

Mass loss and longevity of gravitationally bound oscillating scalar lumps (oscillatons) in D -dimensions

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(Dated: June 20, 2018)

Spherically symmetric *oscillatons* (also referred to as oscillating soliton stars) i.e. gravitationally bound oscillating scalar lumps are considered in theories containing a massive self-interacting real scalar field coupled to Einstein's gravity in $1 + D$ dimensional spacetimes. Oscillations are known to decay by emitting scalar radiation with a characteristic time scale which is, however, extremely long, it can be comparable even to the lifetime of our universe. In the limit when the central density (or amplitude) of the oscillaton tends to zero (small-amplitude limit) a method is introduced to compute the transcendently small amplitude of the outgoing waves. The results are illustrated in detail on the simplest case, a single massive free scalar field coupled to gravity.

I. INTRODUCTION

Numerical simulations of Seidel and Suen[1] have revealed that spatially localized, extremely long living, oscillating configurations evolve from quite general initial data in the spherically symmetric sector of Einstein's gravity coupled to a free, massive real Klein-Gordon field. For example, they observed that initially Gaussian pulses evolve quickly into configurations which appear to be time-periodic. It has been already noted in Ref.[1], that the resulting objects may not be strictly time-periodic, rather they may evolve on a secular time scale many orders of magnitude longer than the observed oscillation period. These interesting objects were first baptized "oscillating soliton stars" in Ref. [1], but somewhat later the same objects have been referred to as "*oscillatons*" by the same authors [2]. This latter name has been by now widely adopted, and we shall also stick to its usage throughout this paper. It has been observed in the numerical simulations of Ref. [1] that oscillatons are *stable* during the time evolution. Moreover it has been argued in Ref.[2] that oscillatons do form in physical processes through a dissipationless gravitational cooling mechanism, making them of great physical importance. For example oscillatons would be good candidates for dark matter in our Universe.

On the other hand, stimulated by the seminal work of Dashen, Hasslacher and Neveu in the one-dimensional ϕ^4 -theory [3], numerical simulations have revealed that in an impressive number of scalar field theories spatially localized structures –*oscillons*– form *from generic initial data* which become very closely time periodic, and live for very long times [4–12]. These objects oscillate nearly periodically in time, resembling "true" (i.e. time-periodic)

breathers. An oscillon possesses a "radiative" tail outside of its core region where its energy is leaking continuously in form of (scalar) radiation. Therefore a simple approximate physical picture of a sufficiently small-amplitude oscillon is the that of a "true" breather whose frequency is increasing on a secular time scale since the amplitude of the outgoing radiation is much smaller than that of the core. It has been shown in Refs. [13], [14], that slowly radiating oscillons can be well described by a special class of exactly time-periodic "quasibreathers" (QB). Being time periodic, QBs are easier to describe mathematically by ordinary Fourier analysis than the long time asymptotics of oscillons. A QB possesses a localized core in space (just like true breathers) which approximates that of the corresponding oscillon very well, but in addition it has a standing wave tail whose amplitude is *minimized*. This is a physically motivated condition, which heuristically singles out "the" solution approximating a true breather as well as possible, for which this amplitude would be identically zero. The amplitude of the standing wave tail of a QB is closely related to that of the oscillon radiation, therefore its computation is of prime interest. Roughly speaking "half" of the standing wave tail corresponds to incoming radiation from spatial infinity. It is the incoming radiation that maintains the time periodicity of the QBs by compensating the energy loss through the outgoing waves. In a series of papers [14–16] a method has been developed to compute the leading part of the exponentially suppressed tail amplitude of QBs, in a large class of scalar theories in various dimensions, in the limit when the QB core amplitude is small. Although oscillons continuously lose energy through radiation, many of them are remarkably stable. The longevity and the ubiquity of oscillons make them of potentially great physical in-

terest [17–21]. Quite importantly oscillons also appear in the course of time evolution when other fields, e.g. vector fields are present [22–24]. There is little doubt that oscillons and oscillatons are closely related objects.

The basic physical mechanism for the anti-intuitively slow radiation of oscillons is that the lowest frequency mode of the scalar field is trapped below the mass threshold and only the higher frequency modes are coupled to the continuum.

In this paper we generalize the method of Refs. [14–16] to compute the mass loss of spherically symmetric oscillations induced by scalar radiation in the limit of small oscillaton amplitudes, ε , in $1 + D$ dimensional spacetimes. These methods have been successfully applied to D -dimensional scalar field theories coupled to a dilaton field [25]. Numerous similarities exist between coupling a theory to a dilaton field and to gravitation: the field configurations are of ε^2 order and the lowest order equations determining the profiles are the Schrödinger-Newton equations. The stability pattern is also analogous. Despite these similarities between the dilaton and the gravitational theory there are some technical and even some conceptual differences. Since there is no time-like Killing vector neither for oscillatons nor for the corresponding QBs, already the very definition of mass and mass loss is less obvious than in flat spacetime. Another conceptual issue is that the spacetime of a time-periodic QB is not asymptotically flat, which is related to the fact that the “total mass” of a QB is infinite. In the case of spherical symmetry considered in this paper a suitable local mass function is the Misner-Sharp energy and the mass loss can be defined with aid of the Kodama vector. The issue of the precise asymptotics of spacetimes can be sidestepped in the limit $\varepsilon \rightarrow 0$ by considering only a restricted, approximatively flat spacetime region containing the core of the QB (having a size of order $\mathcal{O}(1/\varepsilon)$) and part of its oscillating tail. We find that to leading order in the ε expansion the oscillaton core is determined by the D -dimensional analogues of the Schrödinger-Newton equations [26–30] independently of the self-interaction potential. It turns out that exponentially localized oscillatons exist for $2 < D < 6$. These findings show a striking similarity to dilaton-scalar theories as found in Ref. [25]. In the case of spherically symmetric oscillatons no gravitational radiation is expected due to Birkhoff’s theorem. The mass loss of spherically symmetric oscillatons is entirely due to scalar radiation.

The following simple formula gives the mass loss of a small-amplitude oscillaton in D spatial dimensions:

$$\frac{dM}{dt} = -\frac{c_1}{m^{D-3}\varepsilon^{D-1}} \exp\left(-\frac{c_2}{\varepsilon}\right), \quad (1)$$

where m denotes the mass of the scalar field, c_1 is a D -dependent constant, while c_2 depends on both D and the self-interaction scalar potential. The numerical values of c_1 , c_2 in the Einstein-Klein-Gordon (EKG) theory for spatial dimensions $D = 3, 4, 5$ are given in Table VI. We also compute and tabulate the most important physical

properties of oscillatons in the EKG theory (their mass as a function of time, their radii). We would like to stress, that the method is applicable for oscillatons in scalar theories with any self-interaction potential developable into power series.

In the seminal work of Don N. Page [31] both the classical and quantum decay rate of oscillatons has been considered for the case of free massive scalars in the EKG theory (for $D = 3$). We agree with the overall qualitative picture of the oscillaton’s mass loss found in Ref.[31], however, there are also some differences in the quantitative results. For example, the amplitude of the outgoing wave (related to $\sqrt{c_1}$) found by our method differs significantly from that of Ref.[31]. The main source of this discrepancy is due to the fact that this amplitude is given by an infinite series in the ε expansion, where all terms contribute by the same order, whereas in the estimate of Ref.[31] only the lowest order term in this series has been used. Our methods which are based on the work of Segur-Kruskal [32] avoid this difficulty altogether, moreover for the class of self-interaction potentials containing only even powers of the scalar field, Φ , the radiation amplitude can be computed analytically using Borel summation.

We now give a lightning review on previous results scattered in the literature on oscillatons in 3+1 dimensions. For a given scalar field mass, m , there is a one-parameter family of oscillatons, parametrized, for example, by the central amplitude of the field, Φ_c . As Φ_c increases from small values, the mass of the oscillaton, M , is getting larger, while the radius of the configuration decreases. For a critical value of the central amplitude, Φ_{crit} , a maximal mass configuration is reached. Oscillatons with central amplitudes $\Phi_c > \Phi_{\text{crit}}$ are unstable [1]. This behavior is both qualitatively and quantitatively very similar to that of boson stars [26, 33, 34], and also to the behavior of white dwarfs and neutron stars [35]. For reviews of the vast literature on boson stars see for example, Refs. [36] and [37]. In Refs. [38, 39] a one-parameter family of oscillaton-type solutions in an Einstein-scalar theory with two massive, real scalar fields has been presented, which are essentially transitional states between boson stars and oscillatons.

The interaction of weak gravity axion field oscillatons with white dwarfs and neutron stars have been discussed in [40, 41], proposing a possible mechanism for gamma ray bursts [42]. Since for very low mass scalar fields oscillatons may be extremely heavy, it has been suggested that they may be the central object of galaxies [43], or form the dark matter galactic halos [44–50].

Qualitatively good results for various properties of oscillatons has been obtained by Ureña-López [51], truncating the Fourier mode decomposition of the field equations at as low order as $\cos(2\omega t)$, where ω is the fundamental frequency. Then the space and time dependence of the scalar field separates as $\Phi(t, r) = \Phi_1(r) \cos(\omega t)$. Oscillatons with nontrivial self-interaction potentials have also been studied in [51], indicating that similarly to boson

stars, the maximal mass can be significantly larger than in the Klein-Gordon case.

The Fourier mode equations have been studied in [52] up to orders $\cos(10\omega t)$. The obtained value of the maximal mass by this higher order truncation is $0.607/m$ in Planck units. For small-amplitude nearly Minkowskian configurations spatial derivatives are also small, and in Ref. [52] (and independently in [53]) it has been demonstrated that such nearly flat oscillatons can be described by a pair of coupled differential equations, the so called time independent the Schrödinger-Newton equations [26–30]. These equations also describe the weak gravity limit of boson stars. For quantum mechanical motivations leading to the Schrödinger-Newton equations see [54, 55].

The time evolution of perturbed oscillatons has been investigated in detail by [56]. For each mass smaller than the maximal oscillaton mass there are two oscillaton configurations. The one with the larger radius is a stable S-branch oscillaton, and the other is an unstable U-branch oscillaton. Moderately perturbed S-branch oscillatons vibrate with a low frequency corresponding to a quasinormal mode. Perturbed U-branch oscillatons collapse to black holes if the perturbation increases their mass, otherwise they migrate to an S-branch oscillaton. Actually, U-branch oscillatons turn out to be the critical solutions for type I critical collapse of massive scalar fields [57]. Corresponding apparently periodic objects also form in the critical collapse of massive vector fields [58].

There are also excited state oscillatons, indexed by the nodes of the scalar field. The instability and the decay of excited state oscillatons into black holes or S-branch oscillatons is described in [59]. The evolution of oscillatons on a full 3D grid has been also performed in [59], calculating the emitted gravitational radiation. Since $f(R)$ gravity theories are equivalent to ordinary general relativity coupled to a real scalar field, oscillatons naturally form in these theories as well [60]. The geodesics around oscillatons has been investigated in [61].

The plan of the paper is the following. In Section II the general formalism concerning a classical real scalar field coupled to gravitation in D dimensional, spherically symmetric spacetimes is set up. In subsection II C the coupled Einstein-scalar equations are explicated in a spatially conformally flat coordinate system. In Section III the small-amplitude expansion is presented and is carried out in detail. In subsection III C it is shown that in leading order one obtains the Schrödinger-Newton eqs. in D dimensions. In subsection III D the next to leading order results are given. Subsection III E contains an analysis of the singularities in the complexified radial variable. In Section IV the proper mass resp. the total mass of the QB core is evaluated in subsection IV A resp. subsection IV B. In subsection IV D a conjecture for a criterion of oscillaton stability is formulated. In Section V the Fourier analysis of the field equations is related to the small-amplitude expansion, and the amplitude of the standing wave tail of the QB is determined using Borel summation techniques. In subsection V F the mass loss rate of

oscillatons in the EKG theory is computed for $D = 3, 4, 5$ and for various values of the mass of the scalar field.

II. SCALAR FIELD ON CURVED BACKGROUND

A. Field equations

We consider a real scalar field Φ with a self-interaction potential $U(\Phi)$ in a $D + 1$ dimensional curved spacetime with metric g_{ab} . We use Planck units with $G = c = \hbar = 1$. For a free field with mass m the potential is $U(\Phi) = m^2\Phi^2/2$. The total Lagrangian density is

$$\mathcal{L} = \mathcal{L}_G + 16\pi\mathcal{L}_M, \quad (2)$$

where the Einstein Lagrangian density is $\mathcal{L}_G = \sqrt{-g}R$, and the Lagrangian density belonging to the scalar field is

$$\mathcal{L}_M = -\sqrt{-g} \left(\frac{1}{2}\Phi_{,a}\Phi^{,a} + U(\Phi) \right). \quad (3)$$

Variation of the action with respect to Φ yields the wave equation

$$g^{ab}\Phi_{;ab} - U'(\Phi) = 0, \quad (4)$$

while variation with respect to g^{ab} yields Einstein equations

$$G_{ab} = 8\pi T_{ab}, \quad (5)$$

where the stress-energy tensor is

$$T_{ab} = \Phi_{,a}\Phi_{,b} - g_{ab} \left(\frac{1}{2}\Phi_{,c}\Phi^{,c} + U(\Phi) \right). \quad (6)$$

If $D = 1$ then, by definition, the Einstein tensor is traceless, and from the trace of the Einstein equations it follows that $U(\Phi) = 0$. Hence we assume that $D > 1$.

