Proximity-induced sub-gaps in Andreev billiards

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We examine the density of states of an Andreev billiard and show that any billiard with a finite upper cut-off in the path length distribution $P(s)$ will possess an energy gap on the scale of the Thouless energy. An exact quantum mechanical calculation for different Andreev billiards gives good agreement with the semiclassical predictions when the energy dependent phase shift for Andreev reflections is properly taken into account. Based on this new semiclassical Bohr-Sommerfeld approximation of the density of states, we derive a simple formula for the energy gap. We show that the energy gap, in units of Thouless energy, may exceed the value predicted earlier from random matrix theory for chaotic billiards.

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Studies of sub-gap transport in hybrid superconductors are an important starting point for the design and simulation of superconducting nanoscale devices. For a given inhomogeneous structure, a fundamental question is the existence or otherwise of a finite density of states at the Fermi energy. This is important experimentally, since the sub-gap spectrum determines the tunneling conductance of an N-S contact. The ability to address this question is also important theoretically, since a well-posed problem of this kind provides a testing ground for complementary (and occasionally competing) theoretical techniques.

One important class of structures, for which a general analysis might be forthcoming, are known as Andreev billiards. These are formed when a classically-chaotic normal dot is placed in contact with a superconductor [1, 2, 3, 4, 5, 6, 7, 8]. Consider a ballistic two dimensional normal dot of area A, with the mean level spacing of the isolated normal system $\delta = \frac{2\pi\hbar^2}{mA}$ at the Fermi energy E_F. If a superconductor of width W and bulk order parameter Δ is placed in contact with such a billiard (see Fig. 1), then the question of interest is whether or not an energy gap on the scale of the Thouless energy $E_T = \frac{M\delta}{4\pi}$ [9], exists in the sub-gap spectrum $E < \Delta$, when $\delta << E_T < \Delta$. The number of open channels in the S region is the integer part of $M = \frac{k_F W}{\pi}$, and the energy levels of the Andreev billiards are the positive eigenvalues E (measured from the Fermi energy) of the Bogoliubov-de Gennes equation[10].

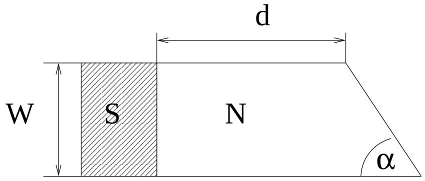

FIG. 1: Geometry.

An initial study of this problem based on random matrix theory[2], concluded that a classically chaotic billiard possesses an energy gap, whereas an integrable system is gapless. As a consequence it was suggested that the existence of such a gap could be used to distinguish between integrable and chaotic systems. Later it was shown that billiards with classically mixed phase space do possess a smaller gap comparing to the fully chaotic systems[6]. Studying the existence of the gap for chaotic billiards Ihra et al. [7] have drawn the attention to the role of the non-diagonal terms of the scattering matrix which could explain the disagreement between the random matrix theory and the Bohr-Sommerfeld semiclassical approximation. The non-universal feature of the excitation spectra of the Andreev billiards has been recently studied by Ihra and Richter in Ref. [8].

The aim of this Letter is to identify for the first time, a pseudo-integrable billiard which does possess an energy gap. Moreover, this gap can be larger than that predicted for chaotic billiards on the scale of E_T. To this end, we study the quasi-particle spectrum of the ballistic structure shown in Fig. 1, for different values of the angle α and length d. Our key result is illustrated in Fig. 2, which shows that for $d = 0$, the counting function or integrated density of states $N(E) = \int_0^E n(E')\,dE'$ (where $n(E)$ is the density of states for the Andreev billiard) exhibits an energy gap $E_{\rm gap}$ varying between $E_{\rm gap} \approx 0.5E_T$ and $E_{\rm gap} \approx 1.5E_T$ as α varies from 80° to 45° (for the details of the calculation see below). In contrast, as shown in Fig. 3, for $d \neq 0$, no such energy gap exists. To understand this result, we start from the formula relating the integrated density of states $N(E)$ to the path length distribution $P(s)$ (which is the classical probability that an electron entering the billiard at the N-S contact exits after a path length s) derived by using

