

A Haar meager set that is not strongly Haar meager

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ABSTRACT. Following Darji, we say that a Borel subset B of an abelian Polish group G is *Haar meager* if there is a compact metric space K and a continuous function $f : K \rightarrow G$ such that the preimage of the translate, $f^{-1}(B + g)$ is meager in K for every $g \in G$. The set B is called *strongly Haar meager* if there is a compact set $C \subseteq G$ such that $(B + g) \cap C$ is meager in C for every $g \in G$. The main open problem in this area is Darji's question asking whether these two notions are the same. Even though there have been several partial results suggesting a positive answer, in this paper we construct a counterexample. More specifically, we construct a G_δ set in \mathbb{Z}^ω that is Haar meager but not strongly Haar meager. We also show that no F_σ counterexample exists, hence our result is optimal.

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1. INTRODUCTION

The notion of Haar meager sets in Polish groups was first introduced by Darji in [2] in 2013 as a topological counterpart to the so-called Haar null sets. Although in this paper we will only study abelian Polish groups, we note that [3] have generalized this notion for arbitrary Polish groups. For a recent survey about the basic properties of Haar null sets and Haar meager sets, see [5].

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Definition 1.1. Let $(G, +)$ be an abelian Polish group. A set $A \subseteq G$ is said to be *Haar meager* if there are a Borel set $B \supseteq A$, a nonempty compact metric space K and a continuous function $f : K \rightarrow G$ such that $f^{-1}(B + g)$ is meager in K for every $g \in G$. A function f satisfying this is called a *witness function* for A .

Haar meager sets form a σ -ideal which is contained in the σ -ideal of meager sets (see [2, Theorems 2.9 and 2.2]). In a locally compact group, these two ideals coincide, but if G is not locally compact, then there exists a closed meager set that is not Haar meager (see [2, Corollary 2.5 and Example 2.6]).

When we prove that a set is Haar meager, it is often possible to use a witness function which is just the identity of G restricted to some compact subset of G . This observation yields the following:

Definition 1.2. Let $(G, +)$ be an abelian Polish group. A set $A \subseteq G$ is said to be *strongly Haar meager* if there are a Borel set $B \supseteq A$ and a nonempty compact set $C \subseteq G$ such that $(B + g) \cap C$ is meager in C for every $g \in G$. A compact set C satisfying this is called a *witness set* for A .

Remark. It is easy to see that a set $A \subseteq G$ is strongly Haar meager if and only if it is Haar meager and has an injective witness function.

One of the most important questions in the topic is [2, Problem 2], which asks if these two notions are equivalent:

Question 1.3 (Darji). *Is every Haar meager set strongly Haar meager?*

We remark that in the case of Haar null sets [1, Theorem 4.3] answers the analogous question affirmatively.

To answer Question 1.3 affirmatively, it would have been a natural idea to prove that if $f : K \rightarrow G$ is a witness function for a Haar meager set, then the choice $C = f(K)$ satisfies the requirements of Definition 1.2. However [3, Example 11] ruled out this by constructing a Haar meager set where the image of one particular witness function does not satisfy the requirements of Definition 1.2:

Example 1.4 (Doležal-Rmoutil-Vejnar-Vlasák). *There exists a G_δ Haar meager set $A \subseteq \mathbb{R}$, a compact metric space K and a witness function $f : K \rightarrow \mathbb{R}$ such that $A \cap f(K)$ is comeager in $f(K)$.*

The result [1, Theorem 5.13] shows that in a certain class of Polish groups the Haar meager sets and the strongly Haar meager sets coincide:

Definition 1.5. A topological group G is called *hull-compact* if each compact subset of G is contained in a compact subgroup of G .

Theorem 1.6 (Banach-Głab-Jabłońska-Swaczyna). *If the abelian Polish group G is hull-compact, then every Haar meager subset of G is strongly Haar meager.*

Example 1.7. *It is not very hard to verify that the abelian Polish group $(\mathbb{Q}/\mathbb{Z})^\omega$ is hull-compact (where we endow \mathbb{Q} with the discrete topology).*

In this paper we answer the question of Darji negatively by constructing a counterexample in the group \mathbb{Z}^ω :

Theorem 1.8. *In the abelian Polish group \mathbb{Z}^ω , there exists a G_δ set R that is Haar meager but not strongly Haar meager.*

In Theorem 5.7 we also prove that this counterexample is as simple as possible: every F_σ Haar meager set is strongly Haar meager. We also note that in Claim 5.5 we prove that our example is in fact a so-called Haar nowhere dense set.

As an additional motivation, we mention that according to part (2) of [1, Theorem 13.8] the *generic* variants of these notions (when we require not only one witness, but comeager many witnesses) do coincide:

Theorem 1.9 (Banach-Głab-Jabłońska-Swaczyna). *Assume that G is an abelian Polish group and $B \subseteq G$ is a Borel set. Then the following are equivalent:*

- (1) *in the Polish space $\mathcal{C}(\{0, 1\}^\omega, G)$ of continuous functions from $\{0, 1\}^\omega$ to G (endowed with the compact-open topology) the set*

$$\{f \in \mathcal{C}(\{0, 1\}^\omega, G) : f \text{ witnesses that } B \text{ is Haar meager}\} \text{ is comeager,}$$

- (2) *in the Polish space $\mathcal{K}(G)$ of nonempty, compact subsets of G*

$$\{C \in \mathcal{K}(G) : C \text{ witnesses that } B \text{ is strongly Haar meager}\} \text{ is comeager.}$$

In (1) we only considered the potential witness functions whose domain is the Cantor set $\{0, 1\}^\omega$. This is a natural restriction, as according to [4, Proposition 3], every Haar meager set has a witness function whose domain is $\{0, 1\}^\omega$.