We shall assume that the self-interaction potential, $U(\Phi)$, has a minimum $U(\Phi) = 0$ at $\Phi = 0$, and expand its derivative as

$$U'(\Phi) = \sum_{k=1}^{\infty} u_k \Phi^k, \quad (7)$$

where u_k are constants. In order to get rid of the 8π factors in the equations we introduce a rescaled scalar field and potential by

$$\phi = \sqrt{8\pi}\Phi, \quad \bar{U}(\phi) = 8\pi U(\Phi). \quad (8)$$

Then

$$\bar{U}'(\phi) = \sum_{k=1}^{\infty} v_k \phi^k, \quad (9)$$

with

$$v_k = \frac{u_k}{(8\pi)^{(k-1)/2}}. \quad (10)$$

The mass of the field is $m \equiv \sqrt{u_1} = \sqrt{v_1}$. If the pair $\phi(x^c)$ and $g_{ab}(x^c)$ solves the field equations with a potential $\bar{U}(\phi)$, then $\hat{\phi}(x^c) = \phi(\gamma x^c)$ and $\hat{g}_{ab}(x^c) = g_{ab}(\gamma x^c)$, for any positive constant γ , is a solution with a rescaled potential $\gamma^2 \bar{U}(\phi)$. It is sufficient to study the problem with potentials satisfying $m^2 = u_1 = v_1 = 1$, since the solutions corresponding to an arbitrary potential can be obtained from the solutions with an appropriate potential with $m = 1$ by applying the transformation

$$\phi(x^c) \rightarrow \phi(mx^c), \quad g_{ab}(x^c) \rightarrow g_{ab}(mx^c). \quad (11)$$

To simplify the expressions, unless explicitly stated, in the following we assume $m = 1$.

B. Spherically symmetric $D + 1$ dimensional spacetime

We consider a spherically symmetric $D + 1$ dimensional spacetime with coordinates $x^\mu = (t, r, \theta_1, \dots, \theta_{D-1})$. The metric can be chosen diagonal with components

$$\begin{aligned} g_{tt} &= -A, & g_{rr} &= B, \\ g_{\theta_1 \theta_1} &= C, & g_{\theta_n \theta_n} &= C \prod_{k=1}^{n-1} \sin^2 \theta_k, \end{aligned} \quad (12)$$

where A , B and C are functions of temporal coordinate t and radial coordinate r . The nonvanishing components of the Einstein tensor and the form of the wave equation are given in Appendix A.

A natural radius function, \hat{r} , can be defined in terms of the area of the symmetry spheres in general spherically symmetric spacetimes. In the metric (12) it is simply

$$\hat{r} = \sqrt{C}. \quad (13)$$

The Kodama vector [62, 63] is defined then by

$$K^a = \epsilon^{ab} \hat{r}_{,b}, \quad (14)$$

where ϵ_{ab} is the volume form in the (t, r) plane. Choosing the orientation such that $\epsilon_{rt} = \sqrt{AB}$ makes K^a future pointing, with nonvanishing components

$$K^t = \frac{\hat{r}_{,r}}{\sqrt{AB}}, \quad K^r = -\frac{\hat{r}_{,t}}{\sqrt{AB}}. \quad (15)$$

It can be checked that, in general, the Kodama vector is divergence free, $K^a_{;a} = 0$. Since contracting with the Einstein tensor, $G^{ab} K_{a;b} = 0$, the current

$$J_a = T_{ab} K^b \quad (16)$$

is also divergence free, $J^a_{;a} = 0$, it defines a conserved charge. Integrating on a constant t hypersurface with

a future oriented unit normal vector n^a , the conserved charge is

$$\begin{aligned} E &= \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int_0^r \hat{r}^{D-1} \sqrt{B} n^a J_a dr \\ &= \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int_0^r \frac{\hat{r}^{D-1}}{A} (T_{tt} \hat{r}_{,r} - T_{tr} \hat{r}_{,t}) dr. \end{aligned} \quad (17)$$

It is possible to show [62, 63], that E agrees with the Misner-Sharp energy (or local mass) function \hat{m} [64], which can be defined for arbitrary dimensions by

$$\hat{m} = \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma(\frac{D}{2})} \hat{r}^{D-2} (1 - g^{ab} \hat{r}_{,a} \hat{r}_{,b}). \quad (18)$$

It can be checked by a lengthy calculation, that the derivative of the mass function is

$$\hat{m}_{,a} = -\frac{2\pi^{\frac{D}{2}} \hat{r}^{D-1}}{\Gamma(\frac{D}{2})} \epsilon_{ab} J^b. \quad (19)$$

For the radial derivative follows that

$$\hat{m}_{,r} = \frac{2\pi^{\frac{D}{2}} \hat{r}^{D-1}}{\Gamma(\frac{D}{2}) A} (T_{tt} \hat{r}_{,r} - T_{tr} \hat{r}_{,t}), \quad (20)$$

which, comparing with (17), gives $E = \hat{m}$. Since for large r the function \hat{m} tends to the total mass, this relation will be important when calculating the mass loss rate caused by the scalar radiation in Section V F. The time derivative of the mass function is

$$\hat{m}_{,t} = \frac{2\pi^{\frac{D}{2}} \hat{r}^{D-1}}{\Gamma(\frac{D}{2}) B} (T_{rt} \hat{r}_{,r} - T_{rr} \hat{r}_{,t}). \quad (21)$$

This equation is according to the expectation, that, because of the spherical symmetry, the mass loss is caused only by the outward energy current of the massive scalar field. If at large distances the metric becomes asymptotically Minkowskian, $A = B = 1$, $C = r^2$ and $\hat{r} = r$, then using (6) and (8),

$$\hat{m}_{,t} = \frac{2\pi^{\frac{D}{2}} r^{D-1}}{\Gamma(\frac{D}{2})} \Phi_{,t} \Phi_{,r} = \frac{2\pi^{\frac{D}{2}} r^{D-1}}{8\pi\Gamma(\frac{D}{2})} \phi_{,t} \phi_{,r}. \quad (22)$$

C. Spatially conformally flat coordinate system

The diffeomorphism freedom of the general spherically symmetric time-dependent metric form (12) can be fixed in various ways. The most obvious choice is the use of Schwarzschild area coordinates by setting $C = r^2$. However, as it was pointed out by Don N. Page in [31], for the oscillaton problem it is more instructive to use the spatially conformally flat coordinate system defined by

$$C = r^2 B, \quad (23)$$

even if some expressions are becoming longer by this choice. As we will see in Sec. III and in Appendix B, inside the oscillaton the spheres described by constant Schwarzschild r coordinates are oscillating with much

larger amplitude than the constant r spheres in the conformally flat coordinate system. In both coordinates, when the functions A and B tend to 1, the spacetime approaches the flat Minkowskian metric.

In the spatially conformally flat coordinate system the Einstein equations take the form

$$(D-1) \left[\frac{D}{4B^2} (B_{,t})^2 - \frac{A}{r^{D-1} B^{\frac{D+2}{4}}} \left(\frac{r^{D-1} B_{,r}}{B^{\frac{6-D}{4}}} \right)_{,r} \right] = (\phi_{,t})^2 + \frac{A}{B} (\phi_{,r})^2 + 2A\bar{U}(\phi), \quad (24)$$

$$(D-1) \left[\frac{(D-2) (r^2 B)_{,r}}{4r^4 A^{\frac{2}{D-2}} B^2} \left(r^2 A^{\frac{2}{D-2}} B \right)_{,r} - \frac{1}{A^{\frac{1}{2}} B^{\frac{D}{4}-1}} \left(\frac{B^{\frac{D}{4}-1} B_{,t}}{A^{\frac{1}{2}}} \right)_{,t} - \frac{D-2}{r^2} \right] = (\phi_{,r})^2 + \frac{B}{A} (\phi_{,t})^2 - 2B\bar{U}(\phi), \quad (25)$$

$$-\frac{D-1}{2} A^{\frac{1}{2}} \left(\frac{B_{,t}}{A^{\frac{1}{2}} B} \right)_{,r} = \phi_{,t} \phi_{,r}, \quad (26)$$

$$\frac{rB}{A^{\frac{1}{2}}} \left(\frac{A_{,r}}{rA^{\frac{1}{2}} B} \right)_{,r} + (D-2)rB^{\frac{1}{2}} \left(\frac{B_{,r}}{rB^{\frac{1}{2}}} \right)_{,r} = 2(\phi_{,r})^2. \quad (27)$$

The right hand sides are equal to $2G_{tt}$, $2G_{rr}$, G_{tr} and $2(G_{\theta_1\theta_1}/r^2 - G_{rr})$, respectively. The wave equation is then

$$\frac{\phi_{,rr}}{B} - \frac{\phi_{,tt}}{A} + \frac{\phi_{,r}}{2r^{2D-2} AB^{D-1}} (r^{2D-2} AB^{D-2})_{,r} - \frac{\phi_{,t}}{2B^D} \left(\frac{B^D}{A} \right)_{,t} - \bar{U}'(\phi) = 0. \quad (28)$$

III. SMALL-AMPLITUDE EXPANSION

The small-amplitude expansion procedure has been applied successfully to describe the core region of one-dimensional flat background oscillons in ϕ^4 scalar theory [3, 32, 65]. Later it has been generalized for $D+1$ dimensional spherically symmetric systems in [14], and to a scalar-dilaton system in [25]. In this section we generalize the method for the case when the scalar field is coupled to gravity.

A. Choice of coordinates

We are looking for spatially localized bounded solutions of the field equations (5) for which ϕ is small and the metric is close to flat Minkowskian. We use the spatially conformally flat coordinate system defined by (23). It turns out, that under this approximation, all configurations that remain bounded as time passes are necessarily periodically oscillating in time. We expect that similarly to flat background oscillons, the smaller the amplitude of an oscillaton is, the larger its spatial extent becomes. Numerical simulation of oscillatons clearly support this expectation. Therefore, we introduce a new radial coordinate ρ by

$$\rho = \varepsilon r, \quad (29)$$

where ε denotes the small-amplitude parameter. We expand ϕ and the metric functions in powers of ε as

$$\phi = \sum_{k=1}^{\infty} \varepsilon^{2k} \phi_{2k}, \quad (30)$$

$$A = 1 + \sum_{k=1}^{\infty} \varepsilon^{2k} A_{2k}, \quad (31)$$

$$B = 1 + \sum_{k=1}^{\infty} \varepsilon^{2k} B_{2k}. \quad (32)$$

Since we intend to use asymptotically Minkowskian coordinates, where far from the oscillaton t measures the proper time and r the radial distances, we look for functions ϕ_{2k} , A_{2k} and B_{2k} that tend to zero when $\rho \rightarrow \infty$. One could initially include odd powers of ε into the expansions (30)-(32), however, it can be shown by the method presented below, that the coefficients of those terms necessarily vanish when we are looking for configurations that remain bounded in time.

The frequency of the oscillaton also depends on its amplitude. Similarly to the flat background case we expect that the smaller the amplitude is, the closer the frequency becomes to the threshold $m=1$. Numerical simulations also show this. Hence we introduce a rescaled time coordinate τ by

$$\tau = \omega t. \quad (33)$$

and expand the square of the ε dependent factor ω as

$$\omega^2 = 1 + \sum_{k=1}^{\infty} \varepsilon^{2k} \omega_{2k}. \quad (34)$$

It is possible to allow odd powers of ε into the expansion of ω^2 , but the coefficients of those terms turn out to be zero when solving the equations arising from the small-amplitude expansion. There is a considerable freedom in choosing different parametrizations of the small-amplitude states, changing the actual form of the function ω . The physical parameter is not ε but the frequency of the periodic states that will be given by ω . Similarly to the dilaton model in [25], we will show, that for spatial dimensions $2 < D < 6$ the parametrization of the small-amplitude states can be fixed by setting $\omega = \sqrt{1 - \varepsilon^2}$.

B. Leading order results

The field equations we solve are the Einstein equations (24)-(27), together with the wave equation (28), using the spatially conformally flat coordinate system $C = r^2 B$. The results of the corresponding calculations in Schwarzschild area coordinates $C = r^2$ are presented in Appendix B. Since we look for spatially slowly varying configurations with an ε dependent frequency, we apply the ε expansion in τ and ρ coordinates. This can be achieved by replacing the time and space derivatives as

$$\frac{\partial}{\partial t} \rightarrow \omega \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial r} \rightarrow \varepsilon \frac{\partial}{\partial \rho}, \quad (35)$$

and substituting $r = \rho/\varepsilon$.

From the ε^2 components of the field equations follows that

$$\phi_2 = p_2 \cos(\tau + \delta), \quad B_2 = b_2, \quad (36)$$

where three new functions, p_2 , δ and b_2 are introduced, depending only on ρ . From the ε^4 part of (26) it follows that δ is a constant. Then by a shift in the time coordinate we set

$$\delta = 0. \quad (37)$$

This shows that the scalar field oscillates simultaneously, with the same phase at all radii.

The ε^4 component of the field equations yield that

$$A_2 = a_2, \quad (38)$$

$$\phi_4 = p_4 \cos \tau + \frac{v_2 p_2^2}{6} [\cos(2\tau) - 3], \quad (39)$$

$$B_4 = b_4 - \frac{p_2^2}{4(D-1)} \cos(2\tau), \quad (40)$$

where a_2 , p_4 and b_4 are three new functions of ρ . If $D \neq 2$, from the ε^4 equations also follows that

$$b_2 = \frac{a_2}{2-D}, \quad (41)$$

and that the functions a_2 and p_2 are determined by the coupled differential equations

$$\frac{d^2 a_2}{d\rho^2} + \frac{D-1}{\rho} \frac{da_2}{d\rho} = \frac{D-2}{D-1} p_2^2, \quad (42)$$

$$\frac{d^2 p_2}{d\rho^2} + \frac{D-1}{\rho} \frac{dp_2}{d\rho} = p_2(a_2 - \omega_2). \quad (43)$$

If $D = 2$ then $a_2 = 0$, and there are no nontrivial localized regular solutions for b_2 and p_2 , so we assume $D > 2$ from now. We note that at all orders $\sin \tau$ terms can be absorbed by a small shift in the time coordinate. After this, no $\sin(k\tau)$ terms appear in the expansion, resulting in the time reflection symmetry at $\tau = 0$.

Since we have already set $m^2 = u_1 = v_1 = 1$, equations (42) and (43) do not depend on the coefficients v_k of the potential $\bar{U}(\phi)$. To order ε^2 the functions ϕ , A and B are the same for any potential. This means that the leading order small-amplitude behavior of oscillatons is always the same as for the Klein-Gordon case.

