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3+1+1 dimensional covariant gravitational dynamics on an asymmetrically embedded brane

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We give the evolution and constraint equations on an asymmetrically embedded brane in the form of average and difference equations.

1 Introduction

In a recent paper [1] we have developed the covariant gravitational dynamics in a 3+1+1 dimensional space-time in the spirit of the general relativistic 3+1 covariant cosmology [2], generalizing both previous approaches to 5-dimensional (5d) gravitational dynamics [3], brane 3+1 covariant cosmology [4] and 2+1+1 covariant dynamics [5]. Such a formalism may turn useful in discussing perturbations on the brane [6]. The singled-out directions are the normal n^a to the hypersurface representing the time-evolving 3-dimensional space (the brane with metric h_{ab} and tension λ) and a temporal direction u^a tangent to this hypersurface. All employed gravitational variables are projected to the brane. They consist of kinematical variables $(\Theta, \sigma_{ab}, \omega_{ab}, A_a, K_a, L_a)$ related to the vector u^a; analogous quantities, carrying an overhat, related to n^a; two kinematical scalars $(K, \hat{K})$ related to both; finally gravito-electro-magnetic quantities $(\mathcal{E}, \mathcal{H}_k, \mathcal{F}_{kl}, \mathcal{E}_k, \widehat{\mathcal{E}}_k, \mathcal{E}_{kl}, \widehat{\mathcal{E}}_{kl}, \mathcal{H}_{kl}, \widehat{\mathcal{H}}_{kl})$. For details on the definitions of these quantities see Ref. [1]. The matter on the brane has been decomposed as $T_{ab} = \rho u_a u_b + q_{(a}u_{b)} + p h_{ab} + \pi_{ab}$, while possible non-standard model fields nesting in the 5d space-time as $\widetilde{T}_{ab} = \widetilde{\rho} u_a u_b + 2\widetilde{q}_{(a}u_{b)} + 2\widetilde{q} u_{(a}n_{b)} + \widetilde{p} h_{ab} + \widetilde{\pi} n_a n_b + 2\widetilde{\pi}_{(a}n_{b)} + \widetilde{\pi}_{ab}$. The gravitational coupling constants $\widetilde{\kappa}^2$ and $\kappa^2 = \widetilde{\kappa}^4 \lambda/6$ act in 5d and on the brane, respectively.

The generic form of the evolution and constraint equations in the 5-dimensional spacetime were given as Appendix C of Ref. [1] and they were specified on a Z_2-symmetrically embedded brane in Subsection IV.D. The embedding however is not necessarely symmetric: an asymmetric embedding is known to generate late time acceleration in a cosmological setup [7]. Therefore here we generalize the formalism to an asymmetrically embedded brane. Following the recipe presented in Subsection IV.D of Ref. [1], we give here the evolution and constraint equations obtained both as averages and differences across the brane. We denote the average over the two sides of the brane of any quantity by angle brackets and the jump by Δ. For the extrinsic curvature components $\mathcal{K} \equiv (\widehat{\Theta}, \widehat{\sigma}_{ab}, \widehat{K}, \widehat{K}_a)$ the latter is directly related to the brane matter variables and brane tension cf. the Israel-Lanczos condition, Eqs. (76)-(79) of Ref. [1]. Angle brackets on indices indicate projection to the 3-space, symmetrization and trace-free character.

2 The average equations

The evolution equations arising as averages are:

$$0=\langle\dot{\widehat{\Theta}}\rangle-D^a\langle\widehat{K}_a\rangle+\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle\Theta-2\langle\widehat{K}^a\rangle A_a+\langle\widehat{\sigma}_{ab}\rangle\sigma^{ab}-\widetilde{\kappa}^2\langle\widetilde{q}\rangle\ ,$$

$$0=\langle\dot{\widehat{K}}_{\langle a\rangle}\rangle-D_a\langle\widehat{K}-\frac{2}{3}\widehat{\Theta}\rangle-D^b\langle\widehat{\sigma}_{ab}\rangle+\frac{4\Theta}{3}\langle\widehat{K}_a\rangle-\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle A_a-\langle\widehat{\sigma}_{ab}\rangle A^b-\omega_{ab}\langle\widehat{K}^b\rangle+\sigma_{ab}\langle\widehat{K}^b\rangle+\widetilde{\kappa}^2\langle\widetilde{\pi}_a\rangle\ ,$$

$$\begin{aligned}
&0=\dot{H}_{\langle kj\rangle}+\varepsilon_{ab\langle k}D^aE^b_{j\rangle}+\frac{1}{2}\varepsilon_{ab\langle k}D^a\langle\widehat{\mathcal{E}}^b_{j\rangle}\rangle+\Theta H_{kj}-3\sigma_{a\langle k}H^a_{j\rangle}-\omega_{a\langle k}H^a_{j\rangle}-2\varepsilon_{\langle k}{}^{ab}E_{j\rangle a}A_b\\
&-\frac{3}{2}\langle\widehat{\mathcal{E}}_{\langle j}\rangle\omega_{k\rangle}-\frac{1}{2}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\langle\widehat{\mathcal{E}}_b\rangle-\frac{1}{2}\varepsilon_{\langle k}{}^{cd}\langle\widehat{\sigma}_{j\rangle c}\rangle D_d\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle+\frac{1}{2}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\varepsilon_{ab\langle k}D^a\langle\widehat{\sigma}^b_{j\rangle}\rangle\\
&+\frac{\langle\widehat{\sigma}^{cb}\rangle}{2}\varepsilon_{ab\langle k}D^a\langle\widehat{\sigma}_{j\rangle c}\rangle+\langle\widehat{\Theta}\rangle\langle\widehat{K}_{\langle k}\rangle\omega_{j\rangle}+\frac{1}{2}\varepsilon_{ab\langle k}\langle\widehat{\sigma}_{j\rangle c}\rangle D^a\langle\widehat{\sigma}^{cb}\rangle+\frac{\langle\widehat{\Theta}\rangle}{3}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\langle\widehat{K}_b\rangle\\
&-\frac{1}{2}\varepsilon_{ab\langle k}\langle\widehat{K}_{j\rangle}\rangle D^a\langle\widehat{K}^b\rangle-\frac{1}{2}\varepsilon_{ab\langle k}\langle\widehat{K}^b\rangle D^a\langle\widehat{K}_{j\rangle}\rangle+\frac{\langle\widehat{K}_c\rangle}{2}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle b}\langle\widehat{\sigma}^c_a\rangle-\frac{3\langle\widehat{K}_a\rangle}{2}\langle\widehat{\sigma}^a_{\langle j}\rangle\omega_{k\rangle}\\
&+\frac{\widetilde{\kappa}^4}{8}\Big[\varepsilon_{ab\langle k}D^a\pi_{j\rangle c}\pi^{cb}+\varepsilon_{ab\langle k}\pi_{j\rangle c}D^a\pi^{cb}-\frac{1}{3}\varepsilon_{\langle k}{}^{cd}\pi_{j\rangle c}D_d\left(\rho+3p\right)-\varepsilon_{ab\langle k}q^bD^aq_{j\rangle}\\
&-\frac{(2\lambda-\rho-3p)}{3}\varepsilon_{ab\langle k}D^a\pi^b_{j\rangle}-\left(\lambda+\rho\right)\omega_{\langle k}q_{j\rangle}-\frac{2\left(\lambda+\rho\right)}{3}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}q_b-\varepsilon_{ab\langle k}q_{j\rangle}D^aq^b\\
&-\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle b}\pi^c_aq_c+3\pi^a_{\langle j}\omega_{k\rangle}q_a\Big]-\frac{\widetilde{\kappa}^2}{3}\varepsilon_{ab\langle k}D^a\langle\widetilde{\pi}^b_{j\rangle}\rangle-\widetilde{\kappa}^2\langle\widetilde{q}_{\langle j}\rangle\omega_{k\rangle}-\frac{\widetilde{\kappa}^2}{3}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\langle\widetilde{q}_b\rangle\ ,
\end{aligned}$$

