C_2 Toda theory in the reduced WZNW framework

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Abstract

We consider the C_2 Toda theory in the reduced WZNW framework. Analysing the classical representation space of the symmetry algebra (which is the corresponding C_2 W algebra) we determine its classical highest weight representations. We quantise the model promoting only the relevant quantities to operators. Using the quantised equation of motion we determine the selection rules for the u field that corresponds to one of the Toda fields and give restrictions for its amplitude functions and for the structure of the Hilbert space of the model.

Introduction

The importance of the Toda models lies in the facts that first they are exactly integrable, furthermore they give a realisation of extended conformal symmetries, which are crucial in the classification of conformal field theories and so equally important in the string theory and in the analysis of statistical physical systems.

Since the discovery of the Toda models their investigation, classically as well as at the quantum level, has attracted a great deal of interest. Analysing the classical Toda models Leznov and Savaliev showed that their equation of motion can be written as zero curvature conditions and in this way they are exactly integrable [1]. Then Bilal and Gervais discovered that the symmetry algebras of the models are the corresponding W algebras [2], extensions of the chiral Virasoro algebra by some higher spin chiral currents [3]. It has been shown recently that Toda theories can be regarded as constrained WZNW models [4]: imposing certain conformal invariant constraints the WZNW model reduces to the appropriate Toda model while its symmetry algebra –the Kac-Moody algebra– becomes the corresponding W algebra. The enormous advantage of this picture is that it shows the relevant variables of the Toda theory explicitly.

Although there have been different approaches to quantise the Toda models [5-8], much of these efforts concentrated only on the symmetry algebra using free field constructions. Concretely in the C_2 case Kausch and Watts have shown that the chiral symmetry algebra is nothing but the commutant of the screening charges [5]. Using this fact they constructed a free field representation for the C_2 (or B_2) W algebra and analysed the representation theory of the algebra via determinant formulae.

The first work in the field of investigating Toda models as reduced WZNW theories was done in ref. [9] for the simplest case, the Liouville theory. Recently (in collaboration with L. Palla and G. Takács) we have carried out the analysis for the A_2 Toda theory [10]. We follow this work here with the next simplest case, C_2. The interest of this model lies in the non simply-laced nature of the corresponding Lie algebra. It is an important task to check that the methods used, and results obtained for simply-laced algebras can be generalised for non simply-laced theories. Let us remark that the B_2 Toda model is equivalent to the C_2 one and even the corresponding W algebras are the same. However, we work with the C_2 model since the computations in the WZNW description are simpler for this. (Although the WZNW model depends on this choice the resulting Toda model is independent of it.)

In this paper we try to make a deeper investigation of the C_2 Toda model using the very natural WZNW framework. As a first step we compute the defining relations of the classical C_2 W algebra. Then the relevant degrees of freedom of the Toda model are identified as the symmetry generators and the exponential form of one of the Toda fields (u). These variables are not independent since u satisfies the classical equation of motion. With the help of these quantities the classical representation space of the symmetry algebra is analysed on the solution space of the classical equation of motion. Characterising the W orbits by their monodromy matrices we determine the classical highest weight representations (h.w.r.). In the quantum case -contrary to other approaches- we promote only the relevant variables to operators, which act on a Hilbert space that is the direct sum of h.w.r. spaces. The quantum Toda theory is investigated in three steps. First we derive the quantum equation of motion - that is the normal ordered analogue of the classical one - from its covariance. Then using this equation we determine the selection rules for the u field. This gives restrictions on the structure of the Hilbert space of the model. At last we look for restrictions on the amplitude functions of u by studying the locality requirement for the four point functions.

The paper is organised as follows : In section 1 the advantages of the reduced WZNW description are summarised. Using these results in section 2 we describe the classical representation space of the symmetry algebra. Section 3 deals with the quantum version of the model. In App. A we show the details of the analysis of the classical h.w. type solutions. App. B illustrates how we determine the quantum equation of motion and in App. C we give the differential equation that the four point function has to satisfy.

1. Classical C_2 Toda theory

The classical C_2 Toda theory is a field theory of two periodic scalar fields $\Phi^j(x^0, x^1) = \Phi^j(x^0, x^1 + 2\pi); j = 1, 2$ in two dimensions with exponential interaction:

$$L = \sum_{i,j=1}^{2} \frac{1}{2|\alpha_i|^2} K_{ij} \partial_+ \Phi^i \partial_- \Phi^j - 2 \sum_{i=1}^{2} \exp[\frac{1}{2} \sum_{j=1}^{2} K_{ij} \Phi^j] \tag{1.1}$$

Here $x^\pm = \frac{1}{2}(x^0 \pm x^1)$ are light cone coordinates, K_{ij} denotes the Cartan matrix and α_is denote the simple roots of the C_2 algebra. The length of the long root of the algebra is 2. The equations of motion are:

$$\partial_+ \partial_- \Phi^1 + 2e^{\Phi^1 - \Phi^2} = 0$$

$$\partial_+ \partial_- \Phi^2 + 2e^{\Phi^2 - \frac{1}{2}\Phi^1} = 0 \tag{1.2}$$

The model is conformally invariant since the improved, Feigin-Fuchs type energy momentum tensor is traceless. However the model is invariant not only under the Virasoro algebra but the C_2 W algebra, too. This algebra, an extension of the Virasoro algebra, is generated by the energy momentum tensor $L(x^+) = W_2(x^+)$ and a spin 4 chiral current $W(x^+) = W_4(x^+)$. It is known for a long time that this model is soluble. The exact solution of the Toda models has been found by Leznov and Savaliev in [1].

Another important step was when J. Balog and his collaborators [4] showed that the Toda models can be regarded as constrained WZNW models modulo gauge transformations generated by appropriate conformal invariant constraints. The main advantages of this framework can be summarised in the following points:

(i) The symmetry algebra - that is the appropriate W algebra - can be computed easily : It is the algebra generated by the gauge invariant polynomials of the constrained currents and their derivatives with respect to the Dirac bracket. The action of this algebra on the phase space of the theory, which is the space of the constrained currents, can be implemented by certain KM transformations.

(ii) It indicates the relevant degrees of freedom.

(iii) The solutions of the Toda model can be obtained by reduction from the WZNW solution.

