# Classification of $k$-nets 

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#### Abstract

A finite $k$-net of order $n$ is an incidence structure consisting of $k \geq 3$ pairwise disjoint classes of lines, each of size $n$, such that every point incident with two lines from distinct classes is incident with exactly one line from each of the $k$ classes. Deleting a line class from a $k$-net, with $k \geq 4$, gives a derived ( $k-1$ )-net of the same order. Finite $k$-nets embedded in a projective plane $\mathrm{PG}(2, \mathbb{K})$ coordinatized by a field $\mathbb{K}$ of characteristic 0 only exist for $k=3,4$, see [11]. In this paper, we investigate 3-nets embedded in $\operatorname{PG}(2, \mathbb{K})$ whose line classes are in perspective position with an axis $r$, that is, every point on the line $r$ incident with a line of the net is incident with exactly one line from each class. The problem of determining all such 3-nets remains open whereas we obtain a complete classification for those coordinatizable by a group. As a corollary, the (unique) 4 -net of order 3 embedded in $\mathrm{PG}(2, \mathbb{K})$ turns out to be the only 4 -net embedded in $\operatorname{PG}(2, \mathbb{K})$ with a derived 3-net which can be coordinatized by a group. Our results hold true in positive characteristic under the hypothesis that the order of the $k$-net considered is smaller than the characteristic of $\mathbb{K}$.


Keywords: $k$-net, projective plane, group

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## 1. Introduction

Finite 3-nets occur naturally in combinatorics since they are geometric representations of important objects as latin squares, quasigroups, loops and strictly transitive permutation sets. Historically, the concept of 3-net arose from classical differential geometry via the combinatorial abstraction of the concept of a 3 -web; see [14]. In recent years finite 3-nets embedded in a projective plane $\operatorname{PG}(2, \mathbb{K})$ coordinatized by a field $\mathbb{K}$ were investigated in algebraic geometry and resonance theory, see [5, 12, 15, 18, 19], and a few infinite families of such 3 -nets were constructed and classified, see [10].

In this paper, we deal with finite 3-nets embedded in $\operatorname{PG}(2, \mathbb{K})$ such that the three line classes of the 3-net appear to be in perspective position with axis $r$, that is, whenever a point $P \in r$ lies on a line of the 3 -net then $P$ lies on exactly one line from each line classes of the 3 -net. If a 3 -net is in perspective position then the corresponding latin square has a transversal, equivalently, at least one of the quasigroups which have the latin square as a multiplicative table has a complete mapping; see [2, Section 1.4]. A group has a complete mapping if and only its 2 -subgroups of Sylow are either trivial or not cyclic. This was conjectured in the 1950's by Hall and Paige [6], see [2, p. 37], and proven only recently by Evans [4].

As in [10], most of the known examples in this paper arise naturally in the dual plane of $\operatorname{PG}(2, \mathbb{K})$, and it is convenient work with the dual concept of a 3-net embedded in $\operatorname{PG}(2, \mathbb{K})$. Formally, a dual 3-net in $\operatorname{PG}(2, \mathbb{K})$ consists of a triple $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ with $\Lambda_{1}, \Lambda_{2}, \Lambda_{3}$ pairwise disjoint point-sets, called components, such that every line meeting two distinct components meets each component in precisely one point. Every component has the same size $n$, the order of the dual 3 -net, and each of the $n^{2}$ lines meeting all components is a line of the dual 3 -net.

A dual 3-net $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ is in perspective position with a center $C$, where $C$ is a point off $\Lambda_{1} \cup \Lambda_{2} \cup \Lambda_{3}$, if every line through $C$ meeting a component is a line of the dual 3 -net, that is, still meets each component in exactly one point. A dual 3-net in perspective position has a transversal. Furthermore, a dual 3-net may be in perspective position with different centers although the number of such centers is bounded by the order of the 3-net. If this bound is attained and every lines through two centers is disjoint from $\Lambda_{1} \cup \Lambda_{2} \cup \Lambda_{3}$, then the set of the centers can be viewed as a new component $\Lambda_{4}$ to add to
$\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ so that the resulting quadruple $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}, \Lambda_{4}\right)$ is a dual 4-net, that is, a 4-net in the dual plane.

From previous work [11, Proposition 3.1], finite dual 4-nets have constant cross-ratio, that is, for every line $\ell$ intersecting the components, the crossratio ( $P_{1}, P_{2}, P_{3}, P_{4}$ ) with $P_{i}=\Lambda_{i} \cap \ell$ is the same. For a dual 3-net in perspective position with a center $C$, this raises the problem whether for all lines $\ell$ through $C$, the cross-ratio of the points $C, \ell \cap \Lambda_{1}, \ell \cap \Lambda_{2}, \ell \cap \Lambda_{3}$ is the same. By our Proposition [2.3, the answer is affirmative. Moreover, in case of more than one centers, the cross-ratio does not depend on which center is referred to. Therefore, any dual 3 -net in perspective position has constant cross-ratio.

