi U(\beta)\chi + U(\beta)\chi\chi). \end{aligned} \quad (3.9)$$

These relations are easy to interpret. $U(\beta)$ is nothing else than the highest weight in the multiplet on which the step operator χ of the quantum group acts creating the lower weights. The action of χ (apart from normalization factors) is the same as the adjoint action of the SU(2) (remember that this is the O(3) vector multiplet and that the action of SU(2) and SU(2)₋₁ on integer spin representations are the same as can be read off from the corresponding coproduct).

A remark on the derivation of the above bosonization formulae is in order. The derivation of the first two expressions is obvious but in the third case one has to consider two screening charges. We consider first the vacuum-vacuum matrix element of the relation. As the labeling of the integration variables for the two charges are irrelevant (since they are integrated out) one may symmetrize the integrand with respect to the rapidity argument of the two screening charges. With this trick one can prove the equality of the integrands. Then one checks that the contours that the contours can be deformed to prove the equality in the manner described in [12]. The last thing is that since eq. (3.9) is an operator relation one must prove it for all matrix elements. This can be achieved by writing down the relation between general multiparticle states. As a straightforward calculation shows, the proof reduces to using the same formulae as for vacuum matrix elements.

The last thing to check is the residue equation for the ZF operators of the O(3) sigma model. Namely, in order to solve Smirnov's form-factor equations one has to reproduce the kinematical poles at the rapidity arguments displaced by $i\pi$. This means that we need

$$\text{res}_{\beta_1=\beta_2+i\pi} \tilde{Z}_I(\beta_1)\tilde{Z}_J(\beta_2) = -iC_{IJ} \quad (3.10)$$

However for the operators defined in (3.9) this relation is not fulfilled. The problem is that if we write out the explicit representation of the O(3) ZF operators with the help of the bootstrap fusion in terms of the SU(2) Thirring model ZF operators, we can see that we do not obtain the correct pole structure. We return to this question later when we discuss an explicit example of the solution for the bootstrap equations.

IV. THE FORM-FACTOR EQUATIONS AND THEIR SOLUTION

We call form-factor the matrix element of a local field operator between the vacuum and some n -particle in-state $|\beta_1, i_1, \dots, \beta_n, i_n\rangle$ (i_1, \dots, i_n denote the possible internal quantum numbers of the corresponding particle):

$$F_{i_1 \dots i_n}(\beta_1, \dots, \beta_n) = \langle \text{vac} | O(0) | \beta_1, i_1, \dots, \beta_n, i_n \rangle. \quad (4.1)$$

Knowledge of these objects permits the reconstruction of the matrix elements of local operators between any many-particle states in the theory by analytic continuation. Also if the theory satisfies asymptotic completeness with this set of many-particle states, then an infinite series can be written down for any correlation function. Here we will be concerned by finding an integral representation for the form-factors. They are constrained by the Smirnov axioms which may be taken as defining the space of local operators. The axioms are the following:

1. The functions F are analytic in the rapidity differences $\beta_{ij} = \beta_i - \beta_j$ inside the strip $0 < \text{Im}\beta < 2\pi$ except for simple poles. The physical matrix elements are the values of the F 's at real rapidity arguments ordered as $\beta_1 < \beta_2 < \dots < \beta_n$.

2. The form-factor of a local operator of Lorentz spin s satisfies

$$F_{i_1, \dots, i_n}(\beta_1 + \Lambda, \dots, \beta_n + \Lambda) = \exp(s\Lambda) F_{i_1, \dots, i_n}(\beta_1, \dots, \beta_n). \quad (4.2)$$

This means that the form-factor of a spin $s = 0$ operator depends only on the rapidity differences.

3. Form factors satisfy Watson's symmetry property:

$$F_{i_1, \dots, i_j, i_{j+1}, \dots, i_n}(\beta_1, \dots, \beta_j, \beta_{j+1}, \dots, \beta_n) = S_{i_j i_{j+1}}^{k_{j+1} k_j} F_{i_1, \dots, k_{j+1}, k_j, \dots, i_n}(\beta_1, \dots, \beta_{j+1}, \beta_j, \dots, \beta_n). \quad (4.3)$$

4. Form factors satisfy the cyclic property

$$F_{i_1, \dots, i_n}(\beta_1, \dots, \beta_n + 2\pi i) = F_{i_n, i_1, \dots, i_{n-1}}(\beta_n, \beta_1, \dots, \beta_{n-1}). \quad (4.4)$$

