Quantum equivalence of σ models related by non Abelian Duality Transformations

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Abstract

Coupling constant renormalization is investigated in 2 dimensional σ models related by non Abelian duality transformations. In this respect it is shown that in the one loop order of perturbation theory the duals of a one parameter family of models, interpolating between the $SU(2)$ principal model and the $O(3)$ sigma model, exhibit the same behaviour as the original models. For the $O(3)$ model also the two loop equivalence is investigated, and is found to be broken just like in the already known example of the principal model.

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1 Introduction

Various target space ('T') duality transformations [1] connecting two seemingly different sigma-models or string-backgrounds are playing an increasingly important role nowadays. It is assumed that models related by these classical transformations are alternative descriptions of the same physical system (also at the quantum level). The duality transfomations were originally formulated in the σ-model description of the Conformal Field Theory underlying string theory (for a review see [2]).

Using the σ-model formulation it has been recently shown that both the Abelian [3] and the non Abelian T-duality [4] transformation rules can be recovered in an elegant way by performing a canonical transformation. This clearly shows that models related by these transformations are *classically* equivalent. By making some formal manipulations in the functional integral without going, however, into the details of regularization, it is not difficult to argue that models which are related by duality transformations correspond to the the same Quantum Field Theory (QFT) [1],[5]. While this may be sufficient for conformal invariant string backgrounds [6] (like the gauged [7] or ungauged WZWN [8] models) when no perturbative quantum corrections are expected, we feel, that from a pure 2d field theory point of view, the question of quantum equivalence between sigma models related by duality transformations deserves further study.

Concentrating mainly on Abelian duality, such a study was initiated in ref.[9], where the various sigma models were treated as "ordinary" (i.e. not necessarily conformally invariant) two dimensional quantum field theories in the framework of perturbation theory. Working in a field theoretic rather than string theoretic framework i.e. working without the dilaton on a flat, non dynamical 2 space, it was shown on a number of examples that the 'naive' (tree level) T-duality transformations in 2d σ-models cannot be exact symmetries of the quantum theory. The 'naive' Abelian duality transformations are correct to one loop in perturbation theory [9],[10], they break down in general, however, at the two loop order, and to promote them to full quantum symmetries some non trivial modifications are needed [11], [12]. These conclusions were reached by analyzing and comparing various β functions in the original and dual theories.

The aim of this paper is to repeat as much of this program as possible for sigma models related by non Abelian duality transformations [13], [14],[15],[16]. Non Abelian duality is a special case of the so called Poisson-Lie T duality [17],[18], which generalizes the concept of T duality for sigma models without isometries. The motivation for this investigation came from several directions. First of all quantum equivalence among sigma models related by non Abelian duality has some problems even in the conformal invariant case, as there are examples [19] where non Abelian duality is mapping a conformal invariant background to a non conformal dual. The second, 'non conformal' motivation is the discovery [9],[20] that the relation between the $SU(2)$ principal model and its non Abelian dual shows the same features as in the case of Abelian duality: in the one loop order the two models are equivalent while at two loops the dual is not renormalizable in the usual, field theoretic sense. The investigation of this problem is also made urgent by one of the results of [21]. In ref.[21] exact S matrices were proposed for a particular class of $2d$ models and in an appropriate limit these S matrices yield a non perturbative S matrix, which can be associated to the non Abelian dual of the $SU(2)$ principal model, and which is *identical* to the well known S matrix of the principal model.

Therefore, in this paper, we consider a one parameter family of sigma models interpolating between the $SU(2)$ principal model and the $O(3)$ sigma model together with the non Abelian dual of this family, and investigate their renormalization. The interest in this one parameter family comes from two sources: on the one hand it provides a convenient laboratory to compare in a more general setting the renormalization of sigma models connected by non Abelian duality, and on the other, by enlarging the parameter space of the principal model it may provide a sufficient generalization where the two loop renormalizability of the dual model is restored. (This phenomenon was recently shown to happen in some $SL(3)$ sigma models and their Abelian duals [22]). As we show in detail, for the general member of this family, the renormalization of the duals in the one loop order leads to the same β functions as in the case of the original models. However for the only case besides the $SU(2)$ principal model, when the complexity of the two loop analyzis becomes tractable, namely for the $O(3)$ model, we find that the dual is not renormalizable in this order.

The paper is organized as follows: in sect.2 we describe in some detail the two sets of models which are related to each other by non Abelian duality transformation. In sect.3 we discuss some aspects of the canonical transformation implementing this duality transformation in the classical theory. In sect.4 we give a short summary of the renormalization procedure used and apply it in detail to our models. We discuss the results and make our conclusions in sect.5. The somewhat complicated expressions for the components of the generalized Ricci tensor, that determine the one loop counterterm, are collected in the Appendix.

2 The dually related models

We choose the "original" model from the class of 'deformed principal' models, the Lagrangian of which can be written as:

$$\mathcal{L}=\frac{1}{2}R_{ab}L^a_{\ i}L^b_{\ j}\partial_\mu\xi^i\partial^\mu\xi^j, \tag{1}$$

where ξ^j, $j=1,...\mathrm{dim}\mathbf{G}$, parametrize the elements, G, of a group, $\mathbf{G}$, and $L^a_{\ i}$ denote the components of the left invariant Maurer Cartan one form:

$$L^a_{\ i}=\frac{1}{\omega}\mathrm{tr}(\lambda^a G^{-1}\frac{\partial G}{\partial\xi^i}). \tag{2}$$

(λ^a stand for the generators of the Lie algebra of $\mathbf{G}$: $[\lambda^a,\lambda^b]=f^{abc}\lambda^c$, normalized according to $\mathrm{tr}(\lambda^a\lambda^b)=\omega\delta^{ab}$. The inverse of $L^a_{\ i}$ is denoted by $L^i_{\ a}$, and we frequently use the abbreviation $L^a_\mu=L^a_{\ i}\partial_\mu\xi^i$). In Eq. (1) R_{ab} is a constant (i.e. ξ^i independent), symmetric $R_{ab}=R_{ba}$ matrix, that describes the deviation of the model from the $\mathbf{G}_L \times \mathbf{G}_R$ invariant principal model, which is obtained if $R_{ab} \sim \delta_{ab}$. In the general case, the presence of R_{ab} breaks the symmetry of the principal model to $\mathbf{G}_L \times \mathbf{H}_R$, where the actual form of the subgroup $\mathbf{H}_R$ depends on the actual form of R_{ab}. R_{ab} also appears in the equations of motion following from Eq. (1):

$$R_{ac}(\partial_\mu L^{a\mu}) - R_{ab} L^a_\mu L^{k\mu} f^{bkc} = 0. \tag{3}$$

Therefore, although even in the general case L^a_μ form curvature free currents on account of the definition, Eq. (2),

$$\partial_\mu L^a_\nu - \partial_\nu L^a_\mu + f^{abc} L^b_\mu L^c_\nu = 0, \tag{4}$$

they are no longer conserved. Note, however, that the Noether currents:

$$C^a_\mu = R_{cd} N^c{}_a L^d_\mu, \qquad N^c{}_a = \frac{1}{\omega} \mathrm{tr}(\lambda^c G^{-1} \lambda_a G), \tag{5}$$

are conserved as a result of $\mathcal{L}$ being invariant under the left $\mathbf{G}$ transformations: $G \to WG$, $W \in \mathbf{G}$.