$$\begin{aligned}
&0=\dot{E}_{\langle kj\rangle}-\frac{1}{2}\langle\dot{\widehat{\mathcal{E}}}_{\langle kj\rangle}\rangle-\varepsilon_{ab\langle k}D^aH^b_{j\rangle}+\frac{1}{2}D_{\langle k}\langle\widehat{\mathcal{E}}_{j\rangle}\rangle+\Theta E_{kj}-\frac{\Theta}{6}\langle\widehat{\mathcal{E}}_{kj}\rangle+\langle\widehat{\mathcal{E}}_{\langle k}\rangle A_{j\rangle}-\frac{2}{3}\langle\mathcal{E}\rangle\sigma_{kj}\\
&-\frac{\langle\widehat{\mathcal{E}}^a_{\langle j}\rangle}{2}\left(\omega_{k\rangle a}+\sigma_{k\rangle a}\right)+E^a_{\langle k}\left(\omega_{j\rangle a}-3\sigma_{j\rangle a}\right)+2\varepsilon_{\langle k}{}^{ab}H_{j\rangle a}A_b-\langle\widehat{\sigma}^a_{\langle j}\rangle\langle\dot{\widehat{\sigma}}_{k\rangle a}\rangle-\frac{\langle\widehat{\sigma}_{kj}\rangle}{2}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle^{\cdot}\\
&+\langle\widehat{K}_{\langle k}\rangle D_{j\rangle}\langle\widehat{K}-\widehat{\Theta}\rangle-\frac{\langle\widehat{\Theta}\rangle}{3}D_{\langle k}\langle\widehat{K}_{j\rangle}\rangle+\frac{\langle\widehat{\sigma}^a_{\langle j}\rangle}{2}D_{k\rangle}\langle\widehat{K}_a\rangle+\frac{\langle\widehat{K}_a\rangle}{2}D_{\langle k}\langle\widehat{\sigma}^a_{j\rangle}\rangle+\frac{\langle\widehat{\Theta}\rangle}{3}\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle\sigma_{kj}\\
&+\langle\widehat{K}_{\langle k}\rangle\Big[D^b\langle\widehat{\sigma}_{j\rangle b}\rangle+\frac{3}{2}\omega_{j\rangle a}\langle\widehat{K}^a\rangle+\langle\widehat{\sigma}_{j\rangle a}\rangle A^a-\frac{1}{2}\sigma_{j\rangle b}\langle\widehat{K}^b\rangle-\frac{7\Theta}{6}\langle\widehat{K}_{j\rangle}\rangle\Big]-\frac{\langle\widehat{\sigma}^c_a\rangle}{2}\langle\widehat{\sigma}^a_{\langle j}\rangle\omega_{k\rangle c}\\
&-\frac{1}{2}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\Big[\langle\dot{\widehat{\sigma}}_{\langle kj\rangle}\rangle+\frac{\Theta}{3}\langle\widehat{\sigma}_{kj}\rangle-2\langle\widehat{K}_{\langle k}\rangle A_{j\rangle}+\langle\widehat{\sigma}^a_{\langle k}\rangle\left(\omega_{j\rangle a}+\sigma_{j\rangle a}\right)\Big]-\frac{\sigma_{jk}}{2}\langle\widehat{K}^a\rangle\langle\widehat{K}_a\rangle\\
&+\langle\widehat{\sigma}_{a\langle j}\rangle A_{k\rangle}\langle\widehat{K}^a\rangle-\frac{\langle\widehat{\sigma}^a_c\rangle}{2}\langle\widehat{\sigma}^c_{\langle k}\rangle\sigma_{j\rangle a}-\frac{\Theta}{6}\langle\widehat{\sigma}^a_{\langle j}\rangle\langle\widehat{\sigma}_{k\rangle a}\rangle+\frac{\widetilde{\kappa}^4}{8}\Big[\frac{2}{3}q_{\langle k}D_{j\rangle}\left(\rho-3p\right)-\frac{(\rho+3p)^{\cdot}}{3}\pi_{kj}\\
&-2\pi^a_{\langle j}\dot{\pi}_{k\rangle a}+\frac{2\left(\lambda+\rho\right)}{3}D_{\langle k}q_{j\rangle}-\pi^a_{\langle j}D_{k\rangle}q_a-q_aD_{\langle k}\pi^a_{j\rangle}+\frac{2\left(\lambda+\rho\right)}{3}\left(\rho+p\right)\sigma_{kj}-\pi^a_{\langle j}\omega_{k\rangle c}\pi^c_a\\
&-\sigma_{jk}q^aq_a-2\pi_{a\langle j}A_{k\rangle}q^a+q_{\langle k}\Big(-2D^b\pi_{j\rangle b}+3\omega_{j\rangle a}q^a-2\pi_{j\rangle a}A^a-\sigma_{j\rangle b}q^b-\frac{7\Theta}{3}q_{j\rangle}\Big)-\pi^a_c\pi^c_{\langle k}\sigma_{j\rangle a}\\
&+\frac{(2\lambda-\rho-3p)}{3}\Big(\dot{\pi}_{\langle kj\rangle}+\frac{\Theta}{3}\pi_{kj}+2q_{\langle k}A_{j\rangle}+\pi^a_{\langle k}\omega_{j\rangle a}+\pi^a_{\langle k}\sigma_{j\rangle a}\Big)-\frac{\Theta}{3}\pi^a_{\langle j}\pi_{k\rangle a}\Big]-\frac{\widetilde{\kappa}^2}{3}\Big[\langle\dot{\widetilde{\pi}}_{\langle kj\rangle}\rangle\\
&+D_{\langle k}\langle\widetilde{q}_{j\rangle}\rangle-\langle\widetilde{\pi}_{\langle k}\rangle\langle\widehat{K}_{j\rangle}\rangle-\frac{\widetilde{\kappa}^2}{4}q_{\langle k}\Delta\widetilde{\pi}_{j\rangle}+4\langle\widetilde{q}_{\langle k}\rangle A_{j\rangle}+\langle\widetilde{\rho}+\widetilde{p}\rangle\sigma_{jk}+\frac{\Theta}{3}\langle\widetilde{\pi}_{jk}\rangle+\langle\widetilde{\pi}^a_{\langle j}\rangle\left(\omega_{k\rangle a}+\sigma_{k\rangle a}\right)\Big]\ ,
\end{aligned}$$