5. Form factors have kinematical singularities which are simple poles at points where two of the rapidity arguments are displaced by $i\pi$. The residue is given by the following formula

$$\begin{aligned} \text{res}_{\beta_i = \beta_n + i\pi} F_{i_1, \dots, i_n}(\beta_1, \dots, \beta_n) &= -i C_{i_n, i_j} F_{i_1, \dots, \hat{i}_j, \dots, i_{n-1}}(\beta_1, \dots, \hat{\beta}_j, \dots, \beta_n) \times \\ &[\delta_{i_1}^{i_1'} \dots \delta_{i_{j-1}}^{i_{j-1}'} S_{i_{n-1} k_1}^{i_{j-1}'} (\beta_{n-1} - \beta_j) S_{i_{n-2} k_2}^{i_{n-2} k_1} (\beta_{n-2} - \beta_j) \dots S_{i_{j+1} i_j}^{i_{j+1} k_{n-j-2}} (\beta_{j+1} - \beta_j) \\ &- S_{k_1, i_1}^{i_j i_1'} (\beta_j - \beta_1) \dots S_{k_{j-2} i_{j-2}}^{k_{j-3} i_{j-2}} (\beta_j - \beta_{j-2}) S_{i_j i_{j-1}}^{k_{j-2} i_{j-1}} (\beta_j - \beta_{j-1}) \delta_{i_{j+1}}^{i_{j+1}'} \dots \delta_{i_{n-1}}^{i_{n-1}'}], \end{aligned} \quad (4.5)$$

where the hat denotes the omission of the corresponding rapidity argument and internal index.

Since in the $O(3)$ sigma model we assume no bound states, there are no more poles. If one has bound states then it is necessary to add a new axiom stating the position and the residue of the bound state poles.

Smirnov proved the following important theorem: if two operators are defined by matrix elements satisfying the axioms (1-5) then they are mutually local.

This is called the locality theorem and it has a profound implication; namely we can take the space of operators defined by the solutions of the axioms (4.5) to generate the local operator algebra of the model.

Now we proceed to give a formula for the solutions of the equations (4.5) which was first written down by Lukyanov. Before doing it we comment on the omission of the conjugate ZF operators $\tilde{Z}^\dagger(\beta)$. They are certainly needed in order to build up the space of asymptotic states. However one can define a representation of the ZF algebra as follows [12]. We take a ground state $|0\rangle$ (this is not the vacuum state of the theory!) and define a module of the ZF algebra by acting with the \tilde{Z}' s on it. Specifically we define a representation π_Z on the space \mathcal{H}_Z which is given by

$$\mathcal{H}_Z = \text{span}\{Z_{i_1}(\beta_1) \dots Z_{i_n}(\beta_n)|0\rangle\}. \quad (4.6)$$

This representation can be interpreted as doing quantum field theory in Rindler space and using angular quantization. This was explained in [17].

Having the representation discussed above one looks for the following structures:

1. The operator of Lorentz boosts K satisfying

$$Z_i(\beta + \Lambda) = \exp(-\Lambda K) Z_i(\beta) \exp(\Lambda K). \quad (4.7)$$

2. A map $O \rightarrow L(O)$ from the space of local operators to the algebra of endomorphisms of π_Z satisfying the following two conditions

$$\begin{aligned} L(O) Z_i(\beta) &= Z_i(\beta) L(O), \\ \exp(\Lambda K) L(O) \exp(-\Lambda K) &= \exp(\Lambda s(O)) L(O). \end{aligned} \quad (4.8)$$

where $s(O)$ denotes the Lorentz spin of the operator O .

If one has these objects then, as shown by Lukyanov [12], the functions

$$F_{i_1, \dots, i_n}(\beta_1, \dots, \beta_n) = \text{Tr}_{\pi_Z} [\exp(2\pi i K) L(O) Z_{i_n}(\beta_n) \dots Z_{i_1}(\beta_1)], \quad (4.9)$$

are solutions of the form-factor axioms.