2. PRELIMINARIES

As usual, \mathbb{N} and ω will both denote the set of nonnegative integers. We will use “ \mathbb{N} ” when we use this set as a topological space (with the discrete topology) and use “ ω ” when we use it as an ordinal or index set.

We will use some notation related to sequences (i.e. functions s whose domain is either ω or $\{0, 1, \dots, n-1\}$ for some natural number n). As usual, s_k denotes the element of the sequence s with index k (i.e. $s_k = s(k)$ for an index k that is in the domain of s). If S is an arbitrary set, then $S^{<\omega} = \bigcup_{n \in \omega} S^n$ denotes the set of finite sequences of elements of S and \emptyset denotes the empty sequence $\emptyset \in S^0$. For $s \in S^{<\omega}$, $|s|$ denotes the length of s .

If s and s' are two sequences and s is finite (s' may be infinite), then $s \hat{\ } s'$ denotes concatenation of s and s' . In particular if $s \in S^{<\omega}$ and $\ell \in S$ is an additional element, then $s \hat{\ } \ell$ denotes the sequence of length $|s| + 1$ which consists of the elements of s followed by ℓ as the last element. If x is a (finite or infinite) sequence of length at least n , then $x \upharpoonright n$ denotes the sequence formed by the first n elements of x .

If s, s' are two sequences then we say that $s \subseteq s'$ if there is a sequence t such that $s' = s \hat{\ } t$ (if we consider functions and sequences as sets of pairs, this is just the usual inclusion relation). For a sequence $s \in S^{<\omega}$, let $[s] \subseteq S^\omega$ be the set of

sequences which have s as an initial segment, i.e.

$$[s] = \{x \in S^\omega : s \subseteq x\}.$$

We say that a set $A \subseteq S^{<\omega}$ is cofinal if for every $s \in S^{<\omega}$ there exists an $a \in A$ such that $s \subseteq a$ (i.e. s is an initial segment of a).

We will use $\mathcal{K}(X)$ to denote the *nonempty* compact sets of a space X , equipped with the Vietoris topology. The well-known result [7, Theorem 4.25] states that if X is Polish, then $\mathcal{K}(X)$ is also Polish. We will also use the following fact:

Fact 2.1 (Michael [8, 4.13.1]). *If X is a zero-dimensional Polish space (that is, X is Polish and has a basis consisting of clopen sets), then $\mathcal{K}(X)$ is also zero-dimensional.*

A standard reference book for notions in descriptive set theory is [7].

3. TRANSLATING THE COMPACT SETS APART

This section is motivated by the results and ideas in [6, 9, 10]. These papers prove slightly weaker claims than our Theorem 3.1, but work in a more general setting. (Our proof relies on the structure of \mathbb{Z}^ω to make the calculations shorter and simpler.)

Let

$$H = \{(K, x) \in \mathcal{K}(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega : x \in K\}.$$

Note that H is a closed set in the product space $\mathcal{K}(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega$.

Theorem 3.1. *There exists a map $t : \mathcal{K}(\mathbb{Z}^\omega) \rightarrow \mathbb{Z}^\omega$ so that the map $T : H \rightarrow \mathbb{Z}^\omega$ defined by*

$$T(K, x) = x + t(K)$$

is a homeomorphism between H and the set

$$F = T(H) = \bigcup_{K \in \mathcal{K}(\mathbb{Z}^\omega)} (K + t(K)),$$

where the union is disjoint. Moreover, F is a closed subset of \mathbb{Z}^ω and satisfies that

$$(K + t(K) + \{-1, 0, 1\}^\omega) \cap F = K + t(K)$$

for each $K \in \mathcal{K}(\mathbb{Z}^\omega)$.

Proof. As Fact 2.1 states that $\mathcal{K}(\mathbb{Z}^\omega)$ is zero-dimensional, we may apply [7, Theorem 7.2] to get an embedding

$$c : \mathcal{K}(\mathbb{Z}^\omega) \rightarrow \{-1, 1\}^\omega.$$

Define the (clearly continuous) function

$$b : \mathcal{K}(\mathbb{Z}^\omega) \rightarrow \omega^\omega, \quad b(K)_n = \max\{|x_n| : x \in K\} + 1.$$

For each $n \in \omega$ let

$$t(K)_n = 3 \cdot b(K)_n \cdot c(K)_n.$$

It is clear that both t and

$$T : H \rightarrow \mathbb{Z}^\omega, \quad T(K, x) = x + t(K)$$

are continuous.