The problem of classifying all 3-nets in perspective position remains open and appears to be difficult. Its solution would indeed imply the answer to the main conjecture on finite 4-nets, namely the non-existence of 4-nets of order greater than three. Our main result in this context is the following theorem that provides a complete classification for those 3-nets in perspective position which are coordinatizable by a group.

Theorem 1.1. Let $\Lambda$ be a dual 3-net of order $n$ which is coordinatized by a group. Assume that $\Lambda$ is embedded in a projective plane $\mathrm{PG}(2, \mathbb{K})$ over an algebraically closed field whose characteristic is either 0 or bigger than $n$. If $\Lambda$ is in perspective position and $n \neq 8$ then one of the following two cases occur:
(i) A component of $\Lambda$ lies on a line while the other two lie on a nonsingular conic. More precisely, $\Lambda$ is projectively equivalent to the dual 3-net given in Lemma 4.2.
(ii) $\Lambda$ is contained in a nonsingular cubic curve $\mathcal{C}$ with zero $j(\mathcal{C})$-invariant, and $\Lambda$ is in perspective position with at most three centers.

Theorem 1.1 provides evidence on the above mentioned conjecture about 4 -nets. In fact, it shows for $n \neq 8$ that the (unique) 4 -net of order 3 embedded in $\operatorname{PG}(2, \mathbb{K})$ is the only 4 -net embedded in $\operatorname{PG}(2, \mathbb{K})$ which has a derived 3 -net coordinatized by a group $G$. This result remains valid in positive characteristic under the hypothesis that that the order $n$ of the $k$-net considered is smaller than the characteristic of $\mathbb{K}$, apart from possibile sporadic cases occurring for $n \in\{12,24,60\}$ and $G \cong \mathrm{Alt}_{4}, \mathrm{Sym}_{4}$, $\mathrm{Alt}_{5}$, respectively.

The proof of Theorem 1.1 follows from Propositions 3.2, 3.3, 4.3 and Theorem 5.3 together with the classification of 3-nets coordinatized by groups,
see [10] and [13], which states that the dual of such a 3-net is either algebraic (that is, contained in a reducible or irreducible cubic curve), or of tetrahedron type, or $n=8$ and $G$ is the quaternion group of order 8 . This classification holds true in positive characteristic if the characteristic of $\mathbb{K}$ exceeds the order $n$ of 3 -net and none of the above mentioned special cases for $n=12,24,60$ occurs.

## 2. The constant cross-ratio property

In 11, Proposition 3.1], the authors showed that (dual) 4-nets have constant cross-ratio, that is, for any line intersecting the components, the crossratio of the four intersection points is constant. In this section we prove a similar result for (dual) 3-nets in perspective positions. Our proof relies on some ideas coming from [11].

Proposition 2.1. Let $F, G$ be homogeneous polynomials of degree $n$ such that the curves $\mathcal{F}: F=0$ and $\mathcal{G}: G=0$ have $n^{2}$ different points in common. Fix nonzero scalars $\alpha, \beta, \alpha^{\prime}, \beta^{\prime} \in \mathbb{K}$, and define the polynomials

$$
H=\alpha F+\beta G, H^{\prime}=\alpha^{\prime} F+\beta^{\prime} G
$$

and the corresponding curves $\mathcal{H}: H=0, \mathcal{H}^{\prime}: H^{\prime}=0$. Then, for all $P \in \mathcal{F} \cap \mathcal{G}$, the tangent lines $t_{P}(\mathcal{F}), t_{P}(\mathcal{G}), t_{P}(\mathcal{H}), t_{P}\left(\mathcal{H}^{\prime}\right)$ have cross-ratio

$$
\kappa=\frac{\alpha \beta^{\prime}}{\alpha^{\prime} \beta} .
$$

Proof. We start with three observations. Notice first that for any $P \in \mathcal{F} \cap$ $\mathcal{G}$, the intersection multiplicity of $\mathcal{F}$ and $\mathcal{G}$ at $P$ must be 1 by Bézout's theorem. This implies that $P$ is a smooth point of both curves, and that the tangent lines $t_{P}(\mathcal{F}), t_{P}(\mathcal{G})$ are different. Second, the polynomials $F, G, H, H^{\prime}$ are defined up to a scalar multiple. Multiplying them by scalars such that the curves $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{H}^{\prime}$ don't change, the value of the cross-ratio $\kappa$ remains invariant as well. And third, the change of the projective coordinate system leaves the homogeneous pairs $(\alpha, \beta),\left(\alpha^{\prime}, \beta^{\prime}\right)$ invariant, hence it does not affect $\kappa$.