Focusing on the C_2 case this means that the Toda model can be obtained by reduction of the $Sp(2,\mathbf{R})$ WZNW model. Let us consider now what the advantages mentioned above mean in this concrete case, when we work in the so called h.w. gauge:

(i) The defining relations of the classical C_2 W algebra are:

$$\begin{aligned}\{W_2(x),W_2(y)\} &= (W_2(x)+W_2(y))\delta^{'}(x-y)-5\delta^{'''}(x-y)\\ \{W_2(x),W_4(y)\} &= -W_4^{'}(y)\delta(x-y)+4W_4(y)\delta^{'}(x-y)\\ \{W_4(x),W_4(y)\} &= \frac{1}{2}\sum_{i=0}^{2}(F_{2i+1}(x)+F_{2i+1}(y))\delta^{(2i+1)}(x-y)\end{aligned} \tag{1.3}$$

where

$$F_5=\frac{7}{25}W_2\ ;\qquad F_3=-\frac{3}{5}W_4-\frac{14}{25}W_2^{''}-\frac{49}{125}W_2^2$$

$$F_1=\frac{14}{25}W_4W_2+\frac{2}{5}W_2^{''}+\frac{72}{625}W_2^3+\frac{59}{125}W_2W_2^{''}+\frac{293}{500}(W_2^{'})^2+\frac{17}{50}W_2^{''''}$$

(ii) The fundamental and proper variables of the Toda theory are the generators of the symmetry algebra and the lower right corner element u of g. g is the general WZNW solution whose currents satisfy the constraints. It can be constructed from the W and u.

$$g=\begin{pmatrix} D_4^+D_4^-u & D_4^+D_3^-u & D_4^+D_2^-u & D_4^+u\\ D_3^+D_4^-u & D_3^+D_3^-u & D_3^+D_2^-u & D_3^+u\\ D_2^+D_4^-u & D_2^+D_3^-u & D_2^+D_2^-u & D_2^+u\\ D_4^-u & D_3^-u & D_2^-u & u\end{pmatrix} \tag{1.4}$$

where

$$D_2^{\pm}=-\partial_{\pm}\quad ;\qquad D_3^{\pm}=-\partial_{\pm}^2+\frac{3}{10}W_2(x^{\pm})$$

$$D_4^{\pm}=-\partial_{\pm}^3+\frac{7}{10}W_2(x^{\pm})\partial_{\pm}+\frac{3}{10}\partial_{\pm}W_2(x^{\pm})$$

Here $W_2(x^+)$ and $W_2(x^-)$ denote the left moving and the right moving conformal generators, respectively. One of the Toda fields now can be written as $u=exp\{-\frac{1}{2}\Phi^2\}$. The other one is given with the help of the 2×2 lower right sub-determinant of g.

Since the symmetry transformations can be implemented by KM transformations their action on g (and thus on u) is explicitly given. u turns out to be a W primary field, since

$$\begin{aligned}\delta u =&a_1 u^{'} - \frac{3}{2}a_1^{'}u + \\ &a_2(-u^{'''} + \frac{41}{50}W_2 u^{'} + \frac{27}{100}W_2^{'}u) + a_2^{'}(\frac{1}{2}u^{''} - \frac{23}{100}W_2 u) - \frac{1}{5}a_2^{''}u^{'} + \frac{1}{20}a_2^{'''}u\end{aligned} \tag{1.5}$$

where a_1 and a_2 are infinitesimal functions parameterising the pure conformal and pure W transformation, respectively [9]. We distinguish two kinds of Toda solutions. If u has no zeroes the solution is called regular, and it can be expressed in terms of the original Toda variables, the fields Φ^i. In the opposite case the solution is called singular, and we really need the u field to express them. We notice that on this level they are allowed solutions since the W densities still remain regular.

(iii) Since g is the general solution of the WZNW model, u must be of the following form:

$$u(x^+, x^-) = \sum_i \psi_i(x^+)\chi_i(x^-) \tag{1.6}$$

As the currents of g satisfy the constraints, the ψ_is satisfy a certain differential equation. The same holds for their right handed counterparts, χ_is. Finally the u built up from them satisfies the same equation:

$$u^{''''} - W_2 u^{''} - W_2^{'}u' + \big(W_4 + \frac{9}{100}(W_2)^2 - \frac{3}{10}W_2^{''}\big)u = 0, \tag{1.7}$$

which we call the classical equation of motion, since one of the Toda equations is an integral of it. This can be seen by expressing u and the symmetry generators in terms of the original Toda fields. (The other Toda equation is equivalent to the definition of Φ^1 with the help of 2×2 lower right sub-determinant of g.) Here the prime means derivative with respect to x^+ . The boundary conditions of (1.7) are such that at some initial point (x_0^+, x_0^-) g, built up from u, is an element of the group. Then the evolution of the system, governed by (1.7), will ensure that g remains in the group. A similar equation and boundary condition hold for the right moving variables.

2. The classical representations of the C_2 W algebra

To find the classical highest weight representations for the W algebra we shall follow the same procedure as we did in ref.[10]. Since the W algebra preserves the form of the constrained current the transformed u field will satisfy the transformed equation of motion. This implies that the corresponding Φ fields satisfy the same Toda equation. So the symmetry transformations map every classical solution into another one. In other words the action of the symmetry algebra can be represented on the solution space of the Toda equation. The solutions connected by W transformations form the so-called W orbit.

We are looking for gauge invariant quantities to parameterise these orbits or representations. The freedom that only the u field must be periodic, and not the ψ and χ fields in(1.5), is coded in the monodromy matrix:

$$\psi_k(z+2\pi) = M_{kl}\psi_l(z), \tag{2.1}$$

(and similarly for χ). The two monodromy matrices are not independent and they must be chosen correctly to ensure the periodicity of u. The monodromy matrices are gauge invariant since the W transformations act linearly on the ψ fields, they transform exactly the same way as u in (1.5). As suggested above they can be used to parameterise the representations. We are interested in the classical highest weight representations of the W algebra. These are the generalisations of the analogue quantum representations since we require the following: The existence of a solution - a highest weight vector - on which L and W are constant, and for which the total energy $\int W_2$ is a minimum on the orbit.

In the rest of this chapter we outline how to find these highest weight representations. In three typical cases we give explicit results.

We consider diagonalisable monodromy matrices only, since all other are unphysical in the sense that they never lead to constant W densities. First we remark that constant W densities are necessary to obtain h.w. representations (see appendix for the details). Furthermore it is not to difficult to show that using constant densities the solutions of

(1.7), which is now a linear differential equation with constant coefficients, exhaust the cases of diagonalisable monodromy matrices.

We classify the monodromy matrices by their eigenvalues. From each class we take a representative, and imposing it as a boundary condition (2.1), we can determine the linearly independent solutions of (1.6). These can be used to compute the W densities. If these densities are periodic and non-singular one has to check whether the representation obtained is h.w. or not. This is done by iterating the W transformations. The calculation is completely analogous to the one explained in [10].

Let us consider the individual cases. If the monodromy matrix has four real eigenvalues it can be written as

$$M = diag(e^{\Lambda\pi}, e^{-\mu\pi}, e^{\mu\pi}, e^{-\Lambda\pi}) \qquad \Lambda \neq \mu \tag{2.2}$$

with Λ and μ arbitrary positive parameters, and let $\mu > \Lambda$ for convenience. The corresponding solutions:

$$\psi_1 = N_1 e^{\Lambda x^+} \ ; \quad \psi_4 = N_4 e^{-\Lambda x^+}$$
$$\psi_2 = N_2 e^{-\mu x^+} \ ; \quad \psi_3 = N_3 e^{\mu x^+}$$

where

$$N_1 N_4 = (2\Lambda(\mu^2 - \Lambda^2))^{-1} \ ; \qquad N_2 N_3 = (2\mu(\mu^2 - \Lambda^2))^{-1}$$

They are normalised to ensure that the matrix built up from them is an element of SP(2,$\mathbf{R}$). These solutions give constant W densities :

$$W_2 = \mu^2 + \Lambda^2 \ ; \qquad W_4 = -\frac{9}{100}(\mu^2 + \Lambda^2)^2 + \mu^2\Lambda^2$$

In the same manner the monodromy matrix of the right moving fields can be written with some other parameters $\tilde{\Lambda}$ and $\tilde{\mu}$. The requirement that u must be periodic identifies Λ with $\tilde{\Lambda}$ and μ with $\tilde{\mu}$. Using (1.5) u can be built up from its left moving and right moving components :

$$u = N_1\tilde{N}_1 e^{\Lambda x^0} + N_2\tilde{N}_2 e^{-\mu x^0} + N_3\tilde{N}_3 e^{\mu x^0} + N_4\tilde{N}_4 e^{-\Lambda x^0}$$

If the normalisation constants are all negative or positive then this is a regular solution in the sense that it never changes its sign, so it can be expressed by the original Toda variables. In the opposite case the solution may be singular. This representation can be shown to be classically h.w. for all possible Λ and μ .