Fact 3.2. *Assume that $(K, x) \in H$, $y = T(K, x)$ and $n \in \omega$. Then it is easy to verify that*

- (1) $|y_n| > b(K)_n$ and
- (2) $c(K)_n = \begin{cases} +1 & \text{if } y_n > 0, \\ -1 & \text{if } y_n < 0. \end{cases}$

Claim 3.3. *T is injective.*

Proof. Assume that $(K, x), (K', x') \in H$ and $T(K, x) = T(K', x')$. Using Fact 3.2 part (2), $c(K) = c(K')$, but then $K = K'$, as c is injective. This also implies that

$$x = T(K, x) - t(K) = T(K', x') - t(K') = x'. \quad \square$$

Claim 3.4. *$F = T(H)$ is closed and the map $T^{-1} : F \rightarrow H$ is continuous.*

Proof. Assume that $y^{(m)} \in F$ for each $m \in \omega$ and this sequence converges to some $y^* \in \mathbb{Z}^\omega$. As $F = T(H)$, there are compact sets $K^{(m)} \in \mathcal{K}(\mathbb{Z}^\omega)$ and sequences $x^{(m)} \in K^{(m)}$ such that $y^{(m)} = T(K^{(m)}, x^{(m)})$ for each $m \in \omega$. It is sufficient to prove that $((K^{(m)}, x^{(m)}))_{m \in \omega}$ converges to some $(K^*, x^*) \in H$. (If this holds, then $y^* = T(K^*, x^*) \in F$ also demonstrates that F is closed.)

Using Fact 3.2 part (2) and the convergence of $y^{(m)}$ yields that $(c(K^{(m)}))_{m \in \omega}$ converges to some element $\gamma \in \{-1, 1\}^\omega$. Our next step is to prove that γ is contained in the image of c .

Notice that

$$f : \omega \rightarrow \omega, \quad f(n) = \sup_{m \in \omega} |y_n^{(m)}|$$

is a well-defined function, because for each $n \in \omega$ the sequence $(|y_n^{(m)}|)_{m \in \omega}$ is convergent and therefore bounded. Using Fact 3.2 part (1),

$$f(n) \geq |y_n^{(m)}| > b(K^{(m)})_n \quad \text{for each } n, m \in \omega.$$

Applying the definition of b , this implies that for each $m \in \omega$,

$$K^{(m)} \in \mathcal{K}(\underbrace{\{z \in \mathbb{Z}^\omega : |z_n| < f(n) \text{ for each } n \in \omega\}}_{\text{compact}}).$$

As the (nonempty) compact subsets of a compact space form a compact space themselves, $(K^{(m)})_{m \in \omega}$ has a subsequence that converges to some compact set K^* . Applying this, the continuity of c and the fact that $\lim_{m \in \omega} c(K^{(m)}) = \gamma$ exists, we obtain that

$$\gamma = \lim_{m \in \omega} c(K^{(m)}) = c(K^*).$$

As c was an embedding, this implies that $K^{(m)}$ converges to K^* (when $m \rightarrow \infty$).

As t is continuous and $y^{(m)}$ is convergent, this implies that $x^{(m)} = y^{(m)} - t(K^{(m)})$ is also convergent to some $x^* \in \mathbb{Z}^\omega$. Finally $(K^*, x^*) \in H$ follows from the fact that $(K^{(m)}, x^{(m)}) \rightarrow (K^*, x^*)$ and H is a closed set. \square

This claim implies that F is closed and T is a homeomorphism between H and F . It is clear that

$$F = \bigcup_{K \in \mathcal{K}(\mathbb{Z}^\omega)} (K + t(K)).$$

To conclude the proof, we will fix an arbitrary $K \in \mathcal{K}(\mathbb{Z}^\omega)$ and show that

$$(K + t(K) + \{-1, 0, 1\}^\omega) \cap F = (K + t(K)).$$

Fix an arbitrary $z \in (K + t(K) + \{-1, 0, 1\}^\omega) \cap F$. There are $x \in K$, $\varepsilon \in \{-1, 0, 1\}^\omega$, $K' \in \mathcal{K}(\mathbb{Z}^\omega)$ and $x' \in K'$ such that

$$z = \underbrace{x + t(K)}_{=H(K,x)} + \varepsilon = \underbrace{x' + t(K')}_{=H(K',x')}.$$

Using both parts of Fact 3.2 and the fact that $b(K)_n \geq 1$ for each $n \in \omega$ it is easy to verify that

$$c(K)_n = 1 \iff z_n > 0 \iff c(K')_n = 1 \text{ for each } n \in \omega.$$

As c is injective, this means that $K' = K$, so $z \in K + t(K)$, as claimed. \square

Remark. An analogous proof would work in the case when $\{-1, 0, 1\}^\omega$ is replaced in the statement by any other compact subset $C \subseteq \mathbb{Z}^\omega$ that contains the all zero sequence.

4. CONSTRUCTION OF THE WITNESS FUNCTION

The goal of this section is to construct a compact metric space K_0 and a function $f_0 : K_0 \rightarrow \mathbb{Z}^\omega$ that will witness the Haar meagerness of our example. In order to do this, we will introduce some structures and prove elementary claims about their properties.

We say that a sequence s is *m-segmented* if it is the concatenation of constant sequences of length 2^m . More formally, this means the following:

Definition 4.1. Let S be an arbitrary set and $m \in \mathbb{N}$ be a nonnegative integer. An infinite sequence $s \in S^\omega$ is *m-segmented* if $s_{q \cdot 2^m + r_1} = s_{q \cdot 2^m + r_2}$ for all integers $q \in \mathbb{N}$ and $0 \leq r_1, r_2 < 2^m$. A finite sequence $s \in S^{<\omega}$ is *m-segmented* if $|s| = Q \cdot 2^m$ for some $Q \in \mathbb{N}$ and $s_{q \cdot 2^m + r_1} = s_{q \cdot 2^m + r_2}$ for all integers $0 \leq q < Q$ and $0 \leq r_1, r_2 < 2^m$.

Notice that an *m-segmented* sequence is also *m'-segmented* for all $0 \leq m' \leq m$.