Let us now fix an arbitrary point $P \in \mathcal{F} \cap \mathcal{G}$ and choose the projective coordinate system such that $P=(0,0,1), t_{P}(\mathcal{F}): X=0, t_{P}(\mathcal{G}): Y=0$. We
set $Z=0$ as the line at infinity and switch to affine coordinates $x=X / Z$, $y=Y / Z$. For the polynomials we have

$$
\left.\begin{array}{lrl}
F(x, y, 1) & =x+f_{2}(x, y), & G(x, y, 1)
\end{array}\right)=y+g_{2}(x, y), ~ 子 \beta^{\prime}(x, y, 1)=\alpha x+\beta y+h_{2}(x, y), ~ H H^{\prime}(x, y, 1)=\alpha^{\prime} x+\beta^{\prime} y+h_{2}^{\prime}(x, y), ~ l
$$

with polynomials $f_{2}, g_{2}, h_{2}, h_{2}^{\prime}$ of lower degree at least 2 . This shows that the respective tangent lines have equations

$$
x=0, y=0, \alpha x+\beta y=0, \alpha^{\prime} x+\beta^{\prime} y=0,
$$

hence, the cross-ratio is indeed $\kappa$.
Let $\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$ be a 3 -net of order $n$, embedded in $\operatorname{PG}(2, \mathbb{K})$. Let

$$
r_{1}=0, \ldots, r_{n}=0, w_{1}=0, \ldots, w_{n}=0, t_{1}=0, \ldots, t_{n}=0
$$

be the equations of lines of $\lambda_{1}, \lambda_{2}, \lambda_{3}$. Define the polynomials

$$
F=r_{1} \cdots r_{n}, G=w_{1} \cdots w_{n}, H=t_{1} \cdots t_{n} ;
$$

these have degree $n$ and the corresponding curves have exactly $n^{2}$ points in common. Moreover, the tangents in the intersection points are different; in fact, they are the lines of the dual 3-net. As explained in [11], there are scalars $\alpha, \beta \in \mathbb{K}$ such that $H=\alpha F+\beta G$.

Let $\ell$ be a transversal line of $\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$, that is, assume that $\ell$ intersects all lines of the 3 -net in the total of $n$ points $P_{1}, \ldots, P_{n}$. Let $Q$ be another point of $\ell$, that is, $Q \neq P_{i}, i=1, \ldots, n$. There are unique scalars $\alpha^{\prime}, \beta^{\prime}$ such that the curve $\mathcal{H}^{\prime}: \alpha^{\prime} F+\beta^{\prime} G=0$ passes through $Q$. As $\mathcal{H}^{\prime}$ has degree $n$ and $\left|\ell \cap \mathcal{H}^{\prime}\right| \geq n+1, \ell$ turns out to be a component of $\mathcal{H}^{\prime}$. This means that $\mathcal{H}^{\prime}=\ell \cup \mathcal{H}_{0}$ for a curve $\mathcal{H}_{0}$ of degree $n-1$. Moreover, since $\mathcal{H}_{0}$ cannot pass through $P_{1}, \ldots, P_{n}$, the tangent lines of $\mathcal{H}^{\prime}$ at these points are equal to $\ell$. Proposition 2.1 implies the following.

Proposition 2.2 (Constant cross-ratio for 3-nets with transversal). Let $\lambda=$ $\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$ be a 3-net of order $n$, embedded in $\mathrm{PG}(2, \mathbb{K})$. Assume that $\ell$ is a transversal to $\lambda$. Then there is a scalar $\kappa$ such that for all $P \in \ell \cap \lambda$, $m_{1} \in \lambda_{1}, m_{2} \in \lambda_{2}, m_{3} \in \lambda_{3}, P=m_{1} \cap m_{2} \cap m_{3}$, the cross-ratio of the lines $\ell, m_{1}, m_{2}, m_{3}$ is $\kappa$.

The dual formulation of the above result is the following

Proposition 2.3 (Constant cross-ratio for dual 3-nets in perspective position). Let $\Lambda=\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ be a dual 3 -net of order $n$, embedded in $\operatorname{PG}(2, \mathbb{K})$. Assume that $\Lambda$ is in perspective position with respect to the point $T$. Then there is a scalar $\kappa$ such that for all lines $\ell$ through $T$, the cross-ratio of the points $T, \ell \cap \Lambda_{1}, \ell \cap \Lambda_{2}, \ell \cap \Lambda_{3}$ is $\kappa$.

In the case when a component of a dual 3-net is contained in a line, the constant cross-ratio property implies a high level of symmetry of the dual 3 -net. The idea comes from the argument of the proof of [11, Theorem 5.4].

Proposition 2.4. Let $\Lambda=\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ be a dual 3 -net embedded in $\operatorname{PG}(2, \mathbb{K})$. Assume that $\Lambda$ is in perspective position with respect to the point $T$, and, $\Lambda_{1}$ is contained in a line $\ell$. Then there is a perspectivity $u$ with center $T$ and axis $\ell$ such that $\Lambda_{2}^{u}=\Lambda_{3}$.

Proof. Let $\kappa$ be the constant cross-ratio of $\Lambda$ w.r.t $T$. Define $u$ as the $(T, \ell)$ perspectivity which maps the point $P$ to $P^{\prime}$ such that the cross-ratio of the points $T, P, P^{\prime}$ and $T P \cap \ell$ is $\kappa$. Then $\Lambda_{2}^{u}=\Lambda_{3}$ holds.

