Classification of k-nets

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Abstract

A finite k-net of order n is an incidence structure consisting of $k \ge 3$ pairwise disjoint classes of lines, each of size n, such that every point incident with two lines from distinct classes is incident with exactly one line from each of the k classes. Deleting a line class from a k-net, with $k \geq 4$, gives a *derived* (k-1)-net of the same order. Finite k-nets embedded in a projective plane $PG(2, \mathbb{K})$ coordinatized by a field \mathbb{K} of characteristic 0 only exist for k = 3, 4, 4see [11]. In this paper, we investigate 3-nets embedded in $PG(2, \mathbb{K})$ whose line classes are in perspective position with an axis r, that is, every point on the line r incident with a line of the net is incident with exactly one line from each class. The problem of determining all such 3-nets remains open whereas we obtain a complete classification for those coordinatizable by a group. As a corollary, the (unique) 4-net of order 3 embedded in $PG(2, \mathbb{K})$ turns out to be the only 4-net embedded in $PG(2, \mathbb{K})$ with a derived 3-net which can be coordinatized by a group. Our results hold true in positive characteristic under the hypothesis that the order of the k-net considered is smaller than the characteristic of \mathbb{K} .

Keywords: k-net, projective plane, group

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1. Introduction

Finite 3-nets occur naturally in combinatorics since they are geometric representations of important objects as latin squares, quasigroups, loops and strictly transitive permutation sets. Historically, the concept of 3-net arose from classical differential geometry via the combinatorial abstraction of the concept of a 3-web; see [14]. In recent years finite 3-nets embedded in a projective plane $PG(2, \mathbb{K})$ coordinatized by a field \mathbb{K} were investigated in algebraic geometry and resonance theory, see [5, 12, 15, 18, 19], and a few infinite families of such 3-nets were constructed and classified, see [10].

In this paper, we deal with finite 3-nets embedded in $PG(2, \mathbb{K})$ such that the three line classes of the 3-net appear to be in perspective position with axis r, that is, whenever a point $P \in r$ lies on a line of the 3-net then Plies on exactly one line from each line classes of the 3-net. If a 3-net is in perspective position then the corresponding latin square has a transversal, equivalently, at least one of the quasigroups which have the latin square as a multiplicative table has a complete mapping; see [2, Section 1.4]. A group has a complete mapping if and only its 2-subgroups of Sylow are either trivial or not cyclic. This was conjectured in the 1950's by Hall and Paige [6], see [2, p. 37], and proven only recently by Evans [4].

As in [10], most of the known examples in this paper arise naturally in the dual plane of $PG(2, \mathbb{K})$, and it is convenient work with the dual concept of a 3-net embedded in $PG(2, \mathbb{K})$. Formally, a *dual 3-net* in $PG(2, \mathbb{K})$ consists of a triple $(\Lambda_1, \Lambda_2, \Lambda_3)$ with $\Lambda_1, \Lambda_2, \Lambda_3$ pairwise disjoint point-sets, called *components*, such that every line meeting two distinct components meets each component in precisely one point. Every component has the same size n, the *order* of the dual 3-net, and each of the n^2 lines meeting all components is a *line* of the dual 3-net.

A dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ is in perspective position with a center C, where C is a point off $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, if every line through C meeting a component is a line of the dual 3-net, that is, still meets each component in exactly one point. A dual 3-net in perspective position has a transversal. Furthermore, a dual 3-net may be in perspective position with different centers although the number of such centers is bounded by the order of the 3-net. If this bound is attained and every lines through two centers is disjoint from $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, then the set of the centers can be viewed as a new component Λ_4 to add to

 $(\Lambda_1, \Lambda_2, \Lambda_3)$ so that the resulting quadruple $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ is a dual 4-net, that is, a 4-net in the dual plane.

From previous work [11, Proposition 3.1], finite dual 4-nets have constant cross-ratio, that is, for every line ℓ intersecting the components, the crossratio (P_1, P_2, P_3, P_4) with $P_i = \Lambda_i \cap \ell$ is the same. For a dual 3-net in perspective position with a center C, this raises the problem whether for all lines ℓ through C, the cross-ratio of the points $C, \ell \cap \Lambda_1, \ell \cap \Lambda_2, \ell \cap \Lambda_3$ is the same. By our Proposition 2.3, the answer is affirmative. Moreover, in case of more than one centers, the cross-ratio does not depend on which center is referred to. Therefore, any dual 3-net in perspective position has *constant cross-ratio*.