We now have to give the definition of the dual operators which give us the map L representing the local operators of the model. We introduce the scalar field $\phi'(\alpha)$ satisfying the same relations as ϕ together with its associated field $\bar{\phi}'(\alpha)$. (We give a closer definition of these fields shortly when we introduce an ultraviolet regularization for the free field system.) With the help of these new fields we define the dual vertex operators as follows:

$$\begin{aligned}\Lambda_-(\alpha) &= U'(\alpha), \\ \Lambda_0(\alpha) &= \frac{i}{\sqrt{2}}[\chi'U'(\alpha) - U'(\alpha)\chi'], \\ \Lambda_+(\alpha) &= \frac{1}{2}[\chi'^2U'(\alpha) - 2\chi'U'(\alpha)\chi' + U'(\alpha)\chi'^2].\end{aligned}\tag{4.10}$$

The dual vertex operators U' and the \bar{V}' required for χ' , satisfy similar relations as the unprimed ones, but with the function \bar{g} changed to

$$\bar{g}'(\alpha) = -k^2 \frac{\alpha(\alpha - i\pi)}{4\pi^2}.\tag{4.11}$$

The normalization of χ' which is the analog of the parameter η for χ is chosen to equal $(i\pi)^{1/2}$.

As can be seen from the above definition, χ' plays the role of the other step operator for the quantum group. With the operator P to be introduced later, χ and χ' satisfy all the algebraic relations of $SU(2)_{-1}$ and obey the correct coproduct rule. The proof of this statement can be found in [12].

One can also calculate all the normal ordering relations involving the primed and unprimed operators merely by using the relation

$$\bar{V}'(\delta)V(\beta) = \rho\rho'u(\beta - \delta) : \bar{V}'(\delta)V(\beta) :, \tag{4.12}$$

where the function u is given by

$$u(\beta) = k \frac{i\beta}{2\pi}.\tag{4.13}$$

Now one writes down the functions

$$\begin{aligned}\mathcal{F}_{j_1, \dots, j_l}^{i_1, \dots, i_k}(\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_n) \\ = Tr_{\pi_z}[\exp(2\pi i K)\Lambda_{i_k}(\alpha_k) \dots \Lambda_{i_1}(\alpha_1)\tilde{Z}_{j_l}(\beta_l) \dots \tilde{Z}_{j_1}(\beta_1)],\end{aligned}\tag{4.14}$$

then these functions have to be expanded as

$$\begin{aligned}\mathcal{F}_{j_1, \dots, j_l}^{i_1, \dots, i_k}(\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_n) \\ = \sum_{\{s_j\}} F^{i_1, \dots, i_k}(\alpha_1, \dots, \alpha_k | \{s_j\}) F_{j_1, \dots, j_l}(\{s_j\} | \beta_1, \dots, \beta_n) \times \\ \exp(s_1\alpha_1) \dots \exp(s_k\alpha_k),\end{aligned}\tag{4.15}$$

where the functions F are the required solutions of the form-factor axioms. This means that the functions \mathcal{F} play the role of generating functions for the form-factors of local operators.

The actual calculation of the matrix elements and the traces proceeds through the procedure described in [12]. The free field construction needs a regularization; we choose an ultraviolet cut-off by taking the rapidity interval finite

$$-\frac{\pi}{\epsilon} < \beta < \frac{\pi}{\epsilon}.\tag{4.16}$$

Then we introduce the mode expansion of the free field ϕ as:

$$\phi_\epsilon(\beta) = \frac{1}{\sqrt{2}}(Q - \epsilon\beta P) + \sum_{k \neq 0} \frac{a_k}{i \sinh(\pi k \epsilon)} \exp(ik\epsilon\beta),\tag{4.17}$$

where the oscillator modes satisfy the commutation relation

$$[a_k, a_l] = \frac{\sinh \frac{\pi k \epsilon}{2} \sinh \pi k \epsilon}{k} \exp \frac{\pi |k| \epsilon}{2} \delta_{k, -l},\tag{4.18}$$

and the zero modes satisfy the canonical commutation relation

$$[P, Q] = -i. \quad (4.19)$$

The two-point function and the commutation relations for this field can be calculated and it can be seen that in the limit $\epsilon \rightarrow 0$ we recover the relations satisfied by the field ϕ (see eq. (2.10)). The dual field is given by

$$\phi'_\epsilon(\alpha) = -\frac{1}{\sqrt{2}}(Q - \epsilon\alpha P) + \sum_{k \neq 0} \frac{a'_k}{i \sinh(\pi k \epsilon)} \exp(ik\epsilon\alpha), \quad (4.20)$$

with the oscillators a'_k defined by the relation

$$a'_k \exp\left(\frac{\pi|k|\epsilon}{4}\right) = a_k \exp\left(-\frac{\pi|k|\epsilon}{4}\right) \quad (4.21)$$