The non Abelian dual of the model in Eq. (1) can be written as [16],[17],[18],[23],[24]:

$$\mathcal{L}^d = \frac{1}{2} \partial_+ \chi^a [M^{-1}]^{ab} \partial_- \chi^b, \tag{6}$$

where χ^a are coordinates on the Lie algebra of $\mathbf{G}$, $\partial_\pm \chi^a = \partial_\tau \chi^a \pm \partial_\sigma \chi^a$, and the matrix M is given as:

$$M^{ab} = R^{ab} + f^{abc} \chi^c. \tag{7}$$

The transformation connecting $\mathcal{L}$ and $\mathcal{L}^d$ is a special case of the Poisson Lie T duality [17],[18], when the dual group is Abelian.

The actual pair of dual models where we investigate the question of their quantum equivalence is obtained from Eqs. (1-7) by choosing $\mathbf{G} = SU(2)$ and

$$R_{ab} = \frac{1}{\lambda} \mathrm{diag}(1, 1, 1 + g), \tag{8}$$

where λ is the coupling constant (which is there even for the principal model) and g is the deformation parameter. Using the Euler angles (ϕ,θ,ψ) to parametrize the elements of $SU(2)$, G is written as

$$G = e^{\phi \tau^3} e^{\theta \tau^1} e^{\psi \tau^3}, \tag{9}$$

(where $\tau^a = \sigma^a/(2i)$, with σ^a being the standard Pauli matrices), and one readily obtains

$$\begin{aligned} L^3_\mu &= \partial_\mu \psi + \partial_\mu \phi \cos\theta, \\ L^1_\mu &= \partial_\mu \theta \cos\psi + \partial_\mu \phi \sin\theta \sin\psi, \\ L^2_\mu &= -\partial_\mu \theta \sin\psi + \partial_\mu \phi \sin\theta \cos\psi. \end{aligned} \tag{10}$$

Then the Lagrangian of the deformed $SU(2)$ model (1) becomes

$$\begin{aligned}\mathcal{L} =& \frac{1}{2\lambda}\{(\partial_\mu\theta)^2 + (\partial_\mu\phi)^2(1+g\cos^2\theta) + \\ & (1+g)(\partial_\mu\psi)^2 + 2(1+g)\partial_\mu\phi\partial^\mu\psi\cos\theta\}\,.\end{aligned} \tag{11}$$

Note that for $g=-1$ the ψ field decouples and Eq. (11) reduces to the Lagrangian of the $O(3)$ sigma model. Thus Eq. (11) describes a one parameter family of models interpolating between the $SU(2)$ principal model ($g=0$) and the $O(3)$ one. Using the explicit form of R_{ab} in Eq. (3) shows that

$$(1+g)\partial_\mu L^{3\mu}=0, \qquad \partial_\mu L^{\,1}_{\,2}{}^{\mu} \pm g L^3_\mu L^{\,2}_{\,1}{}^{\mu}=0; \tag{12}$$

i.e. (for $g\neq -1$) L^3_μ is conserved. This conservation is the manifestation of Eq. (11) being invariant under the $\psi\to\psi+\psi_0$ translation. Since this translation acts on G, Eq. (9), on the right, the total symmetry group of the deformed $SU(2)$ model is $SU(2)_L\times U(1)_R$.

As only R_{ab} is singular for $g=-1$, but M^{ab} is not, the dual model is readily defined for all $g\geq -1$. Rescaling the χ^a fields appropriately and introducing the variables:

$$\chi^1=\rho\cos\alpha,\quad \chi^2=\rho\sin\alpha,\quad \chi^3=z, \tag{13}$$

the Lagrangian of the dual model assumes the form:

$$\begin{aligned}\mathcal{L}^d =& \frac{1}{2\lambda D}\Big((1+g+\rho^2)(\partial_\mu\rho)^2+(1+g)\rho^2(\partial_\mu\alpha)^2+(1+z^2)(\partial_\mu z)^2+ \\ & 2z\rho\partial_\mu z\partial^\mu\rho - 2\epsilon_{\mu\nu}((1+g)z\rho\partial^\mu\rho\partial^\nu\alpha+\rho^2\partial^\mu\alpha\partial^\nu z)\Big)\,, \\ & D=(1+g)(1+z^2)+\rho^2\,.\end{aligned} \tag{14}$$

Note that for $g=0$ the substitution $\rho=r\sin\gamma$, $z=r\cos\gamma$, really converts Eq. (14) into the well known non Abelian dual of the $SU(2)$ principal model, while for $g=-1$, when (after discarding a total derivative) the α field decouples, it reduces to a purely metric model:

$$\mathcal{L}^d_{O(3)}=\frac{1}{2\lambda}\Big((\partial_\mu\rho)^2+\frac{1+z^2}{\rho^2}(\partial_\mu z)^2+2\frac{z}{\rho}\partial_\mu z\partial^\mu\rho\Big)\,, \tag{15}$$

which may be called the non Abelian dual of the $O(3)$ sigma model. (This particular non Abelian dual is a special case of the coset examples discussed in [25]). The only manifest symmetry of the dual Lagrangian, Eq. (14), is $U(1)$, corresponding to the $\alpha\to\alpha+\alpha_0$ translation.

3 The canonical transformation

Before investigating the equivalence of the quantized versions of Eqs. (11) and (14) we review the canonical transformation that connects these models classically. In doing so we clarify the relation between the symmetries of the dually related models and also make a minor observation on the interpretation of the transformation itself.

The canonical transformation implementing non Abelian duality in case of the principal models was described in [4],[26], while for the 'left invariant models' (which Eq. (1) belongs to) it was worked out in [23]. Poisson-Lie T duality is by definition a canonical transformation and an expression for the generating functional is given in [18].