The problem of classifying all 3-nets in perspective position remains open and appears to be difficult. Its solution would indeed imply the answer to the main conjecture on finite 4-nets, namely the non-existence of 4-nets of order greater than three. Our main result in this context is the following theorem that provides a complete classification for those 3-nets in perspective position which are coordinatizable by a group.

Theorem 1.1. Let Λ be a dual 3-net of order n which is coordinatized by a group. Assume that Λ is embedded in a projective plane $PG(2, \mathbb{K})$ over an algebraically closed field whose characteristic is either 0 or bigger than n. If Λ is in perspective position and $n \neq 8$ then one of the following two cases occur:

- (i) A component of Λ lies on a line while the other two lie on a nonsingular conic. More precisely, Λ is projectively equivalent to the dual 3-net given in Lemma 4.2.
- (ii) Λ is contained in a nonsingular cubic curve C with zero j(C)-invariant, and Λ is in perspective position with at most three centers.

Theorem 1.1 provides evidence on the above mentioned conjecture about 4-nets. In fact, it shows for $n \neq 8$ that the (unique) 4-net of order 3 embedded in PG(2, K) is the only 4-net embedded in PG(2, K) which has a derived 3-net coordinatized by a group G. This result remains valid in positive characteristic under the hypothesis that that the order n of the k-net considered is smaller than the characteristic of K, apart from possibile sporadic cases occurring for $n \in \{12, 24, 60\}$ and $G \cong \text{Alt}_4, \text{Sym}_4, \text{Alt}_5$, respectively.

The proof of Theorem 1.1 follows from Propositions 3.2, 3.3, 4.3 and Theorem 5.3 together with the classification of 3-nets coordinatized by groups, see [10] and [13], which states that the dual of such a 3-net is either algebraic (that is, contained in a reducible or irreducible cubic curve), or of tetrahedron type, or n = 8 and G is the quaternion group of order 8. This classification holds true in positive characteristic if the characteristic of \mathbb{K} exceeds the order n of 3-net and none of the above mentioned special cases for n = 12, 24, 60 occurs.

2. The constant cross-ratio property

In [11, Proposition 3.1], the authors showed that (dual) 4-nets have constant cross-ratio, that is, for any line intersecting the components, the crossratio of the four intersection points is constant. In this section we prove a similar result for (dual) 3-nets in perspective positions. Our proof relies on some ideas coming from [11].

Proposition 2.1. Let F, G be homogeneous polynomials of degree n such that the curves $\mathcal{F} : F = 0$ and $\mathcal{G} : G = 0$ have n^2 different points in common. Fix nonzero scalars $\alpha, \beta, \alpha', \beta' \in \mathbb{K}$, and define the polynomials

$$H = \alpha F + \beta G, H' = \alpha' F + \beta' G$$

and the corresponding curves $\mathcal{H} : H = 0, \mathcal{H}' : H' = 0$. Then, for all $P \in \mathcal{F} \cap \mathcal{G}$, the tangent lines $t_P(\mathcal{F}), t_P(\mathcal{G}), t_P(\mathcal{H}), t_P(\mathcal{H}')$ have cross-ratio

$$\kappa = \frac{\alpha \beta'}{\alpha' \beta}.$$

Proof. We start with three observations. Notice first that for any $P \in \mathcal{F} \cap \mathcal{G}$, the intersection multiplicity of \mathcal{F} and \mathcal{G} at P must be 1 by Bézout's theorem. This implies that P is a smooth point of both curves, and that the tangent lines $t_P(\mathcal{F})$, $t_P(\mathcal{G})$ are different. Second, the polynomials F, G, H, H' are defined up to a scalar multiple. Multiplying them by scalars such that the curves $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{H}'$ don't change, the value of the cross-ratio κ remains invariant as well. And third, the change of the projective coordinate system leaves the homogeneous pairs $(\alpha, \beta), (\alpha', \beta')$ invariant, hence it does not affect κ .