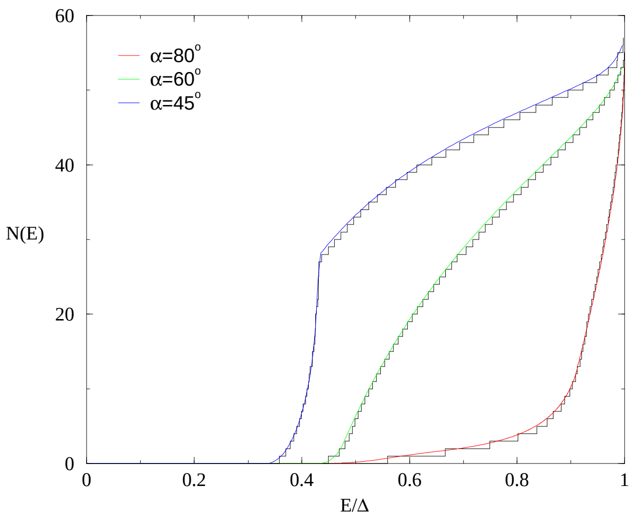

FIG. 2: The integrated density of states, $N(E)$ as a function of E (in units of Δ) for $d = 0$ and angle $\alpha = 80^\circ, 60^\circ, 45^\circ$. For each angle α the stair type lines correspond to the exact diagonalization of the Bogoliubov-de Gennes equation, while the solid lines are obtained by using Eq. (1). The parameters are $M = 55.5$, $\Delta/E_F = 0.015$.

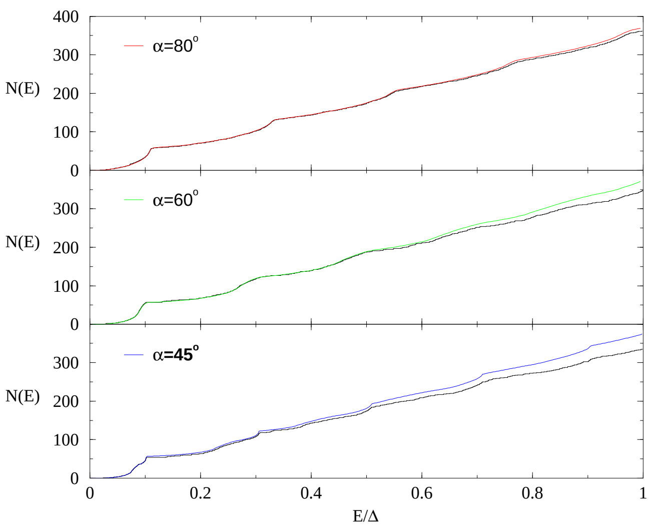

FIG. 3: The integrated density of states, $N(E)$ as a function of E (in units of Δ) for $d \neq 0$ and angle $\alpha = 80^\circ, 60^\circ, 45^\circ$. In each case, d varies such that the area of the normal dot is fixed at $A = 5W^2$ for the angles $\alpha = 80^\circ, 60^\circ, 45^\circ$. This choice results in approximately the same number of states below Δ for different values of α. The meaning of the lines and the parameters are the same as those in Fig. 2.

the Bohr-Sommerfeld approximation,

$$N\left(E\right) = M \sum_{n=0}^{\infty} \int_{s_n(E)}^{\infty} P(s)\, ds, \tag{1}$$

where

$$s_n\left(E\right) = \frac{\left(n + \frac{1}{\pi} \arccos \frac{E}{\Delta}\right) \pi}{E/\Delta}\, \xi_0. \tag{2}$$

Here $\xi_0 = \frac{\hbar v_{\mathrm{F}}}{\Delta} = W\, \frac{2}{\pi M}\, \frac{E_F}{\Delta}$ is the coherence length and the distribution $P(s)$ is normalized to one, $\int_0^\infty P(s)\, ds = 1$. The density of states $n(E) = \frac{dN(E)}{dE}$ can be found from Eq. (1) yielding

$$n\left(E\right) = \frac{M}{E} \sum_{n=0}^{\infty} P\left(s_n\left(E\right)\right) \left[\frac{\xi_0}{\sqrt{1 - \left(E/\Delta\right)^2}} + s_n\left(E\right)\right]. \tag{3}$$

This expression reduces to that of derived in a different way by Melsen et al. [2, 6], Lodder and Nazarov[5], Ihra et al. [7], in the limit $E << \Delta$ and $\xi_0^2 << A$, but more generally incorporates an energy dependent path length correction [11].