C. Schrödinger-Newton equations

Introducing the functions s and S by

$$s = \omega_2 - a_2, \quad S = p_2 \sqrt{\frac{D-2}{D-1}}, \quad (44)$$

equations (42) and (43) can be written into the form which is called the time-independent Schrödinger-Newton (SN) equations in the literature [26–30]:

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + sS = 0, \quad (45)$$

$$\frac{d^2 s}{d\rho^2} + \frac{D-1}{\rho} \frac{ds}{d\rho} + S^2 = 0. \quad (46)$$

Equations (45) and (46) have the scaling invariance

$$(S(\rho), s(\rho)) \rightarrow (\lambda^2 S(\lambda\rho), \lambda^2 s(\lambda\rho)). \quad (47)$$

If $2 < D < 6$ the SN equations have a family of solutions with S tending to zero exponentially as $\rho \rightarrow \infty$, and s tending to a constant $s_0 < 0$ as

$$s \approx s_0 + s_1 \rho^{2-D}. \quad (48)$$

The solutions are indexed by the number of nodes of S . The nodeless solution corresponds to the lowest energy and most stable oscillaton. We use the scaling freedom (47) to make the nodeless solution unique by setting $s_0 = \lim_{\rho \rightarrow \infty} s = -1$. At the same time we change the ε parametrization by requiring

$$\omega_2 = -1 \quad \text{for } 2 < D < 6, \quad (49)$$

ensuring that the limiting value of a_2 vanishes. Then for large ρ

$$a_2 \approx -s_1 \rho^{2-D} \quad \text{for } 2 < D < 6, \quad (50)$$

with only exponentially decaying corrections. Going to higher orders, it can be shown that one can always make the choice $\omega_i = 0$ for $i \geq 3$, thereby fixing the ε parametrization, and setting

$$\omega = \sqrt{1 - \varepsilon^2} \quad \text{for } 2 < D < 6. \quad (51)$$

For $D = 6$ the explicit form of the asymptotically decaying solutions are known

$$s = \pm S = \frac{24\alpha^2}{(1 + \alpha^2\rho^2)^2} \quad \text{for } D = 6, \quad (52)$$

where α is any constant. In this case, since both s and S tend to zero at infinity, we have no method yet to fix the value of α in (52). Moreover, in order to ensure that φ tends to zero at infinity we have to set

$$\omega_2 = 0 \quad \text{for } D = 6. \quad (53)$$

For $D > 6$ there are no solutions of the SN equations representing localized configurations [66].

Motivated by the asymptotic behavior of s , if $D \neq 2$ it is useful to introduce the variables

$$\sigma = \frac{\rho^{D-1}}{2-D} \frac{ds}{d\rho}, \quad \nu = s - \rho^{2-D}\sigma. \quad (54)$$

In $2 < D < 6$ dimensions these variables tend exponentially to the earlier introduced constants

$$\lim_{\rho \rightarrow \infty} \sigma = s_1, \quad \lim_{\rho \rightarrow \infty} \nu = s_0. \quad (55)$$

Then the SN equations can be written into the equivalent form

$$\frac{d\sigma}{d\rho} + \frac{\rho^{D-1}}{2-D} S^2 = 0, \quad (56)$$

$$\frac{d\nu}{d\rho} + \frac{\rho}{D-2} S^2 = 0, \quad (57)$$

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + (\nu + \rho^{2-D}\sigma) S = 0, \quad (58)$$

which is more appropriate for finding high precision numerical solutions. Equation (56) will turn out to be useful when integrating the mass-energy density in Section IV A in order to determine the proper mass.

D. Higher order expansion

From the ε^6 components of the field equations follows the time dependence of A_4 ,

$$A_4 = a_4^{(0)} + a_4^{(2)} \cos(2\tau), \quad (59)$$

where $a_4^{(0)}$ and $a_4^{(2)}$ are functions of ρ . The functions p_4 and $a_4^{(0)}$ are determined by the coupled equations

$$\begin{aligned} \frac{d^2 a_4^{(0)}}{d\rho^2} + \frac{D-1}{\rho} \frac{da_4^{(0)}}{d\rho} &= \frac{2p_2 p_4 (D-2)}{D-1} \\ &+ \left(\frac{da_2}{d\rho} \right)^2 + \omega_2 p_2^2 - \frac{2p_2^2 a_2}{D-1}, \end{aligned} \quad (60)$$

$$\begin{aligned} \frac{d^2 p_4}{d\rho^2} + \frac{D-1}{\rho} \frac{dp_4}{d\rho} &= p_4 (a_2 - \omega_2) \\ &+ \left(a_4^{(0)} - \omega_4 \right) p_2 - \frac{a_2 p_2 (D-1) (a_2 - \omega_2)}{D-2} \\ &- \frac{D p_2^3}{8(D-1)} - \left(\frac{5}{6} v_2^2 - \frac{3}{4} v_3 \right) p_2^3. \end{aligned} \quad (61)$$

We look for the unique solution for which both $a_4^{(0)}$ and p_4 tend to zero as $\rho \rightarrow \infty$. For $2 < D < 6$ the function p_4 goes to zero exponentially, while for large ρ

$$a_4^{(0)} \approx \frac{1}{2} s_1^2 \rho^{4-2D} + s_2 \rho^{2-D} + s_3, \quad (62)$$

where s_1 is defined in (48), and s_2 and s_3 are some constants. If $a_4^{(0)}$ and p_4 are solutions of (60) and (61), then for any constant c

$$\bar{a}_4^{(0)} = a_4^{(0)} + c \left[2(a_2 - \omega_2) + \rho \frac{da_2}{d\rho} \right], \quad (63)$$

$$\bar{p}_4 = p_4 + c \left(2p_2 + \rho \frac{dp_2}{d\rho} \right), \quad (64)$$

are also solutions. This family of solutions is generated by the scaling freedom (47) of the SN equations. If we have any solution of (60) and (61) then by choosing c appropriately we can get another solution for which $s_3 = 0$ in (62).

The equation for b_4 is

$$\begin{aligned} \frac{db_4}{d\rho} &= \frac{1}{2-D} \frac{da_4^{(0)}}{d\rho} \\ &+ \frac{1}{4(D-2)^2} \frac{da_2}{d\rho} \left[\rho \frac{da_2}{d\rho} + 4(D-1)a_2 \right] \\ &+ \frac{\rho}{2(D-1)(D-2)} \left[\left(\frac{dp_2}{d\rho} \right)^2 - p_2^2 (a_2 - \omega_2) \right]. \end{aligned} \quad (65)$$

For large ρ the function b_4 tends to zero as

$$b_4 \approx \frac{6-D}{8(D-2)^2} s_1^2 \rho^{4-2D} + \frac{s_2}{2-D} \rho^{2-D}. \quad (66)$$

The $\cos(2\tau)$ part of A_4 is determined by

$$\begin{aligned} \frac{d^2 a_4^{(2)}}{d\rho^2} - \frac{1}{\rho} \frac{da_4^{(2)}}{d\rho} &= \frac{(D-2)(a_2 - \omega_2) p_2^2}{2(D-1)} \\ &- \frac{D}{2(D-1)} \frac{dp_2}{d\rho} \left(\frac{dp_2}{d\rho} + \frac{D-2}{\rho} p_2 \right). \end{aligned} \quad (67)$$

We remind the reader, that for $2 < D < 6$ the choice $\omega_2 = -1$, $\omega_4 = 0$ is natural, while for $D = 6$ necessarily $\omega_2 = 0$. For a Klein-Gordon field in $D = 6$ the only nonvanishing coefficient is $\omega_4 = -1$.

Summarizing the results, the scalar field and the metric components up to ε^4 order are

$$\phi = \varepsilon^2 p_2 \cos \tau \quad (68)$$

$$+ \varepsilon^4 \left\{ p_4 \cos \tau + \frac{v_2 p_2^2}{6} [\cos(2\tau) - 3] \right\} + \mathcal{O}(\varepsilon^6),$$

$$A = 1 + \varepsilon^2 a_2 + \varepsilon^4 \left[a_4^{(0)} + a_4^{(2)} \cos(2\tau) \right] + \mathcal{O}(\varepsilon^6), \quad (69)$$

$$B = 1 - \varepsilon^2 \frac{a_2}{D-2} \quad (70)$$

$$+ \varepsilon^4 \left[b_4 - \frac{p_2^2}{4(D-1)} \cos(2\tau) \right] + \mathcal{O}(\varepsilon^6).$$

Going to higher orders, the expressions get rather complicated. However, it can be seen that for symmetric potentials, when $v_{2k} = 0$, the scalar field ϕ contains only $\cos(k\tau)$ components with odd k , while A and B only contains even Fourier components.

Some of the higher order expressions simplifies considerably when considering symmetric potentials with $v_{2k} = 0$. Because the first radiating mode proportional to $\cos(3\tau)$ emerges at ε^6 order in ϕ in symmetric potentials, we present its higher order expression for the symmetric case

$$\phi = \varepsilon^2 p_2 \cos \tau + \varepsilon^4 p_4 \cos \tau + \varepsilon^6 p_6 \cos \tau \quad (71)$$

$$+ \varepsilon^6 \left(\frac{D p_2^3}{64(D-1)} + \frac{v_3 p_2^3}{32} + \frac{p_2 a_4^{(2)}}{8} \right) \cos(3\tau) + \mathcal{O}(\varepsilon^8),$$

where p_6 is a function of ρ determined by lengthy differential equations arising at higher orders.

For the Klein-Gordon case in $D = 3$ spatial dimensions we plot the numerically obtained functions p_2 , a_2 , p_4 , $a_4^{(0)}$, $a_4^{(2)}$ and b_4 on Figs. 1 and 2.

Equations (68)-(70) determine a one-parameter family of oscillating configurations depending on the parameter ε . This family solves the field equations with a scalar field mass $m = 1$. By applying the rescaling (11) to the t and r coordinates, we can obtain one-parameter families of solutions with any scalar mass m .

To ε^2 order, the metric is static. This is the biggest advantage of the spatially conformally flat coordinate system $C = r^2 B$ over the Schwarzschild area coordinates $C = r^2$. In the Schwarzschild system the constant r observers “feel” an ε^2 order small oscillation in the metric (see Appendix B). The magnitude of the acceleration of the constant $(r, \theta_1, \theta_2, \dots)$ observers in the general metric (12) is

$$a = \frac{1}{2A\sqrt{B}} \frac{dA}{dr}, \quad (72)$$

which has an ε^3 order oscillating component when using Schwarzschild coordinates, while in spatially conformally

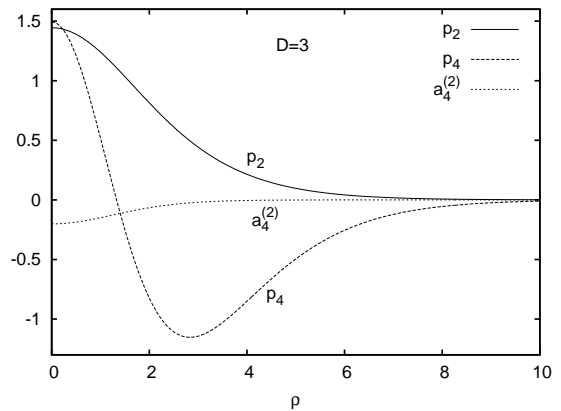


FIG. 1: The exponentially decaying functions p_2 , p_4 , and $a_4^{(2)}$ for the small-amplitude expansion of the Klein-Gordon oscillaton in the $D = 3$ case.

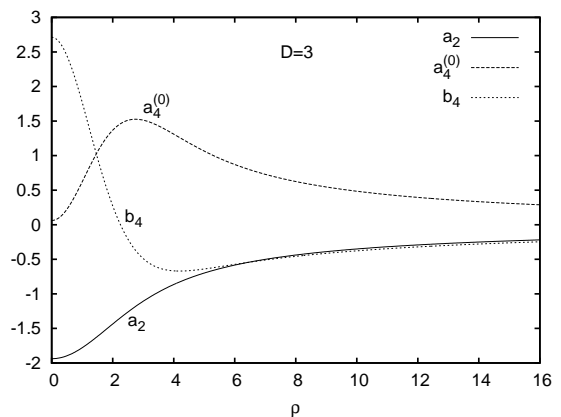


FIG. 2: The functions a_2 , $a_4^{(0)}$, and b_4 for the $D = 3$ Klein-Gordon system. These functions tend to zero according to a power law for $\rho \rightarrow \infty$.

flat coordinates the temporal change in the acceleration is only of order ε^5 .

The function $W = AB^{D-2}$ is equal to 1 to order ε^4 in the conformally flat coordinates. This motivates the metric form choice

$$ds^2 = -Adt^2 \quad (73)$$

$$+ \left(\frac{W}{A} \right)^{\frac{1}{D-2}} (dr^2 + r^2 d\theta_1^2 + r^2 \sin^2 \theta_1 d\theta_2^2 + \dots),$$

which has been employed for the $D = 3$ case in [31].

E. Singularities on the complex plane

As we will see in Section V, in order to determine the energy loss of oscillatons it is advantageous to extend the

functions ϕ , A and B to the complex plane. In the small-amplitude expansion formalism the extension of the coefficient functions ϕ_k , A_k and B_k have symmetrically positioned poles along the imaginary axis, induced by the poles of the SN equations. We consider the closest pair of singularities, located at $\rho = \pm iQ_D$, since these will provide the dominant contribution to the energy loss. The numerically determined location of the pole for the spatial dimensions where there is an exponentially localized core is

$$Q_3 = 3.97736, \quad (74)$$

$$Q_4 = 2.30468, \quad (75)$$

$$Q_5 = 1.23595. \quad (76)$$

The leading order behavior of the functions near the poles can be determined analytically, even if the solution of the SN equations is only known numerically on the real axis. Let us measure distances from the upper singularity by a coordinate R defined as

$$\rho = iQ_D + R. \quad (77)$$

Close to the pole we can expand the SN equations, and obtain that s and S have the same behavior,

$$s = S = -\frac{6}{R^2} - \frac{6i(D-1)}{5Q_D R} - \frac{(D-1)(D-51)}{50Q_D^2} + \mathcal{O}(R), \quad (78)$$

even though they clearly differ on the real axis. We note that for $D > 1$ there are logarithmic terms in the expansion of s and S , starting with terms proportional to $R^4 \ln R$. According to (41) and (44), the expression (78) determines the ε^2 parts of ϕ , A and B near the pole.