Let us now fix an arbitrary point $P \in \mathcal{F} \cap \mathcal{G}$ and choose the projective coordinate system such that $P = (0, 0, 1), t_P(\mathcal{F}) : X = 0, t_P(\mathcal{G}) : Y = 0$. We set Z = 0 as the line at infinity and switch to affine coordinates x = X/Z, y = Y/Z. For the polynomials we have

$$F(x, y, 1) = x + f_2(x, y), \qquad G(x, y, 1) = y + g_2(x, y), H(x, y, 1) = \alpha x + \beta y + h_2(x, y), \qquad H'(x, y, 1) = \alpha' x + \beta' y + h'_2(x, y),$$

with polynomials f_2, g_2, h_2, h'_2 of lower degree at least 2. This shows that the respective tangent lines have equations

$$x = 0, y = 0, \alpha x + \beta y = 0, \alpha' x + \beta' y = 0,$$

hence, the cross-ratio is indeed κ .

Let $(\lambda_1, \lambda_2, \lambda_3)$ be a 3-net of order *n*, embedded in PG(2, K). Let

$$r_1 = 0, \dots, r_n = 0, w_1 = 0, \dots, w_n = 0, t_1 = 0, \dots, t_n = 0$$

be the equations of lines of $\lambda_1, \lambda_2, \lambda_3$. Define the polynomials

$$F = r_1 \cdots r_n, G = w_1 \cdots w_n, H = t_1 \cdots t_n;$$

these have degree n and the corresponding curves have exactly n^2 points in common. Moreover, the tangents in the intersection points are different; in fact, they are the lines of the dual 3-net. As explained in [11], there are scalars $\alpha, \beta \in \mathbb{K}$ such that $H = \alpha F + \beta G$.

Let ℓ be a transversal line of $(\lambda_1, \lambda_2, \lambda_3)$, that is, assume that ℓ intersects all lines of the 3-net in the total of n points P_1, \ldots, P_n . Let Q be another point of ℓ , that is, $Q \neq P_i$, $i = 1, \ldots, n$. There are unique scalars α', β' such that the curve $\mathcal{H}' : \alpha'F + \beta'G = 0$ passes through Q. As \mathcal{H}' has degree nand $|\ell \cap \mathcal{H}'| \geq n + 1$, ℓ turns out to be a component of \mathcal{H}' . This means that $\mathcal{H}' = \ell \cup \mathcal{H}_0$ for a curve \mathcal{H}_0 of degree n - 1. Moreover, since \mathcal{H}_0 cannot pass through P_1, \ldots, P_n , the tangent lines of \mathcal{H}' at these points are equal to ℓ . Proposition 2.1 implies the following.

Proposition 2.2 (Constant cross-ratio for 3-nets with transversal). Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a 3-net of order n, embedded in $PG(2, \mathbb{K})$. Assume that ℓ is a transversal to λ . Then there is a scalar κ such that for all $P \in \ell \cap \lambda$, $m_1 \in \lambda_1, m_2 \in \lambda_2, m_3 \in \lambda_3, P = m_1 \cap m_2 \cap m_3$, the cross-ratio of the lines ℓ, m_1, m_2, m_3 is κ .

The dual formulation of the above result is the following

Proposition 2.3 (Constant cross-ratio for dual 3-nets in perspective position). Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order n, embedded in PG(2, \mathbb{K}). Assume that Λ is in perspective position with respect to the point T. Then there is a scalar κ such that for all lines ℓ through T, the cross-ratio of the points $T, \ell \cap \Lambda_1, \ell \cap \Lambda_2, \ell \cap \Lambda_3$ is κ .

In the case when a component of a dual 3-net is contained in a line, the constant cross-ratio property implies a high level of symmetry of the dual 3-net. The idea comes from the argument of the proof of [11, Theorem 5.4].

Proposition 2.4. Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net embedded in PG(2, K). Assume that Λ is in perspective position with respect to the point T, and, Λ_1 is contained in a line ℓ . Then there is a perspectivity u with center T and axis ℓ such that $\Lambda_2^u = \Lambda_3$.