To test the semiclassical expression, we perform an exact (numerical) diagonalization of the Bogoliubov-de Gennes equation by matching the wave functions at the N-S interface. This results in a secular equation including the scattering matrix, $S_0(E)$ of the normal billiard opened at the N-S interface [12]:

$$\det\left[1 - e^{-2i \arccos \frac{E}{\Delta}}\, S_{\mathrm{eff}}(E)\, S^*_{\mathrm{eff}}(-E)\right] = 0, \tag{4}$$

where

$$\begin{aligned} S_{\mathrm{eff}}(E) &= \left[Q(E) + K(E)D(E)\right]^{-1} \times \\ &\quad \left[Q^*(E) - K(E)D(E)\right], \qquad (5) \\ D(E) &= \left[1 - S_0(E)\right]\left[1 + S_0(E)\right]^{-1}. \end{aligned}$$

Here Q and K are diagonal matrices with elements $Q_{nm}(E) = i\, \delta_{nm}\, q_n(E)$ and $K_{nm}(E) = \delta_{nm}\, k_n(E)$, where $q_n(E) = k_F \sqrt{1 + i\, \frac{\sqrt{\Delta^2 - E^2}}{E_F} - \frac{n^2}{M^2}}$ are the transverse wavenumbers of the electron in the S region and $k_n(E) = k_F \sqrt{1 + \frac{E}{E_F} - \frac{n^2}{M^2}}$ are the transverse wavenumbers of the electron in the S region when $\Delta = 0$. It is assumed that the Fermi wavenumber, $k_F = \sqrt{2mE_F/\hbar^2}$ is the same in the S and N regions. All the matrices are M by M dimensional. Finally, S_0 was calculated using Bessel functions in the wedge part of the normal region, and including evanescent modes. In a different context, the same type of normal billiard was studied by Kaplan and Heller [13]. A secular equation similar to Eq. (4) was derived by Beenakker [14] for SNS systems but there instead of $S_{\mathrm{eff}}(E)$ the scattering matrix $S_0(E)$ of the normal billiard appears. In this sense, the matrix $S_{\mathrm{eff}}(E)$ can be regarded as an effective scattering matrix. Note that in Andreev approximation[15] $q_n \approx k_n$, and one then finds $S_{\mathrm{eff}}(E) = S_0(E)$. In contrast the secular equation (4) for Andreev billiards is valid outside

Andreev approximation. Using the unitarity of $S_0(E)$, one can show that the following equation

$$\det\left(\mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\arccos\frac{\mathrm{E}}{\Delta}}\left[\mathrm{Q(E)}+\mathrm{K(E)D(E)}\right]\times\right.\right.$$
$$\left.\left.[Q(E)+K(E)D^*(-E)]^{-1}\right\}\right) = 0, \quad (6)$$

has the same zeros as those of Eq. (4), and is a more suitable form for numerical calculations (the left hand side of Eq. (6) is a real function instead of a complex one).

Figures 2 and 3 show both exact results and an evaluation of the semiclassical expression of $N(E)$ given by Eq. (1) for Andreev billiards shown in Fig. 1 with different parameters [16]. One can see that the results of the two methods are in good agreement over the entire energy range $E < \Delta$. It is important to note that inclusion of the energy dependence of the phase shift was essential to obtain such a good agreement. A small deviation can be seen only at energies close to the value of Δ and for $d \neq 0$.

Both the semiclassical and the exact calculations reveal the presence of an energy gap when $d = 0$, while no gap appears in case of $d \neq 0$. The origin of this result can be traced to the existence of a finite upper cut-off $s_{\max}$ in the path length distribution $P(s)$. From Eq. (2) it is obvious that if the path length of the electron has a maximum value, then the energy spectrum will possess a gap. The condition for determining $E_{\rm gap}$ is $s_n(E) > s_{\max}$ for $n \geq 0$. Expanding the arccos term in Eq. (2) to first order in E/Δ, one finds that the energy gap is given by the following simple equation:

$$\frac{E_{\rm gap}}{E_{\rm T}} = \pi^2 \frac{A}{W s_{\max}} \frac{1}{1+\xi_0/s_{\max}}, \quad (7)$$

where A is the area of the billiard and W is the width of the superconductor lead.