## 3. Triangular and tetrahedron type dual 3 -nets in perspective positions

With the terminology used in [10], a dual 3-net is regular if each of its three components is linear, that is, contained in a line. Also, a regular dual 3 -net is triangular or of pencil type according as the lines containing the components form a triangle or are concurrent.

Proposition 3.1. Any regular dual 3-net in perspective position is of pencil type.

Proof. Let $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ be a dual 3-net in perspective position with center $P$ such that $\Lambda_{i}$ is contained in a line $\ell_{i}$ for $i=1,2,3$. Let $R=\ell_{1} \cap \ell_{2}$. Take any two points $L_{1}, L_{2} \in \Lambda_{1}$ and define $M_{1}=P L_{1} \cap \Lambda_{2}, M_{2}=P L_{2} \cap \Lambda_{2}$. There exists a perspectivity $\varphi$ with center $R$ and axis $r$ through $P$ which takes $L_{1}$ to $L_{2}$ and $M_{1}$ to $M_{2}$. For $i=1,2$, the line $t_{i}$ through $P, L_{i}, M_{i}$ meets $\Lambda_{3}$ in a point $N_{i}$. From Proposition 2.3 the cross ratios $\left(P L_{1} M_{1} N_{1}\right)$ and $\left(P L_{2} M_{2} N_{2}\right)$ coincide. Therefore, $\varphi$ takes $N_{1}$ to $N_{2}$ and hence the line $s=N_{1} N_{2}$ passes through $R$. If $L_{1}$ is fixed while $L_{2}$ ranges over $\Lambda_{1}$, the point $N_{2}$ hits each point of $\Lambda_{3}$. Therefore $\Lambda_{3}$ is contained in $s$.

From [10, Lemma 3], dual 3-nets of pencil type do not exist in zero characteristic whereas in positive characteristic they only exist when the order of the dual 3 -net is divisible by the characteristic. This, together with Proposition 3.1, give the following result.

Proposition 3.2. No regular dual 3-net in perspective position exists in zero characteristic. This holds true for dual 3-nets in positive characteristic whenever the order of the 3-net is smaller than the characteristic.

Now, let $\Lambda=\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ be tetrahedron type dual 3-net of order $n \geq 4$ embedded in $\operatorname{PG}(2, \mathbb{K})$. From [10, Section 4.4], $n$ is even, say $n=2 m$, and $\Lambda_{i}=\Gamma_{i} \cup \Delta_{i}$ for $i=1,2,3$ such that each triple

$$
\Phi_{1}=\left(\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right), \Phi_{2}=\left(\Gamma_{1}, \Delta_{2}, \Delta_{3}\right), \Phi_{3}=\left(\Delta_{1}, \Gamma_{2}, \Delta_{3}\right), \Phi_{4}=\left(\Delta_{1}, \Delta_{2}, \Gamma_{3}\right)
$$

is a dual 3 -net of triangular type. An easy counting argument shows that each of the $n^{2}$ lines of $\Lambda$ is a line of (exactly) one of the dual 3-nets $\Phi_{i}$ with $1 \leq i \leq 4$.

Now, assume that $\Lambda$ is in perspective position with a center $P$. Then each of the $n$ lines of $\Lambda$ passing through $P$ is also a line of (exactly) one $\Phi_{i}$ with $1 \leq i \leq 4$. Since $n>4$, some of these triangular dual 3-nets, say $\Phi$, has at least two lines through $T$. Let $\ell_{1}, \ell_{2}, \ell_{3}$ denote the lines containing the components of $\Phi$, respectively. Now, the proof of Proposition 3.1 remains valid for $\Phi$ showing that $\Phi$ is of pencil type. But this is impossible as we have pointed out after that proof. Therefore, the following result is proven.

Proposition 3.3. No tetrahedron type dual 3-net in perspective position exists in zero characteristic. This holds true for dual 3-nets in positive characteristic whenever the order of the 3-net is smaller than the characteristic.

Propositions 3.2 and 3.3 have the following corollary.
Corollary 3.4. Let $\mathbb{K}$ be an algebraically closed field whose characteristic is either zero or greater than $n$. Then no dual 4-net of order $n$ embedded in $\mathrm{PG}(2, \mathbb{K})$ has a derived dual 3 -net which is either triangular or of tetrahedron type.

## 4. Conic-line type dual 3 -nets in perspective positions

Let $\Lambda=\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ be dual 3-net of order $n \geq 5$ in perspective position with center $T$, and assume that $\Lambda$ is of conic-line type. Then $\Lambda$ has a component, say $\Lambda_{1}$, contained in a line $\ell$, while the other two components $\Lambda_{2}, \Lambda_{3}$ lie
on a nonsingular conic $\mathcal{C}$. From Proposition 2.3, $\Lambda$ has constant cross-ratio $\kappa$ with respect to $T$.