The monodromy matrix that has two real eigenvalues and a complex conjugate pair can be written in the following form:

$$M = \begin{pmatrix} e^{\Lambda\pi} & 0 & 0 & 0 \\ 0 & \cos(\pi\rho) & \sin(\pi\rho) & 0 \\ 0 & -\sin(\pi\rho) & \cos(\pi\rho) & 0 \\ 0 & 0 & 0 & e^{-\Lambda\pi} \end{pmatrix} \tag{2.3}$$

where Λ and ρ are positive parameters. Since the monodromy matrix is periodic in ρ only the $0 < \rho < 2$ region is relevant. The ψ_is which correspond to this monodromy matrix are:

$$\psi_1 = N_1 e^{-\Lambda x^+} \ ; \quad \psi_4 = N_4 e^{\Lambda x^+}$$

$$\psi_2 = N_2 \sin(\rho x^+) \ ; \quad \psi_3 = N_2 \cos(\rho x^+)$$

$$N_1 N_4 = (2\Lambda(\rho^2 + \Lambda^2))^{-1} \ ; \quad N_2 = (\rho(\rho^2 + \Lambda^2))^{-\frac{1}{2}}$$

They yield constant W densities:

$$W_2 = \Lambda^2 - \rho^2 \ ; \quad W_4 = -\frac{9}{100}(\Lambda^2 - \rho^2)^2 - \rho^2\Lambda^2$$

We require a similar form for the monodromy matrix of the right moving variables except for the changing $\Lambda \mapsto \tilde{\Lambda}$ and $\rho \mapsto \tilde{\rho}$. The u can be obtained from the ψ_is and their counterparts:

$$u = N_1\tilde{N}_1 e^{\Lambda x^0} + N_4\tilde{N}_4 e^{-\Lambda x^0} + N_2\tilde{N}_2 \cos\left(\frac{(\rho - \tilde{\rho})}{2}x^0 + \frac{(\rho + \tilde{\rho})}{2}x^1\right)$$

The periodicity of u connects ρ to $\tilde{\rho}$, namely $\rho + \tilde{\rho} = 2M$ must hold, where M is an integer. It goes without saying that this solution is a singular one. From the analysis of the stability condition we conclude that the representation is h.w. only for $\rho < \frac{1}{2}$, $\Lambda^2 - \rho^2 > -\frac{5}{2}$.

Finally let us consider the case when the monodromy matrix has no real eigenvalues:

$$M = \begin{pmatrix} \cos(\pi\nu) & 0 & 0 & -\sin(\pi\nu) \\ 0 & \cos(\pi\rho) & \sin(\pi\rho) & 0 \\ 0 & -\sin(\pi\rho) & \cos(\pi\rho) & 0 \\ \sin(\pi\nu) & 0 & 0 & \cos(\pi\nu) \end{pmatrix} \tag{2.4}$$

where again let $\nu > \rho > 0$ for convenience. The solutions which satisfy the quasi periodicity conditions are:

$$\psi_1 = -N_1 \sin(\nu x^+) \ ; \quad \psi_4 = N_1 \cos(\nu x^+)$$

$$\psi_2 = N_2 \sin(\rho x^+) \ ; \quad \psi_3 = N_2 \cos(\rho x^+)$$

where

$$N_1 = (\nu(\nu^2 - \rho^2))^{-\frac{1}{2}} \ ; \quad N_2 = (\rho(\nu^2 - \rho^2))^{-\frac{1}{2}}$$

They give constant W densities again:

$$W_2 = -\nu^2 - \rho^2 \ ; \quad W_4 = -\frac{9}{100}(\nu^2 + \rho^2)^2 + \rho^2\nu^2$$

Similarly to the previous case the right moving monodromy matrix is supposed to have the same form but with $\nu \mapsto \tilde{\nu}$ and $\rho \mapsto \tilde{\rho}$. In the usual way the field u can be written as:

$$u = N_1\tilde{N}_1 \cos\Big(\frac{(\nu - \tilde{\nu})}{2}x^0 + \frac{(\nu + \tilde{\nu})}{2}x^1\Big) + N_2\tilde{N}_2 \cos\Big(\frac{(\rho - \tilde{\rho})}{2}x^0 + \frac{(\rho + \tilde{\rho})}{2}x^1\Big)$$

Periodicity links ν to $\tilde{\nu}$ and ρ to $\tilde{\rho}$ as $\nu + \tilde{\nu} = 2K$; $\rho + \tilde{\rho} = 2L$ with K, L integers. As we outline in appendix A this representation is h.w. only for the range of the parameters $\rho < \frac{1}{2}$, $\nu > \frac{1}{2}$, $\nu - \rho < 1$.

This sector contains the classical SL_2 invariant vacuum as its boundary point. Since this solution is singular in the usual sense, it can be described only with the help of the field u, we hope to quantise the model using the variable u. (In the quantum case we are interested at least in those quantum h.w. representations that contain the SL_2 invariant vacuum.) We remark that in the A_1 case the classical vacuum, as here, is on the boundary of the allowed region, however in the A_2 case the vacuum does not belong to the classically h.w. representations. The origin of this difference is not clear yet, although it may indicate that the quantum theory will have a semiclassical limit. Neither the A_1 nor the A_2 quantum theory has a semiclassical limit.

The quantum C_2 Toda theory

We use the WZNW framework to quantise the Toda theory. The reasons why we do this be summarised in the following:

On the one hand we would like to describe not only the regular sector of the classical TT but also the singular one, which can be parameterised only by the u field.

On the other hand in the quantum case we are interested in those h.w.r. which contain the SL_2 invariant vacuum. This sector could be described classically only by the means of u.

Motivated by the classical theory we assume that the relevant degrees of freedom will become operators in the quantum theory namely: the generators of the symmetry algebra, (the quantum C_2 W algebra), and u. Actually they are not independent since u is a primary field with respect to this algebra, and satisfies the quantum equation of motion that is the normal ordered analogue of the classical one. Using this equation, which turns out to describe a grade 4 null state, we could compute the matrix elements of the u field.