Definition 4.2. Let us fix a sequence $b_s \in \{0, 1\}^{<\omega}$ for each $s \in \omega^{<\omega}$ in a way that it satisfies the following properties:

- (1) $b_\emptyset = \emptyset$,
- (2) $|b_s|$ is divisible by $2^{|s|}$,
- (3) if $s \in \omega^{<\omega}$ and β is a nonempty finite $|s|$ -segmented sequence such that $|b_s| + |\beta|$ is divisible by $2^{|s|+1}$, then there is exactly one $\ell \in \omega$ such that $b_{s \frown \ell} = b_s \frown \beta$,
- (4) conversely, if $s \in \omega^{<\omega}$ and $\ell \in \omega$, then $b_{s \frown \ell}$ can be written as $b_{s \frown \ell} = b_s \frown \beta$ where β is a nonempty finite $|s|$ -segmented sequence satisfying that $|b_s| + |\beta|$ is divisible by $2^{|s|+1}$.

It is clear that using recursion we can choose a system $\{b_s\}_{s \in \omega^{<\omega}}$ that satisfies these properties.

Definition 4.3. Define the set $C_s \subseteq \{0, 1\}^\omega \subseteq \mathbb{Z}^\omega$ by

$$C_s = \{b_s \wedge x : x \in \{0, 1\}^\omega \text{ and } x \text{ is } |s|\text{-segmented}\}.$$

Fact 4.4. *The sets C_s have the following properties:*

- (1) $C_\emptyset = \{0, 1\}^\omega$,
- (2) C_s is compact for each $s \in \omega^{<\omega}$,
- (3) if $s \subseteq s'$, then $C_s \supseteq C_{s'}$.

Proof. Properties (1) and (2) are trivial, property (3) can be proved by using induction on the value of $(|s'| - |s|)$ and applying property (4) of Definition 4.2. \square

Claim 4.5. *Assume that $s \in \omega^{<\omega}$ and U is a nonempty relatively open subset of C_s . Then there are infinitely many indices $\ell \in \omega$ such that $C_{s \smallfrown \ell} \subseteq U$.*

Proof. Fix an arbitrary element $u \in U$. As U is relatively open, there exists an $n_0 \in \omega$ such that $C_s \cap [u \upharpoonright n] \subseteq U$ for each $n \geq n_0$. We may also assume that $n_0 > |b_s|$.

Let us define the infinite set

$$N = \{n \in \omega : n \geq n_0 \text{ and } 2^{|s|+1} \text{ divides } n\}$$

and consider an arbitrary $n \in N$.

As $U \subseteq C_s$, the sequence u can be written as $b_s \wedge x$ where $x \in \{0, 1\}^\omega$ is an $|s|$ -segmented sequence. Using property (2) of Definition 4.2, $2^{|s|}$ divides $n - |b_s|$, therefore it is clear that $\beta = x \upharpoonright (n - |b_s|)$ is $|s|$ -segmented. Also notice that β is nonempty because $n \geq n_0 > |b_s|$ and $|b_s| + |\beta| = n$ is divisible by $2^{|s|+1}$, and thus by property (3) of Definition 4.2, there exists an index $\ell_n \in \omega$ such that

$$u \upharpoonright n = b_s \wedge (x \upharpoonright (n - |b_s|)) = b_s \wedge \beta = b_{s \smallfrown \ell_n}.$$

Clearly $C_{s \smallfrown \ell_n} \subseteq [b_{s \smallfrown \ell_n}]$ and according to property (3) of Fact 4.4 $C_{s \smallfrown \ell_n} \subseteq C_s$, therefore $C_{s \smallfrown \ell_n} \subseteq U$. This is sufficient, because property (3) of Definition 4.2 implies that $\ell_n \neq \ell_{n'}$ if $n \neq n'$. \square

We will also use the following encoding of $\omega^{<\omega}$ in $\{0, 1\}^{<\omega}$:

Definition 4.6. For $n \in \omega$, let $h(n) = (0, 0, \dots, 0, 1) \in \{0, 1\}^{n+1}$ be the sequence consisting of n zeroes and then “1” as the last element. Generalizing this, for a finite sequence $s = (s_0, s_1, \dots, s_{k-1}) \in \omega^{<\omega}$, let $h(s) = h(s_0) \wedge h(s_1) \wedge \dots \wedge h(s_{k-1})$.

Let $h_0(s) \in \{0, 1\}^\omega$ denote the infinite sequence $h(s) \wedge (0, 0, \dots)$ (that is, $h(s)$ extended to infinite length by appending zeroes).

Fact 4.7. *The functions $h : \omega^{<\omega} \rightarrow \{0, 1\}^{<\omega}$ and $h_0 : \omega^{<\omega} \rightarrow \{0, 1\}^\omega$ have the following properties:*

- (1) h and h_0 are both injective,
- (2) if $s, s' \in \omega^{<\omega}$, then

$$s \subseteq s' \iff [h(s)] \supseteq [h(s')] \iff [h(s)] \ni h_0(s'),$$

- (3) if $s \in \omega^{<\omega}$, $k \in \omega$ and $\ell \in \omega$ is large enough, then $[h(s \smallfrown \ell)] \subseteq [h_0(s) \upharpoonright k]$.