Lemma 4.1. $\kappa=-1$ and $T$ is the pole of $\ell$ in the polarity arising from $\mathcal{C}$.
Proof. By Proposition [2.4, there is a ( $T, \ell$ )-perspectivity $u$ which takes $\Lambda_{2}$ to $\Lambda_{3}$. The image $\mathcal{C}^{\prime}$ of the conic $\mathcal{C}$ by $u$ contains $\Lambda_{3}$ and hence $\Lambda_{3}$ is in the intersection of $\mathcal{C}$ and $\mathcal{C}^{\prime}$. Since $\Lambda_{3}$ has size $n \geq 5$, these nonsingular conics must coincide. Therefore, $u$ preserves $\mathcal{C}$ and its center $T$ is the pole of its axis $\ell$ is in the polarity arising from $\mathcal{C}$. In particular, $u$ is involution and hence $\kappa=-1$.

Choose our projective coordinate system such that $T=(0,0,1), \ell: Z=0$ and that $\mathcal{C}$ has equation $X Y=Z^{2}$. In the affine frame of reference, $\ell$ is the line at infinity, $T$ is the origin and $\mathcal{C}$ is the hyperbola of equation $x y=1$. Doing so, the map $(x, y) \mapsto(-x,-y)$ is the involutorial perspectivity $u$ introduced in the proof of Proposition 4.1.

As it is shown in [10, Section 4.3], $\Lambda$ has a parametrization in terms of an $n^{\text {th }}$-root of unity provided that the characteristic of $\mathbb{K}$ is either zero or it exceeds $n$. In our setting

$$
\Lambda_{2}=\left\{\left(c, c^{-1}\right),\left(c \xi, c^{-1} \xi^{-1}\right), \ldots,\left(c \xi^{n-1}, c^{-1} \xi^{-n+1}\right)\right\}
$$

where $c \in \mathbb{K}^{*}$ and $\xi$ is an $n$th root of unity in $\mathbb{K}$. Moreover, since $u$ takes $\Lambda_{2}$ to $\Lambda_{3}$,

$$
\Lambda_{3}=\left\{\left(-c,-c^{-1}\right),\left(-c \xi,-c^{-1} \xi^{-1}\right), \ldots,\left(-c \xi^{n-1},-c^{-1} \xi^{-n+1}\right)\right\}
$$

It should be noted that such sets $\Lambda_{2}$ and $\Lambda_{3}$ may coincide, and this occurs if and only if $n$ is even since $\xi^{n / 2}=-1$ for $n$ even. Therefore $n$ is odd and then

$$
\Lambda_{1}=\left\{\left(c^{-2}\right),\left(c^{-2} \xi\right), \ldots,\left(c^{-2} \xi^{n-1}\right)\right\}
$$

where $(m)$ denotes the infinite point of the affine line with slope $m$. Therefore the following results are obtained.

Lemma 4.2. For $n$ odd, the above conic-line type dual 3-net is in perspective position with center $T$.

Proposition 4.3. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero or greater than $n$. Then every conic-line type dual 3-net of order $n$ which is embedded in $\mathrm{PG}(2, \mathbb{K})$ in perspective position is projectively equivalent to the example given in Lemma 4.2. In particular, $n$ is odd.

Proposition 4.3 has the following corollary.
Corollary 4.4. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero or greater than $n$. Then no dual 4 -net of order $n$ embedded in $\mathrm{PG}(2, \mathbb{K})$ has a derived dual 3-net of conic-line type.

Proof. Since the only 4 -net of order 3 contains no linear component, and there exist no 4 -nets of order 4 , we may assume that $n \geq 5$. Let $\Lambda_{1}, \Lambda_{2}, \Lambda_{3}, \Lambda_{4}$ be a dual 4-net with a derived dual 3 -net of conic-line type. Without loss of generality, $\Lambda_{1}$ lies on a line. Then the derived dual 3 -net $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ is of conic-line type. From Proposition 4.3, there exists a non-singular conic $\mathcal{C}$ containing $\Lambda_{2} \cup \Lambda_{3}$. Similarly, $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{4}\right)$ is of conic-line type and $\Lambda_{2} \cup \Lambda_{4}$ is contained in a non-singular conic $\mathcal{D}$. Since $\mathcal{C}$ and $\mathcal{D}$ share $\Lambda_{2}$, the hypothesis $n \geq 5$ yields that $\mathcal{C}=\mathcal{D}$. Therefore, the dual 3-net $\left(\Lambda_{2}, \Lambda_{3}, \Lambda_{4}\right)$ is contained in $\mathcal{C}$. But this contradicts [1, Theorem 6.1].

## 5. Proper algebraic dual 3-nets in perspective positions

With the terminology introduced in [18], see also 10], a dual 3-net $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ embedded in $\operatorname{PG}(2, \mathbb{K})$ is algebraic if its components lie in a plane cubic $\Gamma$ of $\mathrm{PG}(2, \mathbb{K})$. If the plane cubic is reducible, then $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ is either regular or of conic-line type. Both cases have already been considered in the previous sections. Therefore, $\Gamma$ may be assumed irreducible, that is, $\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ is a proper algebraic dual 3-net.