We require the symmetry algebra of the model to be the direct product of the left moving and the right moving C_2 W algebra. These are generated by the energy momentum tensor and the spin 4 current. (The commutation relations can be found in the literature [11-13].) We assume the Hilbert space of the model to be of the form: $\mathcal{H} = \mathcal{W} \bigotimes \bar{\mathcal{W}}$ where $\mathcal{W}$ and $\bar{\mathcal{W}}$ are built up from h.w.r. spaces. (From now on we use only the left moving variables, if it does not lead to confusion.) For every h.w. space there exists a h.w. state $\left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle$, from which the space can be built up using the Laurent coefficients of the generators:

$$W_2(z) = L(z) = \sum_n L_n z^{-n-2}; \qquad W_4(z) = W(z) = \sum_n W_n z^{-n-4} \tag{3.1}$$

Since the action of these operators W_n^j ($W_n^2 = L_n, W_n^4 = W_n$) on any local field $\phi(z,\bar{z})$ can be formulated as:

$$W_n^j \phi(z,\bar{z}) = \oint_z \frac{d\xi}{2\pi i} (\xi - z)^{n+j+1} W_j(\xi)\phi(z,\bar{z}) \tag{3.2}$$

and the h.w. states correspond to W primary fields in the sense of [15], the action on the h.w. states is:

$$\begin{array}{c} W_n^j \\ \bar{W}_n^j \end{array} \left| \begin{array}{cc} h & \bar{h} \\ w & \bar{w} \end{array} \right\rangle = 0 \qquad n > 0 \quad j = 1,2 \tag{3.3}$$

$$\begin{array}{c} L_0 \\ \bar{L}_0 \end{array} \left| \begin{array}{cc} h & \bar{h} \\ w & \bar{w} \end{array} \right\rangle = \begin{array}{c} h \\ \bar{h} \end{array} \left| \begin{array}{cc} h & \bar{h} \\ w & \bar{w} \end{array} \right\rangle \qquad \begin{array}{c} W_0 \\ \bar{W}_0 \end{array} \left| \begin{array}{cc} h & \bar{h} \\ w & \bar{w} \end{array} \right\rangle = \begin{array}{c} w \\ \bar{w} \end{array} \left| \begin{array}{cc} h & \bar{h} \\ w & \bar{w} \end{array} \right\rangle$$

In order to completely describe the model we have to represent the relevant operators on this Hilbert space. Clearly the action of the symmetry generators is given so we are concerned with the representation of the operator u. We require this operator to be the quantised analogue of the classical u field. This means that u must be a primary, spin zero, periodic field, i.e. :

$$L_n u(z,\bar{z}) = \delta_{n,0} \Delta u(z,\bar{z}) \qquad n \geq 0$$

$$W_n u(z,\bar{z}) = \delta_{n,0} \omega u(z,\bar{z}) \qquad n \geq 0$$

where the weights of the field are allowed to differ from their classical values (1.5) because of the normal ordering. On quantising the system we use the short distance OPE to calculate the normal ordered products of the operators. (Normal ordering is nothing but the subtraction of the singular terms from the usual OP.)

Furthermore u must satisfy an equation of motion, that is the quantum equation of motion, the normal ordered form of the classical one. Replacing every term in the classical equation of motion with a normal ordered term and using the primary nature of u we find:

$$AL_{-4}u + BW_{-4}u + CL_{-2}^2 u + DL_{-3}L_{-1}u + EL_{-2}L_{-1}^2 u + FL_{-1}^4 u = 0 \tag{3.4}$$

We remark that although the classical equation (1.7) contains a cubic term and the normal ordering is not associative this ambiguity leads only to an uncertainty of the coefficients. However we have to keep in mind that, due to the normal ordering, the coefficients may alter. These modified values of the parameters $(\Delta, \omega, A, B, C, D, E, F,)$ can be determined from the requirement that the grade 4 null vector defined by eq.(3.4) - the quantum equation of motion - must transform covariantly under the symmetry algebra. Denoting the l.h.s. of eq.(3.4) by χ, this means that χ must be annihilated by L_n, W_n for $n > 0$ or equivalently $L_1\chi = W_1\chi = L_2\chi = 0$ must hold. We carry out this analysis in detail in appendix B, and we find using an appropriate parameterisation that Δ and ω can be expressed in terms of Q as

$$\Delta = \frac{1}{4}(5Q - 6)$$

$$\omega = -\frac{Q}{8}\Delta\sqrt{\frac{(4Q-5)(\frac{8}{Q}-5)(\frac{2}{Q}-3)(\frac{6}{Q}-7)}{(226-75Q-\frac{150}{Q})(Q-3)(3Q-7)}} \tag{3.5}$$

where Q is given by $c = (5Q-6)(\frac{10}{Q}-6)$. Since the central charge of the theory is invariant under the transformation $Q \mapsto 2/Q$ we have two kinds of u in a certain model and the W weights of these us are related to each other by changing every Q to $2/Q$. In this parameterisation $Q < 0$ represents the region where $c > 86 + 60\sqrt{2}$ while $Q > 0$ is the region of $c < 86 - 60\sqrt{2}$, being in accordance with the defining relations of the symmetry algebra [11-13]. Consequently $Q \to_- 0$ corresponds to the classical limit $(\Delta \to -\frac{3}{2}\ , c \to \infty)$.

At this point we are ready to calculate the matrix elements of u. Since the Hilbert space of the model is built up from h.w.r. spaces it is sufficient to determine the matrix elements between h.w. states:

$$\left\langle \begin{matrix} H & \bar{H} \\ \Omega & \bar{\Omega} \end{matrix} \middle| u(z,\bar{z}) \middle| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle = G(H,h,\ldots) z^{H-h-\Delta} \bar{z}^{\bar{H}-\bar{h}-\bar{\Delta}}$$

where G may depend on all the parameters describing the initial and final states. These amplitude functions are restricted by the quantum equation of motion. Sandwiching eq.(3.4) between h.w. states, and using the same method to compute matrix elements as we did in [10], we get that G must vanish unless:

$$
\begin{aligned}
&Fy(y+1)(y+2)(y+3)+E(y+2)(y+3)(y+h)+C(y+h)(y+h+2)+\\
&\quad(D-2E)(y+3)(y+2h)+(A-2D+2E)(y+3h)+\\
&\qquad B\Bigg(w-3\big((\beta^{-3}-\beta^{-2-1})(y+2h)+\beta^{-2-1}(y+h)(y+2)+\\
&\qquad\qquad \beta^{-1-1-1}y(y+1)(y+2)\big)\\
&\qquad\quad +3\big(\beta^{-2}(y+h)+\beta^{-1-1}y(y+1)\big)-\beta^{-1}y\Bigg)=0
\end{aligned}
\tag{3.6}
$$

with $y=h+\Delta-H$ and the β coefficients are as defined in appendix B. It turns out to be very fruitful to introduce the following reparametrisation for the W weights:

$$
\begin{aligned}
h(a,b)=&\frac{Q}{4}(a^2+2ab+2b^2)-\frac{1}{4}(5Q+\frac{10}{Q}-14)\\
w(a,b)=&\bar{B}\Bigg(4Q(Q-3)(27Q-32)b^3(b+2a)-Q(3Q-2)(16Q-27)a^3(a+4b)\\
&+\frac{(Q^2-2)}{Q}\Big(14Q(a^2+2ab+2b^2-6Qa^2b^2)+Q-6+\frac{2}{Q}\Big)\Bigg)
\end{aligned}
\tag{3.7}
$$

where the $\bar{B}$ normalisation constant is such that $\bar{B}\times B=-\frac{4}{Q}$. A similar parameterisation has been used in [5]. In terms of these variables the vacuum corresponds to $a_{\text{vac}}=(1-2Q^{-1})$; $b_{\text{vac}}=\pm(1-Q^{-1})$ while u can be described by $a_u=2-2Q^{-1}$, $b_u=1-Q^{-1}$.