We define the canonical momenta of the original and dual models in the usual way:

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{\xi}^j} = R_{ab} L^a{}_j L^b_\tau, \quad \pi_a = \frac{\partial \mathcal{L}^d}{\partial \dot{\chi}^a} = \frac{1}{2}([M^{-1}]^{ab}\partial_- \chi^b + [M^{-1}]^{ba}\partial_+ \chi^b)\,. \quad (16)$$

Then the canonical transformation $(\xi^i, p_j) \to (\chi^a, \pi_b)$ following from the generating functional:

$$F[\chi^a, \xi^i] = \oint_{S^1} \chi^a L^a{}_i \partial_\sigma \xi^i\,, \quad (17)$$

can be written as:

$$\pi_a = \frac{\delta F}{\delta \chi^a} = L^a{}_i \partial_\sigma \xi^i\,, \quad (18)$$

$$p_j = -\frac{\delta F}{\delta \xi^j} = L^a{}_j \partial_\sigma \chi^a - f^{abc} \chi^a L^b{}_i L^c{}_j \partial_\sigma \xi^i\,. \quad (19)$$

(The Maurer Cartan condition, Eq. (4), is used in writing Eq. (19)). Note – as was pointed out in [23] – that both the generating functional and the canonical transfomations have the same form as in the case of the principal model. However, as we show below, in contrast to the principal model, in the general case, when $R_{ab} \not\sim \delta_{ab}$, they map two curvature free (but not necessarily conserved) currents – that are at the starting point of Poisson Lie duality [17] – into each other. In the original model this current is of course L^a_μ. In the dual model, we define the 'dual current' by

$$J^a_- = [M^{-1}]^{ad}\partial_- \chi^d, \qquad J^a_+ = -\partial_+ \chi^d [M^{-1}]^{da}, \quad (20)$$

after observing that using the simple identity:

$$\frac{\partial [M^{-1}]^{ab}}{\partial \chi^c} = -[M^{-1}]^{ad} f^{dec} [M^{-1}]^{eb}, \quad (21)$$

the equations of motion following from Eq. (6) can be interpreted as the curvature free conditions for J^a_μ.

Using the definition of π_a, Eq. (16), and Eq. (20) we get $\pi_a = -J^a_\sigma$, thus the first of the canonical transformations, Eq.(18), indeed identifies (up to a sign) the spatial components of L^a_μ and J^a_μ. Multiplying both sides of Eq. (19) by $L^i{}_c$ and exploiting Eq. (18) leads to

$$L^i{}_c p_i = \partial_\sigma \chi^c - f^{abc} \chi^a \pi_b\,. \quad (22)$$

It follows from the definition of p_i, Eq. (16), that the left hand side of this equation is nothing but $R_{bc} L^b_\tau$. A simple computation, using the expression of π_c in terms of $J^a_\pm$, as well as the obvious identities

$$R_{ab} = \frac{1}{2}(M^{ab} + M^{ba}); \qquad f^{abc}\chi^c = \frac{1}{2}(M^{ab} - M^{ba}); \quad (23)$$

which follow from the definition, Eq. (7), confirms, that the right hand side of Eq. (22) can indeed be written as $-R_{ac}J^a_\tau$. Thus the canonical transformation really connects the curvature free (but in general non conserved) L^a_μ and J^a_μ:

$$L^a_\sigma = -J^a_\sigma, \qquad R_{bc}L^b_\tau = -R_{bc}J^b_\tau\,. \tag{24}$$

Note also that $R_{ab}J^b_\mu$ is related in an interesting way to the topological current $T^a_\mu = \epsilon_{\mu\nu}\partial^\nu\chi^a$ and the 'would be' Noether current, $V^a_\pm$,

$$V^a_+ = \frac{1}{2}\partial_+\chi^b[M^{-1}]^{bc}f^{cda}\chi^d, \quad V^a_- = \frac{1}{2}f^{cda}\chi^d[M^{-1}]^{cb}\partial_-\chi^b, \tag{25}$$

that corresponds to the $\chi^a \to \chi^a + f^{abc}\chi^b\omega^c$ transformation:

$$-R_{ab}J^b_\pm = 2V^a_\pm + T^a_\pm\,. \tag{26}$$

Eq. (22) also imply that the special combinations of the original variables and momenta, $q_c \equiv L^i{}_c p_i$, (which are nothing but $R_{bc}L^b_\tau$), become local in the dual model. Using the basic Poisson brackets $\{\xi^i(\sigma), p_j(\sigma')\} = \delta_{ij}\delta(\sigma-\sigma')$ it is straightforward to show that the bracket among the q_a-s in the original model has the form $\{q_a(\sigma), q_b(\sigma')\} = f^{abc}q_c(\sigma)\delta(\sigma-\sigma')$. As the transformation between the original and dual models is canonical, we get the same, if we use $\{\chi^a(\sigma), \pi_b(\sigma')\} = \delta_{ab}\delta(\sigma-\sigma')$ and identify q_c with the quantities on the right hand side of Eq. (22). However the integrals $\int d\sigma(\partial_\sigma\chi^c - f^{abc}\chi^a\pi_b)$ generate local, conserved charges in the dual model only for those special values of c, (if there is any), for which $R_{cb}L^b_\mu$ is conserved (see Eq. (3)).

In the deformed $SU(2)$ model only L^3_μ is conserved; and since $L^i{}_3 p_i = p_\psi$, the conserved $U(1)_R$ charge is $\int d\sigma p_\psi$. In the dual model, for $c=3$, in terms of the new variables, Eq.(13), the right hand side of Eq. (22) can be written as $\partial_\sigma z - \pi_\alpha$. Thus (discarding the uninteresting integral of a total derivative) we see that the 'image' of the $U(1)_R$ charge in the dual model is the conserved 'α charge' $\int d\sigma\pi_\alpha$. This explaines the $\alpha \to \alpha + \alpha_0$ symmetry of Eq. (14). The charges of the left Noether currents, $\int C^a_\tau d\sigma$, of the original model become non local in the dual, since C^a_τ can be written as $C^a_\tau = N^c{}_a(\xi)L^i{}_c p_i = N^c{}_a(\xi)(\partial_\sigma\chi^c - f^{dbc}\chi^d\pi_b)$.

It is interesting to understand what happens for $g=-1$, i.e. to study in more details the canonical transformation connecting the $O(3)$ sigma model and its non Abelian dual. The problem with this is that the generating functional, $F[\xi^i, \chi^a]$, Eq. (17), is independent of g, yet for $g=-1$ the ψ field of the $O(3)$ and the α field of the dual models decouple. In the Hamiltonian formalism, these decouplings can be handled in general by the procedure described in [25]. In the present case, the canonical transformation connecting these two models can be described by the following generating functional

$$F = \oint d\sigma\,\{z\psi' + z\phi'\cos\theta + \rho\theta'\cos\psi + \rho\phi'\sin\theta\sin\psi\}\,, \tag{27}$$

which is obtained formally from Eq. (17) by setting $\alpha = 0$. However ψ is not an independent field now, but is rather a functional of the other fields (and their derivatives), as determined from

$$\frac{\delta F}{\delta\psi} = 0. \tag{28}$$

Thus ψ satisfies

$$\rho\phi'\sin\theta\cos\psi - \rho\theta'\sin\psi = z' \tag{29}$$

and could be expressed algebraically in terms of the other fields. However we do not need this explicit expression, as Eq. (28) guarantees that for the canonical transformation of the other fields, ψ appears as an independent field:

$$\begin{aligned}
\pi_z &= \frac{\delta F}{\delta z} = \psi' + \phi'\cos\theta\,, \\
\pi_\rho &= \frac{\delta F}{\delta \rho} = \theta'\cos\psi + \phi'\sin\theta\sin\psi\,, \\
p_\phi &= -\frac{\delta F}{\delta \phi} = (z\cos\theta)' + (\rho\sin\theta\sin\psi)'\,, \\
p_\theta &= -\frac{\delta F}{\delta \theta} = z\phi'\sin\theta + (\rho\cos\psi)' - \rho\phi'\cos\theta\sin\psi\,.
\end{aligned} \tag{30}$$