It is well known that plane cubic curves have quite a different behavior over fields of characteristic 2,3 . Since the relevant case in our work is in zero characteristic or in positive characteristic greater than the order of the 3-net considered, we assume that the characteristic of $\mathbb{K}$ is neither 2 nor 3 . From classical results, $\Gamma$ is either nonsingular or it has at most one singular point, and in the latter case the point is either a node or a cusp. Accordingly, the number of inflection points of $\Gamma$ is 9,3 or 1 . If $\Gamma$ has a cusp then it has an affine equation $Y^{2}=X^{3}$ up to a change of the reference frame. Otherwise, $\Gamma$ may be taken in its Legendre form

$$
Y^{2}=X(X-1)(X-c)
$$

so that $\Gamma$ has an infinite point in $Y_{\infty}=(0,1,0)$ and it is nonsingular if and only if $c(c-1) \neq 0$. The projective equivalence class of a nonsingular cubic is uniquely determined by the $j$-invariant; see [7, Section IV.4]. Recall that
$j$ arises from the cross-ratio of the four tangents which can be drawn to $\Gamma$ from a point of $\Gamma$ and it takes into account the fact that four lines have six different permutations. Formally, let $t_{1}, t_{2}, t_{3}, t_{4}$ be indeterminates over $\mathbb{K}$. The cross-ratio of $t_{1}, t_{2}, t_{3}, t_{4}$ is the rational expression

$$
k=k\left(t_{1}, t_{2}, t_{3}, t_{4}\right)=\frac{\left(t_{3}-t_{1}\right)\left(t_{2}-t_{4}\right)}{\left(t_{2}-t_{3}\right)\left(t_{4}-t_{1}\right)} .
$$

$k$ is not symmetric in $t_{1}, t_{2}, t_{3}, t_{4}$; by permuting the indeterminates, $k$ can take 6 different values

$$
k, \quad \frac{1}{k}, \quad 1-k, \quad \frac{1}{1-k}, \quad \frac{k}{k-1}, \quad 1-\frac{1}{k} .
$$

The maps $k \mapsto 1 / k$ and $k \mapsto 1-k$ generate the anharmonic group of order 6 , which acts regularly on the 6 values of the cross-ratio. It is straightforward to check that the rational expression

$$
u=u(k)=\frac{\left(k^{2}-k+1\right)^{3}}{(k+1)^{2}(k-2)^{2}(2 k-1)^{2}}
$$

is invariant under the substitutions $k \mapsto 1 / k$ and $k \mapsto 1-k$. Hence, $u$ is a symmetric rational function of $t_{1}, t_{2}, t_{3}, t_{4}$. Assume that $t_{1}, t_{2}, t_{3}, t_{4}$ are roots of the quartic polynomial

$$
\varphi(t)=\alpha_{0}+\alpha_{1} t+\alpha_{2} t^{2}+\alpha_{3} t^{3}+\alpha_{4} t^{4}
$$

Then $u$ can be expressed by the coefficients $\alpha_{0}, \ldots, \alpha_{4}$ as

$$
\begin{equation*}
u=\frac{\left(12 \alpha_{0} \alpha_{4}-3 \alpha_{1} \alpha_{3}+\alpha_{2}^{2}\right)^{3}}{\left(72 \alpha_{0} \alpha_{2} \alpha_{4}-27 \alpha_{0} \alpha_{3}^{2}-27 \alpha_{1}^{2} \alpha_{4}-2 \alpha_{2}^{3}+9 \alpha_{1} \alpha_{2} \alpha_{3}\right)^{2}} . \tag{1}
\end{equation*}
$$

It may be observed that if $\Gamma$ is given in Legendre form then its $j$-invariant is

$$
\begin{equation*}
j(\Gamma)=2^{8} \frac{\left(c^{2}-c+1\right)^{3}}{c^{2}(c-1)^{2}} \tag{2}
\end{equation*}
$$

Assume that $j(\Gamma)=0$, that is, $c^{2}-c+1=0$. Then, the Hessian $H$ of $\Gamma$ is

$$
\begin{equation*}
H:\left(X-\frac{c+1}{3}\right)\left(Y+\sqrt{\frac{2 c-1}{3}}\right)\left(Y-\sqrt{\frac{2 c-1}{3}}\right)=0 . \tag{3}
\end{equation*}
$$

This shows that $H$ is the union of three nonconcurrent lines whose intersection points are the corners of $\Gamma$.

To our further investigation we need the following result.

Proposition 5.1. Let $\Gamma$ be an irreducible cubic curve in $\operatorname{PG}(2, \mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic different from 2 and 3. For each $i=1, \ldots, 7$, take pairwise distinct nonsingular points $P_{i}, Q_{i}, R_{i} \in \Gamma$ such that the triple $\left\{P_{i}, Q_{i}, R_{i}\right\}$ is collinear. Assume that there exists a point $T$ off $\Gamma$ such that quadruples $\left\{T, P_{i}, Q_{i}, R_{i}\right\}$ are collinear and that their crossratio $\left(T, P_{i}, Q_{i}, R_{i}\right)$ is a constant $\kappa$. Then $\Gamma$ has $j$-invariant 0 and $T$ is one of the three corners of $\Gamma$.