Using this parameterisation we solved (3.6), and found that u has non-vanishing transition elements if the parameters of the final and initial states are linked to each other as:

$$
A,\ B=\begin{array}{cc}
a+1, & b\\
a+1, & b-1\\
a-1, & b\\
a-1, & b+1
\end{array}
\tag{3.8}
$$

Let us remark that decomposing the tensor product of two irreps of $SP(2,\mathbf{R})$, characterised by Dynkin labels (a,b) and $(1,0)$, we get the irreducible representations appearing in eq.(3.8). A similar result was obtained in ref. [14], but for minimal models.

Let us consider now the chiral counterpart of the equation of motion. Repeating the computation step by step we find the same fusion rules for u but replacing a with

$\bar{a}$ and b with $\bar{b}$. Here $\bar{a}, \bar{b}$ parameterise the h.w.r. of the $\bar{W}$ algebra. We can relate a, b to $\bar{a}, \bar{b}$ by analysing the periodicity requirement for u. From the diagonal transition $\left(\left\langle \begin{matrix} b+1 & a \\ \bar{b}+1 & \bar{a} \end{matrix} \right| u(z,\bar{z}) \left| \begin{matrix} a & b \\ \bar{a} & \bar{b} \end{matrix} \right\rangle \text{ etc.}\right)$ it follows that

$$\bar{b} = b - \frac{2M}{Q} \ ; \qquad \bar{a} = a + \frac{2(M+N)}{Q}$$

where M, N are integers. If M and N were different from zero the non diagonal elements would imply among others that Qb should be integer, which is excluded as we shall see later. In this way we conclude that the Hilbert space of the model may be of the form:

$$\mathcal{H} = \sum_{k,l} \mathcal{W}_{a_0+k, b_0+l} \otimes \bar{\mathcal{W}}_{a_0+k, b_0+l} \tag{3.9}$$

where $\mathcal{W}_{a_0+k,b_0+l}$ is the h.w.r space corresponding to the following h.w. state $\left| \begin{matrix} a_0+k & b_0+l \\ a_0+k & b_0+l \end{matrix} \right\rangle (= |a_0+k, b_0+l\rangle$ for short). This choice is natural in the sense that this Hilbert space may contain the SL_2 invariant vacuum. From now on we will use the following amplitude functions to characterise u in $\mathcal{H}$:

$$G_1(a,b) = \langle a+1, b|u(1,1)|a,b\rangle \ ; \ G_2(a,b) = \langle a+1, b-1|u(1,1)|a,b\rangle$$

$$G_3(a,b) = \langle a-1, b|u(1,1)|a,b\rangle \ ; \ G_4(a,b) = \langle a-1, b+1|u(1,1)|a,b\rangle$$

From the reality of u it follows that:

$$G_3(a,b) = G_1^*(a-1,b) \ ; \qquad G_4(a,b) = G_2^*(a-1,b+1) \tag{3.10}$$

This means that there are essentially only two independent amplitude functions of u.

Let us remark that one can obtain a further restriction for the amplitude functions analysing the following automorphism: $\mathcal{M}|ab\rangle = |-a-b\rangle$. This is almost trivial, since due to the special form of the W weights (3.7), $\mathcal{M}|hw\rangle = |hw\rangle$ and $\mathcal{M}u(\Delta,\omega)\mathcal{M}^{-1} = u(\Delta,\omega)$. Applying these results for the matrix elements of u we get:

$$G_1(a,b) = G_1^*(-a-1,-b) \ ; \ G_2(a,b) = G_2^*(-a-1,-b+1) \tag{3.11}$$

Let us consider the restrictions following from the requirement that u must be local. These can be established by studying the matrix elements of $u(z,\bar{z})u(\zeta,\bar{\zeta})$. Conformal symmetry restricts this expectation value to be of the following form:

$$\left\langle \begin{matrix} H & \bar{H} \\ \Omega & \bar{\Omega} \end{matrix} \middle| u(z,\bar{z})u(\zeta,\bar{\zeta}) \middle| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle = (z\zeta)^{\lambda}(\bar{z}\bar{\zeta})^{\bar{\lambda}} f(x,\bar{x})$$

where $\lambda = \frac{1}{2}(H-h) - \Delta$ and $x = \zeta/z, \bar{x} = \bar{\zeta}/\bar{z}$. The locality of u can be formulated in terms of $f(x,\bar{x})$ as $f(x,\bar{x})$ should be invariant under the $x \to x^{-1}, \bar{x} \to \bar{x}^{-1}$transformation. Since u corresponds to a grade 4 null state $f(x,\bar{x})$ must satisfy a fourth order differential equation. Solving this d.e. -with the boundary conditions to be described below- and requiring the symmetry property of the expectation value we obtain non-linear equations for the amplitude functions. The boundary conditions can be deduced from the $x \to 0$ $(z \to \infty)$ limit:

$$\langle AB|u(z,\bar{z})u(\zeta,\bar{\zeta})|ab\rangle \to \sum_{c,d} \langle AB|u(z,\bar{z})|cd\rangle\langle cd|u(\zeta,\bar{\zeta})|ab\rangle(1+\ldots) =$$

$$= (z\bar{z}\zeta\bar{\zeta})^{\lambda} \sum_{c,d} G(AB;cd)G(cd;ab)(x\bar{x})^{h(c,d)-\frac{1}{2}(h(A,B)+h(a,b))}(1+\ldots) \tag{3.12}$$

where only those h.w. states give contribution which are allowed by the selection rules. The dots note polynomials of x and $\bar{x}$.

To derive the d.e. we use the usual method to compute matrix elements [10]. After a lengthy but straightforward calculation we determine the d.e., which is given in appendix C. In order to solve this equation we analyse the behaviour of the solutions in the vicinity of the singular points : $x = 0, x = 1, x = \infty$.

At the $x = 1$ $(z = \zeta)$ singularity the indices of f $(f \sim (1-x)^{\gamma})$ are independent of the initial and final states and contain information about the short distance OPE of $u(z,\bar{z})u(\zeta,\bar{\zeta})$. From the (C.1) differential equation one can compute the following indices:

$$\gamma_1 = -\frac{1}{2}(5Q-6) \ ; \quad \gamma_2 = -\frac{1}{2}(Q-2) \ ; \quad \gamma_3 = -\frac{1}{2}(Q+4) \ ; \quad \gamma_4 = \frac{Q}{2}$$

They correspond to the presence of operators with conformal weights:

$$\Delta_1 = 0 \ ; \quad \Delta_2 = 2Q-2 \ ; \quad \Delta_3 = 3Q-1 \ ; \quad \Delta_4 = 3Q-3$$

It is easy to see that the first is nothing but the identity operator. The second and the fourth are W primary operators since they can be parameterised by $a = 1 - 2Q^{-1}$, $b = 2 - Q^{-1}$ and $a = 3 - 2Q^{-1}$, $b = 1 - Q^{-1}$, respectively. The operator with conformal weight Δ_3 is a descendant of the fourth because its weight differs from Δ_4 by a positive integer.