¿From these equations, using Eq. (29), one readily obtains the derivatives of the original coordinates:

$$\begin{aligned}
\theta' &= -\frac{z'}{\rho}\sin\psi + \pi_\rho\cos\psi\,, \\
\sin\theta\phi' &= \frac{z'}{\rho}\cos\psi + \pi_\rho\sin\psi\,,
\end{aligned} \tag{31}$$

and the original momenta:

$$\begin{aligned}
p_\phi &= \sin\theta[\cos\psi(\rho\pi_z - z\pi_\rho) + \sin\psi(\rho' + \frac{zz'}{\rho})]\,, \\
p_\theta &= [-\sin\psi(\rho\pi_z - z\pi_\rho) + \cos\psi(\rho' + \frac{zz'}{\rho})]\,.
\end{aligned} \tag{32}$$

Eqs. (31,32) give the required transformation between the $O(3)$ sigma model and its non Abelian dual.

Finally we discuss briefly the question of potential quantum corrections to the canonical transformations. Since the generating functional, Eq. (17), is non linear in the variables of the original model, one may expect that it receives quantum corrections [27], when implemented in the functional integral. In quantum theory, using Schrödinger wave functional techniques, a *formal* equivalence between the two theories can be established if the energy-momentum eigenfunctionals of the dual theory, $\Psi_{E,p}[\chi^a]$, and those of the original theory, $\Phi_{E,p}[\xi^i]$, are related to each other by a non linear functional Fourier transformation [27], [4]:

$$\Psi_{E,p}[\chi^a] = N(E,p)\int \prod_{i=1}^{\dim\mathbf{G}} \mathcal{D}\xi^i e^{i\tilde{F}[\xi^i,\chi^a]}\Phi_{E,p}[\xi^i]. \tag{33}$$

If $\tilde{F}[\xi^i,\chi^a]$ coincides with the classical generating functional, $F[\xi^i,\chi^a]$, then the classical transformation receives no quantum corrections. Let us make the ansatz that indeed Eq. (33) is valid with the classical generating functional, and then verify that, as required, an eigenfunction $\Phi_{E,p}[\xi^i]$ of the original Hamiltonian becomes transformed into an eigenfunction $\Psi_{E,p}[\chi^a]$ of the dual one. As a result of Eq. (22,24), the exponentiated classical generating functional satisfies the functional differential equations:

$$J^a_\sigma\Big[\pi^b \equiv -i\frac{\delta}{\delta\chi^b}\Big]\mathrm{e}^{iF[\xi^i,\chi^a]} = -L^a_\sigma[\xi^i]\mathrm{e}^{iF[\xi^i,\chi^a]},$$
$$R_{bc}J^b_\tau\Big[\chi^a,\pi^b \equiv -i\frac{\delta}{\delta\chi^b}\Big]\mathrm{e}^{iF[\xi^i,\chi^a]} = L^j{}_c[\xi^i]i\frac{\delta}{\delta\xi^j}\mathrm{e}^{iF[\xi^i,\chi^a]} \equiv -R_{bc}L^b_\tau\mathrm{e}^{iF[\xi^i,\chi^a]}\,, \tag{34}$$

and as a consequence, $\Psi_{E,p}$ and $\Phi_{E,p}$ obey e.g.

$$J^a_\sigma\Psi_{E,p}[\chi^a] = -N(E,p)\int\prod_{i=1}^{\dim\mathbf{G}}\mathcal{D}\xi^i L^a_\sigma\mathrm{e}^{iF[\xi^i,\chi^a]}\Phi_{E,p}[\xi^i], \tag{35}$$

(plus a similar equation where the space components of the currents are replaced by $R\otimes$ the corresponding time components). Now, at least for the deformed $SU(2)$ model, Eq. (11), the Hamiltonian is a simple quadratic expression of the L^a_μ-s (which, in turn, are expressed in terms of p_j and ξ^i):

$$H = \frac{1}{2\lambda}\int d\sigma\Big[\sum_{a=1}^{3}((L^a_\tau)^2 + (L^a_\sigma)^2) + g((L^3_\tau)^2 + (L^3_\sigma)^2)\Big]\,. \tag{36}$$

(The Hamiltonian, corresponding to the dual model, Eq. (14), is obtained by using Eq. (24)). Therefore, applying twice the argument leading to Eq. (35), we easily verify that Eq. (33) with $\tilde{F}[\xi^i,\chi^a] = F[\xi^i,\chi^a]$ has the property we wanted to check, just as for the case $g = 0$ [4]. Note, however, that this argument is formal in the sense that it does not take into account the effects of renormalization.

4 Renormalization of the dually related models

4.1 Coupling constant renormalization procedure

To carry out explicitly the renormalization of the original and dual models, Eq .(11) and (14), we use the general strategy developed in [9]. Since it is desribed there in quite some detail here we summarize only the main points. The procedure is based on the well known one and two loop counterterms [28],[29],[30] for the class of general bosonic σ models, obtained by the background field method in the dimensional regularization scheme. The counterterms of the Lagrangian

$$\mathcal{L} = \frac{1}{2\lambda}(g_{ij}\partial_\mu\xi^i\partial^\mu\xi^j + \epsilon_{\mu\nu}b_{ij}\partial^\mu\xi^i\partial^\nu\xi^j) = \frac{1}{\lambda}\tilde{\mathcal{L}}, \tag{37}$$

can be written as

$$\mu^\epsilon\mathcal{L}_1 = \frac{\alpha'}{2\epsilon\lambda}\hat{R}_{ij}(\partial_\mu\xi^i\partial^\mu\xi^j + \epsilon_{\mu\nu}\partial^\mu\xi^i\partial^\nu\xi^j) = \frac{1}{\pi\epsilon}\Sigma_1, \tag{38}$$

and

$$\mu^\epsilon\mathcal{L}_2 = \frac{1}{2\epsilon}(\frac{\alpha'}{2})^2\frac{1}{2\lambda}Y^{lmk}{}_j\hat{R}_{iklm}(\partial_\mu\xi^i\partial^\mu\xi^j + \epsilon_{\mu\nu}\partial^\mu\xi^i\partial^\nu\xi^j) = \frac{\lambda}{8\pi^2\epsilon}\Sigma_2, \tag{39}$$

where $\alpha' = \lambda/(2\pi)$

$$\begin{aligned} Y_{lmkj} =& -2\hat{R}_{lmkj} + 3\hat{R}_{[klm]j} + 2(H^2_{kl}g_{mj} - H^2_{km}g_{lj})\,, \\ H^2_{ij} =& H_{ikl}H_j^{\ kl}\,, \qquad 2H_{ijk} = \partial_i b_{jk} + \text{cyclic}\,, \end{aligned} \tag{40}$$