Substituting into (60), (61), (65) and (67), the ε^4 order contributions $a_4^{(0)}$, p_4 , b_4 and $a_4^{(2)}$ can also be determined around the pole. We give the results for the Klein-Gordon case, when $v_k = 0$ for $k > 1$:

$$a_4^{(0)} = -\frac{9(25D+208)}{52(D-2)R^4} + \frac{324iD(D-1)\ln R}{35Q_D(D-2)R^3} + \frac{a_{-3}}{R^3} + \mathcal{O}\left(\frac{\ln R}{R^2}\right), \quad (79)$$

$$p_4 \sqrt{\frac{D-2}{D-1}} + a_4^{(0)} = \frac{9(43D-104)}{26(D-2)R^4} + \frac{9i(D-1)(3D-8)}{5Q_D(D-2)R^3} + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (80)$$

$$b_4 = \frac{9(333D+832)}{260(D-2)^2R^4} - \frac{324iD(D-1)\ln R}{35Q_D(D-2)^2R^3} - \frac{a_{-3}}{(D-2)R^3} + \frac{18i(D-1)}{5Q_D(D-2)R^3} + \mathcal{O}\left(\frac{\ln R}{R^2}\right), \quad (81)$$

$$a_4^{(2)} = -\frac{9(6-D)}{5(D-2)R^4} + \frac{6i(D-1)(D-6)}{5Q_D(D-2)R^3} + \mathcal{O}\left(\frac{1}{R^2}\right). \quad (82)$$

The constant a_{-3} can only be determined from the specific behavior of the functions on the real axis, namely from the requirement of the exponential decay of p_4 for large real ρ .

IV. PROPER AND TOTAL MASS

A. Proper mass

In this subsection we present the calculation of the proper mass M_p , which is usually obtained by the integral of the mass-energy density over a spatial slice of the corresponding spacetime. In the next subsection the calculation of the total mass M will be performed, by investigating the asymptotic behavior of the metric components. The difference $E_b = M_p - M$ defines the gravitational binding energy, which is expected to be positive.

The mass-energy density is $\mu = T_{ab}u^a u^b$, where the unit timelike vector u^a has the components $(1/\sqrt{A}, 0, \dots, 0)$. In terms of the rescaled scalar field ϕ ,

$$\mu = \frac{1}{8\pi} \left[\frac{1}{2A} \left(\frac{d\phi}{dt} \right)^2 + \frac{1}{2B} \left(\frac{d\phi}{dr} \right)^2 + \bar{U}(\phi) \right]. \quad (83)$$

The total proper mass in the metric (12) is defined by the D dimensional volume integral

$$M_p = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int_0^\infty dr \mu \sqrt{BC^{D-1}}. \quad (84)$$

Applying this for the small-amplitude expansion of oscillators in spatially conformally flat coordinates, and using that $\rho = \varepsilon r$ and $\omega^2 = 1 - \varepsilon^2$, we can write

$$M_p = \frac{2\pi^{\frac{D}{2}}}{8\pi\Gamma(\frac{D}{2})} \int_0^\infty d\rho \rho^{D-1} \varepsilon^{-D} (1 + \varepsilon^2 b_2)^{\frac{D}{2}} \times \left\{ \frac{1}{2(1 + \varepsilon^2 a_2)} \left(\varepsilon^2 p_2 \omega \sin \tau + \varepsilon^4 p_4 \sin \tau + \varepsilon^4 \frac{v_2 p_2^2}{3} \sin(2\tau) \right)^2 + \frac{1}{2} \left(\varepsilon^3 \cos \tau \frac{dp_2}{d\rho} \right)^2 + \frac{1}{2} \left[\varepsilon^2 p_2 \cos \tau + \varepsilon^4 p_4 \cos \tau + \varepsilon^4 \frac{v_2 p_2^2}{3} (\cos(2\tau) - 3) \right]^2 + \frac{v_2}{3} (\varepsilon^2 p_2 \cos \tau)^3 \right\}. \quad (85)$$

Using (41) and (43), for the proper mass we obtain

$$M_p = \varepsilon^{4-D} M_p^{(1)} + \varepsilon^{6-D} M_p^{(2)} + \mathcal{O}(\varepsilon^{8-D}), \quad (86)$$

where

$$M_p^{(1)} = \frac{\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} \int_0^\infty d\rho \rho^{D-1} p_2^2, \quad (87)$$

$$M_p^{(2)} = \frac{\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} \int_0^\infty d\rho \rho^{D-1} \left(2p_2 p_4 - p_2^2 - \frac{3D-4}{2(D-2)} a_2 p_2^2 \right). \quad (88)$$

The result turns out to be time independent to this order. Although the coefficients v_i of the potential also drop out from (88), the dependence on the form of the potential still comes in through (61). The leading order behavior, (87), only depends on the scalar field mass m , which has been rescaled to 1 for simplicity. Applying (11) to obtain solutions with $m \neq 1$, an m^2 factor appears in (83) for the mass-energy density μ . Since the volume element in the integral contains a m^{-D} factor, in all the presented proper and total mass formulas an m^{2-D} factor appears.

Using (44), (55) and (56), the leading order coefficient is

$$M_p^{(1)} = \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} s_1. \quad (89)$$

The numerically calculated values for $M_p^{(1)}$ and $M_p^{(2)}$ for the Klein-Gordon field case in various spatial dimensions are listed in Table I.

	$D = 3$	$D = 4$	$D = 5$
$M^{(1)} = M_p^{(1)}$	1.75266	9.06533	21.7897
$M^{(2)}$	-2.11742	-43.5347	-533.732
$M_p^{(2)}$	-1.53319	-39.0020	-555.521

TABLE I: Coefficients of the ε expansion of the total mass M and proper mass M_p for the $m = 1$ Klein-Gordon case in $D = 3, 4, 5$ spatial dimensions.

B. Total mass

Since the scalar field tends to zero exponentially, at large distances the metric should approach the static Schwarzschild-Tangherlini metric [67]. In Schwarzschild area coordinates, with $C = r^2$, this metric has the form

$$ds^2 = - \left(1 - \frac{r_0^{D-2}}{r^{D-2}} \right) dt^2 + \frac{1}{1 - \frac{r_0^{D-2}}{r^{D-2}}} dr^2 + r^2 d\Omega_{D-1}^2, \quad (90)$$

while in the spatially conformally flat coordinate system, $C = r^2 B$, it can be written as

$$ds^2 = - \left(\frac{4r^{D-2} - r_0^{D-2}}{4r^{D-2} + r_0^{D-2}} \right)^2 dt^2 + \left(1 + \frac{r_0^{D-2}}{4r^{D-2}} \right)^{\frac{4}{D-2}} (dr^2 + r^2 d\Omega_{D-1}^2), \quad (91)$$

where r_0 is a constant related to the mass. In general spherically symmetric spacetimes it is possible to define the natural radius function \hat{r} by (13), and the mass function \hat{m} by (18). In both the Schwarzschild and conformally flat coordinates, for the Schwarzschild-Tangherlini metric \hat{m} is constant,

$$\hat{m} = M = \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} r_0^{D-2}. \quad (92)$$

For the small-amplitude expansion of oscillatons in the spatially conformally flat coordinate system the radius function is $\hat{r} = r\sqrt{B}$, where B is expanded according to (32). Using the rescaled radial coordinate $\rho = \varepsilon r$, the mass function can be expanded as

$$\hat{m} = \varepsilon^{4-D} \hat{m}^{(1)} + \varepsilon^{6-D} \hat{m}^{(2)} + \mathcal{O}(\varepsilon^{8-D}), \quad (93)$$

where

$$\hat{m}^{(1)} = - \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} \rho^{D-1} \frac{dB_2}{d\rho}, \quad (94)$$

and

$$\hat{m}^{(2)} = - \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} \rho^{D-1} \left[\frac{dB_4}{d\rho} + \frac{\rho}{4} \left(\frac{dB_2}{d\rho} \right)^2 + \frac{D-4}{2} B_2 \frac{dB_2}{d\rho} \right]. \quad (95)$$

The total mass is the limit at $r \rightarrow \infty$,

$$M = \varepsilon^{4-D} M^{(1)} + \varepsilon^{6-D} M^{(2)} + \mathcal{O}(\varepsilon^{8-D}). \quad (96)$$

Since $B_2 = a_2/(2-D)$, using the asymptotic form (50) of a_2 , we get

$$M^{(1)} = \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} s_1, \quad (97)$$

agreeing with the leading order coefficient of the proper mass, $M_p^{(1)}$, given in (89). Using (65) for the derivative of b_4 ,

$$M^{(2)} = \lim_{\rho \rightarrow \infty} \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma\left(\frac{D}{2}\right)} \frac{\rho^{D-1}}{D-2} \left[\frac{da_4^{(0)}}{d\rho} - \frac{\rho}{2(D-2)} \left(\frac{da_2}{d\rho} \right)^2 - \frac{3}{2} a_2 \frac{da_2}{d\rho} \right], \quad (98)$$

which can be easily calculated numerically, since the expression in the limit tends to a constant exponentially. The numerical results for the Klein-Gordon field are presented in Table I. The proper mass and the total mass agree to leading order. However, taking into account the next term in the expansion, it turns out that, as it can be expected, the gravitational binding energy $E_b = M_p - M$ is positive.

It is instructive to write the expressions for the total mass in natural units, where the oscillaton mass M is measured in kilograms while the mass of the scalar field m in units of eV/c^2 :

$$M_{(D=3)} = \varepsilon (4.66 - 5.63 \varepsilon^2) 10^{20} kg \frac{eV}{mc^2}, \quad (99)$$

$$M_{(D=4)} = (2.94 - 14.1 \varepsilon^2) 10^{49} kg \left(\frac{eV}{mc^2} \right)^2, \quad (100)$$

$$M_{(D=5)} = \frac{1}{\varepsilon} (8.63 - 211 \varepsilon^2) 10^{77} kg \left(\frac{eV}{mc^2} \right)^3. \quad (101)$$

Since $\mathcal{O}(\varepsilon^4)$ terms were dropped, these expressions are precise only for small values of ε . However, comparing to the 3 + 1 dimensional numerical results obtained by solving the Fourier mode equations in [52], it can be inferred that these total mass expressions give a reasonable estimate even when $\varepsilon \approx 0.5$.

C. Size of oscillatons

Although oscillatons are exponentially localized, they do not have a definite outer surface. A natural definition for their size is to take the radius r_n inside which n percentage of the mass can be found. It is usual to take, for example, $n = 95$. The mass inside a given radius r can be defined either by the integral (84) replacing the upper limit by r , or by taking the local mass function \hat{m} in (18). To leading order in ε both definitions give

$$M(r) = \frac{(D-1)\pi^{\frac{D}{2}}}{8\pi\Gamma(\frac{D}{2})} \sigma(\varepsilon r), \quad (102)$$

where σ has been introduced in (54) as a function of $\rho = \varepsilon r$. The rescaled radius ρ_n can be defined by

$$\frac{\sigma(\rho_n)}{\sigma(\infty)} = \frac{n}{100}. \quad (103)$$

The numerical values of ρ_n for various n in $D = 3, 4, 5$ dimensions are listed in Table II. Restoring the scalar field mass m into the expression, the physical radius is

$$r_n = \frac{\rho_n}{\varepsilon m}. \quad (104)$$

In natural units, measuring mc^2 in electron volts and r_n in meters (Roman m),

$$r_n = 1.97 \cdot 10^{-7} m \frac{\rho_n}{\varepsilon} \frac{eV}{mc^2}. \quad (105)$$

	$D = 3$	$D = 4$	$D = 5$
ρ_{50}	2.240	1.778	1.317
ρ_{90}	3.900	3.013	2.284
ρ_{95}	4.471	3.455	2.652
ρ_{99}	5.675	4.410	3.478
$\rho_{99.9}$	7.239	5.692	4.634

TABLE II: The radius inside which given percentage of the mass is contained for various spatial dimensions.

Similarly to the total mass expressions in the previous subsection, this result is still a reasonable approximation for as large ε values as 0.5.

D. Stability

The stability properties of oscillatons are in many respects very similar to cold neutron and boson stars. Perfect fluid stars are known to be stable for small μ_c central densities. As μ_c increases, the total mass M also increases, until it reaches a maximal value M_{\max} , where according to a theorem in [35], an unstable radial mode sets in. Boson stars have analogous stability properties [68]. For oscillatons in the EKG system (for $D = 3$) this behavior has also been observed in Refs. [1] and [56]. Oscillatons are closely related to flat background oscillons, which also behave very similarly to neutron and boson stars. Since we use a small-amplitude expansion for oscillons and oscillatons it is more instructive to use the magnitude of the oscillating central amplitude Φ_c , instead of the central density μ_c . The central density is expected to be a monotonically increasing function of the central amplitude.

Given the very close analogy with oscillons we formulate a general conjecture on the stability of oscillatons. We recall that in all known examples the stability pattern of oscillons is the same, namely if $dE/d\varepsilon > 0$ oscillons are stable, while when $dE/d\varepsilon < 0$ oscillons are unstable, where $E = E(\varepsilon)$ is the total energy of the oscillon [14–16, 25]. Therefore we conjecture that the same stability pattern holds true for oscillatons, except that the energy, E is replaced by the total mass $M = M(\varepsilon)$ of the oscillaton. In other words if the time evolution (i.e. energy/mass loss) of an oscillon/oscillaton leads to spreading of the core, the oscillon/oscillaton is stable, while oscillons/oscillatons are unstable if they have to contract with time evolution. Therefore if the conjecture is true, the first two terms in the expansion of M enables us to determine the stability of oscillatons.

Taking into account the first two terms in (96), for the $D = 3$ Klein-Gordon field the total mass has a maximum at

$$\varepsilon_{\max} = \sqrt{\frac{-M^{(1)}}{3M^{(2)}}} \approx 0.525, \quad (106)$$

corresponding to the value of the frequency, $\omega_{\min} \approx$

0.851. Although this is just a leading order result, it agrees reasonably well with the frequency 0.864 obtained by the numerical solution of the Fourier mode equations in [52]. The value of the mass at the maximum is

$$M_{\max} = \frac{2}{3}\varepsilon_{\max}M^{(1)} \approx 0.614, \quad (107)$$

which is also quite close to the number 0.607 given in [52].