Let us turn to the study of the $x = 0$ singular point. The indices at this point can be classified by the possible intermediate states in (3.12).

If there is only one intermediate state, i.e. we are dealing with one of the cases $A, B = a+2, b$; $a+2, b-2$; $a-2, b$; $a-2, b+2$, the index is $Q/4$. Combining this index with the $Q/2$ index of the $x = 1$ singularity we can build up the following trial function:

$$\big(x^{-1}(1-x)^2\big)^{Q/4}\big(\bar{x}^{-1}(1-\bar{x})^2\big)^{Q/4} \tag{3.13}$$

Using the FORM program [17] we checked that this function solves the appropriate differential equation. Multiplying it with the corresponding amplitude functions we can ensure the right $x \to 0$ behaviour. Since this solution is invariant under the $x \to x^{-1}, \bar{x} \to \bar{x}^{-1}$ transformation the locality requirement does not give any restriction for the amplitude functions.

Let us consider the case when there are two intermediate states. Since there are essentially two independent amplitude functions it is enough to investigate the following possibilities: $A, B = a, b+1$ and $A, B = a+2, b-1$. The corresponding indices are $\pm aQ/4$ and $\pm(a+2b)Q/4$. Motivated by our earlier results [10] we look for the trial function in the form $(1-x)^{\frac{Q}{2}} x^{\text{index}} F(\alpha, \beta, \gamma; x)$. Here F is the well known hypergeometric function which is analytic around $x = 0$. Its parameters α, β, γ can be determined by requiring the correct asymptotic behaviour around the singularities, and by demanding the appropriate transformation properties under $x \to x^{-1}, \bar{x} \to \bar{x}^{-1}$. Combining this form with the corresponding amplitude functions the trial functions are:

$$\begin{aligned} \langle a, b+1|uu|ab\rangle : \quad & \sigma(x)\sigma(\bar{x})\big(G_4(a,b)G_1(a-1,b+1)\Psi_a(x)\Psi_a(\bar{x}) \\ & + G_1(a,b)G_4(a+1,b)\Psi_{-a}(x)\Psi_{-a}(\bar{x})\big) \end{aligned} \tag{3.14}$$

$$\langle a+2,b-1|uu|ab\rangle: \quad \begin{aligned}&\sigma(x)\sigma(\bar{x})\big(G_1(a,b)G_2(a+1,b)\Psi_{a+2b}(x)\Psi_{a+2b}(\bar{x})\\ &+G_2(a,b)G_1(a+1,b-1)\Psi_{-a-2b}(x)\Psi_{-a-2b}(\bar{x})\big)\end{aligned} \tag{3.15}$$

where $\sigma(x) = \big(x^{-1}(1-x)^2\big)^{\frac{Q}{4}}$ and

$$\Psi_a(x) = x^{\frac{Q}{4}(1+a)} F(\frac{Q}{2}, \frac{Q}{2}(1+a), 1+\frac{Q}{2}a; x).$$

Using FORM again as in ref.[10] we checked that these solutions solve the corresponding differential equations.

Let us consider the restrictions given by locality. From the $x \to x^{-1}$ transformation property of the hypergeometric functions it follows that:

$$\Psi_a(x) = B_1(a)\frac{x^{\frac{Q}{2}(a+1)}}{(-x)^{\frac{Q}{2}(a+1)}}\Psi_a(1/x) + B_2(a)\frac{x^{-\frac{Q}{2}}}{(-x)^{-\frac{Q}{2}}}\Psi_{-a}(1/x) \tag{3.16}$$

where $B_1(a) = \frac{\Gamma(1+\frac{Q}{2}a)\Gamma(-\frac{Q}{2}a)}{\Gamma(1-\frac{Q}{2})\Gamma(\frac{Q}{2})}$; $B_2(a) = \frac{\Gamma(1+\frac{Q}{2}a)\Gamma(\frac{Q}{2}a)}{\Gamma(\frac{Q}{2}(a-1)+1)\Gamma(\frac{Q}{2}(a+1))}$ This implies for the amplitude functions that:

$$\frac{G_1(a,b)G_2(a+1,b)}{G_2(a,b)G_1(a+1,b-1)} = \phi(a+2b)$$

$$\frac{G_1(a,b)G_4(a+1,a)}{G_4(a,b)G_1(a-1,b+1)} = \phi(a)$$

where

$$\phi(a) = -\frac{\Gamma^2(-\frac{Q}{2}a)\Gamma(\frac{Q}{2}(a+1))\Gamma(1+\frac{Q}{2}(a-1))}{\Gamma^2(\frac{Q}{2}a)\Gamma(\frac{Q}{2}(1-a))\Gamma(1-\frac{Q}{2}(a+1))} = \frac{s(a+1)}{s(a-1)}\frac{\Gamma^2(-\frac{Q}{2}a)\Gamma^2(\frac{Q}{2}(a+1))}{\Gamma^2(\frac{Q}{2}a)\Gamma^2(\frac{Q}{2}(1-a))}$$

with $s(x) = \sin(\pi\frac{Q}{2}x)$. Using these equations one can show that the amplitude functions must have the following form:

$$|G_1(a,b)|^2 = f_2(a,b)\frac{\Gamma(\frac{Q}{2}(a+2b+1))\Gamma(-\frac{Q}{2}(a+2b))\Gamma(-\frac{Q}{2}a)\Gamma(\frac{Q}{2}(a+1))}{\Gamma(1-\frac{Q}{2}(a+2b+1))\Gamma(1+\frac{Q}{2}(a+2b))\Gamma(1+\frac{Q}{2}a)\Gamma(1-\frac{Q}{2}(a+1))}$$

$$|G_2(a,b)|^2 = f_1(a,b)\frac{\Gamma(\frac{Q}{2}(a+2b))\Gamma(-\frac{Q}{2}(a+2b-1))\Gamma(\frac{Q}{2}(a+1))\Gamma(-\frac{Q}{2}a)}{\Gamma(1-\frac{Q}{2}(a+2b))\Gamma(1+\frac{Q}{2}(a+2b-1))\Gamma(1-\frac{Q}{2}(a+1))\Gamma(1+\frac{Q}{2}a)} \tag{3.17}$$

where $f_1(a,b)$ is invariant under the transformation $a \to a+1$ -essentially independent of a- and $f_2(a,b)$ is invariant under the transformation $a \to a+1, b \to b-1$ (i.e. $f_2(a,b)$ essentially depends on $a+b$).