and $\hat{R}_{ij}$ resp. $\hat{R}_{iklm}$ denote the 'generalized' (i.e. containing torsion) Ricci resp. Riemann tensors of the background g_{ij} and b_{ij}. If the metric, g_{ij}, and the torsion potential, b_{ij}, depend also on a parameter, g; $g_{ij} = g_{ij}(\xi, g)$, $b_{ij} = b_{ij}(\xi, g)$, then the counterterms are converted into coupling, parameter and (in general non-linear) field renormalizations:

$$\begin{aligned} \lambda_0 &= \mu^\epsilon \lambda\Big(1 + \frac{\zeta_1(g)\lambda}{\pi\epsilon} + \frac{\zeta_2(g)\lambda^2}{8\pi^2\epsilon} + ...\Big) = \mu^\epsilon \lambda Z_\lambda(g, \lambda), \\ g_0 &= g + \frac{x_1(g)\lambda}{\pi\epsilon} + \frac{x_2(g)\lambda^2}{8\pi^2\epsilon} + ... = gZ_g(g, \lambda), \end{aligned} \tag{41}$$

$$\xi^j_0 = \xi^j + \frac{\xi^j_1(\xi^k, g)\lambda}{\pi\epsilon} + \frac{\xi^j_2(\xi^k, g)\lambda^2}{8\pi^2\epsilon} + ..., \tag{42}$$

if the $\zeta_i(g)$, $x_i(g)$ and $\xi^j_i(\xi^k, g)$ quantities solve the "conversion equation":

$$-\zeta_i(g)\tilde{\mathcal{L}} + \frac{\partial\tilde{\mathcal{L}}}{\partial g}x_i(g) + \frac{\delta\tilde{\mathcal{L}}}{\delta\xi^k}\xi^k_i(\xi, g) + (\text{gauge}) = \Sigma_i, \qquad i = 1, 2. \tag{43}$$

The appearance of (gauge) in Eqs. (43) implies that the equality of the two sides is required up to a (perturbative) gauge transformation of the antisymmetric tensor field b_{ij}:

$$b_{ij} \to b_{ij} + (\text{gauge}) = b_{ij} + \frac{\lambda}{\pi\epsilon}\partial_{[i}W^{(1)}_{j]}(\xi^k, g) + \frac{\lambda^2}{8\pi^2\epsilon}\partial_{[i}W^{(2)}_{j]}(\xi^k, g), \tag{44}$$

which changes $\tilde{\mathcal{L}}$, but leaves the torsion invariant. ($[ij] = ij - ji$) We emphasize that it is not a priori guaranteed that Eqs. (43) may be solved at all for the unkown quantities. If Eqs. (43) do not have a solution, then the renormalization of the model is not possible within the restricted subspace characterized by the coupling λ and the parameter g, only in the (infinite dimensional) space of all metrics and torsion potentials. On the other hand, if Eqs. (43) admit a solution, then the model is renormalizable in the restricted field theoretic sense, and writing $Z_\lambda = 1 + \frac{y_\lambda(\lambda,x)}{\epsilon} + ...$ and $Z_g = 1 + \frac{y_g(\lambda,x)}{\epsilon} + ...$, the β functions of λ and g, defined in the standard way, $\beta_\lambda = \mu\frac{d\lambda}{d\mu}$, $\beta_g = \mu\frac{dg}{d\mu}$, are readily obtained [9]:

$$\beta_\lambda = \lambda^2\frac{\partial y_\lambda}{\partial\lambda}, \qquad \beta_g = g\lambda\frac{\partial y_g}{\partial\lambda}. \tag{45}$$

In ref.[9] it was shown that the deformed $SU(2)$ model, Eq. (11), is renormalizable in the ordinary sense both in the one and in two loop order: there is no wave function renormalization for θ, ϕ and ψ, while the coupling constant and the parameter get renormalized as in Eq. (41); the solutions of Eq. (43) finally lead to

$$\begin{aligned} \beta_\lambda =& -\frac{\lambda^2}{4\pi}(1 - g + \frac{\lambda}{8\pi}(1 - 2g + 5g^2))\,, \\ \beta_g =& \frac{\lambda}{2\pi}g(1 + g)(1 + \frac{\lambda}{4\pi}(1 - g))\,. \end{aligned} \tag{46}$$

Note that the $g=0$ resp. the $g=-1$ lines are fixed lines under the renormalization group, and β_λ reduces to the β function of the $SU(2)$ principal model, resp. of the $O(3)$ σ-model on them. In the most interesting ($\lambda \geq 0$, $g<0$) quarter of the (λ,g) plane the renorm trajectories run into $\lambda=0$, $g=-1$; while for $g>0$ they run to infinity.

4.2 One loop renormalization of the dual models

To use Eqs. (37 -43) in the renormalization of the dual model, Eq. (14), we index the ρ, α and z fields as ξ^1, ξ^2 and ξ^3. First we work in the one loop order only. The various components of the generalized Ricci tensor, that determine the one loop counterterm, Σ_1, are collected in the Appendix. Inspecting them we deduce the following:

• as $\hat{R}_{ij}$ do not reproduce the polynomial form of the metric, g_{ij}, and torsion potential, b_{ij}, of the dual model, Eq. (14), to abstract the coupling constant renormalization we have to assume that the ρ, α and z fields undergo a (possibly nonlinear) renormalization like in Eq. (42) and also the gauge transformations may be present. We denote the $\xi^k_1(\xi,g)$ one loop corrections to ρ, α and z as $F(\rho,z,\alpha,g)$, $\alpha Y(\rho,z,\alpha,g)$ and $G(\rho,z,\alpha,g)$, respectively and also delete the index 1 from $W_i(\rho,z,\alpha,g)$.

• as $\hat{R}_{ij}$ do not depend on α, (a manifestation, that the background field method preserves the symmetry translating α), none of the F, Y and G functions may depend on α.

• from the (anti)symmetry properties of $\hat{R}_{ij}$ it follows, that Σ_1 contains no new derivative couplings that are not present in $\mathcal{L}^d$, Eq. (14). In particular - as $\hat{R}_{12}=-\hat{R}_{21}$ and $\hat{R}_{32}=-\hat{R}_{23}$ - it contains no $\partial_\mu\rho\partial^\mu\alpha$ and $\partial_\mu z\partial^\mu\alpha$ terms. Therefore $Y(\rho,z,g)$ may depend only on g, $Y=Y(g)$, as the only source of e.g. $\partial_\mu\rho\partial^\mu\alpha$ on the left hand side of Eq. (43) is proportional to $\partial_\rho Y$. Furthermore, as $\hat{R}_{31}=\hat{R}_{13}$ we must have $\partial_1W_3-\partial_3W_1=0$; combining this with the α independence we put as an Ansatz $W_1=W_3=0$, $W_2=W(\rho,z,g)$.