The space where these operators live is defined to be

$$\pi_Z^\epsilon = \bigoplus_{k \in \mathbb{Z}} F_{k/\sqrt{2}}, \quad (4.22)$$

where F_p denotes the Fock space built up with the help of the creation operators (as usual, the oscillator modes with negativ indices) from the ground state $|p\rangle$ which satisfies:

$$P|p\rangle = p|p\rangle. \quad (4.23)$$

The operator K , of the Lorentz boost is given by

$$K_\epsilon = i\epsilon H - i\frac{\sqrt{2}}{4}\epsilon P, \quad (4.24)$$

where

$$H = \frac{P^2}{2} + \sum_{k=1}^{\infty} \frac{k^2}{\sinh \frac{\pi k \epsilon}{2} \sinh \pi k \epsilon} a'_{-k} a_k. \quad (4.25)$$

Formally we can write $\pi_Z = \lim_{\epsilon \rightarrow 0} \pi_Z^\epsilon$. This should be understood in the following sense: we make all calculations with the regularized operators and in the end take the limit $\epsilon \rightarrow 0$.

V. CALCULATION OF THE TRACES AND THE PROBLEM OF THE INTEGRATION

We should calculate the traces required for obtaining the integrand of the representation for the form-factors. In [12] it was proposed that one should use the technique of Clavelli and Shapiro which they invented for the calculation of string scattering amplitudes. The prescription is to introduce another copy of the oscillators a_k which commute with a_k and satisfies the same commutation among themselves. Then we make the following definitions:

$$\begin{aligned} \tilde{a}_k &= \frac{a_k}{1 - \exp(-2\pi k \epsilon)} + b_{-k} \quad (k > 0), \\ \tilde{a}_k &= a_k + \frac{b_{-k}}{1 - \exp(-2\pi k \epsilon)} \quad (k < 0). \end{aligned} \quad (5.1)$$

For an operator $O(a_k)$ on the Fock space $F(a)$ we introduce the operator $\tilde{O} = O(\tilde{a}_k)$ by substituting \tilde{a}_k for a_k . Then we can calculate the regularized trace over the non-zero modes as (Tr_F denotes the trace over the Fock module of non-zero modes):

$$Tr_F[\exp(2\pi i K_\epsilon) O] = \frac{\langle 0 | \tilde{O} | 0 \rangle}{\prod_{k=1}^{\infty} (1 - \exp(-2\pi k \epsilon))}, \quad (5.2)$$

and then take the limit $\epsilon \rightarrow 0$. The state $|0\rangle$ is nothing else but the ground state in the product Fock module $F[a] \otimes F[b]$. The trace over the zero modes just gives the $SU(2)$ superselection rules for the matrix elements. Using the results in [12] we get for the general trace

$$Tr_{\pi_Z}[\exp(2\pi i K)U'(\alpha_k) \dots U'(\alpha_1)\bar{V}'(\delta_p) \dots \bar{V}'(\delta_1)U(\beta_n) \dots U(\beta_1)\bar{V}(\gamma_r) \dots \bar{V}(\gamma_1)] \text{ nonnumber} \quad (5.3)$$

$$(5.4)$$

the following result:

$$\begin{aligned} & \mathcal{C}_1^{-n} \mathcal{C}_2^{\frac{n+r}{2}} \mathcal{C}_2'^{\frac{p+k}{2}} 2^{-r-n} i^{n+p+k} \eta^r \eta'^{-p-k} \delta_{2n-2r+2p-2k,0} \prod_{i<j} \bar{G}(\beta_i - \beta_j) \prod_{i<j} \bar{G}(\gamma_i - \gamma_j) \times \\ & \prod_{i,j} \bar{G}^{-1}(\gamma_i - \beta_j) \prod_{i<j} \bar{G}'(\alpha_i - \alpha_j) \prod_{i<j} \bar{G}'(\delta_i - \delta_j) \prod_{i,j} \bar{G}'^{-1}(\delta_i - \alpha_j) \prod_{i,j} \bar{H}(\beta_i - \alpha_j) \times \\ & \prod_{i,j} \bar{H}^{-1}(\beta_i - \delta_j) \prod_{i,j} \bar{H}(\gamma_i - \delta_j) \prod_{i,j} \bar{H}^{-1}(\gamma_i - \alpha_j). \end{aligned} \quad (5.5)$$