Proof. Let κ be the constant cross-ratio of Λ w.r.t T. Define u as the (T, ℓ) perspectivity which maps the point P to P' such that the cross-ratio of the
points T, P, P' and $TP \cap \ell$ is κ . Then $\Lambda_2^u = \Lambda_3$ holds.

3. Triangular and tetrahedron type dual 3-nets in perspective positions

With the terminology used in [10], a dual 3-net is *regular* if each of its three components is linear, that is, contained in a line. Also, a regular dual 3-net is *triangular* or of *pencil type* according as the lines containing the components form a triangle or are concurrent.

Proposition 3.1. Any regular dual 3-net in perspective position is of pencil type.

Proof. Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net in perspective position with center P such that Λ_i is contained in a line ℓ_i for i = 1, 2, 3. Let $R = \ell_1 \cap \ell_2$. Take any two points $L_1, L_2 \in \Lambda_1$ and define $M_1 = PL_1 \cap \Lambda_2, M_2 = PL_2 \cap \Lambda_2$. There exists a perspectivity φ with center R and axis r through P which takes L_1 to L_2 and M_1 to M_2 . For i = 1, 2, the line t_i through P, L_i, M_i meets Λ_3 in a point N_i . From Proposition 2.3 the cross ratios $(P L_1 M_1 N_1)$ and $(P L_2 M_2 N_2)$ coincide. Therefore, φ takes N_1 to N_2 and hence the line $s = N_1 N_2$ passes through R. If L_1 is fixed while L_2 ranges over Λ_1 , the point N_2 hits each point of Λ_3 . Therefore Λ_3 is contained in s.

From [10, Lemma 3], dual 3-nets of pencil type do not exist in zero characteristic whereas in positive characteristic they only exist when the order of the dual 3-net is divisible by the characteristic. This, together with Proposition 3.1, give the following result.

Proposition 3.2. No regular dual 3-net in perspective position exists in zero characteristic. This holds true for dual 3-nets in positive characteristic whenever the order of the 3-net is smaller than the characteristic. \Box

Now, let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be tetrahedron type dual 3-net of order $n \ge 4$ embedded in PG(2, K). From [10, Section 4.4], n is even, say n = 2m, and $\Lambda_i = \Gamma_i \cup \Delta_i$ for i = 1, 2, 3 such that each triple

$$\Phi_1 = (\Gamma_1, \Gamma_2, \Gamma_3), \ \Phi_2 = (\Gamma_1, \Delta_2, \Delta_3), \ \Phi_3 = (\Delta_1, \Gamma_2, \Delta_3), \ \Phi_4 = (\Delta_1, \Delta_2, \Gamma_3)$$

is a dual 3-net of triangular type. An easy counting argument shows that each of the n^2 lines of Λ is a line of (exactly) one of the dual 3-nets Φ_i with $1 \le i \le 4$.

Now, assume that Λ is in perspective position with a center P. Then each of the n lines of Λ passing through P is also a line of (exactly) one Φ_i with $1 \leq i \leq 4$. Since n > 4, some of these triangular dual 3-nets, say Φ , has at least two lines through T. Let ℓ_1, ℓ_2, ℓ_3 denote the lines containing the components of Φ , respectively. Now, the proof of Proposition 3.1 remains valid for Φ showing that Φ is of pencil type. But this is impossible as we have pointed out after that proof. Therefore, the following result is proven.

Proposition 3.3. No tetrahedron type dual 3-net in perspective position exists in zero characteristic. This holds true for dual 3-nets in positive characteristic whenever the order of the 3-net is smaller than the characteristic. \Box

Propositions 3.2 and 3.3 have the following corollary.

Corollary 3.4. Let \mathbb{K} be an algebraically closed field whose characteristic is either zero or greater than n. Then no dual 4-net of order n embedded in $PG(2, \mathbb{K})$ has a derived dual 3-net which is either triangular or of tetrahedron type.

4. Conic-line type dual 3-nets in perspective positions

Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be dual 3-net of order $n \geq 5$ in perspective position with center T, and assume that Λ is of conic-line type. Then Λ has a component, say Λ_1 , contained in a line ℓ , while the other two components Λ_2, Λ_3 lie on a nonsingular conic C. From Proposition 2.3, Λ has constant cross-ratio κ with respect to T.