Proof. We explicitly present the proof for nonsingular cubics and for cubics with a node, as the cuspidal case can be handled with similar, even much simpler, computation. Therefore, $\Gamma$ has 9 or 3 inflection points. Pick an inflection point off the tangents to $\Gamma$ through $T$. Fix a reference frame such that this inflection point is $Y_{\infty}=(0,1,0)$ and that $\Gamma$ is in Legendre form

$$
Y^{2}=X(X-1)(X-c)
$$

Then $T=(a, b)$ is an affine point and so are $P_{i}, Q_{i}, R_{i}$. Let $\ell$ be the generic line through $T$ with parametric equation $x=a+t, y=b+m t$. The parameters of the points of $\ell$ which also lie in $\Gamma$, say $P, Q, R$, are the roots $\tau, \tau^{\prime}, \tau^{\prime \prime}$ of the cubic polynomial

$$
h_{1}(t)=(a+t)(a+t-1)(a+t-c)-(b+t m)^{2}
$$

The cross-ratio $k$ of $T, P, Q, R$ is equal to the cross-ratio of $0, \tau, \tau^{\prime}, \tau^{\prime \prime}$, and its $u(k)$ value can be computed from (1) by substituting

$$
\begin{aligned}
& \alpha_{0}=0, \\
& \alpha_{1}=a^{3}-a^{2} c-a^{2}+a c-b^{2}, \\
& \alpha_{2}=3 a^{2}-2 a-2 a c+c-2 b m, \\
& \alpha_{3}=3 a-1-c-m^{2}, \\
& \alpha_{4}=1 .
\end{aligned}
$$

From this, $u(k)=f(m)^{3} / g(m)^{2}$, where $f, g$ have $m$-degree 2 and 3 , respectively. Let $m_{i}$ be the slope of the line containing the points $T, P_{i}, Q_{i}, R_{i}$ ( $i=1, \ldots, 7$ ). By our assumption,

$$
u(\kappa)=\frac{f\left(m_{1}\right)^{3}}{g\left(m_{1}\right)^{2}}=\cdots=\frac{f\left(m_{7}\right)^{3}}{g\left(m_{7}\right)^{2}}
$$

which implies $\frac{f(m)^{3}}{g(m)^{2}}=u(\kappa)$ for all $m$, and this holds true even if one of the $m_{i}$ 's is infinite. If the rational function $\frac{f(m)^{3}}{g(m)^{2}}$ is constant then either its derivative is constant zero, or $f(m) \equiv 0$, or $g(m) \equiv 0$. In any of these cases,

$$
\begin{equation*}
3 f^{\prime}(m) g(m)-2 f(m) g^{\prime}(m) \equiv 0 \tag{4}
\end{equation*}
$$

By setting

$$
\begin{aligned}
& \beta_{0}=2 b\left(c^{2}-c+1\right), \\
& \beta_{1}=2 a c-2 a c^{2}-2 a+3 b^{2}+c^{2}+c, \\
& \beta_{2}=-2 b(3 a-1-c), \\
& \beta_{3}=3 a^{2}-2 a c+c-2 a,
\end{aligned}
$$

(4) becomes

$$
54\left(b^{2}-a(a-1)(a-c)\right)^{2}\left(\beta_{0}+\beta_{1} m+\beta_{2} m^{2}+\beta_{3} m^{3}\right) \equiv 0
$$

As $T$ is not on $\Gamma, b^{2}-a(a-1)(a-c) \neq 0$ and $\beta_{0}=\cdots=\beta_{3}=0$ holds. Define

$$
\begin{aligned}
& \gamma_{0}=-3 b(c-2)(2 c-1)(c+1), \\
& \gamma_{1}=-2\left(c^{2}-c+1\right)\left(6 a-4+3 c+6 a c^{2}+3 c^{2}-4 c^{3}-6 a c\right), \\
& \gamma_{2}=-6 b\left(c^{2}-c+1\right)^{2} \\
& \gamma_{3}=-8\left(c^{2}-c+1\right)^{3}
\end{aligned}
$$

Then

$$
0=\beta_{0} \gamma_{0}+\beta_{1} \gamma_{1}+\beta_{2} \gamma_{2}+\beta_{3} \gamma_{3}=18 c^{2}(c-1)^{2}\left(c^{2}-c+1\right)
$$

If $c(c-1)=0$ then $\beta_{0}=2 b=0$ and $\beta_{1}=2(a-c)=0$, which implies $T \in \Gamma$, a contradiction. This proves $c^{2}-c+1=0$ and $j(\Gamma)=0$ by (2). Moreover, $\beta_{1}=3 b^{2}+c^{2}+c=0$ amd $\beta_{3}=\frac{1}{3}(3 a-c-1)^{2}=0$ imply

$$
a=\frac{c+1}{3}, \quad b=\sqrt{\frac{1-2 c}{3}}
$$

therefore by (3), the lines $X=a$ and $Y=b$ through $T$ are components of the Hessian curve of $\Gamma$. This shows that $T$ is a corner point of $\Gamma$.