For an axion with $m = 10^{-5}eV/c^2$ the maximal mass is $M_{\max} = 1.63 \cdot 10^{25}kg$, which is about three times the mass of the Earth. The radius of this oscillaton, according to the leading order approximation (105), is $r_{95} = 16.8$ cm, while its Schwarzschild radius is 2.42 cm.

The mass maximum is very important concerning the stability of oscillatons. According to [1, 56], for the Klein-Gordon field in $D = 3$ small-amplitude oscillatons with $\varepsilon < \varepsilon_{\max}$ are stable, while those with $\varepsilon > \varepsilon_{\max}$ are unstable, presumably having a single decay mode. For $D = 4$ and $D = 5$ the mass is a monotonically decreasing function of ε , and all oscillatons are expected to be unstable. The situation may be totally different for scalar fields with a nontrivial potential $U(\phi)$. For example, for $v_2 = 0$ and $v_3 = 4$, the coefficient $M^{(2)}$ becomes positive, and there is no maximum on the energy curve in $D = 3$ dimensions, consequently even large amplitude oscillatons can be expected to be stable. Using this potential in $D = 4$ dimensions, for small ε the mass will be a monotonically increasing function, so small-amplitude configurations should be stable. For $D = 5$, in this case, there is a minimum in the energy curve, above which stable oscillatons can be expected.

V. RADIATION LAW OF OSCILLATONS

The methods that we apply in this section for the calculation of the radiation law of oscillatons have been already applied for oscillons formed by scalar fields on flat background. The extension of the Fourier mode equations to the complex plane has been first used for the one-dimensional ϕ^4 theory by Segur and Kruskal [32]. The Borel summation method to calculate the small correction near the pole has been introduced by Pomeau, Ramani and Grammaticos [69]. The results has been extended to higher dimensional oscillons in [16] and to a scalar-dilaton system in [25].

A. Fourier expansion

Since all terms in the expansion (30)-(32) are exponentially decaying, the small-amplitude expansion can only be applied to the core region of oscillatons. It cannot describe the exponentially small radiative tail responsible for the energy loss. This is closely related to the fact that the expansion is not convergent, it is an asymptotic expansion. Instead of studying a radiating oscil-

aton configuration with slowly varying frequency, it is simpler to consider exactly periodic solutions having a relatively large amplitude core and a very small amplitude standing wave tail. We Fourier expand the scalar and the metric components as

$$\phi = \sum_{k=0}^{N_F} \bar{\phi}_k \cos(k\omega t), \quad (108)$$

$$A = 1 + \sum_{k=0}^{N_F} \bar{A}_k \cos(k\omega t), \quad (109)$$

$$B = 1 + \sum_{k=0}^{N_F} \bar{B}_k \cos(k\omega t), \quad (110)$$

where $\bar{\phi}_k$, \bar{A}_k and \bar{B}_k only depend on r , and solve the Fourier mode equations obtained from Einstein's equations and the wave equation. Although, in principle, the Fourier truncation order N_F should tend to infinity, one can expect very good approximation for moderate values of N_F . We assume that the frequency is approaching from below the mass threshold $m = 1$, and in this context, define the ε parameter by $\varepsilon = \sqrt{1 - \omega^2}$.

Regularity at the center require finite values for $\bar{\phi}_k$, \bar{A}_k and \bar{B}_k for $r = 0$, together with

$$\left. \frac{d\bar{\phi}_k}{dr} \right|_{r=0} = 0, \quad \left. \frac{d\bar{A}_k}{dr} \right|_{r=0} = 0, \quad \left. \frac{d\bar{B}_k}{dr} \right|_{r=0} = 0. \quad (111)$$

Concerning the boundary conditions at $r \rightarrow \infty$, it is a natural but quite restrictive requirement to assume that the metric is asymptotically flat, with the t coordinate tending to the proper time for large radii. This implies that $\bar{A}_k \rightarrow 0$ and $\bar{B}_k \rightarrow 0$ for $r \rightarrow \infty$. The Fourier components of the wave equation (28) for large r decouple, and in this case can be written as

$$\frac{d^2 \bar{\phi}_n}{dr^2} + \frac{D-1}{r} \frac{d\bar{\phi}_n}{dr} + (n^2\omega^2 - 1)\bar{\phi}_n = 0. \quad (112)$$

In the relevant frequency range $1/2 < \omega < 1$, if $n \geq 2$ these equations have oscillatory solutions, behaving asymptotically as

$$\begin{aligned} \bar{\phi}_n &= \frac{\gamma_n^{(s)}}{r^{(D-1)/2}} \sin\left(r\sqrt{n^2\omega^2 - 1}\right) \\ &+ \frac{\gamma_n^{(c)}}{r^{(D-1)/2}} \cos\left(r\sqrt{n^2\omega^2 - 1}\right), \end{aligned} \quad (113)$$

where $\gamma_n^{(s)}$ and $\gamma_n^{(c)}$ are some constants. According to (83), this oscillating tail has a mass-energy density, μ , proportional to r^{1-D} . This implies that if $\gamma_n^{(s)}$ or $\gamma_n^{(c)}$ is nonzero for any $n \geq 2$, the total proper mass of the spacetime is infinite. The requirement of the vanishing of all these coefficients together with the central boundary conditions are clearly too many conditions to satisfy for the given number of second order differential equations. In general, regular finite mass exactly periodic solutions are not expected to exist.

If we require that $\gamma_n^{(s)} = 0$ and $\gamma_n^{(c)} = 0$ for all n then all ϕ_n tend to zero exponentially, and we have a finite mass asymptotically flat configuration. However, in general, this solution will be singular at the center, hence we name this solution *singular breather* (SB). For a given frequency, ω , the singular breather solution is unique by parameter counting.

Because of their close similarity to oscillatons, it is important to study another, presumably unique periodic solution, the so called *quasibreather* (QB) solution, which is regular at the center, but which has a minimal energy density standing wave tail. Our aim is to construct the quasibreather solution from the singular breather solution. It is important to point out that the quasibreather picture is only valid inside some large but finite radius. However small the energy density of the oscillating tail is, going to very large distances its contribution to the mass will not be negligible anymore. Consequently, the assumption that \bar{A}_k and \bar{B}_k tends to zero will not remain true for arbitrarily large values of r , and consequently Eq. (112) will also change. For sufficiently large values of r , the metric function A increases until it causes to change the first radiating mode (either ϕ_2 or ϕ_3) from oscillating to exponentially decaying. Increasing r further, all modes will stop oscillating one by one. This way we obtain the exactly time-periodic but infinite mass “breathers” which are described in details in Section IX of the paper of Don N. Page [31]. The quasibreather can be considered to be the part of such an infinite mass “breather” containing the core and a large portion of the tail where the first radiating mode oscillates, requiring that the mass inside this region is dominated by that of the oscillon core. Since the core amplitude is of the order ε^2 , while the tail is exponentially suppressed in ε , the quasibreather picture is valid in a sufficiently large volume.

Since the amplitude of the oscillating tail of the quasibreather is very small, apart from a small region around the center the core of the QB is very close to the corresponding singular breather solution. In particular, the SB and the QB solutions have the same ε expansions. For the SB solution the small-amplitude expansion will not be valid in a region near the center $r = 0$, while for the QB solution it will fail for large radii where the oscillating tail becomes dominant. The size of the region, $(0, r_{\text{diff}})$, around $r = 0$ where the difference between the SB and the regular core becomes relevant is $r_{\text{diff}} = \mathcal{O}(e^{-\delta/\varepsilon})$ (with δ being a constant), whereas the size of the SB or QB core is proportional to $1/\varepsilon$. Outside of this region, i.e. for $r > r_{\text{diff}}$ the difference between the SB and the QB will be very small since the singular mode turns out to be proportional to the tail amplitude, which is exponentially small in terms of the small parameter ε , while the core amplitude is of order ε^2 .

For potentials $U(\phi)$ which are symmetric around their minima, i.e. $v_{2k} = 0$ for integer k , the Fourier expansion of the scalar contains only odd, while that of the metric

components only even terms,

$$\bar{\phi}_{2k} = 0, \quad \bar{A}_{2k+1} = 0, \quad \bar{B}_{2k+1} = 0. \quad (114)$$

For symmetric potentials the first radiating mode is $\bar{\phi}_3$. In this section we will concentrate mainly on the Klein-Gordon scalar field with $v_k = 0$ for $k > 1$. It is straightforward to generalize the results for symmetric potentials.

For small-amplitude quasibreather or singular breather configurations we can establish the connection between the Fourier expansion (108)-(110) and the small-amplitude expansion (30)-(32) by comparing to (68)-(71). For symmetric potentials we obtain:

$$\bar{\phi}_1 = \varepsilon^2 p_2 + \varepsilon^4 p_4 + \mathcal{O}(\varepsilon^6), \quad (115)$$

$$\bar{\phi}_3 = \varepsilon^6 \left(\frac{D p_2^3}{64(D-1)} + \frac{v_3 p_2^3}{32} + \frac{p_2 a_4^{(2)}}{8} \right) + \mathcal{O}(\varepsilon^8), \quad (116)$$

$$\bar{A}_0 = \varepsilon^2 a_2 + \varepsilon^4 a_4^{(0)} + \mathcal{O}(\varepsilon^6), \quad (117)$$

$$\bar{A}_2 = \varepsilon^4 a_4^{(2)} + \mathcal{O}(\varepsilon^6), \quad (118)$$

$$\bar{B}_0 = -\varepsilon^2 \frac{a_2}{D-2} + \varepsilon^4 b_4 + \mathcal{O}(\varepsilon^6), \quad (119)$$

$$\bar{B}_2 = -\varepsilon^4 \frac{p_2^2}{4(D-1)} + \mathcal{O}(\varepsilon^6). \quad (120)$$

B. Expansion near the pole

As $\varepsilon \rightarrow 0$ the amplitude of all Fourier coefficients tend to zero. However, extending them to the complex plane, for small ε they all have pole singularities on the imaginary axis at $r = \pm iQ_D/\varepsilon$, corresponding to the poles of the Schrödinger-Newton equations at $\rho = \pm iQ_D$, as it was discussed in Sec. III E. As ε tends to zero, the poles move further and further away from the real axis, but close to them the Fourier components $\bar{\phi}_k$, \bar{A}_k and \bar{B}_k are not getting small, in fact they have ε independent parts. We introduce a shifted radial coordinate, y , for an “inner region” around the upper pole by

$$r = \frac{iQ_D}{\varepsilon} + y. \quad (121)$$

The coordinate y is related to the one (R) defined in Eq. (77) by $R = \varepsilon y$. Substituting the small-amplitude expansion results (78)-(82) into (115)-(120), and taking the limit $\varepsilon \rightarrow 0$, it follows that in the Klein-Gordon case,

near the upper pole

$$\bar{\phi}_1 = \left(-\frac{6}{y^2} + \frac{999D}{52(D-2)y^4} + \dots \right) \sqrt{\frac{D-1}{D-2}}, \quad (122)$$

$$\bar{\phi}_3 = \left(-\frac{27(7D-12)}{40(D-2)y^4} + \dots \right) \sqrt{\frac{D-1}{D-2}}, \quad (123)$$

$$\bar{A}_0 = \frac{6}{y^2} - \frac{9(25D+208)}{52(D-2)y^4} + \dots, \quad (124)$$

$$\bar{A}_2 = -\frac{9(6-D)}{5(D-2)y^4} + \dots, \quad (125)$$

$$\bar{B}_0 = -\frac{6}{(D-2)y^2} + \frac{9(333D+832)}{260(D-2)^2y^4} + \dots, \quad (126)$$

$$\bar{B}_2 = -\frac{9}{(D-2)y^4} + \dots \quad (127)$$

We note that since (122)-(127) are expansions in $1/y^2$, they are valid for large y values. In contrast, (78)-(82) were calculated assuming small R . Both of these conditions can hold simultaneously, since $R = \varepsilon y$.

Expressions (122)-(127) can also be obtained by looking for the solution of the Fourier mode equations in the $\varepsilon \rightarrow 0$ limit near the pole as a power series expansion in $1/y^2$,

$$\bar{\phi}_{2k+1} = \sum_{j=k+1}^{\infty} \psi_{2k+1}^{(j)} \frac{1}{y^{2j}}, \quad (128)$$

$$\bar{A}_{2k} = \sum_{j=k+1}^{\infty} \alpha_{2k}^{(j)} \frac{1}{y^{2j}}, \quad (129)$$

$$\bar{B}_{2k} = \sum_{j=k+1}^{\infty} \beta_{2k}^{(j)} \frac{1}{y^{2j}}, \quad (130)$$

where $\psi_{2k+1}^{(j)}$, $\alpha_{2k}^{(j)}$ and $\beta_{2k}^{(j)}$ are constants. The mode equations that we have to solve can be obtained from the Einstein equations (24)-(27) and from the wave equation (28) by substituting (108)-(110). Equations (24)-(28) are not independent. The wave equation follows from the Einstein equations by the contracted Bianchi identities, and the (t, r) component (26) is a constraint. The truncation of the Fourier expansion at a finite N_F order makes the mode equations mutually contradictory. However, if we choose any three field equations from (24)-(28), the arising mode equations will clearly have solutions. We have checked that our results for the energy loss rate of oscillatons are the same for different choices of the three field equations. We have also tested that the violation of the mode equations obtained from the other two field equation tends to zero quickly as N_F increases.

Substituting (121) into the field equations (24)-(28) and taking the $\varepsilon \rightarrow 0$ limit close to the pole, some terms with lower powers of r can be neglected. Then, inserting the $1/y^2$ expansion (128)-(130) into the resulting mode equations, because of the omission of odd powers of $1/y$, the only ambiguity arises at the choice of the signature of $\psi_1^{(1)}$. The calculation of (122)-(127) using the Fourier

mode equations is technically more simple than using the small-amplitude expansion method, and can be done by algebraic manipulation programs to quite high orders in $1/y$.