$$0=\langle\dot{\widehat{\mathcal{E}}}_{\langle k\rangle}\rangle+\frac{4\Theta}{3}\langle\widehat{\mathcal{E}}_k\rangle-\frac{1}{3}D_k\langle\mathcal{E}\rangle-\frac{4\langle\mathcal{E}\rangle}{3}A_k-D^a\langle\widehat{\mathcal{E}}_{ka}\rangle-\langle\widehat{\mathcal{E}}_{ka}\rangle A^a-(\omega_{ka}-\sigma_{ka})\,\langle\widehat{\mathcal{E}}^a\rangle+\langle\widehat{K}^a\rangle$$
$$\times\left[\langle\dot{\widehat{\sigma}}_{\langle ka\rangle}\rangle+\sigma_{ck}\langle\widehat{\sigma}^c_a\rangle-2D_k\langle\widehat{K}_a\rangle+D_a\langle\widehat{K}_k\rangle-2A_{\langle k}\langle\widehat{K}_{a\rangle}\rangle-\sigma_{ba}\langle\widehat{\sigma}^b_k\rangle+\varepsilon_{cab}\omega^b\langle\widehat{\sigma}^c_k\rangle\right]+\langle\widehat{\sigma}^{ab}\rangle$$
$$\times\left[D_k\langle\widehat{\sigma}_{ab}\rangle+\frac{2}{3}\langle\widehat{K}_k\rangle\sigma_{ab}\right]+\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle\left[\frac{2}{3}D_k\langle\widehat{\Theta}\rangle-D^b\langle\widehat{\sigma}_{kb}\rangle+\frac{2\Theta}{3}\langle\widehat{K}_k\rangle\right]-\langle\widehat{\sigma}^a_b\rangle D^b\langle\widehat{\sigma}_{ka}\rangle$$
$$+\frac{\langle\widehat{K}_k\rangle}{3}D^a\langle\widehat{K}_a\rangle+\varepsilon_k{}^{ab}\langle\widehat{K}_c\rangle\langle\widehat{\sigma}^c_b\rangle\omega_a-\frac{\langle\widehat{\sigma}^a_k\rangle}{3}D_a\langle\widehat{\Theta}\rangle+\frac{\widetilde{\kappa}^4q^a}{4}\big[D_aq_k-2D_kq_a-\dot{\pi}_{\langle ka\rangle}-\sigma_{ck}\pi^c_a-2A_{\langle k}q_{a\rangle}$$
$$+\sigma_{ba}\pi^b_k-\varepsilon_{cab}\omega^b\pi^c_k\big]-\frac{\widetilde{\kappa}^4\pi^{ab}}{4}\left[\frac{2q_k}{3}\sigma_{ab}-D_k\pi_{ab}\right]+\frac{\widetilde{\kappa}^4\,(\rho+p)}{6}\left[D_k\rho-\frac{3}{2}D^b\pi_{kb}-\Theta q_k\right]+\frac{\widetilde{\kappa}^4}{12}$$
$$\times\left[q_kD^aq_a-3\pi^a_bD^b\pi_{ka}-\pi^a_kD_a\rho-3\varepsilon_k{}^{ab}q_c\omega_a\pi^c_b\right]+\frac{2\widetilde{\kappa}^2}{3}\left[\frac{D_k\langle\widetilde{\rho}+3\widetilde{\pi}-3\widetilde{p}\rangle}{4}+\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\langle\widetilde{\pi}_k\rangle\right.$$
$$-\langle\widetilde{\pi}'_{\langle k\rangle}\rangle-\langle K\rangle\langle\widetilde{q}_k\rangle-\langle\widetilde{q}\rangle\langle2\widehat{K}_k+K_k\rangle+\langle\widetilde{\pi}_{ka}\rangle\langle\widehat{A}^a\rangle-\langle\widetilde{\pi}-\widetilde{p}\rangle\langle\widehat{A}_k\rangle-\frac{5\langle\widehat{\sigma}_{ka}\rangle}{2}\langle\widetilde{\pi}^a\rangle-\frac{\widetilde{\kappa}^2q_k}{2}\Delta\widetilde{q}$$
$$\left.+\frac{\widetilde{\kappa}^2(2\lambda-\rho-3p)}{12}\Delta\widetilde{\pi}_k+\frac{5\widetilde{\kappa}^2\pi_{ka}}{8}\Delta\widetilde{\pi}^a-\frac{\Delta K}{4}\Delta\widetilde{q}_k-\frac{\Delta K_k}{4}\Delta\widetilde{q}+\frac{\Delta\widehat{A}^a}{4}\Delta\widetilde{\pi}_{ka}-\frac{[\Delta\,(\widetilde{\pi}-\widetilde{p})]}{4}\Delta\widehat{A}_k\right],$$

$$0=\langle\dot{\mathcal{E}}\rangle-D^a\langle\widehat{\mathcal{E}}_a\rangle+\frac{4}{3}\Theta\langle\mathcal{E}\rangle+\langle\widehat{\mathcal{E}}_{ab}\rangle\sigma^{ab}-2\langle\widehat{\mathcal{E}}_a\rangle A^a+\langle\widehat{\sigma}^{ab}\rangle\left[\langle\dot{\widehat{\sigma}}_{\langle ab\rangle}\rangle-D_a\langle\widehat{K}_b\rangle+\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle\sigma_{ab}\right.$$
$$\left.-2A_b\langle\widehat{K}_a\rangle+\frac{\Theta}{3}\langle\widehat{\sigma}_{ab}\rangle+\sigma_{ca}\langle\widehat{\sigma}^c_b\rangle\right]+\langle\widehat{K}^a\rangle\left[\frac{2}{3}D_a\langle\widehat{\Theta}\rangle-D^b\langle\widehat{\sigma}_{ab}\rangle+\frac{2\Theta}{3}\langle\widehat{K}_a\rangle-\langle\widehat{K}^b\rangle\sigma_{ab}\right]+\frac{\widetilde{\kappa}^4\pi^{ab}}{4}$$
$$\times\left[\dot{\pi}_{\langle ab\rangle}+D_aq_b+2A_bq_a+\frac{\Theta}{3}\pi_{ab}+(\rho+p)\,\sigma_{ab}+\sigma_{ca}\pi^c_b\right]+\frac{\widetilde{\kappa}^4q^a}{4}\left[D^b\pi_{ab}-\frac{2}{3}D_a\rho+\frac{2\Theta}{3}q_a-\sigma_{ab}q^b\right]$$
$$+\frac{\widetilde{\kappa}^4\,(\lambda+\rho)}{6}\Delta\widetilde{q}-\frac{2\widetilde{\kappa}^2}{3}\left[\frac{3\langle\widetilde{\rho}-\widetilde{\pi}+\widetilde{p}\rangle^{\cdot}}{4}+D^a\langle\widetilde{q}_a\rangle+\Theta\langle\widetilde{\rho}+\widetilde{p}\rangle+\langle\widehat{\Theta}\rangle\langle\widetilde{q}\rangle+2\langle\widetilde{q}_a\rangle A^a+\langle\widetilde{\pi}_{ab}\rangle\sigma^{ab}\right].$$