Intuitively we will use $h_0(s)$ as a “height” assigned to the sequence s : when we construct the compact metric space K_0 , we will “lift” a copy of C_s to height $h_0(s)$ for each $s \in \omega$. More precisely, this means the following:

Definition 4.8. Let K_0 be the closure of the set

$$\bigcup_{s \in \omega^{<\omega}} C_s \times \{h_0(s)\}$$

in the space $\mathbb{Z}^\omega \times \{0, 1\}^\omega$ and let $f_0 : K_0 \rightarrow \mathbb{Z}^\omega$ be the restriction of the projection $\mathbb{Z}^\omega \times \{0, 1\}^\omega \rightarrow \mathbb{Z}^\omega$ to the set K_0 .

Claim 4.9. K_0 is compact.

Proof. According to Definition 4.3, $C_s \subseteq \{0, 1\}^\omega \subseteq \mathbb{Z}^\omega$ for each $s \in \omega^{<\omega}$. This clearly implies that the closed set K_0 is a subset of $\{0, 1\}^\omega \times \{0, 1\}^\omega$, a compact set. \square

To study the structure of the set K_0 , we introduce

$$D_s = \bigcup_{\substack{\sigma \in \omega^{<\omega} \\ \sigma \supseteq s}} C_\sigma \times \{h_0(\sigma)\} \subseteq \mathbb{Z}^\omega \times \{0, 1\}^\omega.$$

Using this notation, the definition of K_0 can be written as $K_0 = \overline{D_\emptyset}$. It is clear that if $s \supseteq s'$ then $D_s \subseteq D_{s'}$, and in particular $D_s \subseteq D_\emptyset \subseteq K_0$ holds for each $s \in \omega^{<\omega}$.

Claim 4.10. For each $s \in \omega^{<\omega}$,

$$\overline{D_s} = K_0 \cap (\mathbb{Z}^\omega \times [h(s)]) = K_0 \cap (C_s \times [h(s)])$$

and therefore $\overline{D_s}$ is relatively clopen in K_0 .

Proof. By property (2) of Fact 4.7, $s \subseteq \sigma$ if and only if $h_0(\sigma) \in [h(s)]$, so

$$D_\emptyset \cap (\mathbb{Z}^\omega \times [h(s)]) = D_s.$$

Elementary calculations show that if A, B are two subsets of a topological space and B is clopen, then $\overline{A \cap B} = \overline{A} \cap B$. Using this for the clopen set $\mathbb{Z}^\omega \times [h(s)]$ yields that

$$K_0 \cap (\mathbb{Z}^\omega \times [h(s)]) = \overline{D_s}.$$

This clearly shows that $\overline{D_s}$ is relatively open in K_0 .

To prove the second equality, notice that $C_\sigma \subseteq C_s$ for any sequence $\sigma \supseteq s$ and thus $D_s \subseteq C_s \times \{0, 1\}^\omega$. Taking closure and then intersecting with $\overline{D_s} = K_0 \cap (\mathbb{Z}^\omega \times [h(s)])$ yields that $\overline{D_s} = K_0 \cap (C_s \times [h(s)])$, as stated. \square

We will use the following lemma to prove that certain subsets of K_0 are meager (and in fact, nowhere dense):

Lemma 4.11. If $X \subseteq \mathbb{Z}^\omega$ satisfies that

$$\{s' \in \omega^{<\omega} : X \cap C_{s'} = \emptyset\} \text{ is cofinal in } \omega^{<\omega},$$

then $f_0^{-1}(X)$ is nowhere dense in K_0 .

Proof. Let U be an arbitrary nonempty, relatively open subset of K_0 . Then U can be written as $U = K_0 \cap V$ where V is open (in $\mathbb{Z}^\omega \times \{0, 1\}^\omega$).

$K_0 = \overline{D_\emptyset}$ and K_0 intersects the open set V , therefore D_\emptyset also intersects V . This implies that there is a sequence $s \in \omega^{<\omega}$ and a point $x \in C_s$ such that $(x, h_0(s)) \in V$.

As V is an open set in the product space $\mathbb{Z}^\omega \times \{0, 1\}^\omega$, there are $n, k \in \omega$ such that $[x \upharpoonright n] \times [h_0(s) \upharpoonright k] \subseteq V$.

As $C_s \cap [x \upharpoonright n]$ is a nonempty relatively open subset of C_s , we may apply Claim 4.5 to get an infinite set $L \subseteq \omega$ of indices such that $C_{s \smallfrown \ell} \subseteq C_s \cap [x \upharpoonright n]$ for each $\ell \in L$. Property (3) of Fact 4.7 implies we may choose an index $\ell \in L$ which also satisfies that $[h(s \smallfrown \ell)] \subseteq [h_0(s) \upharpoonright k]$.

Using the condition of the lemma we can find an $s' \in \omega^{<\omega}$ such that $s' \supseteq s \smallfrown \ell$ and $X \cap C_{s'} = \emptyset$. Claim 4.10 states that $\overline{D_{s'}} = K_0 \cap (C_{s'} \times [h(s')])$ is a relatively clopen subset of K_0 . This implies that $f_0(\overline{D_{s'}}) \subseteq C_{s'}$, but then applying $X \cap C_{s'} = \emptyset$ yields $f_0^{-1}(X) \cap \overline{D_{s'}} = \emptyset$. Also notice that

$$\begin{aligned} \overline{D_{s'}} &= K_0 \cap (C_{s'} \times [h(s')]) \subseteq K_0 \cap (C_{s \smallfrown \ell} \times [h(s \smallfrown \ell)]) \subseteq \\ &\subseteq K_0 \cap ([x \upharpoonright n] \times [h_0(s) \upharpoonright k]) \subseteq K_0 \cap V = U. \end{aligned}$$