Apart from an overall factor, the leading order behavior of the coefficients $\psi_k^{(n)}$, $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ for large n can be obtained by studying the structure of the mode equations. It turns out that for large n , $\psi_3^{(n)}$ dominates among the coefficients. For the third Fourier mode of the Klein-Gordon field,

$$\psi_3^{(n)} = k_D (-1)^n \frac{(2n-1)!}{8^n} \left[1 + \frac{3(9D-10)}{2(D-2)n} + \frac{3(9D-10)(7D-8)}{2(D-2)^2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \right], \quad (131)$$

where k_D is a factor depending on D and N_F . All other coefficients grow slower with n asymptotically. Although the $1/n$ and $1/n^2$ correction terms may depend on the choice of the scalar potential, the leading order behavior is the same as in (131) for any symmetric potential. The value of the constant k_D will turn out to be crucial for the determination of the energy loss rate of oscillatons. Calculating the coefficients up to order $n = 100$ and taking into account Fourier modes up to order $N_F = 6$, in the Klein-Gordon case we obtain

$$k_3 = -0.301, \quad (132)$$

$$k_4 = -0.134, \quad (133)$$

$$k_5 = -0.0839. \quad (134)$$

C. The singular breather solution near the pole

Expansion (128)-(130) gives an asymptotic series representation of the Fourier components $\bar{\phi}_k$, \bar{A}_k and \bar{B}_k . The results (122)-(127) can be considered as boundary conditions for the Fourier mode equations for

$$|y| \rightarrow \infty, \quad -\pi/2 < \arg y < 0, \quad (135)$$

ensuring a unique solution for the ‘‘inner problem’’. This corresponds to the requirement that ϕ decays to zero without any oscillating tail for $r \rightarrow \infty$ along the positive half of the real axis, i.e. we consider a singular breather solution.

The Fourier components of the wave equation (28) can be written as

$$\frac{d^2 \bar{\phi}_n}{dr^2} + \frac{D-1}{r} \frac{d \bar{\phi}_n}{dr} + (n^2 \omega^2 - 1) \bar{\phi}_n = F_n, \quad (136)$$

where F_n contain nonlinear polynomial terms in $\bar{\phi}_k$, \bar{A}_k , \bar{B}_k and their derivatives for $k \leq N_F$. Using the y coordinate near the pole and taking the $\varepsilon \rightarrow 0$ limit,

$$\frac{d^2 \bar{\phi}_n}{dy^2} + (n^2 - 1) \bar{\phi}_n = \tilde{F}_n, \quad (137)$$

where \tilde{F}_n denotes the $\varepsilon \rightarrow 0$ limit of F_n . On the imaginary axis the $1/y^2$ expansion gives real valued functions to all orders. As singular breather solutions of the mode equations, with boundary conditions (122)-(127) in the region given by (135), the functions $\bar{\phi}_n^{SB}$ can have small imaginary parts on the imaginary axis, satisfying the left hand side of (137) to a good approximation. For symmetric potentials the first radiating component is $\bar{\phi}_3$. The singular breather solution can have an exponentially decaying small imaginary part on the imaginary axis,

$$\text{Im } \bar{\phi}_3^{SB} = \nu_3 \exp(-i\sqrt{8}y) \quad \text{for } \text{Re } y = 0, \quad (138)$$

where ν_3 is some constant. On the other hand, since the quasibreather solution of the mode equations is regular and symmetric, $\bar{\phi}_3^{QB}$ has zero imaginary part on the imaginary axis.

For symmetric potentials the value of ν_3 can be obtained by Borel summing the series (128) for $\bar{\phi}_3$ [69]. The first step is to define a Borel transformed series by

$$V(z) = \sum_{n=2}^{\infty} \frac{\psi_3^{(n)}}{(2n)!} z^{2n}. \quad (139)$$

Then the Laplace transform of $V(z)$ will give us the Borel summed series of $\bar{\phi}_3^{SB}(y)$ which we denote by $\hat{\phi}_3^{SB}(y)$,

$$\hat{\phi}_3^{SB}(y) = \int_0^{\infty} dt e^{-t} V\left(\frac{t}{y}\right). \quad (140)$$

We are only interested in the imaginary part of $\hat{\phi}_3^{SB}(y)$ on the negative imaginary axis, $y = -iy_i$, where $y_i > 0$ real. Then the argument of V is $z = t/y = it/y_i$, which is pure imaginary with positive imaginary part. Since all terms in (139) contain even powers of z , no individual term gives a contribution to $\text{Im } \hat{\phi}_3^{SB}$ on the imaginary axis. The value of $\text{Im } \hat{\phi}_3^{SB}$ is determined there by the leading order large n behavior of the series (139). Using (131) and including a term proportional to z^2 ,

$$V(z) \sim \sum_{n=1}^{\infty} k_D \frac{(-1)^n}{2n} \left(\frac{z}{\sqrt{8}}\right)^{2n} = -\frac{k_D}{2} \ln\left(1 + \frac{z^2}{8}\right), \quad (141)$$

where the sign \sim denotes equality up to terms that do not give contribution to the imaginary part of $\hat{\phi}_3^{SB}$ on the imaginary axis. Transforming the argument of the logarithm into product form, only one of the factors gives a contribution,

$$V(z) \sim -\frac{k_D}{2} \ln\left(1 + \frac{iz}{\sqrt{8}}\right). \quad (142)$$

For purely imaginary y ,

$$V\left(\frac{t}{y}\right) \sim -\frac{k_D}{2} \ln\left(1 - \frac{t}{y_i\sqrt{8}}\right). \quad (143)$$

In this case, for $t > y_i\sqrt{8}$ we have to integrate along the branch cut of the logarithm function. In order to see

how to go around the singularity at $t = y_i\sqrt{8}$ we note that according to (135), the $1/y^2$ expansion (128)-(130) has been applied for $y = y_r - iy_i$, where y_r and y_i are positive and real. This corresponds to the requirement of exponential decay for $r > 0$ along the real r axis. Then

$$iz = \frac{t}{y_r^2 + y_i^2}(-y_i + iy_r), \quad (144)$$

which shows that the argument of the logarithm in (142) has to go around the singularity in the upper half of the complex plane. This means that we approach the branch cut of the logarithm at the negative part of the real axis from above, where its imaginary part is π . Then for purely imaginary y we can evaluate the imaginary part of the integral (140) by integrating on the branch cut,

$$\begin{aligned} \text{Im } \hat{\phi}_3^{SB}(y) &= -\int_{i\sqrt{8}y}^{\infty} dt e^{-t} \frac{k_D\pi}{2} \\ &= -\frac{k_D\pi}{2} \exp(-i\sqrt{8}y). \end{aligned} \quad (145)$$

The logarithmic singularity of $V(t/y)$ does not contribute to the integral. Comparing with (138),

$$\nu_3 = -\frac{1}{2}k_D\pi. \quad (146)$$

For asymmetric potentials the leading order radiating component will be in $\bar{\phi}_2$, and

$$\text{Im } \bar{\phi}_2^{SB} = \nu_2 \exp(-i\sqrt{3}y) \quad \text{for } \text{Re } y = 0. \quad (147)$$

Because the dominant behavior of $\bar{\phi}_0$, it is not possible to determine the constant ν_2 by the Borel summation. Its value can be calculated by numerical integration of the Fourier mode equations, following the method presented in [32] and [15].

It is reassuring that even though we work with a truncated set of mode equations Birkhoff's theorem still holds in the following sense. Neither $\alpha_k^{(n)}$ nor $\beta_k^{(n)}$ has an appropriately singular behavior so that they generate an imaginary correction on the imaginary axis for \bar{A}_k and \bar{B}_k . The Borel summation procedure does not produce gravitational radiation.

D. Construction of the quasibreather

As we have already discussed, both the singular breather (SB), and the quasibreather (QB) solutions are well approximated in a large domain by the small-amplitude expansion. Since the tail is exponentially suppressed in ε , apart from a small central region around $r = 0$, where the SB solution gets too large, the QB and SB solutions are extremely close to each other. We denote the difference in the first radiating Fourier component $\bar{\phi}_3$ of the two solutions by

$$\bar{\phi}_3^w = \bar{\phi}_3^{QB} - \bar{\phi}_3^{SB}. \quad (148)$$

The small function $\bar{\phi}_3^w$ solves the linearization of the wave equation around the singular breather solution. To leading order in ε this reduces to the flat background wave equation

$$\frac{d^2 \bar{\phi}_3^w}{dr^2} + \frac{D-1}{r} \frac{d\bar{\phi}_3^w}{dr} + 8\bar{\phi}_3^w = 0. \quad (149)$$

The general solution of (149) can be written as

$$\bar{\phi}_3^w = \frac{\sqrt[4]{2}\sqrt{\pi}}{r^{D/2-1}} \left[\alpha_D Y_{D/2-1}(\sqrt{8}r) + \beta_D J_{D/2-1}(\sqrt{8}r) \right], \quad (150)$$

where J and Y are Bessel functions of the first and second kinds, and α_D, β_D are constants. The asymptotic behavior of the Bessel functions is

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (151)$$

$$Y_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (152)$$

for $\arg x < \pi$ and $|x| \rightarrow \infty$. The constants α_D and β_D describe the amplitude of the standing wave tails in $\bar{\phi}_3^w$, since for large distances from the center,

$$\bar{\phi}_3^w \approx \frac{1}{r^{(D-1)/2}} \left\{ \alpha_D \sin\left[\sqrt{8}r - (D-1)\frac{\pi}{4}\right] + \beta_D \cos\left[\sqrt{8}r - (D-1)\frac{\pi}{4}\right] \right\}. \quad (153)$$

Since the SB solution is exponentially decaying, this will also be the tail of the QB configuration.

The second term in (150) gives a purely real contribution to $\bar{\phi}_3^w$ on the imaginary axis. However, converting (152) into exponential form,

$$Y_\nu(x) \approx \frac{1}{\sqrt{2\pi x}} \left\{ \exp\left[ix - \frac{i\pi}{4}(2\nu+3)\right] + \exp\left[-ix + \frac{i\pi}{4}(2\nu+3)\right] \right\}, \quad (154)$$

we see that the first term in (150) yields an exponentially behaving imaginary part along the imaginary axis. Close to the upper pole at iQ_D/ε , using the coordinate y defined in (121), to leading order in ε we obtain that

$$\text{Im} \bar{\phi}_3^w = \frac{\alpha_D}{2} \left(\frac{\varepsilon}{Q_D}\right)^{\frac{D-1}{2}} \exp\left(\frac{\sqrt{8}Q_D}{\varepsilon} - i\sqrt{8}y\right), \quad (155)$$

for $\text{Re} y = 0$. Since our aim is to obtain a non-singular QB solution which is symmetric for $r \rightarrow -r$ for real r , (155) must cancel the exponential behavior of $\text{Im} \bar{\phi}_3^{SB}$ given by (138). This fixes the amplitude α_D ,

$$\alpha_D = -2\nu_3 \left(\frac{Q_D}{\varepsilon}\right)^{\frac{D-1}{2}} \exp\left(-\frac{\sqrt{8}Q_D}{\varepsilon}\right). \quad (156)$$

Substituting the value of ν_3 from (146), obtained by the Borel summation,

$$\alpha_D = k_D \pi \left(\frac{Q_D}{\varepsilon}\right)^{\frac{D-1}{2}} \exp\left(-\frac{\sqrt{8}Q_D}{\varepsilon}\right). \quad (157)$$

For any value of β_D the second term in (150) gives a regular symmetric contribution to $\bar{\phi}_3^w$, which does not change the behavior of the imaginary part on the real axis. However, as it is apparent from (153), any nonzero β_D necessarily increases the tail amplitude, and consequently the energy density in the tail as well. Hence, in order to obtain the minimal tail quasibreather, we set

$$\beta_D = 0. \quad (158)$$

The standing wave tail of the quasibreather in the asymptotic region is given by the first term of (150),

$$\begin{aligned} \phi^{QB} &= \sqrt[4]{2}\sqrt{\pi} \frac{\alpha_D}{r^{D/2-1}} Y_{D/2-1}(\sqrt{8}r) \cos(3\tau) \\ &\approx \frac{\alpha_D}{r^{(D-1)/2}} \sin\left[\sqrt{8}r - (D-1)\frac{\pi}{4}\right] \cos(3\tau). \end{aligned} \quad (159)$$

Subtracting the regular solution involving the Bessel function J_ν with a phase shift in time, we cancel the incoming radiating component, and obtain the radiative tail of the oscillaton,

$$\begin{aligned} \phi^{osc} &= \sqrt[4]{2}\sqrt{\pi} \frac{\alpha_D}{r^{D/2-1}} \left[Y_{D/2-1}(\sqrt{8}r) \cos(3\tau) \right. \\ &\quad \left. - J_{D/2-1}(\sqrt{8}r) \sin(3\tau) \right] \\ &\approx \frac{\alpha_D}{r^{(D-1)/2}} \sin\left[\sqrt{8}r - (D-1)\frac{\pi}{4} - 3\tau\right]. \end{aligned} \quad (160)$$

Equations (159) and (160) are valid for symmetric potentials. In both cases, the amplitude of the tail of ϕ at large r is given by α_D . According to (8), the physical amplitude is $\alpha_D/\sqrt{8\pi}$. Since the transformation (11) changes the coordinates, α_D scales as $m^{(1-D)/2}$ with the scalar field mass m .

E. Tail amplitude

The scalar field tail calculated in the previous subsection is so small that it is not surprising that it has not been detected by numerically solving the Fourier mode equations in [1] and [52]. In order to relate the magnitude of the oscillating tail to the central amplitude, we represent ϕ in the core region by $\phi = \varepsilon^2 p_2 \cos \tau$, and in the tail by (159). The tail starts to dominate at a radius $r = r_t$ where

$$\phi(\tau = 0, r) = \varepsilon^2 p_2(\varepsilon r) = \varepsilon^2 S(\varepsilon r) \sqrt{\frac{D-1}{D-2}} \quad (161)$$

equals to $\alpha_D r^{(1-D)/2}$. Since $s \approx -1 + s_1 \rho^{2-D}$ for large ρ , the asymptotic behavior of S in the relevant dimensions

is

$$S_{D=3}(\rho) = S_t e^{-\rho} \rho^{s_1/2-1} \left[1 - \frac{s_1(s_1-2)}{8\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right) \right], \quad (162)$$

$$S_{D=4}(\rho) = S_t \frac{e^{-\rho}}{\rho^{3/2}} \left[1 - \frac{4s_1-3}{8\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right) \right], \quad (163)$$

$$S_{D=5}(\rho) = S_t \frac{e^{-\rho}}{\rho^2} \left[1 + \frac{1}{\rho} - \frac{s_1}{4\rho^2} + \mathcal{O}\left(\frac{1}{\rho^4}\right) \right], \quad (164)$$

where the constants s_1 and S_t are given in Table III. The

	$D=3$	$D=4$	$D=5$
s_1	3.505	7.695	10.40
S_t	3.495	88.24	23.39

TABLE III: The numerical values of the constants s_1 and S_t in 3, 4 and 5 spatial dimensions.

values of r_t and the amplitude of the tail at that radius, $\Phi_t = \alpha_D r_t^{(1-D)/2} / \sqrt{8\pi}$ for several ε are given in Table IV.