As examples, Fig. 4 shows $P(s)$ for $\alpha = 80^\circ, 60^\circ, 45^\circ$ along with the corresponding semiclassical density of states $n(E)$ given by Eq. (3). From the figure it is seen that the upper cut-off $s_{\max}$ of $P(s)$ equals to $2W$ for $\alpha = 80^\circ, 60^\circ$, and $s_{\max} = 2\sqrt{2}\,W$ for $\alpha = 45^\circ$. It is interesting to note that for $\alpha = 45^\circ$ (more generally for $\alpha = 90^\circ/k$ with an integer $k > 0$) the density of states $n(E)$ has a singularity at some energy $E < \Delta$ while for other values of α no pronounced singularities exist. In Bohr-Sommerfeld approximation, it can be shown that this singularity is related to the singularity of the path length distribution $P(s)$. The classical trajectories of the electron resulting in a singularity for $\alpha = 45^\circ$ are similar to those moving in a lead with length $d+W$ and $\alpha = 90^\circ$. The nature of the singularities of the density of states in this rectangular shape of Andreev billiard was studied in Ref. 17. The reason for the disappearance of these singularities for $\alpha \neq 90^\circ/k$ ($k = 1, 2, 3, \ldots$) is consistent with the points made in this reference.

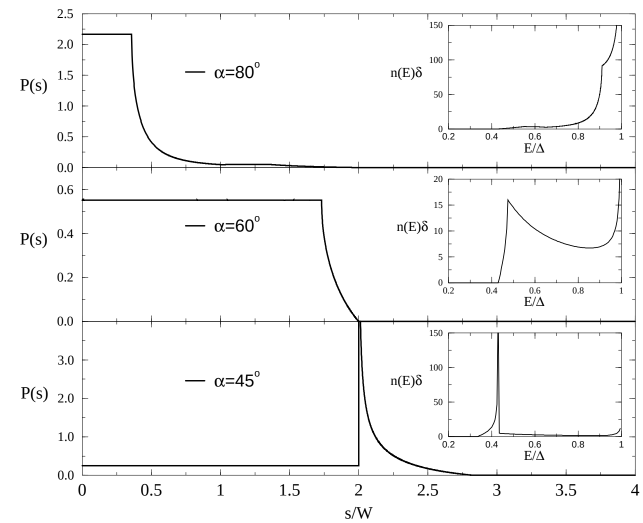

FIG. 4: The path length distribution $P(s)$ for Andreev billiard of Fig. 1 when $d = 0$ for $\alpha = 80^\circ, 60^\circ, 45^\circ$. Insets show the corresponding semiclassical density of states $n(E)$ (in units of $1/\delta$) using Eq. (3). The same parameters was used as those in Fig. 2.

To check the expression (7) we have determined the energy gap from exact calculations for several angles α. For the billiard of Fig. 1 with $d = 0$, one can show analytically that the upper cut-off $s_{\max}$ is

$$\frac{s_{\max}}{W} = \begin{cases} -2\,\frac{\sin k\alpha \sin[(k+1)\alpha]}{\sin\alpha\cos[(2k+1)\alpha]} & \text{if} \quad \frac{\pi/2}{k+1} < \alpha \leq \frac{\pi/2}{k+\frac{2}{3}}, \\ 2\,\frac{\sin k\alpha}{\sin\alpha} & \text{if} \quad \frac{\pi/2}{k+\frac{2}{3}} < \alpha \leq \frac{\pi/2}{k}, \end{cases} \quad (8)$$

where $k = 1, 2, 3, \ldots$. Using Eqs. (7) and (8) the gap is plotted as a function of the angle α in Fig. 5 together with the exact results. The numerical results (full circles in Fig. 5) agree very well with the predictions of Eq. (7). One can see that decreasing α the energy gap in units of Thouless energy increases and tends to a finite value as $\alpha \to 0$. From Eq. (8) and Eq. (2) it can be seen that $s_{\max}$ increases with decreasing α, whereas $E_{\rm gap}/\Delta$ decreases. The Thouless energy E_T also decreases, since the mean level spacing δ becomes smaller for larger area A of the normal region. The two effect together result in a finite value of $E_{\rm gap}/E_T$ at $\alpha = 0$.