This means that in an arbitrary nonempty, relatively open subset U of K_0 we found a (clearly nonempty) subset $\overline{D_{s'}}$ that is relatively open in K_0 and disjoint from $f_0^{-1}(X)$. This implies that $f_0^{-1}(X)$ is nowhere dense in K_0 . \square

5. CONSTRUCTION OF THE EXAMPLE AND PROOF OF THE MAIN RESULT

Now we are ready to prove our main result:

Theorem 1.8. *In the abelian Polish group \mathbb{Z}^ω , there exists a G_δ set R that is Haar meager but not strongly Haar meager.*

Proof. Fix a map $t : \mathcal{K}(\mathbb{Z}^\omega) \rightarrow \mathbb{Z}^\omega$ which satisfies the conditions of Theorem 3.1. Recall that

$$H = \{(K, x) \in \mathcal{K}(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega : x \in K\}$$

is a closed set and according to Theorem 3.1 the map

$$T : H \rightarrow \mathbb{Z}^\omega, \quad T(K, x) = x + t(K)$$

is a homeomorphism between H and the closed set $F = T(H) \subseteq \mathbb{Z}^\omega$, and this set F satisfies that

$$(K + t(K) + \{-1, 0, 1\}^\omega) \cap F = K + t(K)$$

for each $K \in \mathcal{K}(\mathbb{Z}^\omega)$. As

$$\{-1, 0, 1\}^\omega = \{0, 1\}^\omega - \{0, 1\}^\omega = C_\emptyset - C_\emptyset,$$

we can reformulate this fact:

Fact 5.1. *The set F satisfies that*

$$(K + t(K) + C_\emptyset - C_\emptyset) \cap F = K + t(K) \quad \text{for each } K \in \mathcal{K}(\mathbb{Z}^\omega).$$

We will use the following definition:

Definition 5.2. If $s \in \omega^{<\omega}$, then let $T_s = \{t \in \mathbb{Z}^\omega : t + C_s \subseteq F\}$ denote the set of translations which move C_s into F .

Claim 5.3. T_s is closed for each $s \in \omega^{<\omega}$.

Proof. We prove that for an arbitrary $s \in \omega^{<\omega}$, the set T_s contains all of its limit points. Assume that $t_i \in T_s$ for each $i \in \omega$ and $t^* = \lim_{i \rightarrow \infty} t_i$ exists. We know that if $c \in C_s$ and $i \in \omega$, then $t_i + c \in F$. As F is closed, this implies that if $c \in C_s$, then $\lim_{i \rightarrow \infty} (t_i + c) = t^* + c \in F$. This shows that $t^* \in T_s$, concluding the proof. \square

Using these sets of translations, we can define the set R :

Definition 5.4. Let

$$R = F \setminus \bigcup_{s \in \omega^{<\omega}} \bigcup_{\ell \in \omega} (T_s + C_{s \smallfrown \ell}).$$

It is clear from this definition that R is G_δ and $R \subseteq F$.

The rest of the proof consists of two parts: We will first prove Claim 5.5, which will imply that R is Haar meager, then we will prove Claim 5.6, which will imply that R is not strongly Haar meager.

Claim 5.5. For the compact metric space K_0 and function f_0 defined in Definition 4.8, if $g \in \mathbb{Z}^\omega$, then $f_0^{-1}(R + g)$ is a nowhere dense subset of K_0 . (Using the terminology of [1], this states that R is Haar nowhere dense.)

Proof. Fix an arbitrary $g \in \mathbb{Z}^\omega$. According to Lemma 4.11, it is sufficient to prove that

$$\{s' \in \omega^{<\omega} : (R + g) \cap C_{s'} = \emptyset\} \text{ is cofinal in } \omega^{<\omega}.$$

We fix an arbitrary $s \in \omega^{<\omega}$ and prove that there exists a $s' \in \omega^{<\omega}$ such that $s' \supseteq s$ and $(R + g) \cap C_{s'} = \emptyset$. Clearly $(R + g) \cap C_{s'} = \emptyset$ if and only if $R \cap (C_{s'} - g) = \emptyset$.

We distinguish two cases:

Case 1: $C_s - g \subseteq F$.

In this case Definition 5.2 implies that $-g \in T_s$. Pick an arbitrary $\ell \in \omega$ (for example, let $\ell = 0$) and let $s' = s \smallfrown \ell$. Then

$$C_{s'} - g \subseteq T_s + C_{s'} = T_s + C_{s \smallfrown \ell} \subseteq \bigcup_{\sigma \in \omega^{<\omega}} \bigcup_{\ell \in \omega} (T_\sigma + C_{\sigma \smallfrown \ell})$$

and therefore Definition 5.4 implies that $(C_{s'} - g) \cap R = \emptyset$.

Case 2: $C_s - g \not\subseteq F$.

In this case the set $U = C_s \setminus (F + g)$ is nonempty and relatively open in C_s (because F is a closed subset of \mathbb{Z}^ω). Applying Claim 4.5 we can select an index $\ell \in \omega$ such that $C_{s \smallfrown \ell} \subseteq U$. This means that $s' = s \smallfrown \ell$ is a good choice:

$$C_{s'} - g \subseteq U - g = (C_s - g) \setminus F \subseteq (C_s - g) \setminus R$$

using the fact that $R \subseteq F$. \square

Claim 5.6. If $K \subseteq \mathbb{Z}^\omega$ is a nonempty compact set, then there is an element $g \in \mathbb{Z}^\omega$ such that $(R + g) \cap K$ is a comeager subset of K .