ε	$D=3$		$D=4$		$D=5$	
	r_t	Φ_t	r_t	Φ_t	r_t	Φ_t
0.1	1160	$8.96 \cdot 10^{-52}$	648	$2.76 \cdot 10^{-32}$	346	$4.42 \cdot 10^{-20}$
0.2	302	$4.63 \cdot 10^{-27}$	168	$1.06 \cdot 10^{-17}$	92.6	$6.01 \cdot 10^{-12}$
0.3	140	$9.28 \cdot 10^{-19}$	78.2	$9.45 \cdot 10^{-13}$	45.0	$3.83 \cdot 10^{-9}$
0.4	81.6	$1.40 \cdot 10^{-14}$	46.4	$3.07 \cdot 10^{-10}$	27.9	$1.03 \cdot 10^{-7}$
0.5	54.3	$4.68 \cdot 10^{-12}$	31.5	$1.03 \cdot 10^{-8}$	19.8	$7.56 \cdot 10^{-7}$
0.6	39.2	$2.30 \cdot 10^{-10}$	23.2	$1.09 \cdot 10^{-7}$	15.1	$2.87 \cdot 10^{-6}$
0.7	29.9	$3.76 \cdot 10^{-9}$	18.0	$5.91 \cdot 10^{-7}$	12.2	$7.42 \cdot 10^{-6}$
0.8	23.8	$3.08 \cdot 10^{-8}$	14.6	$2.12 \cdot 10^{-6}$	10.3	$1.51 \cdot 10^{-5}$

TABLE IV: The radius r_t where the oscillating tail starts to dominate, and its amplitude Φ_t there.

The tail amplitude should be compared to the central amplitude

$$\Phi_c = \varepsilon^2 \Phi_{1c}, \quad \Phi_{1c} = \frac{S_c}{\sqrt{8\pi}} \sqrt{\frac{D-1}{D-2}}, \quad (165)$$

where the constants S_c and Φ_{1c} are given in Table V. The radius r_t where the tail starts to dominate is much

	$D=3$	$D=4$	$D=5$
S_c	1.021	3.542	14.02
Φ_{1c}	0.288	0.865	3.229
ρ_h	2.218	1.357	0.763

TABLE V: The numerical values of the constants S_c , Φ_{1c} and ρ_h , which determine the central amplitude Φ_c and the characteristic size r_h .

larger than the characteristic radius of the core, which

can be defined as the radius r_h where $\Phi = \Phi_c/2$. Clearly, $r_h = \rho_h/\varepsilon$, where ρ_h is the value of ρ for which $S = S_c/2$. The value of ρ_h for various spatial dimensions D is also given in Table V. Clearly, there is only some chance to numerically observe the tail for as large ε values as 0.5, which for $D=3$ is close to the maximum value $\varepsilon_{\max} \approx 0.525$. It can also be observed from Table IV, that for larger spatial dimensions the radiation is significantly stronger. It would be reasonable to first make the numerical analysis for $D=5$, since then the tail has the largest amplitude. Even if the Klein-Gordon oscillations are unstable in $D=5$, as we have seen in Section IV B, there are scalar potentials, for which large amplitude oscillations are stable. Since the exponent in (157) is potential independent, in general, we expect to get a tail amplitude of similar magnitude as for the Klein-Gordon field.

F. Mass loss rate

Since at large distances from the center the mass function \hat{m} agrees with the total mass M , the mass change rate of the oscillaton can be calculated from the energy current carried by the wave (160) using (22). Averaging for an oscillation period,

$$\frac{dM}{dt} = -\frac{c_1}{m^{D-3}\varepsilon^{D-1}} \exp\left(-\frac{c_2}{\varepsilon}\right), \quad (166)$$

where the D dependent constants are

$$c_1 = 3\sqrt{2}k_D^2 Q_D^{D-1} \frac{\pi^{D/2+1}}{4\Gamma\left(\frac{D}{2}\right)}, \quad c_2 = 2\sqrt{8}Q_D. \quad (167)$$

The numerical values of c_1 and c_2 for various spatial dimensions are listed in Table VI. The values of c_2 are the

	$D=3$	$D=4$	$D=5$
c_1	30.0	7.23	0.720
c_2	22.4993	13.0372	6.99159

TABLE VI: The constants c_1 and c_2 in the mass loss rate expression (166) for $D=3, 4, 5$ spatial dimensions.

same for any symmetric potential, but the numbers given for c_1 are valid only for the Klein-Gordon field.

The higher ε is, the more chance we have to observe the presumably tiny energy loss. As we have seen in subsection IV B, for $D=3$ spatial dimensions oscillons are stable for $\varepsilon < \varepsilon_{\max} \approx 0.525$. The total mass is maximal at ε_{\max} . Restoring the scalar field mass m into the expressions, the maximal mass value is $M_{\max} = 0.614/m$. Substituting into (166), for the maximal mass Klein-Gordon oscillaton we get

$$\left(\frac{1}{M} \frac{dM}{dt}\right)_{M=M_{\max}} = -4.3 \cdot 10^{-17} m. \quad (168)$$

This expression is valid in Planck units. Expressing M in kilograms, and mc^2 in electron volts the maximal oscillaton mass is

$$M_{\max} = 0.614 m_P E_P / m = 1.63 \cdot 10^{20} kg \frac{eV}{mc^2}, \quad (169)$$

where the Planck mass is $m_P = 2.18 \cdot 10^{-8} kg$ and the Planck energy is $E_P = 1.22 \cdot 10^{28} eV$. Expressing t in seconds, since $t_p = 5.39 \cdot 10^{-44} s$, for the maximal mass configuration we get

$$\left(\frac{1}{M} \frac{dM}{dt} \right)_{M=M_{\max}} = -\frac{0.066}{s} \frac{mc^2}{eV}. \quad (170)$$

Note that we only determined the leading order result for the radiation amplitude. We saw hints that the small-amplitude results give sensible answers for moderate ε values, however we do not have a good control over non-leading terms in the radiation amplitude. In general for such high ε values we expect to have the same exponential factor, but with a different prefactor c_1 [15, 16, 25]. The order of magnitude should be nevertheless correct.

According to (96), for $D = 3$ spatial dimensions, to leading order the total mass is proportional to the amplitude. $M = \varepsilon M^{(1)} / m$, where from Table I, $M^{(1)} = 1.75266$. Substituting into (166), for the Klein-Gordon case this yields

$$\frac{dM}{dt} = -\frac{c_3}{m^2 M^2} \exp\left(-\frac{c_4}{mM}\right). \quad (171)$$

where

$$c_3 = 92.2, \quad c_4 = 39.4337. \quad (172)$$

Expression (171) has the same form as the classical mass loss formula (122) of [31], although the constant corresponding to c_3 is much larger there, it is 3797437.776. This means that the amplitude of the radiating tail of the scalar field Φ is overestimated by a factor 202.9 in [31]. In order to understand the reason for this large difference, and why it is necessary to use our more complicated approach to obtain a correct mass loss rate, we first describe the method of [31] in our formalism.

For a $D = 3$ Klein-Gordon system let us consider the third Fourier component (136) of the wave equation (28) in the $\omega \rightarrow 1$ limit. Taking the results (115)-(120) of the small-amplitude expansion, and substituting into the nonlinear terms on right hand side of (136), we obtain an inhomogeneous linear differential equation for $\bar{\phi}_3$,

$$\frac{d^2 \bar{\phi}_3}{dr^2} + \frac{2}{r} \frac{d\bar{\phi}_3}{dr} + 8\bar{\phi}_3 = P(r). \quad (173)$$

Here the function $P(r)$ is given by the small-amplitude expansion, in a power series form in ε ,

$$P(r) = \sum_{k=3}^{\infty} P_{2k}(r) \varepsilon^{2k}. \quad (174)$$

For the $D = 3$ Klein-Gordon system the leading order term is

$$P_6(r) = \frac{3}{16} p_2^3(\varepsilon r) + p_2 a_4^{(2)}(\varepsilon r). \quad (175)$$

Equation (173) with $P(r) = \varepsilon^6 P_6(r)$, but setting $a_4^{(2)} = 0$, corresponds to equation (52) of [31]. Since there the Fourier modes are defined in terms of exponentials instead of cosine functions, the coefficient of the p_2^3 term is 3/4 in [31] instead of 3/16. The term containing $a_4^{(2)}$ is missing there because of the assumption that the $g_{tt} = -A$ metric component is time independent. However, at ε^4 order either g_{tt} becomes oscillatory, or the spatial metric ceases to be conformally flat.

The oscillating tail responsible for the radiation loss in the $\bar{\phi}_3$ mode can be estimated by integrating (173) using the Green function method,

$$\begin{aligned} \bar{\phi}_3(r) &= \frac{\cos(\sqrt{8}r)}{\sqrt{8}r} \int_0^r \bar{r} \sin(\sqrt{8}\bar{r}) P(\bar{r}) d\bar{r} \\ &+ \frac{\sin(\sqrt{8}r)}{\sqrt{8}r} \int_r^\infty \bar{r} \cos(\sqrt{8}\bar{r}) P(\bar{r}) d\bar{r}. \end{aligned} \quad (176)$$

The oscillaton core is exponentially localized, hence in the tail region it is a very good approximation to write

$$\bar{\phi}_3(r) = \bar{\alpha} \frac{\cos(\sqrt{8}r)}{r}, \quad (177)$$

where the constant determining the amplitude is

$$\bar{\alpha} = \frac{1}{\sqrt{8}} \int_0^\infty r \sin(\sqrt{8}r) P(r) dr. \quad (178)$$

Since the functions describing the oscillaton by the ε expansion are symmetric around $r = 0$, and since the sine function can be written as the difference of two exponentials,

$$\bar{\alpha} = \frac{1}{2i\sqrt{8}} \int_{-\infty}^\infty r \exp(\sqrt{8}ir) P(r) dr. \quad (179)$$

The functions in the small-amplitude expansion depend directly on the rescaled radial coordinate $\rho = \varepsilon r$, so it is natural to write the integral into the form

$$\bar{\alpha} = \frac{1}{2i\sqrt{8}\varepsilon^2} \int_{-\infty}^\infty \rho \exp\left(\frac{\sqrt{8}i\rho}{\varepsilon}\right) P\left(\frac{\rho}{\varepsilon}\right) d\rho. \quad (180)$$

This can be replaced by a contour integral around the upper plane, and can be approximated by taking into account the pole which is closest to the real axis. The position of the closest pole is the same as that of the SN equations (45) and (46), it is at $\rho = iQ_3$ on the imaginary axis. Let us first calculate the contribution from the leading P_6 term, given by (175). Then, since for three spatial dimensions $p_2 = \sqrt{2}S$, using (78) and (82), the behavior near the pole is

$$P\left(\frac{\rho}{\varepsilon}\right) \approx \varepsilon^6 P_6\left(\frac{\rho}{\varepsilon}\right) \approx -\frac{3^5 \sqrt{2} \varepsilon^6}{5R^6}, \quad (181)$$

where $R = iQ_3 - \rho$. The omission of the term containing $a_4^{(2)}$ from (175) results in a value which is 5/3 times that of (181). The residue can be calculated by integrating by parts five times,

$$\bar{\alpha} = -\frac{2^3 3^4 \sqrt{2} \pi Q_3}{5^2 \varepsilon} \exp\left(-\frac{\sqrt{8} Q_3}{\varepsilon}\right). \quad (182)$$

This amplitude is much larger than the amplitude α_3 calculated by the Borel summation method in (157),

$$\frac{\bar{\alpha}}{\alpha_3} = -\frac{2^4 3^4 \sqrt{2}}{5^2 k_3} \approx 121.8. \quad (183)$$

If we omit the term containing $a_4^{(2)}$ from (175), we have a 5/3 factor, and get $\bar{\alpha}/\alpha_3 = 202.9$, which is the ratio of the tail amplitude of [31] to our value, as we have mentioned after Eq. (172).

The fundamental problem with the above calculated tail amplitude $\bar{\alpha}$ is that it is just the first term of an infinite series, of which all terms give contributions which are the same order in ε . This can be illustrated by calculating the contribution of the next term in $P(r)$. Although the small-amplitude expansion yields a rather complicated expression for $P_8(r)$, it contains terms proportional to $p_2^3 a_2$, $p_4 a_4^{(2)}$ and $p_2 a_2 a_4^{(2)}$, which have eighth order poles. When calculating the integral (180) one has to integrate by parts seven times, so the result will have the same ε order as the earlier result calculated from the $P_6(r)$ term. Even if we could calculate higher order contributions, we have no reason to expect that the series converges, and even if it would be convergent it may not give a correct result for the mass loss. This has already been demonstrated for the simpler system of a real scalar field with a nontrivial interaction potential on flat Minkowski background. It was first pointed out in [70] that there are too many boundary conditions to satisfy when solving the Fourier mode equations in order to find periodic localized breather solutions. In [70] the energy loss rate of the long living oscillon configurations was estimated by a method analogous to that of [31]. However, calculating higher order contributions, it turned out that the method gives an incorrect nonzero result even for the periodic sine-Gordon breather. Moreover, the expansion is not convergent for the ϕ^4 scalar theory. The proper approach to calculate the lifetime of oscillons has been worked out by [32] and [69], using complex extension and Borel summation. As a result of the above arguments, the correct value of c_3 in the mass loss rate expression (171) for the $D = 3$ Klein-Gordon field is $c_3 = 92.2$.