Lemma 4.1. $\kappa = -1$ and T is the pole of ℓ in the polarity arising from C.

Proof. By Proposition 2.4, there is a (T, ℓ) -perspectivity u which takes Λ_2 to Λ_3 . The image \mathcal{C}' of the conic \mathcal{C} by u contains Λ_3 and hence Λ_3 is in the intersection of \mathcal{C} and \mathcal{C}' . Since Λ_3 has size $n \geq 5$, these nonsingular conics must coincide. Therefore, u preserves \mathcal{C} and its center T is the pole of its axis ℓ is in the polarity arising from \mathcal{C} . In particular, u is involution and hence $\kappa = -1$.

Choose our projective coordinate system such that $T = (0, 0, 1), \ell : Z = 0$ and that \mathcal{C} has equation $XY = Z^2$. In the affine frame of reference, ℓ is the line at infinity, T is the origin and \mathcal{C} is the hyperbola of equation xy = 1. Doing so, the map $(x, y) \mapsto (-x, -y)$ is the involutorial perspectivity uintroduced in the proof of Proposition 4.1.

As it is shown in [10, Section 4.3], Λ has a parametrization in terms of an n^{th} -root of unity provided that the characteristic of \mathbb{K} is either zero or it exceeds n. In our setting

$$\Lambda_2 = \{ (c, c^{-1}), (c\xi, c^{-1}\xi^{-1}), \dots, (c\xi^{n-1}, c^{-1}\xi^{-n+1}) \},\$$

where $c \in \mathbb{K}^*$ and ξ is an *n*th root of unity in \mathbb{K} . Moreover, since *u* takes Λ_2 to Λ_3 ,

$$\Lambda_3 = \{(-c, -c^{-1}), (-c\xi, -c^{-1}\xi^{-1}), \dots, (-c\xi^{n-1}, -c^{-1}\xi^{-n+1})\}.$$

It should be noted that such sets Λ_2 and Λ_3 may coincide, and this occurs if and only if n is even since $\xi^{n/2} = -1$ for n even. Therefore n is odd and then

 $\Lambda_1 = \{ (c^{-2}), (c^{-2}\xi), \dots, (c^{-2}\xi^{n-1}) \},\$

where (m) denotes the infinite point of the affine line with slope m. Therefore the following results are obtained.

Lemma 4.2. For n odd, the above conic-line type dual 3-net is in perspective position with center T. \Box

Proposition 4.3. Let \mathbb{K} be an algebraically closed field of characteristic zero or greater than n. Then every conic-line type dual 3-net of order n which is embedded in $PG(2, \mathbb{K})$ in perspective position is projectively equivalent to the example given in Lemma 4.2. In particular, n is odd.

Proposition 4.3 has the following corollary.

Corollary 4.4. Let \mathbb{K} be an algebraically closed field of characteristic zero or greater than n. Then no dual 4-net of order n embedded in $PG(2, \mathbb{K})$ has a derived dual 3-net of conic-line type.

Proof. Since the only 4-net of order 3 contains no linear component, and there exist no 4-nets of order 4, we may assume that $n \geq 5$. Let $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ be a dual 4-net with a derived dual 3-net of conic-line type. Without loss of generality, Λ_1 lies on a line. Then the derived dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ is of conic-line type. From Proposition 4.3, there exists a non-singular conic \mathcal{C} containing $\Lambda_2 \cup \Lambda_3$. Similarly, $(\Lambda_1, \Lambda_2, \Lambda_4)$ is of conic-line type and $\Lambda_2 \cup \Lambda_4$ is contained in a non-singular conic \mathcal{D} . Since \mathcal{C} and \mathcal{D} share Λ_2 , the hypothesis $n \geq 5$ yields that $\mathcal{C} = \mathcal{D}$. Therefore, the dual 3-net $(\Lambda_2, \Lambda_3, \Lambda_4)$ is contained in \mathcal{C} . But this contradicts [1, Theorem 6.1].

5. Proper algebraic dual 3-nets in perspective positions

With the terminology introduced in [18], see also [10], a dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ embedded in PG(2, K) is algebraic if its components lie in a plane cubic Γ of PG(2, K). If the plane cubic is reducible, then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is either regular or of conic-line type. Both cases have already been considered in the previous sections. Therefore, Γ may be assumed irreducible, that is, $(\Lambda_1, \Lambda_2, \Lambda_3)$ is a proper algebraic dual 3-net.