Proof. Fix a nonempty compact set $K \in \mathcal{K}(\mathbb{Z}^\omega)$. We will prove that $g = -t(K)$ satisfies that $(R - t(K)) \cap K$ is a comeager subset of K . (Recall that we fixed a map t that satisfies the conditions of Definition 5.2.) As $x \mapsto x + t(K)$ is a homeomorphism from K to $K + t(K)$, it is enough to prove that $R \cap (K + t(K))$ is a comeager subset of $K + t(K)$.

Recall that according to Definition 5.4,

$$R = F \setminus \bigcup_{s \in \omega^{<\omega}} \bigcup_{\ell \in \omega} (T_s + C_{s \frown \ell}).$$

It is clear that $K + t(K) \subseteq F$, therefore

$$\begin{aligned} R \cap (K + t(K)) &= (K + t(K)) \setminus \bigcup_{s \in \omega^{<\omega}} \bigcup_{\ell \in \omega} (T_s + C_{s \frown \ell}) = \\ &= \bigcap_{s \in \omega^{<\omega}} \bigcap_{\ell \in \omega} ((K + t(K)) \setminus (T_s + C_{s \frown \ell})). \end{aligned}$$

To prove that this countable intersection is comeager in $K + t(K)$, it is enough to prove that if we fix $s \in \omega^{<\omega}$ and $\ell \in \omega$, then the set $(K + t(K)) \setminus (T_s + C_{s \frown \ell})$ is comeager in $K + t(K)$. This set is clearly relatively open, because $C_{s \frown \ell}$ is compact and Claim 5.3 states that T_s is closed. Therefore it is enough to prove that this set is dense in $K + t(K)$. If we fix an arbitrary nonempty relatively open subset U of $K + t(K)$, then we need to check that

$$U \cap ((K + t(K)) \setminus (T_s + C_{s \frown \ell})) = U \setminus (T_s + C_{s \frown \ell}) \neq \emptyset.$$

Fix an arbitrary $u \in U$. As U is relatively open, we may find an index $n \in \omega$ such that $[u \upharpoonright n] \cap (K + t(K)) \subseteq U$. We will use recursion to define a sequence $x \in K + t(K) \subseteq \mathbb{Z}^\omega$. First let

$$(1) \quad x \upharpoonright n = u \upharpoonright n.$$

After this we will select the elements of the sequence x one by one. Assume that we already defined $x \upharpoonright j$ for an index $j \geq n$ and

$$(2) \quad \text{if } j \equiv 0 \pmod{2^{|s|+1}}, \text{ then let}$$

$$x_j = \min\{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\},$$

$$(3) \quad \text{if } j \equiv 2^{|s|} \pmod{2^{|s|+1}}, \text{ then let}$$

$$x_j = \max\{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\},$$

$$(4) \quad \text{otherwise, choose an arbitrary element } x_j \text{ which satisfies that}$$

$$x_j \in \{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\}.$$

It is easy to check that $(K + t(K)) \cap [x \upharpoonright j]$ remains nonempty during this procedure and therefore $x \in K + t(K)$. Note that in conditions (2) and (3) the minimum and maximum are well-defined because $(K + t(K)) \cap [x \upharpoonright j]$ is compact.

Property (1) implies that $x \in U$, we wish to prove that $x \notin T_s + C_{s \frown \ell}$. Assume for the contrary that $x = t^* + c^*$ for some elements $t^* \in T_s$ and $c^* \in C_{s \frown \ell}$.

Recall that

$$T_s = \{t \in \mathbb{Z}^\omega : t + C_s \subseteq F\} \quad (\text{Definition 5.2})$$

and

$$C_s = \{b_s \wedge x : x \in \{0, 1\}^\omega \text{ and } x \text{ is } |s|\text{-segmented}\} \quad (\text{Definition 4.3}).$$

$t^* \in T_s$ means that $t^* + C_s = x - c^* + C_s \subseteq F$. As $x \in K + t(K)$, $c^* \in C_{s \smallfrown \ell} \subseteq C_\emptyset$ and $C_s \subseteq C_\emptyset$, we know that $x - c^* + C_s \subseteq K + t(K) + C_\emptyset - C_\emptyset$. According to Fact 5.1,

$$(K + t(K) + C_\emptyset - C_\emptyset) \cap F = K + t(K),$$

therefore $t^* + C_s = x - c^* + C_s \subseteq K + t(K)$.

For each index $j \in \omega$, consider the sequences

$$r(j, 0) = (c^* \upharpoonright j) \wedge (0, 0, \dots) \quad \text{and} \quad r(j, 1) = (c^* \upharpoonright j) \wedge (1, 1, \dots)$$

consisting of the first j elements of c^* , followed by zeroes and ones respectively. If $j \geq |b_s|$ and j is divisible by $2^{|s|}$ then it is straightforward to check that $r(j, 0), r(j, 1) \in C_s$ (using Definition 4.3 and the fact that $c^* \in C_{s \smallfrown \ell} \subseteq C_s$).