Here we defined the new functions as:

$$\begin{aligned} \bar{G}(\beta) &= -\frac{\mathcal{C}_2}{4}(\beta + i\pi) \sinh \beta, \\ \bar{G}'(\alpha) &= -\mathcal{C}_2' \frac{\sinh \alpha}{\alpha + i\pi}, \\ \bar{H}(\alpha) &= -\frac{2}{\cosh \alpha}, \end{aligned} \quad (5.6)$$

and the constants are given by:

$$\begin{aligned} \mathcal{C}_1 &= \exp \left[-\int_0^\infty dt \frac{\sinh^2 \frac{t}{2} \exp(-t)}{t \sinh 2t \cosh t} \right], \\ \mathcal{C}_2 &= \mathcal{C}_2'^{-1} = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3}. \end{aligned} \quad (5.7)$$

(see [12]). The trace over the zero modes however contains an infinite constant which we have set equal to unity.

We also have to specify the integration over the variables of the screening charges. We face the following problem: when we take the integration contour as given in the $SU(2)$ Thirring model (see in [12]) and try to take the limit which gives the $O(3)$ sigma model, new double poles arise in the integration over the γ_i variables whose positions depend on the β_j variables. This phenomenon was already noticed by Smirnov [6]. The contour gets pinched by the two poles approaching each other and the integrals diverge. This means that the limit to the $O(3)$ model is singular. To deal with the integral we introduce the prescription that one should take the coefficient of the most divergent term in the integral. This is equivalent to the procedure that Smirnov took. The other integrals cause no problems: the contours are specified as earlier for the calculation of vacuum matrix elements but they must lie within the strip $-i\pi < \gamma_i, \delta_j < i\pi$. In the following section we show the calculation for the simplest nontrivial case.

VI. AN EXPLICIT EXAMPLE

What we will calculate using the method outlined above is the two-particle form-factor for the $O(3)$ current. This was known long before but what we obtain is the generating function for the two-particle form-factors of a sequence of isovector operators with different Lorentz spin. The current is just the lowest element of the sequence.

The two-particle form-factor of the current is the matrix element:

$$f_\mu^{ABC}(\beta_1, \beta_2) = \langle 0 | j_\mu^A(0) | \beta_1, B, \beta_2, C \rangle. \quad (6.1)$$

The tensor structure of the form-factor is given by the Levi-Civita symbol ϵ^{ABC} . Since this matrix element has only one independent component it is enough to calculate only one function. This is given by the following:

$$\mathcal{F}(\alpha, \beta_1, \beta_2) = \text{Tr}_{\pi_Z}[\exp(2\pi i K) Z'_-(\alpha) Z_+(\beta_1) Z_0(\beta_2)]. \quad (6.2)$$

This is the simplest matrix element allowed by the superselection rules which possesses the required isospin properties. The trace we have to calculate is:

$$\text{Tr}_{\pi_Z}[\exp(2\pi i K) U'(\alpha) U(\beta_1) (\bar{V}(\gamma) U(\beta_2) - U(\beta_2) \bar{V}(\gamma))], \quad (6.3)$$

which gives the following result (up to a constant multiplier, which is unknown since the trace of the zero modes, as noted above, contains an infinite constant):

$$\frac{1}{\cosh(\beta_1 - \alpha) \cosh(\beta_2 - \alpha)} (\beta_{21} + i\pi) \sinh(\beta_{21}) \left[\frac{\cosh(\gamma - \alpha)}{(\gamma - \beta_1 + i\pi) \sinh(\gamma - \beta_1) (\beta_2 - \gamma + i\pi) \sinh(\beta_2 - \gamma)} - \frac{\cosh(\gamma - \alpha)}{(\gamma - \beta_1 + i\pi) \sinh(\gamma - \beta_1) (\gamma - \beta_2 + i\pi) \sinh(\gamma - \beta_2)} \right]. \quad (6.4)$$