In the limit $\xi_0 << s_{\max}$ (for example, in the semiclassical limit of large M), the expression (7) of the gap reduces to

$$\frac{E_{\rm gap}}{E_{\rm T}} = \pi^2 \frac{A}{W s_{\max}}. \quad (9)$$

In Fig. 5 the dashed curve is the result from this formula. One can see that for all angles α this limiting result for the energy gap is larger than the finite M value. Note that the area is $A = \frac{1}{2}W^2 \cot\alpha$, and therefore, $E_{\rm gap}/E_{\rm T}$ is *only* a function of α and independent of W when $\xi_0 << s_{\max}$. It can also be shown that $E_{\rm gap}/E_{\rm T} \to \pi^2/4 \approx 2.47$

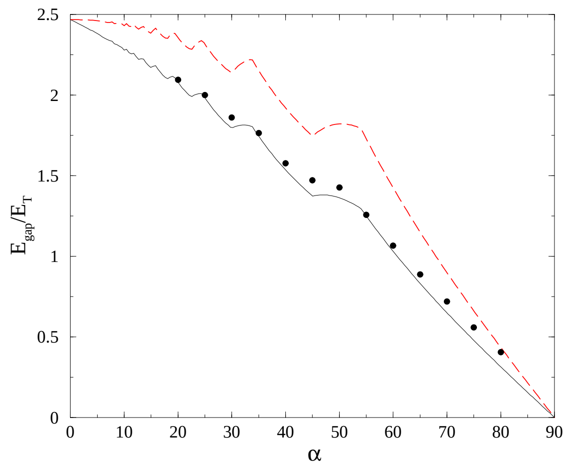

FIG. 5: The gap $E_{\rm gap}$ (in units of Thouless energy) as a function of angle α for Andreev billiards with $d = 0$. The solid and the dashed curves are the semiclassical results from Eq. (7) when $M = 55.5$, and from Eq. (9) i.e. when $M \to \infty$, respectively. The circles are the results of the exact diagonalization when $M = 55.5$.

as $\alpha \to 0$. Finally, it is worth mentioning that for Andreev billiard of Fig. 1 when $d = 0$ the energy gap can be much larger for small angle α than the value of $0.6E_T$ predicted by Melsen et al. [2] for the chaotic billiard.

It is interesting to mention that besides the billiard studied in this work (see Fig. 1 for $d = 0$) there are a number of different shapes of normal dots in which $P(s)$ exhibits finite upper cut-off resulting in a sub-gap. Examples for such dots are shown in Fig. 6 (provided $\alpha + \beta < 180°$ in the case of quadrangle).

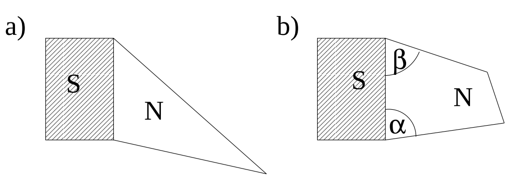

FIG. 6: A triangle (a), and quadrangle (b) shape of normal billiard having a finite upper cut-off of the length distribution $P(s)$.

In conclusion, we have shown that a billiard with a finite upper cut-off in the path length distribution $P(s)$ will possess an energy gap in the density of states on the scale of the Thouless energy. By including the energy dependent phase shift of the Andreev reflection, a new expression for the density of states has been given within the frame work of the semiclassical Bohr-Sommerfeld approximation. We have also derived a formula for the energy gap. To check these results we have performed an exact diagonalization of the Bogoliubov-de Gennes equation for different Andreev billiards. The results of the two methods agrees very well both for the integrated density of states (even for energy levels close to the value of Δ) and for the energy gap. Finally we have shown that the energy gap on the scale of the Thouless energy can be much larger than the value $0.6E_T$ predicted from random matrix theory[2] for chaotic billiards.

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