Finally let us consider the diagonal transition. In this case there are four intermediate states: $a+1, b$; $a+1, b-1$; $a-1, b$; $a-1, b+1$. The corresponding indices around $x=0$ are:

$$\nu_1 = \frac{Q}{4}(1-2a-2b) \ ; \ \nu_2 = \frac{Q}{4}(1-2b) \ ; \ \nu_3 = \frac{Q}{4}(1+2a+2b) \ ; \ \nu_4 = \frac{Q}{4}(1+2b)$$

which are the same as those that come from the differential equation. In order to define a periodic u it is necessary to avoid the appearance of logarithmically singular solutions. Since our differential equation is of the Fuchs type if we want to avoid the logarithmic singularities of its solutions we have to demand that no pairs of the indices differ by an integer. This is exactly the same restriction that we used to establish the Hilbert space of the model (3.9). The next step would be to construct the solutions of the d.e.. However the d.e. is so complicated that we have not succeeded in solving it in the usual way. Using a free field representation for the C_2W algebra [5] we found solutions (with the appropriate asymptotic behaviour for $x=0$) in terms of triple contour integrals. Unfortunately we could determine neither the transformation property of the solutions under $x \mapsto 1/x$ nor the monodromy matrix corresponding to the singular points. In this way the f functions remain arbitrary (except for the fact that they have to satisfy (3.11) since the factors occurring in (3.17) satisfy it.) In order to determine them one has to either carry out a complete analysis of the four point functions (that we have not succeeded in), or analyse higher point functions.

Conclusions

In this paper we successfully applied the reduced WZNW framework for the C_2 Toda theory. At the classical level we could generalise the result obtained previously for A_2: We derived the classical C_2 W algebra in the highest weight gauge. Identifying the relevant degrees of freedom and declaring the connection between them (the classical equation of motion) we established a framework in which we analysed the classical representation space of the symmetry algebra. Parameterising the W orbits by their monodromy matrices we identified those which correspond to classically h.w.r.. We showed that, contrary to the A_2 case, the orbit of the classical SL_2 invariant vacuum is of the h.w. type and is on the boundary of the allowed region.

In the quantum case we quantised the construction above (which was successful classically), in contrast to other approaches. Promoting only the relevant variables to operators we required the symmetry algebra to be the C_2 quantum W algebra [11-13]. We supposed the Hilbert space of the model to contain only h.w.r. spaces and demanded u to be a periodic W primary field, which satisfied the quantum equation of motion that is the normal ordered analogue of the classical one. Using this q.e.m. we obtained the selection rules for the u field and partly determined the relevant amplitude functions, analysing the locality requirement of the four point functions. This means that at the quantum level we could not completely generalise the result obtained previously. However we should remark that the problems are purely technical. One can go further analysing higher point functions or considering other operators in the theory appearing for example in the OPE of u with itself. This operator is nothing but the field which corresponds to the B_2 theory [16]. Representing this field on the same Hilbert space and considering its four point functions with u simultaneously a complete analysis could be made. This may lead to constraints on the value of the central charge and on the structure of the Hilbert space of the model.

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Appendix A

Stability analysis

In this appendix we investigate the condition for an orbit to correspond to classical h.w.r. As we defined earlier, the h.w. solutions -described by the various monodromy matrices- must give constant W densities ($L(z) = L_0, W(z) = W_0$ respectively) and the total energy of the system has to increase moving along the orbit. In order to check that the energy does increase we iterate the W transformations. In the first approximation the variation of the W densities cancel since L_0, W_0 are constant and the a_i transformations functions are periodic:

$$\delta L = \int_0^{2\pi} (2a_1^{'} L_0 - 5a_1^{'''} + 4a_2^{'} W_0) dz = 0$$

$$\delta W = \int_0^{2\pi} \Bigg(4a_1^{'} W_0 + a_2{'} (\frac{14}{25} L_0 W_0 + \frac{72}{625} L_0^3) + \\ a_2{'''} (-\frac{3}{5} W_0 - \frac{49}{125} L_0^2) + \frac{7}{25} a_2^{(V)} L_0 - \frac{1}{20} a_2^{(VII)} \Bigg) dz = 0$$

We remark that in the case of non-constant W densities we can always choose such a_1, a_2, st. δL is positive or negative, which means that non-constant W densities never give minima or maxima. Going back to the formula the point described by L_0, W_0 turns out to be a stationary point of the orbit, and it is necessary to iterate further. Keeping in mind that the W densities are no longer constants (due to the arbitrary a_i functions in δW_i) we could write $\delta\delta L$:

$$\delta\delta L = \int_0^{2\pi} (a_1^{'} \delta L + a_2^{'} \delta W) dz$$

which can be rewritten as

$$\delta\delta L = \int_0^{2\pi} \begin{pmatrix} a_1^{'} & a_2^{'} \end{pmatrix} \begin{pmatrix} 2L_0 - 5\frac{d^2}{dz^2} & 4W_0 \\ 4W_0 & \mathcal{D} \end{pmatrix} \begin{pmatrix} a_1^{'} \\ a_2^{'} \end{pmatrix} dz \qquad (A.1)$$

$$\mathcal{D} = (\frac{14}{25} L_0 W_0 + \frac{72}{625} L_0^3) + (-\frac{3}{5} W_0 - \frac{49}{125} L_0^2) \frac{d^2}{dz^2} + \frac{7}{25} L_0 \frac{d^4}{dz^4} - \frac{1}{20} \frac{d^6}{dz^6}$$

where we have dropped the total derivatives. Let us analyse this matrix in a convenient basis of the form:

$$\begin{pmatrix} a_1^{\prime} \\ a_2^{\prime} \end{pmatrix} = \mathbf{q} e^{inz}, \qquad n \neq 0$$

which span the space of the possible transformations.

For a solution to belong to a h.w.r. it is sufficient and necessary that its matrix (in terms of n)

$$M(L_0, W_0) = \begin{pmatrix} 2L_0 + 5n^2 & 4W_0 \\ 4W_0 & \mathcal{D}(n) \end{pmatrix}$$

$$\mathcal{D}(n) = (\frac{14}{25} L_0 W_0 + \frac{72}{625} L_0^3) + (\frac{3}{5} W_0 + \frac{49}{125} L_0^2) n^2 + \frac{7}{25} L_0 n^4 + \frac{1}{20} n^6$$

be positive definite. This condition can be formulated as the positivity of the left upper component and the determinant of M. It is evident that it is sufficient to consider the $n = 1$ case. The first requirement is nothing but the positivity of the energy. The second can be transformed into the following inequality:

$$\big((4a+1) + 16b\big)\big(\frac{1}{4}(a+1)^2 - b\big) > 0 \qquad (A.2)$$

where we parameterised L_0 and W_0 as:

$$L_0 = a \qquad W_0 = -\frac{9}{100} a^2 + b$$

Fortunately all three cases discussed in chapter 2 are included in this equation with an appropriate choice of the region of the parameters. Analysing the inequality one can show that in the first case $a > 0, b > 0$ -described by the monodromy matrix (2.3) - there is no restriction on the parameters Λ and μ. In such a way the representation is h.w.r. for all Λ and μ. The monodromy matrix with two real eigenvalues corresponds to the case of $a > 0, b < 0$, in which case the positive definiteness condition implies that ρ must be smaller then $\frac{1}{2}$. From the positivity of the energy, $\Lambda^2 - \rho^2 > \frac{5}{2}$ must hold. The representation is h.w. only for these values of the parameters. The last case, when $a < 0$, $b > 0$, describes the monodromy matrix with no real eigenvalues. Using both requirements (for the energy and the determinant) one can show that

$\rho < \frac{1}{2}$ $\nu > \frac{1}{2}$ $\nu - \rho < 1$ is needed. Since the classical SL_2 invariant vacuum, described by $L_0 = -\frac{5}{2}$, $W_0 = 0$, is on the boundary of the possible region, the quantum theory may have a classical limit in contrast to the A_2 case. Finally we notice that the minima obtained are the only possible minima since they are the only points of the orbits which give constant W densities and in this way lead to minima. Clearly the W densities are uniquely determined by the various gauge invariant monodromy matrices.