Thus there are three functions of ρ, z and g (namely F, G and W) and three fuctions of g ($\zeta_1(g)$, $x_1(g)$ and $Y(g)$) at our disposal to solve the algebro-differential system of equations originating from Eq. (43). Introducing the notation

$$\partial_\rho F = FR, \quad \partial_z F = FZ, \quad \partial_\rho G = GR, \quad \partial_z G = GZ, \tag{47}$$

$$\partial_1W_2-\partial_2W_1=\partial_\rho W = WR, \quad \partial_2W_3-\partial_3W_2=-\partial_z W=-WZ, \tag{48}$$

and comparing the coefficients of $(\partial_\mu\rho)^2$, $\epsilon_{\mu\nu}\partial^\mu\rho\partial^\nu\alpha$, $(\partial_\mu\alpha)^2$, $\partial_\mu\rho\partial^\mu z$, $\epsilon_{\mu\nu}\partial^\mu\alpha\partial^\nu z$ and $(\partial_\mu z)^2$ on the two sides of Eq. (43) one finds indeed:

$$\begin{aligned}&-\frac{\zeta_1\,(1+\,g+\,\rho^2)}{2\,D}-\frac{(1+\,g+\,\rho^2)\,(2\,\rho F+x_1\;(1+z^2)+2\;(1+g)\,zG)}{2\,D^2}\\&+\frac{2\,x_1\;+4\,\rho F+4\;(1+g+\rho^2)\,FR+4\,z\rho GR}{4\,D}=\frac{\hat{R}_{11}}{4},\end{aligned} \tag{49}$$

$$\begin{aligned}&\frac{\zeta_1 z\rho\,(1+g)}{2\,D}+\frac{z\rho\,(1+g)\,\left(2\,\rho F+x_1\,\left(1+z^2\right)+2\,\left(1+g\right)zG\right)}{2\,D^2}\\&-\frac{\left(G\rho+zF+z\rho FR+z\rho Y\right)(1+g)+z\rho x_1-\rho^2 GR}{2\,D}+\frac{WR}{2}=\frac{\hat{R}_{12}}{4},\end{aligned} \tag{50}$$

$$\begin{aligned}&-\frac{\zeta_1\,(1+g)\,\rho^2}{2\,D}-\frac{(1+g)\,\rho^2\,\left(2\,\rho F+x_1\,\left(1+z^2\right)+2\,\left(1+g\right)zG\right)}{2\,D^2}\\&+\frac{2\,x_1\,\rho^2+4\,\left(1+g\right)\rho F+4\,\left(1+g\right)\rho^2\,Y}{4\,D}=\frac{\hat{R}_{22}}{4},\end{aligned} \tag{51}$$

$$\begin{aligned}&-\frac{\zeta_1 z\rho}{2\,D}-\frac{z\rho\,\left(2\,\rho F+x_1\,\left(1+z^2\right)+2\,\left(1+g\right)zG\right)}{2\,D^2}+\\&\frac{\left(1+g+\rho^2\right)FZ+\left(1+z^2\right)GR+G\rho+zF+z\rho FR+z\rho GZ}{2\,D}=\frac{\hat{R}_{13}}{4},\end{aligned} \tag{52}$$

$$\begin{aligned}&\frac{\zeta_1\rho^2}{2\,D}+\frac{\rho^2\,\left(2\,\rho F+x_1\,\left(1+z^2\right)+2\,\left(1+g\right)zG\right)}{2\,D^2}\\&+\frac{z\rho\,(1+g)\,FZ-2\,\rho F-\rho^2\,Y-\rho^2\,GZ}{2\,D}-\frac{WZ}{2}=\frac{\hat{R}_{23}}{4},\end{aligned} \tag{53}$$

$$\begin{aligned}&-\frac{\zeta_1\,\left(1+\,z^2\right)}{2\,D}-\frac{\left(1+\,z^2\right)\left(2\,\rho F+x_1\,\left(1+z^2\right)+2\,\left(1+g\right)zG\right)}{2\,D^2}\\&+\frac{4\,zG+4\,\left(1+z^2\right)GZ+4\,z\rho FZ}{4\,D}=\frac{\hat{R}_{33}}{4},\end{aligned} \tag{54}$$

(where $D = \rho^2 + (1+g)(1+z^2)$). Note that this system contains F, FR, FZ, G, GR and GZ linearly, thus - apart from some pathological cases - these quantities may be determined from Eqs.(49-54) algebraically. Then we require that Eq.s (47) hold; i.e. that FR be indeed $\partial_\rho F$, etc.. These requirements yield four equations that should determine $\zeta_1(g)$, $x_1(g)$ and $Y(g)$. Note however, that these four equations should be satisfied for all values of ρ and z, thus it is not clear at all that a choice of ζ_1, x_1 and Y, depending only on g, exists that guarantees this. This is the point where the gauge transformation described by W_i plays an essential role, as we must try to choose it in such a way that the four equations yield a ρ and z independent solution. We emphasize that the imposition of Eq.s (47) on the algebraic solution of Eq.s (49-54) is the essential step of our renormalization program, as this step guarantees that the emerging new couplings can be accounted for by a nonlinear field redefinition.

After some effort the procedure just described yields the following solution:

$$\begin{aligned}F&=\frac{\rho(1-gz^2)}{2D}+\frac{g-1}{4}\rho,\quad Y=0,\\G&=\frac{z[(1+g)^2+g\rho^2]}{2D}+\frac{g-1}{4}z,\quad W=-\frac{zg\rho^2}{4D}\end{aligned} \tag{55}$$

$$\zeta_1(g)=-\frac{1-g}{4},\qquad x_1(g)=\frac{g(1+g)}{2}. \tag{56}$$

Therefore, in the one loop order, $\mathcal{L}^d$ may be renormalized in the restricted, field theoretic sense, i.e. it is really possible to convert the counterterm Σ_1, into coupling constant and parameter renormalizations. Furthermore using the

ζ_1 and x_1 in Eq.s (41) and (45), reproduces the one loop β_λ of the original model in Eq. (46). Thus as far as one loop coupling constant renormalization is concerned the equivalence between the non Abelian dual and the original sigma model is established.

Note that for $g = 0$ the gauge contribution vanishes in Eq.s (55,56) while the F and G one loop corrections to ρ and z are the same as the ones obtained by the $\rho = r\cos\gamma$, $z = r\sin\gamma$ substitution from the renormalization of the 'spherically symmetric' non Abelian dual of the $SU(2)$ principal model [9],[20].