$$0=\dot{\omega}_{\langle a\rangle}-\frac{1}{2}\varepsilon_a{}^{cd}D_cA_d+\frac{2\Theta}{3}\omega_a-\sigma_{ab}\omega^b\,,$$

$$0=\dot{\Theta}-D^aA_a+\frac{\Theta^2}{3}-A^aA_a-2\omega_a\omega^a+\sigma_{ab}\sigma^{ab}+\frac{\kappa^2}{2}\,(\rho+3p)-\Lambda+\frac{\widetilde{\kappa}^4\rho}{12}\,(2\rho+3p)$$
$$-\frac{\widetilde{\kappa}^4}{4}q^aq_a+\langle\widehat{\Theta}\rangle\langle\widehat{K}\rangle-\langle\widehat{K}^a\rangle\langle\widehat{K}_a\rangle-\langle\mathcal{E}\rangle+\frac{\widetilde{\kappa}^2}{2}\langle\widetilde{\rho}+\widetilde{\pi}+\widetilde{p}\rangle\,,$$

$$0=\dot{\sigma}_{\langle ab\rangle}-D_{\langle a}A_{b\rangle}+\frac{2\Theta}{3}\sigma_{ab}-A_{\langle a}A_{b\rangle}+\omega_{\langle a}\omega_{b\rangle}+\sigma_{c\langle a}\sigma^c_{b\rangle}+E_{ab}+\frac{1}{2}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\langle\widehat{\sigma}_{ab}\rangle+\frac{\langle\widehat{\mathcal{E}}_{ab}\rangle}{2}$$
$$-\frac{1}{2}\langle\widehat{K}_{\langle a}\rangle\langle\widehat{K}_{b\rangle}\rangle+\frac{1}{2}\langle\widehat{\sigma}_{c\langle a}\rangle\langle\widehat{\sigma}^c_{b\rangle}\rangle-\frac{\widetilde{\kappa}^2}{3}\langle\widetilde{\pi}_{ab}\rangle+\frac{\widetilde{\kappa}^4}{8}\left[\pi_{c\langle a}\pi^c_{b\rangle}-q_{\langle a}q_{b\rangle}-\frac{(2\lambda-\rho-3p)}{3}\pi_{ab}\right],$$

The constraint equations arising as averages are:

$$
\begin{aligned}
0=\,& D^a H_{ak}-\frac{4\langle\mathcal{E}\rangle}{3}\omega_k+3E_{ka}\omega^a+\frac{\omega^a}{2}\langle\widehat{\mathcal{E}}_{ak}\rangle+\frac{\varepsilon_k{}^{ab}}{2}\Big[D_a\langle\widehat{\mathcal{E}}_b\rangle-2E_{ac}\sigma^c_b+\langle\widehat{\mathcal{E}}_{ac}\rangle\sigma^c_b-\langle\widehat{\sigma}^c_a\rangle D_b\langle\widehat{K}_c\rangle\\
&-\frac{2\langle\widehat{K}_b\rangle}{3}D_a\langle\widehat{\Theta}\rangle-\frac{2\langle\widehat{\Theta}\rangle}{3}D_a\langle\widehat{K}_b\rangle+\sigma_{ac}\langle\widehat{K}^c\rangle\langle\widehat{K}_b\rangle+\langle\widehat{\sigma}_{da}\rangle\langle\widehat{\sigma}^d_c\rangle\sigma^c_b-\langle\widehat{K}^c\rangle D_b\langle\widehat{\sigma}_{ac}\rangle\Big]-\frac{\omega_c}{2}\langle\widehat{K}_k\rangle\langle\widehat{K}^c\rangle\\
&+\frac{1}{2}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\big[\langle\widehat{\sigma}^c_k\rangle\omega_c+\varepsilon_k{}^{ab}\langle\widehat{\sigma}_{ac}\rangle\sigma^c_b\big]+\frac{2\langle\widehat{\Theta}\rangle}{3}\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle\omega_k-\frac{\omega_k}{2}\Big[\langle\widehat{K}_a\rangle\langle\widehat{K}^a\rangle+\langle\widehat{\sigma}_{ab}\rangle\langle\widehat{\sigma}^{ab}\rangle\Big]\\
&+\frac{\omega^a}{2}\langle\widehat{\sigma}_{ca}\rangle\langle\widehat{\sigma}^c_k\rangle+\frac{\widetilde{\kappa}^4\varepsilon_k{}^{ab}}{8}\Big[\frac{2}{3}q_bD_a\rho+\frac{2(\lambda+\rho)}{3}D_aq_b+q^cD_b\pi_{ac}+\sigma_{ac}q^cq_b+\pi_{da}\pi^d_c\sigma^c_b+\pi^c_aD_bq_c\Big]\\
&+\frac{\widetilde{\kappa}^4}{8}\Big[\frac{4(\lambda+\rho)}{3}(\rho+p)\omega_k-\frac{(2\lambda-\rho-3p)}{3}\big(\pi^c_k\omega_c+\varepsilon_k{}^{ab}\pi_{ac}\sigma^c_b\big)-q_aq^a\omega_k-q^c\omega_cq_k\\
&-\pi_{ab}\pi^{ab}\omega_k+\pi_{ca}\pi^c_k\omega^a\Big]-\frac{\widetilde{\kappa}^2}{3}\langle\widetilde{\pi}_{ka}\rangle\omega^a+\frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}D_a\langle\widetilde{q}_b\rangle+\frac{2\widetilde{\kappa}^2}{3}\langle\widetilde{\rho}+\widetilde{p}\rangle\omega_k-\frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}\langle\widetilde{\pi}^c_a\rangle\sigma_{bc}\,,
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& D^b\sigma_{ab}-\frac{2D_a\Theta}{3}+\varepsilon_a{}^{ck}D_c\omega_k+2\varepsilon_a{}^{ck}A_c\omega_k+\frac{\widetilde{\kappa}^4(\lambda+\rho)}{6}q_a\\
&-\frac{\widetilde{\kappa}^4q^b}{4}\pi_{ab}-\frac{2\langle\widehat{\Theta}\rangle}{3}\langle\widehat{K}_a\rangle+\langle\widehat{\sigma}_{ab}\rangle\langle\widehat{K}^b\rangle+\langle\widehat{\mathcal{E}}_a\rangle+\frac{2\widetilde{\kappa}^2}{3}\langle\widetilde{q}_a\rangle\,,\\
0=\,& D^a\omega_a-A_a\omega^a\,,\\
0=\,& D_{\langle c}\omega_{k\rangle}+\varepsilon_{ab\langle k}D^b\sigma^a_{c\rangle}+2A_{\langle c}\omega_{k\rangle}+H_{ab}\,,
\end{aligned}
$$