The integration over the arguments of the screening charges gives:

$$\frac{1}{\cosh(\beta_1 - \alpha) \cosh(\beta_2 - \alpha)} (\beta_{21} + i\pi) \sinh(\beta_{21}) \left[\int_{C_1} \frac{d\gamma}{2\pi} \frac{\cosh(\gamma - \alpha)}{(\gamma - \beta_1 + i\pi) \sinh(\gamma - \beta_1) (\beta_2 - \gamma + i\pi) \sinh(\beta_2 - \gamma)} - \int_{C_2} \frac{d\gamma}{2\pi} \frac{\cosh(\gamma - \alpha)}{(\gamma - \beta_1 + i\pi) \sinh(\gamma - \beta_1) (\gamma - \beta_2 + i\pi) \sinh(\gamma - \beta_2)} \right], \quad (6.5)$$

where the contours run on the complex γ plane from $\Im\gamma = -\infty$ to $\Im\gamma = \infty$ and pass through the double poles which are $\gamma = \beta_1 - i\pi$ and $\gamma = \beta_2 + i\pi$ in the case of C_1 and $\gamma = \beta_1 - i\pi$ and $\gamma = \beta_2 - i\pi$ in the case of C_2 . As we said before we interpret this integral as the limit of two coinciding poles giving a double pole and calculate the coefficient of the most singular term. This is a procedure that can be easily seen to be consistent with the form-factor axioms 2-4 (see section 4) but it does not yield the correct analytic structure. It was already noted when we discussed the singularities of the operator product of the ZF operators (see end of section III) that there will be problems with kinematical singularities. Performing this calculation yields the following:

$$\frac{(\beta_{21} + i\pi) \sinh(\beta_{21})}{\cosh(\beta_1 - \alpha) \cosh(\beta_2 - \alpha)} \left[\frac{\cosh(\beta_2 - \alpha)}{(\beta_{21} + 2i\pi) \sinh(\beta_{21})} - \frac{\cosh(\beta_1 - \alpha)}{\beta_{21} \sinh(\beta_{21})} \right], \quad (6.6)$$

which simplifies to

$$\frac{2i\pi(\beta_{21} + i\pi)}{(\beta_{21} + 2i\pi)\beta_{21}} \left[\frac{1}{\cosh(\beta_1 - \alpha)} + \frac{1}{\cosh(\beta_2 - \alpha)} \right]. \quad (6.7)$$

Here we see the problem with the analytic structure: the function above does not have the correct kinematical pole at $\beta_2 = \beta_1 + i\pi$ but it does have poles at $\beta_2 = \beta_1$ and $\beta_2 = \beta_1 + 2i\pi$. This can be dealt with by introducing a factor which does not destroy the symmetry properties of the function but restores the correct analyticity. The factor must be a function of rapidity difference only, must satisfy periodicity and symmetry and it must have the correct analytical properties. A function with these properties is

$$f(\beta_{21}) = \tanh^2 \frac{\beta_{21}}{2}. \quad (6.8)$$

In the case of general $2n$ -particle form-factor the correct procedure is to introduce such factors for each pairing of particle rapidities. This is true since the analytic behaviour in all two-particle rapidity differences is the same (because the symmetric construction of the form-factor solutions). This appears to be a general prescription for curing the bad analytic behaviour (see [6]). So the final generating function is:

$$\frac{2i\pi(\beta_{21} + i\pi)}{(\beta_{21} + 2i\pi)\beta_{21}} \left[\frac{1}{\cosh(\beta_1 - \alpha)} + \frac{1}{\cosh(\beta_2 - \alpha)} \right] \tanh^2 \frac{\beta_{21}}{2}. \quad (6.9)$$

Now expanding the generating function around $\exp(\alpha) = 0$ and $\exp(-\alpha) = 0$, respectively, we find the first coefficient to be

$$\frac{2i\pi(\beta_{21} + i\pi)}{(\beta_{21} + 2i\pi)\beta_{21}} [\exp(\beta_1) + \exp(\beta_2)] \tanh^2 \frac{\beta_{21}}{2}, \quad (6.10)$$

and

$$\frac{-2i\pi(\beta_{21} + i\pi)}{(\beta_{21} + 2i\pi)\beta_{21}} [\exp(-\beta_1) + \exp(-\beta_2)] \tanh^2 \frac{\beta_{21}}{2}. \quad (6.11)$$

These are the two-particle form-factors of two operators with isospin 1 and Lorentz spin ± 1 , respectively, which can be identified as the two light-cone components of the $O(3)$ current. Expanding the generating function to higher orders the form-factors of infinitely many operators can be found.

In the literature one can find the result for this form-factor [13] which we quote for comparison:

$$\begin{aligned} \langle 0 | j_\omega^A(0) | \beta_1, B, \beta_2, C \rangle &= -\frac{i\pi}{8} \epsilon^{ABC} \frac{\beta_{12} - i\pi}{\beta_{12}(2i\pi - \beta_{12})} \tanh^2 \frac{\beta_{12}}{2} \\ &(-m\omega) (\exp(-\omega\beta_1) + \exp(-\omega\beta_2)). \end{aligned} \quad (6.12)$$

This agrees with the result derived here up to a normalization factor. This factor can be fixed if one requires that the charge obtained from the $O(3)$ current should take the correct value on the many-particle states. Here $\omega = \pm$ denotes the Lorentz components of the current.