It is well known that plane cubic curves have quite a different behavior over fields of characteristic 2, 3. Since the relevant case in our work is in zero characteristic or in positive characteristic greater than the order of the 3-net considered, we assume that the characteristic of \mathbb{K} is neither 2 nor 3. From classical results, Γ is either nonsingular or it has at most one singular point, and in the latter case the point is either a node or a cusp. Accordingly, the number of inflection points of Γ is 9, 3 or 1. If Γ has a cusp then it has an affine equation $Y^2 = X^3$ up to a change of the reference frame. Otherwise, Γ may be taken in its Legendre form

$$Y^2 = X(X - 1)(X - c)$$

so that Γ has an infinite point in $Y_{\infty} = (0, 1, 0)$ and it is nonsingular if and only if $c(c-1) \neq 0$. The projective equivalence class of a nonsingular cubic is uniquely determined by the *j*-invariant; see [7, Section IV.4]. Recall that j arises from the cross-ratio of the four tangents which can be drawn to Γ from a point of Γ and it takes into account the fact that four lines have six different permutations. Formally, let t_1, t_2, t_3, t_4 be indeterminates over \mathbb{K} . The cross-ratio of t_1, t_2, t_3, t_4 is the rational expression

$$k = k(t_1, t_2, t_3, t_4) = \frac{(t_3 - t_1)(t_2 - t_4)}{(t_2 - t_3)(t_4 - t_1)}.$$

k is not symmetric in t_1, t_2, t_3, t_4 ; by permuting the indeterminates, k can take 6 different values

$$k, \quad \frac{1}{k}, \quad 1-k, \quad \frac{1}{1-k}, \quad \frac{k}{k-1}, \quad 1-\frac{1}{k}.$$

The maps $k \mapsto 1/k$ and $k \mapsto 1-k$ generate the anharmonic group of order 6, which acts regularly on the 6 values of the cross-ratio. It is straightforward to check that the rational expression

$$u = u(k) = \frac{(k^2 - k + 1)^3}{(k+1)^2(k-2)^2(2k-1)^2}$$

is invariant under the substitutions $k \mapsto 1/k$ and $k \mapsto 1-k$. Hence, u is a symmetric rational function of t_1, t_2, t_3, t_4 . Assume that t_1, t_2, t_3, t_4 are roots of the quartic polynomial

$$\varphi(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4.$$

Then u can be expressed by the coefficients $\alpha_0, \ldots, \alpha_4$ as

$$u = \frac{(12\alpha_0\alpha_4 - 3\alpha_1\alpha_3 + \alpha_2^2)^3}{(72\alpha_0\alpha_2\alpha_4 - 27\alpha_0\alpha_3^2 - 27\alpha_1^2\alpha_4 - 2\alpha_2^3 + 9\alpha_1\alpha_2\alpha_3)^2}.$$
 (1)

It may be observed that if Γ is given in Legendre form then its *j*-invariant is

$$j(\Gamma) = 2^8 \frac{(c^2 - c + 1)^3}{c^2(c - 1)^2}.$$
(2)

Assume that $j(\Gamma) = 0$, that is, $c^2 - c + 1 = 0$. Then, the Hessian H of Γ is

$$H:\left(X-\frac{c+1}{3}\right)\left(Y+\sqrt{\frac{2c-1}{3}}\right)\left(Y-\sqrt{\frac{2c-1}{3}}\right)=0.$$
 (3)

This shows that H is the union of three nonconcurrent lines whose intersection points are the *corners* of Γ .

To our further investigation we need the following result.

Proposition 5.1. Let Γ be an irreducible cubic curve in $PG(2, \mathbb{K})$ defined over an algebraically closed field \mathbb{K} of characteristic different from 2 and 3. For each i = 1, ..., 7, take pairwise distinct nonsingular points $P_i, Q_i, R_i \in \Gamma$ such that the triple $\{P_i, Q_i, R_i\}$ is collinear. Assume that there exists a point T off Γ such that quadruples $\{T, P_i, Q_i, R_i\}$ are collinear and that their crossratio (T, P_i, Q_i, R_i) is a constant κ . Then Γ has j-invariant 0 and T is one of the three corners of Γ .