$$
\begin{aligned}
0=\,& D^aE_{ak}-\frac{1}{2}D^a\langle\widehat{\mathcal{E}}_{ak}\rangle+\frac{1}{3}D_k\langle\mathcal{E}\rangle-3H_{ka}\omega^a+\varepsilon_k{}^{ab}H_{ac}\sigma^c_b+\frac{\Theta}{3}\langle\widehat{\mathcal{E}}_k\rangle-\frac{1}{2}(3\omega_{ka}+\sigma_{ka})\langle\widehat{\mathcal{E}}^a\rangle\\
&-\frac{2\langle\widehat{\Theta}\rangle}{9}D_k\langle\widehat{\Theta}\rangle-\frac{\langle\widehat{\sigma}_{ak}\rangle}{2}D^a\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle-\frac{1}{2}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle D^a\langle\widehat{\sigma}_{ak}\rangle-\frac{\langle\widehat{\sigma}^c_a\rangle}{2}D^a\langle\widehat{\sigma}_{ck}\rangle+\frac{\langle\widehat{\Theta}\rangle}{3}\langle\widehat{K}^b\rangle\sigma_{kb}\\
&-\frac{\langle\widehat{\sigma}^b_a\rangle}{2}\langle\widehat{K}^a\rangle\sigma_{kb}+\frac{2\langle\widehat{\sigma}^{ab}\rangle}{3}D_k\langle\widehat{\sigma}_{ab}\rangle-\frac{\langle\widehat{K}^a\rangle}{3}D_k\langle\widehat{K}_a\rangle+\frac{\langle\widehat{K}_a\rangle}{2}D^a\langle\widehat{K}_k\rangle+\frac{\langle\widehat{K}_k\rangle}{2}D^a\langle\widehat{K}_a\rangle\\
&+\frac{3}{2}\varepsilon_k{}^{ad}\langle\widehat{K}_c\rangle\langle\widehat{\sigma}^c_d\rangle\omega_a+\langle\widehat{\Theta}\rangle\varepsilon_k{}^{cd}\langle\widehat{K}_c\rangle\omega_d+\frac{\Theta}{3}\langle\widehat{\sigma}^a_k\rangle\langle\widehat{K}_a\rangle-\frac{2\Theta}{9}\langle\widehat{\Theta}\rangle\langle\widehat{K}_k\rangle-\langle\widehat{\sigma}_{kb}\rangle D^a\langle\widehat{\sigma}^b_a\rangle-\frac{\widetilde{\kappa}^4}{8}\\
&\times\Big[\frac{4(\lambda+\rho)}{9}D_k\rho+\frac{\pi_{ak}}{3}D^a(\rho+3p)-\frac{(2\lambda-\rho-3p)}{3}D^a\pi_{ak}+\pi^c_aD^a\pi_{ck}-\frac{4\pi^{ab}}{3}D_k\pi_{ab}+\pi_{kb}D^a\pi^b_a\\
&-q_aD^aq_k-q_kD^aq_a+3\varepsilon_k{}^{ad}q_c\omega_a\pi^c_d-\pi^b_aq^a\sigma_{kb}+\frac{2(\lambda+\rho)}{3}\Big(\sigma_{kb}q^b-\frac{2\Theta}{3}q_k+3\varepsilon_k{}^{cd}q_c\omega_d\Big)\\
&+\frac{2q^a}{3}D_kq_a+\frac{2\Theta}{3}\pi^a_kq_a\Big]+\frac{\widetilde{\kappa}^2}{3}D^a\langle\widetilde{\pi}_{ak}\rangle-\frac{\widetilde{\kappa}^2}{6}D_k\langle\widetilde{\rho}-\widetilde{\pi}+\widetilde{p}\rangle+\frac{2\widetilde{\kappa}^2}{9}\Theta\langle\widetilde{q}_k\rangle-\frac{\widetilde{\kappa}^2}{3}\langle\widetilde{q}^a\rangle(3\omega_{ka}+\sigma_{ka})\,.
\end{aligned}
$$

3 The difference equations

The evolution equations arising as differences are:

$$
\dot{\rho}+(\rho+p)\,\Theta+D^aq_a+2q^aA_a+\pi_{ab}\sigma^{ab}=-\Delta\widetilde{q}\,,
$$

$$
\dot{q}_{\langle a\rangle}+D_ap+D^b\pi_{ab}+\frac{4}{3}\Theta q_a+\sigma_{ab}q^b-\omega_{ab}q^b+(\rho+p)\,A_a+\pi_{ab}A^b=-\Delta\widetilde{\pi}_a\,,
$$

$$0=\Delta\dot{\mathcal{E}}-D^a\Delta\widehat{\mathcal{E}}_a+\frac{4}{3}\Theta\Delta\mathcal{E}+\sigma^{ab}\Delta\widehat{\mathcal{E}}_{ab}-2A^a\Delta\widehat{\mathcal{E}}_a+\widetilde{\kappa}^2\Big[\pi^{ab}D_a\langle\widehat{K}_b\rangle-\langle\widehat{\sigma}^{ab}\rangle D_aq_b-q^aD^b\langle\widehat{\sigma}_{ab}\rangle$$
$$+\langle\widehat{K}^a\rangle D^b\pi_{ab}+\frac{2q^a}{3}D_a\langle\widehat{\Theta}\rangle-\frac{2\langle\widehat{K}^a\rangle}{3}D_a\rho-\pi^{ab}\langle\dot{\widehat{\sigma}}_{\langle ab\rangle}\rangle-\dot{\pi}_{\langle ab\rangle}\langle\widehat{\sigma}^{ab}\rangle-2A_aq_b\langle\widehat{\sigma}^{ab}\rangle+2A_a\pi^{ab}\langle\widehat{K}_b\rangle$$
$$-\frac{2\Theta}{3}\pi_{ab}\langle\widehat{\sigma}^{ab}\rangle-(\rho+p)\,\sigma_{ab}\langle\widehat{\sigma}^{ab}\rangle-\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle\sigma_{ab}\pi^{ab}-2\sigma_{ca}\pi^{ab}\langle\widehat{\sigma}^c_b\rangle+\frac{4\Theta}{3}q^a\langle\widehat{K}_a\rangle-2\sigma_{ab}q^b\langle\widehat{K}^a\rangle$$
$$-\frac{[\Delta(\widetilde{\rho}-\widetilde{\pi}+\widetilde{p})]^{\cdot}}{2}-\frac{2}{3}D^a\Delta\widetilde{q}_a-\frac{2\,[\Delta(\widetilde{\rho}+\widetilde{p})]}{3}\Theta+\frac{2\widetilde{\kappa}^2(\lambda+\rho)}{3}\langle\widetilde{q}\rangle-\frac{2\langle\widehat{\Theta}\rangle}{3}\Delta\widetilde{q}-\frac{4A^a}{3}\Delta\widetilde{q}_a-\frac{2\sigma^{ab}}{3}\Delta\widetilde{\pi}_{ab}\Big],$$