VII. CONCLUSIONS

We would like now to summarize the results briefly. We succeeded in writing down a free field representation for the $O(3)$ ZF algebra and obtained a representation for the solutions of the form-factor equation. We saw that the representation obtained by the fusion procedure is not completely satisfactory since a hand-made, although simple correction, is needed to achieve the necessary analytical structure of the form-factors. First we would like to comment on the origin and the possible way of correcting for this problem.

What lies behind all the construction is that one divides the S-matrix into two factors: a scalar function, which is nothing else but the scattering amplitude in the highest isospin channel (see the function $S_0(\beta)$ in the case of $SU(2)$ Thirring model) and a tensor part, which carries all of the isospin structure of the S-matrix. Then we solve a Riemann-Hilbert problem and find the function which obeys the equation

$$g(-\beta) = S_0(\beta)g(\beta). \quad (7.1)$$

With the help of this function a free field can be defined which has g as the two-point function. But the solution of the above equation is not unique; in fact, there is an ambiguity up to an even function as can be seen easily. The solution must be fixed by requiring some analyticity conditions, which in turn fixes the singularities in the operator products and the pole structure of the form-factors. This suggests that the factorization of the $O(3)$ S-matrix obtained from the bootstrap is not the correct one and this necessitates the introduction of the extra \tanh^2 factors in the form-factor expression. A possible way out would be to find a new factorization of the $O(3)$ S-matrix, which gives a new free field representation satisfying the required analytic properties. We hope to investigate this possibility further.

The above discussion also shows that there may be a generalization of this free field construction to at least a class of further integrable models; it seems to be important to clear up whether such a generalization exists and try to construct it. Another relevant question is whether the family of solutions that can be obtained from the free field representation is complete. We mean by this completeness the question of whether the operators defined by all of these solutions form a closed algebra of local operators. If the answer is yes then the constructed set of operators should be taken as the definition of the theory in the sense of algebraic quantum field theory and then the question of the dynamics can be thoroughly investigated. Namely one can try to show that the fields constructed in this way satisfy the equations of motion. For example, the constraint equation for the $O(3)$ field ($n^a n^a = 1$) can be proved if it can be shown that there exists no Lorentz and isoscalar operator with canonical dimension zero other than the unit operator in the model. This can be checked by showing that the two and higher particle form-factors of such an operator necessarily vanish. There is some evidence for that (the two-particle form-factor vanishes and within the frame of a polynomial ansatz this is also true for the four-particle one [18]). The other field equation is just the conservation of the $O(3)$ current, which is satisfied for all many-particle form-factors of the current (see [6]). However one should also prove the relation between the field n^a and the current j_μ^a which looks

$$j_\mu^a = \epsilon^{abc} n^b \partial_\mu n^c \quad (7.2)$$

A confirmation for this relation would be to show that there is no SU(2) singlet conserved current in the operator algebra since that would mean $j^a n^a = 0$.

The discussion of the form-factors is relevant to the question of correlation functions as well, since the correlation functions can be obtained in the model from the form-factors in terms of infinite series expansions. Up to now, however, there is no nice way to handle such series expansions and deciding whether they are convergent and how fast they converge. However some recent studies [13] show that they can be compared to the perturbation theory results and they show a reasonable agreement also with lattice simulation results. With a better understanding of the properties of form-factors one may get closer to the solution of these problems as well.

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APPENDIX A: A DIRECT PROOF FOR THE ZF RELATIONS

We would like to present a method for proving the ZF scattering relations for the O(3) model. These relations are the analogues of (2.5) with the O(3) S-matrix instead of the one for the SU(2) Thirring model. The reasoning given above used the fusion procedure. There is also a direct way of demonstrating the O(3) ZF relations without relying on the connection between the SU(2) Thirring model and the O(3) sigma model.