Although the expression (171) is correct for small values of M , since it is based on the assumption that M depends linearly on ε , one should not apply it to mass values close to M_{\max} . For example, substituting the value of the maximal mass $M_{\max} = 0.614/m$ into (171), we obtain

$$\left(\frac{1}{M} \frac{dM}{dt}\right)_{M=M_{\max}} = -5.0 \cdot 10^{-26} m, \quad (184)$$

which is 9 magnitudes smaller than the maximal mass loss rate obtained in (168). The reason for this huge difference is that according to the linear expression $M = \varepsilon M^{(1)}$, to the mass value $M_{\max} = 0.614/m$ belongs an ε value of 0.350. At that ε we obviously get a significantly lower radiation than at $\varepsilon_{\max} \approx 0.525$, because of the exponential dependence. Since expression (168) does not involve this approximation, we expect it to give a more reliable result.

G. Time dependence

Instead of using (171) to determine the time dependence of the oscillaton mass, in order to obtain results that are valid for larger mass values, we work out a method involving a higher order approximation for the ε dependence of the mass. Since the first two terms of (96) determine the mass maximum to a good precision, we expect it to be a reasonable approximation for close to maximal ε values. Including the scalar field mass m , we use

$$M = \varepsilon^{4-D} m^{2-D} \left(M^{(1)} + \varepsilon^2 M^{(2)} \right). \quad (185)$$

Taking the time derivative and comparing with (166),

$$\frac{dt}{d\varepsilon} = -\frac{\varepsilon^2}{m} (\beta_1 + \beta_2 \varepsilon^2) \exp\left(\frac{c_2}{\varepsilon}\right), \quad (186)$$

where

$$\beta_1 = \frac{4-D}{c_1} M^{(1)}, \quad \beta_2 = \frac{6-D}{c_1} M^{(2)}. \quad (187)$$

This can be integrated in terms of the exponential integral function,

$$\begin{aligned} t - t_0 = & -\frac{\varepsilon}{120m} [20\beta_1 (c_2^2 + c_2\varepsilon + 2\varepsilon^2) \\ & + \beta_2 (c_2^4 + c_2^3\varepsilon + 2c_2^2\varepsilon^2 + 6c_2\varepsilon^3 + 24\varepsilon^4)] \exp\left(\frac{c_2}{\varepsilon}\right) \\ & + \frac{c_2^3}{120m} (20\beta_1 + \beta_2 c_2^2) \text{Ei}\left(\frac{c_2}{\varepsilon}\right). \end{aligned} \quad (188)$$

Taking the expansion of the result, for small ε ,

$$\begin{aligned} t - t_0 = & \frac{\varepsilon^4}{m} \left[\frac{\beta_1}{c_2} + \frac{4\beta_1}{c_2^2} \varepsilon \right. \\ & \left. + \frac{20\beta_1 + \beta_2 c_2^2}{c_2^3} \left(\varepsilon^2 + \frac{6\varepsilon^3}{c_2} + \mathcal{O}(\varepsilon^4) \right) \right] \exp\left(\frac{c_2}{\varepsilon}\right). \end{aligned} \quad (189)$$

Although the correction from the subleading term $M^{(2)}$ only appears in the third term in the bracket, its influence for $\varepsilon \approx 0.5$ is not negligible. It can be also seen that for $D = 4$ we have $\beta_1 = 0$, and (189) starts with an ε^6 term. The elapsed time as a function of the oscillaton mass can be obtained by expressing ε from (185) and substituting into (189).

For $D = 3$ spatial dimensions it is natural to start with a maximal mass configuration $M = M_{\max}$, and wait for the mass to decrease until the ratio M/M_{\max} reaches a given value. Since the elapsed time t is inversely proportional to the scalar field mass m , in table VII we list the product tm .

$\frac{M_{\max}-M}{M_{\max}}$	ε	tm	$\frac{t}{\text{year}} \frac{mc^2}{eV}$
0.01	0.482	$5.35 \cdot 10^{16}$	$1.12 \cdot 10^{-6}$
0.1	0.383	$3.00 \cdot 10^{21}$	$6.26 \cdot 10^{-2}$
0.2	0.320	$1.50 \cdot 10^{26}$	$3.12 \cdot 10^3$
0.3	0.269	$4.42 \cdot 10^{31}$	$9.22 \cdot 10^8$
0.31884	0.260	$6.57 \cdot 10^{32}$	$1.37 \cdot 10^{10}$
0.4	0.224	$3.99 \cdot 10^{38}$	$8.32 \cdot 10^{15}$
0.5	0.182	$1.22 \cdot 10^{48}$	$2.55 \cdot 10^{25}$
0.6	0.144	$1.28 \cdot 10^{62}$	$2.67 \cdot 10^{39}$
0.7	0.107	$1.94 \cdot 10^{85}$	$4.04 \cdot 10^{62}$

TABLE VII: The time necessary for the oscillaton mass to decrease to M from the value M_{\max} at $t = 0$. The value of tm is given in Planck units, and also when the time is measured in years and the scalar mass in electron volts.

Next we address the question that how much of its mass an initially maximal mass oscillaton loses during the age of the universe, which we take to be $1.37 \cdot 10^{10}$ years. In Table VIII we list the resulting oscillaton masses in units of solar masses (M_{\odot}), as a function of the scalar field mass in eV/c^2 units. In order to facilitate com-

$\frac{mc^2}{eV}$	$\varepsilon_{\max} - \varepsilon$	$\frac{M}{M_{\odot}}$	$\frac{M_{\max}-M}{M_{\max}}$
10^{-35}	$5.09 \cdot 10^{-20}$	$8.20 \cdot 10^{24}$	$1.41 \cdot 10^{-38}$
10^{-30}	$5.09 \cdot 10^{-15}$	$8.20 \cdot 10^{19}$	$1.41 \cdot 10^{-28}$
10^{-25}	$5.09 \cdot 10^{-10}$	$8.20 \cdot 10^{14}$	$1.41 \cdot 10^{-18}$
10^{-20}	$5.08 \cdot 10^{-5}$	$8.20 \cdot 10^9$	$1.40 \cdot 10^{-8}$
10^{-15}	0.0704	$7.99 \cdot 10^4$	0.0258
10^{-10}	0.163	$7.14 \cdot 10^{-1}$	0.129
10^{-5}	0.223	$6.30 \cdot 10^{-6}$	0.232
1	0.266	$5.58 \cdot 10^{-11}$	0.319
10^5	0.297	$5.00 \cdot 10^{-16}$	0.390
10^{10}	0.322	$4.52 \cdot 10^{-21}$	0.449
10^{15}	0.342	$4.12 \cdot 10^{-26}$	0.498

TABLE VIII: Mass M of an initially maximal mass oscillaton after a period corresponding to the age of the universe for various scalar field masses. The decrease in ε from $\varepsilon_{\max} = 0.525$, and the relative mass change rate $(M_{\max} - M)/M_{\max}$ is also given.

parison, we have chosen the same scalar field masses as in Eq. (178) of [31]. The first two orders of the small-amplitude expansion yielded $mM_{\max} = 0.614$ in Planck

units for the maximal mass of the oscillaton. Taking the scalar mass in electron volts, this corresponds to $M_{\max} = 8.20 \cdot 10^{-11} M_{\odot} eV/(mc^2)$, which was used in Table VIII. The value $mM_{\max} = 0.607$ from the numerical solution of the Fourier mode equations calculated in [51] corresponds to $M_{\max} = 8.11 \cdot 10^{-11} M_{\odot} eV/(mc^2)$ in natural units, which is the value used in [31]. Comparing our Table VIII to the numbers in (178) of [31], after compensating for the shift in the initial mass, it is apparent, that for small scalar field masses, i.e. for $m \leq 10^{-10} eV/c^2$, oscillatons decay more slowly in [31]. The reason for this is that [31] uses a linear dependence of the mass on the small parameter, and consequently underestimates the radiation rate close to the maximum mass, similarly as we did in (184). For $m \geq 10^{-5} eV/c^2$ oscillatons radiate faster in [31], which is a consequence of the much larger value of the constant c_3 in the mass loss law (171) used there. In spite of the differences, the overall picture remains essentially the same. For all scalar field masses that appear physically reasonable, a maximal mass oscillaton loses a significant part of its mass during the lifetime of the universe. This mass decrease is greater than 10% if $m > 4.57 \cdot 10^{-12} eV/c^2$, but it remains below 50% if $m < 1.85 \cdot 10^{15} eV/c^2$. The above results support the possibility that provided a scalar field exist in Nature, at least some of the dark matter content of our Universe would be in the form of oscillatons.

VI. CONCLUSIONS

We have derived an infinite set of radial ODEs determining the spatial field profiles of bounded solutions of time-dependent, spherically symmetric Einstein-scalar field equations in the limit when the scalar field amplitude tends to zero. The lowest order equations are nothing but the D-dimensional generalization of the Schrödinger-Newton (SN) eqs. The SN eqs. admit globally regular, exponentially decreasing solutions for spatial dimensions $2 < D < 6$. The eqs. corresponding to higher orders in the expansion are linear inhomogenous ODEs. The class of solutions we are interested in are oscillatons, which loose slowly their mass by scalar radiation. In the small-amplitude expansion we have obtained an asymptotic series for the spatially well localized core of oscillatons and related their radiation amplitude to that of the standing wave tail of exactly time-periodic quasibreathers. For the class of symmetric scalar potentials we have determined the amplitude of the standing wave tail of time-periodic quasibreathers analytically adapting the method of Segur-Kruskal and using Borel summation. We have explicitly computed the mass loss rate for the Einstein-Klein-Gordon system in $D = 3, 4, 5$.

Appendix A: Einstein tensor and the wave equation

The components of the Einstein tensor in the general spherically symmetric coordinate system (12) are

$$G_{tt} = (D-1) \left\{ \frac{B,t C,t}{4BC} - \frac{A}{4C,r} \left[\frac{(C,r)^2}{BC} \right]_{,r} + (D-2) \frac{A}{2C} \left[1 + \frac{(C,t)^2}{4AC} - \frac{(C,r)^2}{4BC} \right] \right\}, \quad (\text{A1})$$

$$G_{rr} = (D-1) \left\{ \frac{A,r C,r}{4AC} - \frac{B}{4C,t} \left[\frac{(C,t)^2}{AC} \right]_{,t} - (D-2) \frac{B}{2C} \left[1 + \frac{(C,t)^2}{4AC} - \frac{(C,r)^2}{4BC} \right] \right\}, \quad (\text{A2})$$

$$G_{tr} = -(D-1) \left[\frac{A}{4\sqrt{C}} \left(\frac{C,t}{A\sqrt{C}} \right)_{,r} + \frac{B}{4\sqrt{C}} \left(\frac{C,r}{B\sqrt{C}} \right)_{,t} \right], \quad (\text{A3})$$

$$G_{\theta_1\theta_1} = \frac{C}{4A,r} \left[\frac{(A,r)^2}{AB} \right]_{,r} - \frac{C}{4B,t} \left[\frac{(B,t)^2}{AB} \right]_{,t} - (D-2) \left\{ 1 + \frac{1}{4BC,t} \left[\frac{B(C,t)^2}{A} \right]_{,t} \right. \\ \left. - \frac{1}{4AC,r} \left[\frac{A(C,r)^2}{B} \right]_{,r} + \frac{1}{2}(D-5) \left[1 + \frac{(C,t)^2}{4AC} - \frac{(C,r)^2}{4BC} \right] \right\}, \quad (\text{A4})$$

$$G_{\theta_n\theta_n} = G_{\theta_1\theta_1} \prod_{k=1}^{n-1} \sin^2 \theta_k. \quad (\text{A5})$$

The wave equation (4) takes the form

$$\frac{\phi_{,rr}}{B} - \frac{\phi_{,tt}}{A} + \frac{\phi_{,r}}{2AC^{D-1}} \left(\frac{AC^{D-1}}{B} \right)_{,r} - \frac{\phi_{,t}}{2BC^{D-1}} \left(\frac{BC^{D-1}}{A} \right)_{,t} - \bar{U}'(\phi) = 0. \quad (\text{A6})$$

Appendix B: Small-amplitude expansion in Schwarzschild coordinates

In the main part of the paper we have used the spatially conformally flat coordinate system $C = r^2 B$. In this appendix we present the results of the ε expansion in $C = r^2$ Schwarzschild area coordinates, in order to compare and to point out the disadvantages. The time dependence of the scalar field ϕ and the metric components A and B up to ε^2 order are

$$\phi = \varepsilon^2 p_2 \cos \tau + \mathcal{O}(\varepsilon^4), \quad (\text{B1})$$

$$A = 1 + \varepsilon^2 a_2 + \varepsilon^2 a_2^{(2)} \cos(2\tau) + \mathcal{O}(\varepsilon^4), \quad (\text{B2})$$

$$B = 1 + \varepsilon^2 b_2 + \mathcal{O}(\varepsilon^4), \quad (\text{B3})$$

where p_2 , a_2 , $a_2^{(2)}$ and b_2 are functions of ρ . The functions a_2 and p_2 are again determined by the coupled differential equations (42) and (43), resulting in the Schrödinger-Newton equations. However, b_2 is determined as

$$b_2 = \frac{\rho}{D-2} \frac{da_2}{d\rho}, \quad (\text{B4})$$

instead of (41). The most important difference is the appearance of the $\cos(2\tau)$ term in (B2), causing an ε^2 or-

der oscillation in the metric component g_{tt} . In spatially conformally flat coordinates there are only ε^4 order oscillating terms in the metric components. The amplitude of the oscillation is determined by the field equations as

$$a_2^{(2)} = -a_2 - b_2. \quad (\text{B5})$$

Substituting into the expression (72) of the magnitude of the acceleration of constant $(r, \theta_1, \theta_2, \dots)$ observers, to leading order we get

$$a = \frac{\varepsilon^3}{2} \left(\frac{da_2}{d\rho} + \frac{da_2^{(2)}}{d\rho} \cos(2\tau) \right). \quad (\text{B6})$$

Acknowledgments

This research has been supported by OTKA Grants No. K61636, NI68228, and by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FG 0205ER41360.

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