$$0=\Delta\dot{\widehat{\mathcal{E}}}_{\langle k\rangle}+\frac{4}{3}\Theta\Delta\widehat{\mathcal{E}}_k-\frac{1}{3}D_k\Delta\mathcal{E}-\frac{4}{3}A_k\Delta\mathcal{E}-D^a\Delta\widehat{\mathcal{E}}_{ka}-\Delta\widehat{\mathcal{E}}_{ka}A^a-(\omega_{ka}-\sigma_{ka})\,\Delta\widehat{\mathcal{E}}^a+\widetilde{\kappa}^2$$
$$\times\Big[(\rho+p)\,D^b\langle\widehat{\sigma}_{kb}\rangle+\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle D^b\pi_{kb}-\frac{2\,(\rho+p)}{3}D_k\langle\widehat{\Theta}\rangle-\frac{2}{3}\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle D_k\rho-2q^aD_k\langle\widehat{K}_a\rangle$$
$$-2\langle\widehat{K}^a\rangle D_kq_a+q^aD_a\langle\widehat{K}_k\rangle+\langle\widehat{K}^a\rangle D_aq_k+\frac{\langle\widehat{K}_k\rangle}{3}D^aq_a-\pi^{ab}D_k\langle\widehat{\sigma}_{ab}\rangle-\langle\widehat{\sigma}^{ab}\rangle D_k\pi_{ab}$$
$$+\pi^a_bD^b\langle\widehat{\sigma}_{ka}\rangle+\langle\widehat{\sigma}^a_b\rangle D^b\pi_{ka}+\frac{\pi^a_k}{3}D_a\langle\widehat{\Theta}\rangle+\frac{\langle\widehat{\sigma}^a_k\rangle}{3}D_a\rho-\dot{\pi}_{\langle ka\rangle}\langle\widehat{K}^a\rangle+q^a\langle\dot{\widehat{\sigma}}_{\langle ka\rangle}\rangle+q^a\sigma_{ck}\langle\widehat{\sigma}^c_a\rangle$$
$$-\sigma_{ck}\pi^c_a\langle\widehat{K}^a\rangle-\frac{2\,(\rho+p)}{3}\Theta\langle\widehat{K}_k\rangle+\frac{2\Theta}{3}\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle q_k-2q^aA_{\langle k}\langle\widehat{K}_{a\rangle}\rangle-2A_{\langle k}q_{a\rangle}\langle\widehat{K}^a\rangle+\sigma_{ba}\pi^b_k\langle\widehat{K}^a\rangle$$
$$-\sigma_{ba}q^a\langle\widehat{\sigma}^b_k\rangle+\frac{2}{3}q_k\sigma_{ab}\langle\widehat{\sigma}^{ab}\rangle-\frac{2}{3}\sigma_{ab}\pi^{ab}\langle\widehat{K}_k\rangle+\varepsilon_{cab}q^a\omega^b\langle\widehat{\sigma}^c_k\rangle-\varepsilon_{cab}\omega^b\pi^c_k\langle\widehat{K}^a\rangle+\varepsilon_k{}^{ab}q_c\omega_a\langle\widehat{\sigma}^c_b\rangle$$
$$-\varepsilon_k{}^{ab}\omega_a\pi^c_b\langle\widehat{K}_c\rangle-\frac{2}{3}\Delta\widetilde{\pi}'_{\langle k\rangle}+\frac{1}{6}D_k\Delta\,(\widetilde{\rho}+3\widetilde{\pi}-3\widetilde{p})+\frac{2\widetilde{\kappa}^2}{9}\,(2\lambda-\rho-3p)\langle\widetilde{\pi}_k\rangle+\frac{2}{3}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\Delta\widetilde{\pi}_k$$
$$+\frac{q_k}{3}D^a\langle\widehat{K}_a\rangle-\frac{2\langle\widetilde{q}_k\rangle}{3}\Delta K-\frac{2\langle K\rangle}{3}\Delta\widetilde{q}_k-\frac{2}{3}\langle2\widehat{K}_k+K_k\rangle\Delta\widetilde{q}-\frac{2\langle\widetilde{q}\rangle}{3}\big(2\widetilde{\kappa}^2q_k+\Delta K_k\big)+\frac{2\langle\widehat{A}^a\rangle}{3}\Delta\widetilde{\pi}_{ka}$$
$$+\frac{2\langle\widetilde{\pi}_{ka}\rangle}{3}\Delta\widehat{A}^a-\frac{2\langle\widehat{A}_k\rangle}{3}\Delta\,(\widetilde{\pi}-\widetilde{p})-\frac{2\langle\widetilde{\pi}-\widetilde{p}\rangle}{3}\Delta\widehat{A}_k+\frac{5\widetilde{\kappa}^2}{3}\pi_{ka}\langle\widetilde{\pi}^a\rangle-\frac{5\langle\widehat{\sigma}_{ka}\rangle}{3}\Delta\widetilde{\pi}^a\Big]\,,$$

$$0=\Delta\dot{\widehat{\mathcal{E}}}_{\langle kj\rangle}-D_{\langle k}\Delta\widehat{\mathcal{E}}_{j\rangle}+\frac{\Theta}{3}\Delta\widehat{\mathcal{E}}_{kj}-2A_{\langle k}\Delta\widehat{\mathcal{E}}_{j\rangle}+\frac{4\sigma_{kj}}{3}\Delta\mathcal{E}-\big(\omega_{a\langle k}-\sigma_{a\langle k}\big)\Delta\widehat{\mathcal{E}}^a_{j\rangle}+\widetilde{\kappa}^2\Big\{\frac{2\langle\widehat{\Theta}\rangle}{3}D_{\langle k}q_{j\rangle}$$
$$-\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle^{\cdot}\pi_{kj}-2\pi^a_{\langle j}\langle\dot{\widehat{\sigma}}_{k\rangle a}\rangle-2\langle\widehat{\sigma}^a_{\langle j}\rangle\dot{\pi}_{k\rangle a}-\frac{(\rho+3p)^{\cdot}}{3}\langle\widehat{\sigma}_{kj}\rangle-2q_{\langle k}D_{j\rangle}\langle\widehat{K}-\widehat{\Theta}\rangle-\frac{2\langle\widehat{K}_{\langle k\rangle}\rangle}{3}D_{j\rangle}(\rho-3p)$$
$$+\pi^a_{\langle j}D_{k\rangle}\langle\widehat{K}_a\rangle-\langle\widehat{\sigma}^a_{\langle j}\rangle D_{k\rangle}q_a-2q_{\langle k}D^b\langle\widehat{\sigma}_{j\rangle b}\rangle+2\langle\widehat{K}_{\langle k\rangle}D^b\pi_{j\rangle b}+\langle\widehat{K}_a\rangle D_{\langle k}\pi^a_{j\rangle}+\frac{14\Theta}{3}q_{\langle k}\langle\widehat{K}_{j\rangle}\rangle$$
$$-\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\Big[\dot{\pi}_{\langle kj\rangle}+\frac{\Theta}{3}\pi_{kj}+\pi^a_{\langle k}\big(\omega_{j\rangle a}+\sigma_{j\rangle a}\big)+2q_{\langle k}A_{j\rangle}\Big]-\pi^a_{\langle j}\omega_{k\rangle c}\langle\widehat{\sigma}^c_a\rangle+q_{\langle k}\sigma_{j\rangle b}\langle\widehat{K}^b\rangle$$
$$+\frac{(2\lambda-\rho-3p)}{3}\Big[\langle\dot{\widehat{\sigma}}_{\langle kj\rangle}\rangle+\frac{\Theta}{3}\langle\widehat{\sigma}_{kj}\rangle+\langle\widehat{\sigma}^a_{\langle k}\rangle\big(\omega_{j\rangle a}+\sigma_{j\rangle a}\big)-2\langle\widehat{K}_{\langle k}\rangle A_{j\rangle}\Big]+\frac{2\,(\rho+p)}{3}\langle\widehat{\Theta}\rangle\sigma_{kj}$$
$$+\langle\widehat{K}_{\langle k}\rangle\sigma_{j\rangle b}q^b-\frac{2\Theta}{3}\pi^a_{\langle j}\langle\widehat{\sigma}_{k\rangle a}\rangle-\frac{2\,(\lambda+\rho)}{3}\Big[D_{\langle k}\langle\widehat{K}_{j\rangle}\rangle-\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle\sigma_{kj}\Big]-q_aD_{\langle k}\langle\widehat{\sigma}^a_{j\rangle}\rangle-\langle\widehat{\sigma}^a_{\langle j}\rangle\omega_{k\rangle c}\pi^c_a$$
$$+2\sigma_{jk}q^a\langle\widehat{K}_a\rangle-3q_{\langle k}\omega_{j\rangle a}\langle\widehat{K}^a\rangle-3\langle\widehat{K}_{\langle k}\rangle\omega_{j\rangle a}q^a-2q_{\langle k}\langle\widehat{\sigma}_{j\rangle a}\rangle A^a+2\langle\widehat{K}_{\langle k}\rangle\pi_{j\rangle a}A^a+2\pi_{a\langle j}A_{k\rangle}\langle\widehat{K}^a\rangle$$
$$-2\langle\widehat{\sigma}_{a\langle j}\rangle A_{k\rangle}q^a-\pi^a_c\langle\widehat{\sigma}^c_{\langle k}\rangle\sigma_{j\rangle a}-\langle\widehat{\sigma}^a_c\rangle\pi^c_{\langle k}\sigma_{j\rangle a}+\frac{2}{3}\Delta\dot{\widetilde{\pi}}_{\langle kj\rangle}+\frac{8A_{\langle k}}{3}\Delta\widetilde{q}_{j\rangle}+\frac{2}{3}D_{\langle k}\Delta\widetilde{q}_{j\rangle}$$
$$-\frac{2\langle\widehat{K}_{\langle k\rangle}}{3}\Delta\widetilde{\pi}_{j\rangle}-\frac{2\widetilde{\kappa}^2}{3}\langle\widetilde{\pi}_{\langle k\rangle}q_{j\rangle}+\frac{2[\Delta\,(\widetilde{\rho}+\widetilde{p})]}{3}\sigma_{jk}+\frac{2\Theta}{9}\Delta\widetilde{\pi}_{jk}-\frac{2}{3}\,\big(\omega_{a\langle k}-\sigma_{a\langle k}\big)\Delta\widetilde{\pi}^a_{j\rangle}\Big\}\,.$$