Appendix B

The covariance of the quantum equation of motion

The covariance of equation (3.5) (χ) - i.e. $L_n\chi = 0$, $W_n\chi = 0$ for $n > 0$ implies that there must exist a null state,ϕ, on grade 3 of the following form:

$$\phi = \beta^{-3}L_{-3}u + W_{-3}u + \beta^{-2-1}L_{-2}L_{-1}u + \beta^{-1-1-1}L_{-1}^3u = 0 \qquad (B.1)$$

From the requirement that the generators with positive indices annihilate χ and ϕ it follows that there are two independent null states on grade 2 and grade 1:

$$\beta^{-1}L_{-1}u + W_{-1}u = 0 \qquad (B.2)$$

$$\beta^{-2}L_{-2}u + W_{-2}u + \beta^{-1-1}L_{-1}^2u = 0, \qquad (B.3)$$

respectively. Since all these states have to be null states one can compute the various β coefficients, iteratively. We find for the first two coefficients that

$$\beta^{-1} = -2\frac{\omega}{\Delta}$$

$$\beta^{-2} = 2(23 - 10Q)N \qquad \beta^{-1-1} = 4(13 - \frac{25}{Q})N \qquad (B.4)$$

with

$$N = \frac{\omega}{\Delta}\frac{1}{(4Q-5)(\frac{8}{Q}-5)}$$

and these conditions fix the W weights of the u operator, $(\Delta, \omega\ )$.

Furthermore

$$\beta^{-3} = -6(20Q^3 - 17Q^2 - 116Q + 108)M$$

$$\beta^{-2-1} = -24(34Q^2 - 113Q + 82)M \qquad (B.5)$$

$$\beta^{-1-1-1} = -16(226 - 75Q - \frac{150}{Q})M$$

where

$$M = \frac{\omega}{\Delta}\frac{1}{(4Q-5)(\frac{8}{Q}-5)(7Q-6)(3Q-2)}$$

As a consequence of these results the coefficients of the quantum equation of motion are:

$$A = -\frac{2}{Q}(30Q^2 - 23Q^3 + 178Q^2 - 936Q + 720)$$

$$B = -8Q\sqrt{(4Q-5)(\frac{8}{Q}-5)(\frac{2}{Q}-3)(\frac{6}{Q}-7)(226-75Q-\frac{150}{Q})(Q-3)(3Q-7)}$$

$$C = 4(16Q - 27)(3Q - 2)$$

$$D = -\frac{8}{Q}(27Q^3 + 37Q^2 - 356Q + 300) \qquad (B.6)$$

$$E = 16(226 - 75Q - \frac{150}{Q})$$

$$F = -\frac{16}{Q}(226 - 75Q - \frac{150}{Q})$$

Appendix C

The differential equation for $f(x, \bar{x})$

In this appendix we describe the differential equation which the four point function, f, has to satisfy. Sandwiching the quantum equation of motion and using the freedom to deform the contour in the integral representation of the W generators, (3.2), we get the following equation:

$$\Big\{F\mathcal{L}_{-1}^{(4)} + (D-2E)(\mathcal{L}_{-1}^{(1)}-3)\mathcal{L}_{-3} + E(\mathcal{L}_{-1}^{(2)} - 4\mathcal{L}_{-1}^{(1)} + 6)\mathcal{L}_{-2} +$$
$$C(\mathcal{L}_{-2}+2)\mathcal{L}_{-2} + (A{-}2D+2E)\mathcal{L}_{-4} +$$
$$B\Big(\frac{\omega}{(1-x)^4} + w + (\frac{1}{(1-x)^3} - 1)(-\beta^{-1}\hat{\mathcal{L}}_{-1}^{(1)}) + \beta^{-1}\mathcal{L}_{-1}^{(1)}$$
$$+\big(-2x-1+\frac{1}{(1-x)^2}\big)(-\beta^{-2}\hat{\mathcal{L}}_{-2} - \beta^{-1-1}\hat{\mathcal{L}}_{-1}^{(2)}) +$$
$$3\beta^{-2}\mathcal{L}_{-2} + 3\beta^{-1-1}\mathcal{L}_{-1}^{(2}) +$$
$$\big(\frac{1}{(1-x)} - (x-1)^2 - 3x\big)\big(-(\beta^{-3}-\beta^{-2-1})\hat{\mathcal{L}}_{-3} - \beta^{-2-1}\hat{\mathcal{L}}_{-1}^{(1)}\hat{\mathcal{L}}_{-2}$$
$$-\beta^{-1-1-1}\hat{\mathcal{L}}_{-1}^{(3)}\big)$$
$$+3(\beta^{-3}-\beta^{-2-1})\mathcal{L}_{-3} + 3\beta^{-2-1}(\mathcal{L}_{-1}^{(1)}-2)\mathcal{L}_{-2} + 3\beta^{-1-1-1}\mathcal{L}_{-1}^{(3)}\Big)\Big\}f = 0 \qquad (C.1)$$

where

$$\mathcal{L}_{-n} = (n-1)\big(\frac{\Delta}{(x-1)^n} + (-1)^n h\big) - \frac{1}{(x-1)^{n-1}}(\frac{\lambda}{x} + d_x)$$
$$- (-1)^n\big(\lambda(1+\frac{1}{x}) + (1-x)d_x\big)$$

$$\hat{\mathcal{L}}_{-n} = (n-1)\big(\frac{\Delta}{(1-x)^n} + (-1)^n\frac{h}{x^n}\big) - \frac{1}{(1-x)^{n-1}}(\lambda - xd_x)$$
$$- \frac{(-1)^n}{x^{n-1}}\big(\lambda(1+\frac{1}{x}) + (1-x)d_x\big)$$

and

$$\mathcal{L}_{-1}^{(n)} = \sum_{k=0}^{n}(-1)^k\lambda^{(n-k)}x^k d_x^k \qquad \lambda^{(k)} = \prod_{i=k}^{n-1}(\lambda - i)$$

$$\hat{\mathcal{L}}_{-1}^{(n)} = \sum_{k=0}^{n}\lambda^{(n-k)}\frac{1}{x^k}d_x^k \qquad \lambda^{(k)} = \prod_{i=0}^{k}(\lambda - i)$$

and d_x means derivative with respect to x.

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