Proof. We explicitly present the proof for nonsingular cubics and for cubics with a node, as the cuspidal case can be handled with similar, even much simpler, computation. Therefore, Γ has 9 or 3 inflection points. Pick an inflection point off the tangents to Γ through T. Fix a reference frame such that this inflection point is $Y_{\infty} = (0, 1, 0)$ and that Γ is in Legendre form

$$Y^{2} = X(X - 1)(X - c).$$

Then T = (a, b) is an affine point and so are P_i, Q_i, R_i . Let ℓ be the generic line through T with parametric equation x = a + t, y = b + mt. The parameters of the points of ℓ which also lie in Γ , say P, Q, R, are the roots τ, τ', τ'' of the cubic polynomial

$$h_1(t) = (a+t)(a+t-1)(a+t-c) - (b+tm)^2.$$

The cross-ratio k of T, P, Q, R is equal to the cross-ratio of $0, \tau, \tau', \tau''$, and its u(k) value can be computed from (1) by substituting

$$\begin{aligned} \alpha_0 &= 0, \\ \alpha_1 &= a^3 - a^2 c - a^2 + a c - b^2, \\ \alpha_2 &= 3a^2 - 2a - 2a c + c - 2bm, \\ \alpha_3 &= 3a - 1 - c - m^2, \\ \alpha_4 &= 1. \end{aligned}$$

From this, $u(k) = f(m)^3/g(m)^2$, where f, g have *m*-degree 2 and 3, respectively. Let m_i be the slope of the line containing the points T, P_i, Q_i, R_i (i = 1, ..., 7). By our assumption,

$$u(\kappa) = \frac{f(m_1)^3}{g(m_1)^2} = \dots = \frac{f(m_7)^3}{g(m_7)^2},$$

which implies $\frac{f(m)^3}{g(m)^2} = u(\kappa)$ for all m, and this holds true even if one of the m_i 's is infinite. If the rational function $\frac{f(m)^3}{g(m)^2}$ is constant then either its derivative is constant zero, or $f(m) \equiv 0$, or $g(m) \equiv 0$. In any of these cases,

$$3f'(m)g(m) - 2f(m)g'(m) \equiv 0.$$
 (4)

By setting

$$\beta_0 = 2b(c^2 - c + 1),$$

$$\beta_1 = 2ac - 2ac^2 - 2a + 3b^2 + c^2 + c,$$

$$\beta_2 = -2b(3a - 1 - c),$$

$$\beta_3 = 3a^2 - 2ac + c - 2a,$$

(4) becomes

$$54(b^2 - a(a-1)(a-c))^2(\beta_0 + \beta_1 m + \beta_2 m^2 + \beta_3 m^3) \equiv 0.$$

As T is not on Γ , $b^2 - a(a-1)(a-c) \neq 0$ and $\beta_0 = \cdots = \beta_3 = 0$ holds. Define

$$\begin{split} \gamma_0 &= -3b(c-2)(2c-1)(c+1), \\ \gamma_1 &= -2(c^2-c+1)(6a-4+3c+6ac^2+3c^2-4c^3-6ac), \\ \gamma_2 &= -6b(c^2-c+1)^2, \\ \gamma_3 &= -8(c^2-c+1)^3. \end{split}$$

Then

$$0 = \beta_0 \gamma_0 + \beta_1 \gamma_1 + \beta_2 \gamma_2 + \beta_3 \gamma_3 = 18c^2(c-1)^2(c^2 - c + 1).$$

If c(c-1) = 0 then $\beta_0 = 2b = 0$ and $\beta_1 = 2(a-c) = 0$, which implies $T \in \Gamma$, a contradiction. This proves $c^2 - c + 1 = 0$ and $j(\Gamma) = 0$ by (2). Moreover, $\beta_1 = 3b^2 + c^2 + c = 0$ and $\beta_3 = \frac{1}{3}(3a - c - 1)^2 = 0$ imply

$$a = \frac{c+1}{3}, \qquad b = \sqrt{\frac{1-2c}{3}},$$

therefore by (3), the lines X = a and Y = b through T are components of the Hessian curve of Γ . This shows that T is a corner point of Γ .