Notice, however, that for $g = -1$, i.e. for the dual of the $O(3)$ model, the gauge contribution does not vanish in spite of the decoupling of the α field in Eq. (14). Therefore it is not entirely clear, that starting with the simpler (purely metric) form of the dual Lagrangian, Eq. (15), one would end up with the solution in (55,56): using $\mathcal{L}^d_{O(3)}$ and the $g \equiv -1$ form of $\hat{R}_{ij}$ in eq. (43) yields a system consisting of three equations only, instead of the six ones in (49)-(54), and the gauge contribution is *absent* as is the antisymmetric tensor field. To clarify this question and to confirm the results in (55,56) we repeat the renormalization of the non Abelian dual of the $O(3)$ sigma model using $\mathcal{L}^d_{O(3)}$, Eq. (15). Since the emerging formulae become much simpler this way, this also makes possible to extend the analysis in this particular example to two loops.

4.3 The dual of the $O(3)$ model at two loops

Since the target space of this model is two dimensional, it is possible to find new variables, $x = x(\tau,\sigma)$, $y = y(\tau,\sigma)$ instead of ρ and z,

$$\rho = \Psi(x,y), \qquad z = \Gamma(x,y), \tag{57}$$

such that the target space metric is manifestly conformal to the flat one:

$$ds^2 = d\rho^2 + \frac{1+z^2}{\rho^2}dz^2 + 2\frac{z}{\rho}d\rho dz = f^2(x,y)(dx^2+dy^2)\,. \tag{58}$$

In terms of these fields, the Lagrangian, $\mathcal{L}^d_{O(3)}$, assumes the form

$$\mathcal{L} = \frac{1}{2\lambda}f^2(x,y)[(\partial_\mu x)^2 + (\partial_\mu y)^2], \tag{59}$$

and the (ordinary) Ricci tensor, that determines Σ_1 in the absence of torsion, is readily obtained:

$$\tilde{R}_{ij} = -\delta_{ij}(\partial_x^2 + \partial_y^2)\ln f. \tag{60}$$

Notice, that $\tilde{R}_{ij}$, just like g_{ij} is proportional to δ_{ij}. Therefore there is no $\partial_\mu x\partial^\mu y$ term in Σ_1, and if we denote by $X = X(x,y)$ and $Y = Y(x,y)$ the $\xi_1^k(\xi)$ one loop corrections (eq. (42)) to x and y respectively, then the vanishing of $\partial_\mu x\partial^\mu y$ on the left hand side of Eq. (43), using Eq. (59), requires:

$$\partial_y X = -\partial_x Y. \tag{61}$$

As $\tilde{R}_{ij} \sim \delta_{ij}$ the coefficients of $(\partial_\mu x)^2$ and $(\partial_\mu y)^2$ must be equal on the left hand side of Eq. (43) too; from this it follows that:

$$\partial_x X = \partial_y Y. \tag{62}$$

Eq.s (61) and (62) imply that X and Y are harmonic functions, and are the real and imaginary parts of a holomorphic function $\epsilon(x,y)=X+iY$. Finally, writing $f=\exp(\Phi)$, the only 'non trivial' equation, following from Eq. (43) and Eq. (59) is:

$$-\frac{\zeta_1}{2}+\partial_x X+(X\partial_x\Phi+Y\partial_y\Phi)=-\frac{1}{4}\mathrm{e}^{-2\Phi}(\partial_x^2+\partial_y^2)\Phi\,. \tag{63}$$

Thus the question of one loop renormalizability of $\mathcal{L}^d_{O(3)}$ can be formulated whether for the given Φ (see explicitely below) it is possible to choose the ζ_1 constant such that (63) admits harmonic solutions X and Y.

To obtain the explicit form of Φ we have to find the mapping $\Psi(x,y)$ and $\Gamma(x,y)$, Eq. (57). The second equality in Eq. (58) can be transformed into a system of differential equations for

$$\frac{1}{2}\Psi^2(x,y)=K(x,y);\qquad \frac{1}{2}\Gamma^2(x,y)=H(x,y); \tag{64}$$

which admits the solution:

$$H=a_0x^2,\qquad K=-a_0x^2+y\epsilon\sqrt{2a_0}, \tag{65}$$

and leads finally to

$$f^2(x,y)=\frac{1}{-x^2+y\epsilon\sqrt{2/a_0}};\qquad \Phi=-\frac{1}{2}\ln(-x^2+y\epsilon\sqrt{2/a_0})\,. \tag{66}$$

(Here $a_0>0$ is a constant of integration and the sign, $\epsilon=\pm$, is chosen so as to guarantee the positivity of f^2 in some domain. Note that we use this mapping only locally, in an appropriate domain of (x,y), the questions about the shape of this domain, its boundary etc. are beyond the scope of this paper). Plugging this Φ into Eq. (63) leads to

$$(-\frac{\zeta_1}{2}+\partial_x X)(-x^2+y\epsilon\sqrt{\frac{2}{a_0}})+Xx-\frac{\epsilon}{\sqrt{2a_0}}Y=-\frac{1}{4}(x^2+y\epsilon\sqrt{\frac{2}{a_0}}+\frac{1}{a_0})\,. \tag{67}$$

This equation admits the solution:

$$\zeta_1=-\frac{1}{2};\qquad X=-x;\qquad Y=-y+\frac{1}{2\epsilon\sqrt{2a_0}}\,. \tag{68}$$

Note that this ζ_1 is the same as the one in Eq. (56) for $g=-1$. However, using the definition, $\rho=\Psi(x,y)=\sqrt{2K}$, $z=\Gamma(x,y)=\sqrt{2H}$, it is easy to show that the non linear redefinition described by this X and Y corresponds to an F and G different from the one in Eq. (55), reflecting the absence of torsion and the gauge transformation. Since only the coupling constant and the parameter have physical significance we can state that the one loop renormalizability of the $O(3)$ model's non Abelian dual and Eq. (56) are confirmed.

The next logical step is to investigate the renormalizability of this model in the two loop order. (The two loop non renomalizability of the non Abelian dual of the $SU(2)$ principal model – which is the only other case of the deformed sigma models when the complexity of this problem becomes tractable – is discussed in [9],[20]). Computing $Y^{lmk}_{\ \ j}\tilde{R}_{iklm}$ for the metric $f^2(x,y)[(dx)^2+(dy)^2]$ one finds it to be proportional to δ_{ij}, thus Σ_2 containes no $\partial_\mu x\partial^\mu y$ term either, and Eq.s (61 ,62) are also valid for the $\tilde{X}(x,y)$, $\tilde{Y}(x,y)$ two loop corrections to x and y. Using the explicit form of Σ_2, the two loop equation following from Eq. (43) can be written as:

$$(-\frac{\zeta_2}{2}+\partial_x\tilde{X})(-x^2+ya)^2+(-x^2+ya)(x\tilde{X}-\frac{a}{2}\tilde{Y})=\frac{1}{8}(a^2+2x^2+2ya)^2\,, \quad (69)$$

