Non-perturbative construction of counterterms for 2PI-approximation

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Abstract

A concise method is presented for the non-perturbative computation of the counterterms renormalising 2PI-actions. The procedure is presented for a real scalar field up to $\mathcal{O}(\lambda^2)$ order in the skeleton truncation of Γ_{2PI} with respect to the self-coupling, and in a constant symmetry breaking background. The method is easily generalizable to field theories with arbitrary global symmetry.

The aim of this contribution is to provide practical tools for the renormalisation program of the 2PI-approximate treatment of quantum field theories. While our approach reproduces the results of previous investigations [1,2,3] it allows in a different scheme explicit determination of the counterterms. In a broad sense the proposed method is close to the scheme of minimal subtraction. Its detailed discussion appears in Refs. [4,5]. Related results were presented by Refs. [6,7] and also at the conference SEWM 2008.

1. Generic 2PI-equations

The 2PI equations for the propagator $G(p)$ and the constant field expectation v of a self-interacting real scalar field adequately represent the generic structures appearing in theories of arbitrary global symmetry:

$$2\frac{\delta\Gamma_{2PI}}{\delta G(p)} = iG^{-1}(p) - \left[(1+\delta Z)p^2 - m^2 - \delta m_2^2 - \frac{\lambda+\delta\lambda_2}{2}v^2 - 2\frac{\delta\Gamma_2[G,v]}{\delta G(p)}\right] = 0,$$

$$\frac{\delta\Gamma_{2PI}}{\delta v} = v\left[m + \delta m_0^2 + \frac{\lambda+\delta\lambda_4}{6}v^2 + \frac{\lambda+\delta\lambda_2}{2}\int_p G(p) + 2\frac{\delta\Gamma_2[G,v]}{\delta v^2}\right] = 0. \qquad (1)$$

At finite truncation of the set of two-particle irreducible skeleton diagrams $\Gamma_2[G,v]$ different counterterms are allowed for each operator compatible with the symmetry of the model [3]. At $\mathcal{O}(\lambda^2)$ skeleton truncation one has: $\delta m_0^2, \delta m_2^2, \delta\lambda_0, \delta\lambda_2, \delta\lambda_4$, and δZ.

Method of renormalisation. In both equations one first identifies divergent coefficients of v^0, v^2 and of the finite environment dependent function $T_F[G] = \int_q G(q)\big|_{\text{finite}}$. Next, by requiring separate cancellation of these divergent coefficients one finds the counterterms. In particular, the vanishing of the divergent pieces $\sim v^0$ determine δm_i^2, that of v^2 determine the counterterms not related to $\Gamma_2[G,v]$ (*i.e.* $\delta\lambda_2, \delta\lambda_4$). Finally, the conditions imposed on coefficients of $T_F[G]$ determine counterterms defined in $\Gamma_2[G,v]$ (*i.e.* $\delta\lambda_0$, see Eq. (4)).

Here we note that the method does not include discussion of overall and sub-divergences which is an essential part of the iterative renormalisation method. On the other hand, the consistency of this renormalisation method must be checked explicitly, either analytically or numerically, since the number of divergence cancellation conditions is larger than the number of counterterms.

Structure of the renormalised propagator. The expression of the self-consistent propagator contains the self-energy split into two pieces:

$$iG^{-1}(p) = p^2 - M^2 - \Pi(p), \qquad\qquad \Pi(p) = \Pi_a(p) + \Pi_0(p) + \Pi_r(p), \qquad (2)$$

where $M^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{\lambda}{2}T_F[G]$ is the momentum independent local part, and the momentum dependent (non-local) part $\Pi(p)$ itself is further decomposed into pieces displaying different asymptotics for large p: $\Pi_a(p) \sim p^2(\ln p)^{c_1}, \Pi_0(p) \sim (\ln p)^{c_2}, \Pi_r(p) \sim p^{-2}$.

The divergences are separated with help of an *auxiliary propagator* $G_a(p)$ which has the same asymptotic behaviour as $G(p)$ and invokes also an arbitrarily chosen IR regulator M_0^2 in its definition:

$$iG_a^{-1}(p) = p^2 - M_0^2 - \Pi_a(p). \qquad (3)$$

2. 2-loop truncation of the effective action: $\Gamma_2^{(2l)}[G,v]$

Renormalisation of this truncation is simpler and still might have several physically interesting applications [8,9]. The contribution of the following 2PI diagrams [10]

$$\Gamma_2^{(2l)} = \frac{\lambda+\delta\lambda_0}{8}\bigcirc\!\!\!\bigcirc + \frac{\lambda^2}{12}v^2\bigcirc\!\!\!\!\bigcirc \qquad (4)$$

to the propagator and field equations is

$$2\frac{\delta\Gamma_2^{(2l)}[G,v]}{\delta G(p)} = \frac{1}{2}(\lambda+\delta\lambda_0)T[G] + \frac{1}{2}\lambda^2v^2 I(p,G), \qquad 2\frac{\delta\Gamma_2^{(2l)}[G,v]}{\delta v^2} = \frac{\lambda^2}{6}S(0,G), (5)$$

2

where we define $I(p,G) = -i \int_q G(q)G(p+q)$, $S(0,G) = \int_p G(p)I(p,G)$.