Theorem 5.2. Under the hypotheses of Proposition 5.1,

- (i) the affine reference frame in $PG(2,\mathbb{K})$ can be chosen such that Γ has affine equation $X^3 + Y^3 = 1$ and that T = (0, 0);
- (ii) there is a perspectivity u of order 3 with center T leaving Γ invariant;
- (iii) the constant cross-ratio κ is a root of the equation $X^2 X + 1 = 0$.

Proof. (i) According to [7, Example IV.4.6.2], the plane cubic Γ' of equation $X^3 + Y^3 = Z^3$ has *j*-invariant 0. Therefore, Γ is projectively equivalent to Γ' . As the Hessian of Γ' is H': XYZ = 0, the corner points of Γ' are (1,0,0), (0,1,0) and (0,0,1). The projectivity $(x,y,z) \mapsto (-y,z,x)$, preserves Γ' , and permutes the corners, thus T = (0, 0, 1) can be assumed w.l.o.g.

(ii) u is the map $(x, y) \to (\varepsilon x, \varepsilon y)$, where ε is a third root of unity in K. (iii) The line Y = mX intersects Γ' in the points

$$\left(\frac{1}{\sqrt[3]{1+m^3}}, \frac{m}{\sqrt[3]{1+m^3}}\right), \left(\frac{\varepsilon}{\sqrt[3]{1+m^3}}, \frac{\varepsilon m}{\sqrt[3]{1+m^3}}\right), \left(\frac{\varepsilon^2}{\sqrt[3]{1+m^3}}, \frac{\varepsilon^2 m}{\sqrt[3]{1+m^3}}\right),$$

whose cross-ratio with $(0,0)$ is $\kappa = -\varepsilon$.

whose cross-ratio with (0,0) is $\kappa = -\varepsilon$.

Therefore the following result holds.

Theorem 5.3. Let \mathbb{K} be an algebraically closed field of characteristic different from 2 and 3. Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \ge 7$ embedded in $PG(2, \mathbb{K})$ which lies on an irreducible cubic curve Γ . If Γ is either singular or is nonsingular with $j(\Gamma) \neq 0$ then Λ is not in perspective position. If $j(\Gamma) = 0$ then there are at most three points T_1, T_2, T_3 such that Λ is in perspective position with center T_i .

Proof. Let T be a point such that Λ is in perspective position with center T. By Proposition 2.3, Λ has a constant cross-ratio κ , hence Theorem 5.2 applies over the algebraic closure of \mathbb{K} .

Theorem 5.3 has the following corollary.

Corollary 5.4. Let \mathbb{K} be an algebraically closed field of characteristic different from 2 and 3. Then no dual 4-net of order $n \geq 7$ embedded in $PG(2, \mathbb{K})$ has a derived dual 3-net lying on a plane cubic. Finally, we show the existence of proper algebraic dual 3-nets in perspective position.

Example 5.5. Let \mathbb{K} be a field of characteristic different from 2 and 3, $\Gamma : X^3 + Y^3 = Z^3$, T = (0,0,1) and $u : (x,y,z) \mapsto (\varepsilon x, \varepsilon y, z)$ with third root of unity ε . The infinite point O(1,-1,0) is an inflection point of Γ , left invariant by u. Since u leaves Γ invariant as well, u induces an automorphism of the abelian group of $(\Gamma, +, O)$; we denote the automorphism by u, too. As the line TP contains the points P^u , P^{u^2} for any $P \in \Gamma$, one has $P + P^u + P^{u^2} = O$. Let H be a subgroup of $(\Gamma, +)$ of finite order n such that $H^u = H$. For any $P \in \Gamma$ with $P - P^u \notin H$, the cosets

$$\Lambda_1 = H + P, \qquad \Lambda_2 = H + P^u, \qquad \Lambda_3 = H + P^{u^2}$$

form a dual 3-net which is in perspective position with center T. Indeed, for any $A_1 + P \in \Lambda_1$ and $A_2 + P^u \in \Lambda_2$, the line joining them passes through $-A_1 - A_2 + P^{u^2} \in \Lambda_3$.

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