Fix a $j \in \omega$ such that $j \equiv 0 \pmod{2^{|s|+1}}$ and $j > \max\{n, |b_{s \smallfrown \ell}|\}$. According to property (2) of x ,

$$x_j = \min\{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\}.$$

Notice that $y = x - c^* + r(j, 0)$ satisfies that $y \in x - c^* + C_s \subseteq K + t(K)$ (because $r(j, 0) \in C_s$) and $y \in [x \upharpoonright j]$ (because $c^* \upharpoonright j = r(j, 0) \upharpoonright j$). This implies that

$$x_j \leq y_j = (x - c^* + r(j, 0))_j = x_j - c_j^* + 0 \quad \Rightarrow \quad c_j^* \leq 0.$$

Now apply an analogous argument for the index $j' = j + 2^{|s|}$: According to property (3) of x ,

$$x_{j'} = \max\{y_{j'} : y \in (K + t(K)) \cap [x \upharpoonright j']\}.$$

Notice that $y = x - c^* + r(j', 1)$ satisfies that $y \in x - c^* + C_s \subseteq K + t(K)$ (because $r(j', 1) \in C_s$) and $y \in [x \upharpoonright j']$ (because $c^* \upharpoonright j' = r(j', 1) \upharpoonright j'$). This implies that

$$x_{j'} \geq y_{j'} = (x - c^* + r(j', 1))_{j'} = x_{j'} - c_{j'}^* + 1 \quad \Rightarrow \quad c_{j'}^* \geq 1.$$

As $c^* = b_{s \smallfrown \ell} \wedge z$ for some $(|s|+1)$ -segmented sequence $z \in \{0, 1\}^\omega$ and the length of $b_{s \smallfrown \ell}$ is divisible by $2^{|s|+1}$, we know that $c_j^* = c_{j+2^{|s|}}^* = c_{j'}^*$. This contradicts that $c_j^* \leq 0$ and $c_{j'}^* \geq 1$, proving that our indirect assumption was incorrect.

We proved that $x \in U$ and $x \notin (T_s + C_{s \smallfrown \ell})$, and this implies that $(K + t(K)) \setminus (T_s + C_{s \smallfrown \ell})$ is indeed a dense subset of $K + t(K)$. \square

This concludes the proof of Theorem 1.8. \square

The following theorem shows that our G_δ counterexample is as simple as possible:

Theorem 5.7. *If G is an abelian Polish group and $A \subseteq G$ is an F_σ Haar meager subset, then A is strongly Haar meager.*

Proof. Assume that the continuous map $f : K \rightarrow G$ witnesses that A is Haar meager (where K is a nonempty compact metric space). We show that the compact set $C = f(K) \subseteq G$ witnesses that A is strongly Haar meager. Assume that the set B is a translate of A (that is, $B = A + g$ for some $g \in G$). It is clearly enough to prove that $B \cap C$ is meager in C .

We will use the following facts which are all well-known and easy to prove:

- (1) an F_σ set is meager if and only if it has empty interior,
- (2) the preimage of an F_σ set under a continuous function is also F_σ ,
- (3) if the preimage of a set X under a continuous function has empty interior, then the set X itself has empty interior relative to the image of the function.

As B is an F_σ set, (2) yields that $f^{-1}(B)$ is also F_σ . As f was a witness function, $f^{-1}(B)$ is meager, but then (1) yields that $f^{-1}(B)$ has empty interior. But $f^{-1}(B) = f^{-1}(B \cap C)$ because C is the image of f , therefore (3) yields that $B \cap C$ has empty interior relative to C . Using (1) again ($B \cap C$ is an F_σ subset of C), this yields that $B \cap C$ is indeed a meager subset of C . \square

Remark. This proof also works when the Polish group G is not necessarily abelian. For the definition of Haar meager and strongly Haar meager sets in this more general setting, see e.g. the survey paper [5].

6. OPEN QUESTIONS

As we answered Question 1.3 negatively, constructing a Haar meager but not strongly Haar meager set in \mathbb{Z}^ω , only the following open-ended question remains of Question 1.3:

Question 6.1. *What can we say about the (abelian) Polish groups where every Haar meager set is strongly Haar meager?*

Notice that both Example 1.7 and Theorem 1.8 studied groups that can be written as countable products of countable discrete groups. In fact, our ideas allow us to describe the situation in this frequently studied, simple class of Polish groups:

Claim 6.2. *Consider a group $G = \prod_{i \in \omega} G_i$ where each $(G_i, +)$ is a countable abelian group endowed with the discrete topology. Then the following are equivalent:*

- (1) every Haar meager subset of G is strongly Haar meager,
- (2) every compact subset of G is contained in a locally compact subgroup,
- (3) for all but finitely many $i \in \omega$ the group G_i is a torsion group.

We do not include a proof of this claim, as our proof involves proving generalizations of Theorem 3.1 and Claim 5.6 that are more complicated and harder to understand, but do not require additional interesting ideas.

Remark. In the previous claim the implication (2) \Rightarrow (1) remains true for any abelian Polish group G . This can be proved by slightly modifying the proof of [1, Theorem 5.13].

In addition to Question 1.3 the paper [2] contains another problem about strongly Haar meager sets, [2, Problem 3]:

Question 6.3 (Darji). *Does the collection of strongly Haar meager sets form a σ -ideal?*

In fact, even the following variant of this question seems to be open:

Question 6.4. *Does the collection of strongly Haar meager sets form an ideal? (Or equivalently, is the union of two strongly Haar meager sets strongly Haar meager?)*

As the Haar meager sets form a σ -ideal, the interesting case for these questions is studying groups that contain Haar meager, but not strongly Haar meager sets.

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