The constraint equations arising as differences are:

$$
\begin{aligned}
0 = {} & D^a\Delta\widehat{\mathcal{E}}_{ak}-\frac{2}{3}D_k\Delta\mathcal{E}-\frac{2\Theta}{3}\Delta\widehat{\mathcal{E}}_k+(3\omega_{ka}+\sigma_{ka})\,\Delta\widehat{\mathcal{E}}^a+\widetilde{\kappa}^2\Big[-\frac{4\,(\lambda+\rho)}{9}D_k\langle\widehat{\Theta}\rangle-\frac{4}{9}\langle\widehat{\Theta}\rangle D_k\rho \\
& -\pi_{ak}D^a\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle-\frac{\langle\widehat{\sigma}_{ak}\rangle}{3}D^a\,(\rho+3p)+\frac{(2\lambda-\rho-3p)}{3}D^a\langle\widehat{\sigma}_{ak}\rangle-\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle D^a\pi_{ak}-\pi^c_aD^a\langle\widehat{\sigma}_{ck}\rangle \\
& -\langle\widehat{\sigma}^c_a\rangle D^a\pi_{ck}+\frac{4\pi^{ab}}{3}D_k\langle\widehat{\sigma}_{ab}\rangle+\frac{4\langle\widehat{\sigma}^{ab}\rangle}{3}D_k\pi_{ab}-\pi_{kb}D^a\langle\widehat{\sigma}^b_a\rangle+3\varepsilon_k{}^{ad}\left[\langle\widehat{K}_c\rangle\omega_a\pi^c_d-q_c\omega_a\langle\widehat{\sigma}^c_d\rangle\right] \\
& -q_aD^a\langle\widehat{K}_k\rangle-\pi^b_a\langle\widehat{K}^a\rangle\sigma_{kb}-\langle\widehat{K}_a\rangle D^aq_k-q_kD^a\langle\widehat{K}_a\rangle-\langle\widehat{K}_k\rangle D^aq_a-\langle\widehat{\sigma}_{kb}\rangle D^a\pi^b_a \\
& +\frac{2q^a}{3}D_k\langle\widehat{K}_a\rangle+\frac{2\langle\widehat{K}^a\rangle}{3}D_kq_a+\langle\widehat{\sigma}^b_a\rangle q^a\sigma_{kb}+\langle\widehat{\Theta}\rangle\left[\frac{4\Theta}{9}q_k-\frac{2}{3}\sigma_{kb}q^b-2\varepsilon_k{}^{cd}q_c\omega_d\right] \\
& -\frac{2\,(\lambda+\rho)}{3}\left[\frac{2\Theta}{3}\langle\widehat{K}_k\rangle-\sigma_{kb}\langle\widehat{K}^b\rangle-3\varepsilon_k{}^{cd}\langle\widehat{K}_c\rangle\omega_d\right]+\frac{2\Theta}{3}\pi^a_k\langle\widehat{K}_a\rangle-\frac{2\Theta}{3}\langle\widehat{\sigma}^a_k\rangle q_a \\
& -\frac{2}{3}D^a\Delta\widetilde{\pi}_{ak}+\frac{1}{3}D_k\Delta\,(\widetilde{\rho}-\widetilde{\pi}+\widetilde{p})-\frac{4\Theta}{9}\Delta\widetilde{q}_k+\frac{2}{3}\,(3\omega_{ka}+\sigma_{ka})\,\Delta\widetilde{q}^a\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \frac{\widetilde{\kappa}^2}{3}\,(\lambda-2\rho-3p)\,\langle\widehat{\Theta}\rangle-\widetilde{\kappa}^2\,(\lambda+\rho)\,\langle\widehat{K}\rangle-2\widetilde{\kappa}^2q^a\langle\widehat{K}_a\rangle-\Delta\mathcal{E}-\frac{\Delta\widetilde{\Lambda}}{2}+\frac{\widetilde{\kappa}^2}{2}\Delta\,(\widetilde{\rho}+\widetilde{\pi}+\widetilde{p})\,, \\
0 = {} & \frac{2\widetilde{\kappa}^2}{3}\,(\lambda+\rho)\,\langle\widehat{K}_a\rangle-\frac{2\widetilde{\kappa}^2}{3}\langle\widehat{\Theta}\rangle q_a-\widetilde{\kappa}^2\pi_{ab}\langle\widehat{K}^b\rangle+\widetilde{\kappa}^2q^b\langle\widehat{\sigma}_{ab}\rangle+\Delta\widehat{\mathcal{E}}_a+\frac{2\widetilde{\kappa}^2}{3}\Delta\widetilde{q}_a\,, \\
0 = {} & \frac{\widetilde{\kappa}^2}{6}\,(2\lambda-\rho-3p)\,\langle\widehat{\sigma}_{ab}\rangle-\frac{\widetilde{\kappa}^2}{2}\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\pi_{ab}-\widetilde{\kappa}^2q_{\langle a}\langle\widehat{K}_{b\rangle}\rangle-\widetilde{\kappa}^2\pi_{c\langle a}\langle\widehat{\sigma}^c_{b\rangle}\rangle+\frac{\Delta\widehat{\mathcal{E}}_{ab}}{2}-\frac{\widetilde{\kappa}^2}{3}\Delta\widetilde{\pi}_{ab}\,,
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \varepsilon_k{}^{ab}D_a\Delta\widehat{\mathcal{E}}_b-\frac{8\omega_k}{3}\Delta\mathcal{E}+\omega^a\Delta\widehat{\mathcal{E}}_{ak}+\varepsilon_k{}^{ab}\sigma^c_b\Delta\widehat{\mathcal{E}}_{ac}+\widetilde{\kappa}^2\Big[\frac{2}{3}\varepsilon_k{}^{ab}\langle\widehat{K}_b\rangle D_a\rho-\frac{2}{3}\varepsilon_k{}^{ab}q_bD_a\langle\widehat{\Theta}\rangle \\