In the present case the form of the auxiliary propagator is simple, since the leading asymptotics of the self-consistent propagator is unchanged: $iG_a^{-1} = p^2 - M_0^2$, ($\Pi_a \equiv 0$). The separation of divergences is achieved by expanding the self-consistent propagator $G(p)$ around $G_a(p)$:

$$I_{\text{div}}^{(2l)}(p,G) = T_a^{(0)} := \bigcirc\Big|_{\text{div}}, \quad T_{\text{div}}^{(2l)}[G] = T_a^{(2)} + (M^2 - M_0^2)T_a^{(0)} + \frac{\lambda^2}{2}v^2 T_a^{(I)}, \ (6)$$

$$S_{\text{div}}^{(2l)}(0,G) = \bigcirc\Big|_{\text{div}} + \frac{3}{2}\lambda^2 v^2 \left[T_a^{(0)}T_a^{(I)} + T_a^{(I,2)}\right]$$

$$+ T_a^{(0)}T_F + 3(M^2 - M_0^2)\left[\left(T_a^{(0)}\right)^2 + T_a^{(I)}\right]. \tag{7}$$

The lines in the diagrams refer to G_a with the following graphical notations (the "bubble" with letter 'F' inside denotes the finite part of the auxiliary bubble integral):

$$T_a^{(2)} = \bigcirc\Big|_{\text{div}}, \quad T_a^{(I)} = \bigcirc\Big|_{\text{div}}, \quad T_a^{(I,2)} = \bigcirc\Big|_{\text{div}}. \tag{8}$$

Since M^2 includes $T_F[G]$, its presence leads to the danger of *environment dependent divergences* in $T_{\text{div}}^{(2l)}[G]$ and $S_{\text{div}}^{(2l)}(0,G)$. The conditions for the cancellation of such divergences in the propagator and the field equation are, respectively:

$$\delta\lambda_0 + \frac{\lambda}{2}(\lambda + \delta\lambda_0)T_a^{(0)} = 0, \quad \delta\lambda_2 + \frac{\lambda}{2}(3\lambda + \delta\lambda_2)T_a^{(0)} + \frac{\lambda^3}{2}\left[\left(T_a^{(0)}\right)^2 + T_a^{(I)}\right] = 0. \ (9)$$

On the other hand, the v^2-dependent divergence cancellation in the propagator gives a relation between $\delta\lambda_0$ and $\delta\lambda_2$:

$$\delta\lambda_2 + \frac{1}{2}\lambda(\lambda + \delta\lambda_0)\left(T_a^{(0)} + \lambda T_a^{(I)}\right) = 0. \tag{10}$$

By the previously determined $\delta\lambda_0$ and $\delta\lambda_2$ this *consistency relation* is satisfied.

In turn the cancellation of the further two v^0-dependent plus one v^2-dependent divergences determines δm_0^2, δm_2^2, and $\delta\lambda_4$.

3. Adding the basket-ball to the effective action

Now one completes the 2PI-part of the action with the "basket-ball" diagram:

$$\Gamma_2^{\lambda^2}[G,v] = \Gamma_2^{(2l)}[G,v] + \frac{\lambda^2}{48}\bigoplus. \tag{11}$$

The form of the equation of state remains unchanged, while the self-energy receives the contribution: $\Pi_2(p) = \frac{\lambda^2}{6}S_F(p,G)$, which results in $\Pi_a(p) \equiv \Pi_{2,a}(p) \neq 0$ term.

The analysis reveals, for instance, the following form for the divergent part of the tadpole integral:

$$T_{\text{div}}[G] = T_{\text{div}}^{(2l)}[G] - i\int_k G_a^2(k)\Pi_{2,0}(k). \tag{12}$$

3

Similar (but more complicated) expression is derived for $S_{\mathrm{div}}(p, G)$. Therefore, the crucial object to be found for the explicit counterterm construction is the subleading asymptotic piece: $\Pi_{2,0}$, which is the logarithmic part of $\Pi_2(p)$. An integral representation can be derived for it:

$$\Pi_{2,0}(p) = -\frac{i}{2} \int_k G_a^2(k) K(p,k) \left[M^2 - M_0^2 + \frac{\lambda^2}{2} v^2 I_a^F(k) + \Pi_{2,0}(p) \right], \tag{13}$$

with the kernel defined with help of the renormalised bubble-integrals [5]:

$$K(p,k) = \frac{\lambda^2}{2} \left[I_a^F(k+p) + I_a^F(k-p) - 2I_a^F(k) \right]. \tag{14}$$

Solving (13) for $\Pi_{2,0}(p)$ one finds a linear combination of $M^2 - M_0^2$ and $\lambda^2 v^2$:

$$\Pi_{2,0}(p) = \frac{1}{2}(M^2 - M_0^2)\Gamma_0(p) + \frac{1}{4}\lambda^2 v^2 \Gamma_1(p), \tag{15}$$

where $\Gamma_0(p) = -i \int_k \Gamma(p,k) G_a^2(k), \Gamma_1(p) = -i \int_k \Gamma(p,k) G_a^2(k) I_a^F(k)$. The kernel which determines the coefficient functions $\Gamma_0(p), \Gamma_1(p)$ satisfies a finite Bethe–Salpeter-type equation:

$$\Gamma(p,k) = K(p,k) - \frac{i}{2} \int_q G_a^2(q) K(p,q) \Gamma(q,k). \tag{16}$$

This result demonstrates that by adding the basket-ball the types of the occurring divergent coefficients do not change (e.g. $\sim v^0, v^2, T_F$). The same procedure, as described for the 2-loop truncation allows also the construction for the counterterms up to $\mathcal{O}(\lambda^2)$ accuracy. The verification of the consistency of the counterterm determination with the redundant conditions becomes more cumbersome. Applicability of this renormalisation procedure together with the analytical check of some of the consistency relations to multicomponent scalar models was illustrated on the $O(N)$ model in Ref. [5].

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