Theorem 5.2. Under the hypotheses of Proposition 5.1,
(i) the affine reference frame in $\mathrm{PG}(2, \mathbb{K})$ can be chosen such that $\Gamma$ has affine equation $X^{3}+Y^{3}=1$ and that $T=(0,0)$;
(ii) there is a perspectivity $u$ of order 3 with center $T$ leaving $\Gamma$ invariant;
(iii) the constant cross-ratio $\kappa$ is a root of the equation $X^{2}-X+1=0$.

Proof. (i) According to [7, Example IV.4.6.2], the plane cubic $\Gamma^{\prime}$ of equation $X^{3}+Y^{3}=Z^{3}$ has $j$-invariant 0 . Therefore, $\Gamma$ is projectively equivalent to $\Gamma^{\prime}$. As the Hessian of $\Gamma^{\prime}$ is $H^{\prime}: X Y Z=0$, the corner points of $\Gamma^{\prime}$ are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The projectivity $(x, y, z) \mapsto(-y, z, x)$, preserves $\Gamma^{\prime}$, and permutes the corners, thus $T=(0,0,1)$ can be assumed w.l.o.g.
(ii) $u$ is the map $(x, y) \rightarrow(\varepsilon x, \varepsilon y)$, where $\varepsilon$ is a third root of unity in $\mathbb{K}$. (iii) The line $Y=m X$ intersects $\Gamma^{\prime}$ in the points

$$
\left(\frac{1}{\sqrt[3]{1+m^{3}}}, \frac{m}{\sqrt[3]{1+m^{3}}}\right),\left(\frac{\varepsilon}{\sqrt[3]{1+m^{3}}}, \frac{\varepsilon m}{\sqrt[3]{1+m^{3}}}\right),\left(\frac{\varepsilon^{2}}{\sqrt[3]{1+m^{3}}}, \frac{\varepsilon^{2} m}{\sqrt[3]{1+m^{3}}}\right)
$$

whose cross-ratio with $(0,0)$ is $\kappa=-\varepsilon$.
Therefore the following result holds.
Theorem 5.3. Let $\mathbb{K}$ be an algebraically closed field of characteristic different from 2 and 3. Let $\Lambda=\left(\Lambda_{1}, \Lambda_{2}, \Lambda_{3}\right)$ be a dual 3-net of order $n \geq 7$ embedded in $\mathrm{PG}(2, \mathbb{K})$ which lies on an irreducible cubic curve $\Gamma$. If $\Gamma$ is either singular or is nonsingular with $j(\Gamma) \neq 0$ then $\Lambda$ is not in perspective position. If $j(\Gamma)=0$ then there are at most three points $T_{1}, T_{2}, T_{3}$ such that $\Lambda$ is in perspective position with center $T_{i}$.

Proof. Let $T$ be a point such that $\Lambda$ is in perspective position with center $T$. By Proposition [2.3, $\Lambda$ has a constant cross-ratio $\kappa$, hence Theorem 5.2 applies over the algebraic closure of $\mathbb{K}$.

Theorem 5.3 has the following corollary.
Corollary 5.4. Let $\mathbb{K}$ be an algebraically closed field of characteristic different from 2 and 3. Then no dual 4 -net of order $n \geq 7$ embedded in PG(2, $\mathbb{K})$ has a derived dual 3-net lying on a plane cubic.

Finally, we show the existence of proper algebraic dual 3-nets in perspective position.

Example 5.5. Let $\mathbb{K}$ be a field of characteristic different from 2 and 3, $\Gamma: X^{3}+Y^{3}=Z^{3}, T=(0,0,1)$ and $u:(x, y, z) \mapsto(\varepsilon x, \varepsilon y, z)$ with third root of unity $\varepsilon$. The infinite point $O(1,-1,0)$ is an inflection point of $\Gamma$, left invariant by $u$. Since u leaves $\Gamma$ invariant as well, $u$ induces an automorphism of the abelian group of $(\Gamma,+, O)$; we denote the automorphism by $u$, too. As the line TP contains the points $P^{u}, P^{u^{2}}$ for any $P \in \Gamma$, one has $P+P^{u}+P^{u^{2}}=O$. Let $H$ be a subgroup of $(\Gamma,+)$ of finite order $n$ such that $H^{u}=H$. For any $P \in \Gamma$ with $P-P^{u} \notin H$, the cosets

$$
\Lambda_{1}=H+P, \quad \Lambda_{2}=H+P^{u}, \quad \Lambda_{3}=H+P^{u^{2}}
$$

form a dual 3-net which is in perspective position with center T. Indeed, for any $A_{1}+P \in \Lambda_{1}$ and $A_{2}+P^{u} \in \Lambda_{2}$, the line joining them passes through $-A_{1}-A_{2}+P^{u^{2}} \in \Lambda_{3}$.

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