& +\frac{2\,(\lambda+\rho)}{3}\varepsilon_k{}^{ab}D_a\langle\widehat{K}_b\rangle-\frac{2\langle\widehat{\Theta}\rangle}{3}\varepsilon_k{}^{ab}D_aq_b-\varepsilon_{abk}q^cD^b\langle\widehat{\sigma}^a_c\rangle+\varepsilon_{abk}\langle\widehat{K}^c\rangle D^b\pi^a_c+\varepsilon_k{}^{ab}\pi^c_aD_b\langle\widehat{K}_c\rangle \\
& -\varepsilon_k{}^{ab}\langle\widehat{\sigma}^c_a\rangle D_bq_c+\frac{(2\lambda-\rho-3p)}{3}\big[\langle\widehat{\sigma}^c_k\rangle\omega_c+\varepsilon_k{}^{ab}\langle\widehat{\sigma}_{ac}\rangle\sigma^c_b\big]-\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle\big[\pi^c_k\omega_c+\varepsilon_k{}^{ab}\pi_{ac}\sigma^c_b\big] \\
& -\left[\frac{4\,(\lambda+\rho)}{3}\langle\widehat{K}+\frac{\widehat{\Theta}}{3}\rangle+\frac{4\,(\rho+p)}{3}\langle\widehat{\Theta}\rangle+2q_a\langle\widehat{K}^a\rangle-2\pi_{ab}\langle\widehat{\sigma}^{ab}\rangle\right]\omega_k-q^c\omega_c\langle\widehat{K}_k\rangle-\langle\widehat{K}^c\rangle\omega_cq_k \\
& +\varepsilon_k{}^{ac}\sigma_{ab}q^b\langle\widehat{K}_c\rangle+\varepsilon_k{}^{ac}\sigma_{ab}\langle\widehat{K}^b\rangle q_c-\pi_{ca}\langle\widehat{\sigma}^c_k\rangle\omega^a-\langle\widehat{\sigma}_{ca}\rangle\pi^c_k\omega^a-\varepsilon_k{}^{ab}\pi_{da}\langle\widehat{\sigma}^d_c\rangle\sigma^c_b-\varepsilon_k{}^{ab}\langle\widehat{\sigma}_{da}\rangle\pi^d_c\sigma^c_b \\
& -\frac{2\omega^a}{3}\Delta\widetilde{\pi}_{ka}+\frac{2}{3}\varepsilon_k{}^{ab}D_a\Delta\widetilde{q}_b+\frac{4\omega_k}{3}\Delta\,(\widetilde{\rho}+\widetilde{p})-\frac{2}{3}\varepsilon_k{}^{ab}\sigma_{bc}\Delta\widetilde{\pi}^c_a\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \varepsilon_{ab\langle k}D^a\Delta\widehat{\mathcal{E}}^b_{j\rangle}-\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\Delta\widehat{\mathcal{E}}_b-3\omega_{\langle k}\Delta\widehat{\mathcal{E}}_{j\rangle}+\widetilde{\kappa}^2\varepsilon_{\langle k}{}^{ab}\left[\pi_{j\rangle a}D_b\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle+\frac{\langle\widehat{\sigma}_{j\rangle a}\rangle}{3}D_b\,(\rho+3p)\right. \\
& \left.-\sigma_{j\rangle b}\pi^c_a\langle\widehat{K}_c\rangle+\sigma_{j\rangle b}\langle\widehat{\sigma}^c_a\rangle q_c-\frac{2\,(\lambda+\rho)}{3}\sigma_{j\rangle a}\langle\widehat{K}_b\rangle+\frac{2\langle\widehat{\Theta}\rangle}{3}\sigma_{j\rangle a}q_b-\frac{2\sigma_{j\rangle a}}{3}\Delta\widetilde{q}_b\right] \\
& +\widetilde{\kappa}^2\varepsilon_{ab\langle k}\left[\frac{(2\lambda-\rho-3p)}{3}D^a\langle\widehat{\sigma}^b_{j\rangle}\rangle-\langle\widehat{K}-\frac{\widehat{\Theta}}{3}\rangle D^a\pi^b_{j\rangle}-D^a\pi_{j\rangle c}\langle\widehat{\sigma}^{cb}\rangle-D^a\langle\widehat{\sigma}_{j\rangle c}\rangle\pi^{cb}\right. \\
& -\pi_{j\rangle c}D^a\langle\widehat{\sigma}^{cb}\rangle-\langle\widehat{\sigma}_{j\rangle c}\rangle D^a\pi^{cb}-q_{j\rangle}D^a\langle\widehat{K}^b\rangle-\langle\widehat{K}_{j\rangle}\rangle D^aq^b-q^bD^a\langle\widehat{K}_{j\rangle}\rangle-\langle\widehat{K}^b\rangle D^aq_{j\rangle} \\
& \left.-\frac{2}{3}D^a\Delta\widetilde{\pi}^b_{j\rangle}\right]+\widetilde{\kappa}^2\omega_{\langle k}\left[3\pi^a_{j\rangle}\langle\widehat{K}_a\rangle-3\langle\widehat{\sigma}^a_{j\rangle}\rangle q_a-2\,(\lambda+\rho)\,\langle\widehat{K}_{j\rangle}\rangle+2\langle\widehat{\Theta}\rangle q_{j\rangle}-2\Delta\widetilde{q}_{j\rangle}\right].
\end{aligned}
$$

4 Concluding remarks

Both the average and the difference equations reduce to the corresponding equations given in Subsection IV.D of Ref [1], by taking into account that in the particular case of a symmetric embedding for quantities defined with an odd (even) number of n^a, the conditions $\Delta f = 2f, \langle f \rangle = 0$ ($\Delta f = 0, \langle f \rangle = f$) hold. In particular, the extrinsic curvature components $\mathcal{K}$ belong to the first group.

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