The relation we want to prove is

$$\tilde{Z}_I(\beta_1) \tilde{Z}_J(\beta_2) = \tilde{S}_{I,J}^{K,L}(\beta_{12}) \tilde{Z}_L(\beta_2) \tilde{Z}_K(\beta_1). \quad (A1)$$

This means nine equations to prove because the indices I, J, K, L take on 3 distinct values. We will classify these equations by the number of the screening charges appearing in them. The only relation without screening charges is:

$$\tilde{Z}_+(\beta_1) \tilde{Z}_+(\beta_2) = \tilde{S}_{+,+}^{+,+}(\beta_{12}) \tilde{Z}_+(\beta_2) \tilde{Z}_+(\beta_1). \quad (A2)$$

Substituting here the expression (3.9) for \tilde{Z}_+ and normal ordering the two side of the equation with the help of eq. (3.8) one can see that the above relation is trivially satisfied.

There are two relations with one screening charge. We discuss one of them since the other one is similarly treated. We should prove that

$$\tilde{Z}_+(\beta_1) \tilde{Z}_0(\beta_2) = \tilde{S}_{+,0}^{0,+}(\beta_{12}) \tilde{Z}_+(\beta_2) \tilde{Z}_0(\beta_1) + \tilde{S}_{+,0}^{+,0}(\beta_{12}) \tilde{Z}_0(\beta_2) \tilde{Z}_+(\beta_1). \quad (A3)$$

We substitute here eq. (3.9) to get

$$U(\beta_1)(\chi U(\beta_2) - U(\beta_2)\chi) = \tilde{S}_{+,0}^{0,+}(\beta_{12}) U(\beta_2)(\chi U(\beta_1) - U(\beta_1)\chi) + \tilde{S}_{+,0}^{+,0}(\beta_{12})(\chi U(\beta_2) - U(\beta_2)\chi) U(\beta_1). \quad (A4)$$

The screening charge contains contour integration. We can write out the integrands and after normal ordering according to eq. (3.8) we get the following for the equality of the integrands:

$$\begin{aligned} & \bar{g}(\beta_2 - \beta_1) \bar{g}^{-1}(\gamma - \beta_1) (\bar{g}^{-1}(\beta_2 - \gamma) - \bar{g}^{-1}(\gamma - \beta_2)) = \\ & \tilde{S}_{+,0}^{0,+}(\beta_{12}) \bar{g}(\beta_1 - \beta_2) \bar{g}^{-1}(\gamma - \beta_2) (\bar{g}^{-1}(\beta_1 - \gamma) - \bar{g}^{-1}(\gamma - \beta_1)) + \\ & \tilde{S}_{+,0}^{+,0}(\beta_{12}) \bar{g}(\beta_1 - \beta_2) \bar{g}^{-1}(\beta_1 - \gamma) (\bar{g}^{-1}(\beta_2 - \gamma) - \bar{g}^{-1}(\gamma - \beta_2)), \end{aligned} \quad (A5)$$

which is a simple algebraic identity that can be proven by a bit of calculation. The variable γ is the integration variable for the screening charge χ . Then one can integrate over γ . Using the rules for the contours given in section II one can draw the different contours on the complex γ plane and show that they can be deformed into each other without encountering any poles. This completes the proof of this particular case. The other case with one screening charge corresponding to the scattering relation $\tilde{Z}_0 \tilde{Z}_+ \rightarrow \tilde{Z}_0 \tilde{Z}_+ + \tilde{Z}_+ \tilde{Z}_0$ (in a selfexplanatory notation where 0, +, - denote the SU(2) quantum number of the corresponding ZF operators) can be proven similarly.

The case of two screening charges is much more involved so we just sketch the method. There are 3 such identities: the $\tilde{Z}_0\tilde{Z}_0 \rightarrow \tilde{Z}_0\tilde{Z}_0 + \tilde{Z}_+\tilde{Z}_- + \tilde{Z}_-\tilde{Z}_+$, the $\tilde{Z}_+\tilde{Z}_- \rightarrow \tilde{Z}_0\tilde{Z}_0 + \tilde{Z}_+\tilde{Z}_- + \tilde{Z}_-\tilde{Z}_+$ and the $\tilde{Z}_-\tilde{Z}_+ \rightarrow \tilde{Z}_0\tilde{Z}_0 + \tilde{Z}_+\tilde{Z}_- + \tilde{Z}_-\tilde{Z}_+$ case. Since we have two integration variables we can symmetrize in them since the integral is independent of which is named γ_1 and which is named γ_2 . We verified the identities by doing calculations with Maple. The case of three and four screening charges is in principle also similar, but quite tedious so it is better to argue with the help of the bootstrap fusion idea.

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