(where $a=\epsilon\sqrt{2/a_0}$). Taking into account the polynomial nature of this equation, (i.e. the fact that the coefficients of $\tilde{X}$, $\tilde{Y}$, $\partial_x\tilde{X}$, and also the terms independent of $\tilde{X}$ and $\tilde{Y}$ are finite, well defined polynomials in x and y), it is not difficult to see that there is no choice of ζ_2 that would make possible to find a pair of harmonic $\tilde{X}$ and $\tilde{Y}$ solving Eq. (69). Notice e.g. that the absence of terms containig x^l, $l>4$ and y^m, $m>2$ among the $\tilde{X}$, $\tilde{Y}$ independent terms in Eq. (69) makes the coefficient of all the $k>1$ terms vanish in the natural polynomial Ansatz: $\tilde{\epsilon}(x,y)=\tilde{X}+i\tilde{Y}=\sum\limits_{k=0}^{N}b_kw^k$; $w=x+iy$. The possibility of $\tilde{\epsilon}$ being linear in x and y is eliminated by realizing that the matching of the various x^ly^m terms on the two sides of (69) leads to mutually inconsistent expressions for ζ_2. (The case of a rational $\tilde{\epsilon}$, $\tilde{\epsilon}=\frac{P_N(w)}{Q_M(w)}$, can be ruled out by a similar argument). Therefore we conclude that $\mathcal{L}^d_{O(3)}$ is not renormalizable in the two loop order in the restricted, field theoretic sense.

5 Discussion and conclusions

In this paper we considered a one parameter family of sigma models interpolating between the $SU(2)$ principal model and the $O(3)$ sigma model together with the non Abelian dual of this family, and investigated the renormalization of the coupling constant and the deformation parameter in the two families of models. The interest of the $O(3)$ sigma model and its non Abelian dual stems from the fact that the fields of the former parametrize a coset space while usually non Abelian duality is formulated for sigma models defined on group manifolds; see however ref.[25] for Poisson Lie duality in case of cosets.

Classically these two sets of models are related by a canonical transformation, thus they are equivalent. If this equivalence persisted for the quantized models then the coupling constant and the parameter of the original and dual models should be renormalized in the same way - apart from some potential change in the renormalization scheme.

We found that in the *one loop* order of perturbation theory this expected equivalence shows up for the complete family of models. However, for the two particular models (for $g=0$ and $g=-1$, that describe the $SU(2)$ principal model and the $O(3)$ sigma model respectively), when the two loop analysis becomes tractable we found the equivalence broken in this order. We came to these conclusions by establishing that the system of equations, guaranteeing that the coupling and parameter renormalizations can be extracted in the ususal, field theoretic sense from the counterterms of the sigma models, have such a solution for the *dual* model in the one loop order that leads to the same β functions as the original model. However, in the two loop order for $g = 0$ or -1 at least, these equations have no solutions for the duals, thus the duals are not even renormalizable, hence the equivalence is obviously broken. We emphasize that the essential point is the non equivalence of the dually related models at the two loop level and we are using renormalizability only as a tool to show this.

One may think that the reason behind the two loop discrepancy between the original and dual theories is the same as in the case of Abelian duality [11], namely, that the bare and renormalized quantities do not transfrom in the same way under duality transformations. In this respect the fact that the dual Lagrangian, $\mathcal{L}^d$, is equivalent to the original one for the complete family of deformed sigma models gives support to the idea that as far as coupling constant renormalization is considered non Abelian duality is similar to Abelian duality; as shown in ref.s [9],[10], [31], for models connected by these transformations, in the one loop order, (after carrying out the required, usually highly non trivial field renormalizations in the duals), the coupling constants and the parameters are renormalized in the same way. If, for non Abelian duality, this indeed turns out to be true in general, then one can go on and look for the required modifications of the transformation rules for the *renormalized* quantities in the framework outlined in [11].

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6 Appendix

Here we collect the components of the generalized Ricci tensor of the dual model, Eq. (14).

$$\begin{aligned}\hat{R}_{11} = &-\frac{1}{2\,(1+z^2+g+gz^2+\rho^2)^3}\Big(-3+2\,gz^2\rho^4+8\,\rho^2z^2g^3\\&+3\,g^2z^4\rho^2+g^3z^4\rho^2+3\,z^4g\rho^2-6\,g+12\,gz^2+6\,\rho^2z^2+3\,g\rho^2+2\,z^4g\\&+28\,g^2z^2+8\,g^2z^4-3\,g^2\rho^2+20\,g^3z^2+6\,g^3z^4+20\,gz^2\rho^2+22\,g^2z^2\rho^2\\&-z^4-4\,g^2+7\,\rho^4-2\,g^3+3\,\rho^2+4\,\rho^4z^2g^2+\rho^2z^4\\&+2\,\rho^4z^2-3\,g^3\rho^2-3\,g^2\rho^4+4\,z^2g^4-g\rho^6+g^4z^4-g^4+\rho^6\Big)\end{aligned}\tag{70}$$

$$\hat{R}_{12} = -\hat{R}_{21} = -\frac{2\,\rho\,(1+g)\,z\,\left((1+g)^3+2\,g^2\rho^2+gz^2\rho^2+3\,g\rho^2+g\rho^4\right)}{(1+z^2+g+gz^2+\rho^2)^3}\tag{71}$$

$$\begin{aligned}\hat{R}_{22} = &-\frac{(1+g)^2\,\rho^2}{2\,(1+z^2+g+gz^2+\rho^2)^3}(-g^2+4\,g^2z^2+g^2z^4+4\,z^4g+12\,gz^2\\&-2\,g\rho^2+4\,gz^2\rho^2-3-z^4-2\,\rho^2z^2-\rho^4)\end{aligned}\tag{72}$$

$$\hat{R}_{13} = \hat{R}_{31} = -\frac{z\rho}{2\left(1+z^2+g+gz^2+\rho^2\right)^3}(5\,g+8\,\rho^2+8\,z^2 + 20\,gz^2+2\,\rho^2z^2+g^2+2\,g\rho^2+3\,z^4g+16\,g^2z^2+3\,g^2z^4+2\,gz^2\rho^2 + z^4+\rho^4-g^3-2\,g^2\rho^2+4\,g^2z^2\rho^2+4\,g^3z^2+g^3z^4-g\rho^4+3) \tag{73}$$

$$\hat{R}_{23} = -\hat{R}_{32} = \frac{2\,(1+g)\,\rho^2}{\left(1+z^2+g+gz^2+\rho^2\right)^3}\left(g^2z^2+z^4g+gz^2\rho^2+3\,gz^2-1\right) \tag{74}$$

$$\hat{R}_{33} = -\frac{1}{2\left(1+z^2+g+gz^2+\rho^2\right)^3}(-3-gz^2\rho^4+4\,g^2z^4\rho^2+2\,z^4g\rho^2 - 9\,g+9\,gz^2+6\,\rho^2z^2-10\,g\rho^2+21\,z^4g+9\,g^2z^2+21\,g^2z^4-6\,g^2\rho^2 + 3\,g^3z^2+7\,g^3z^4-3\,g\rho^4+2\,g^2z^2\rho^2+7\,z^4-9\,g^2-\rho^4-3\,g^3 + 3\,z^2+3\,z^6g+3\,z^6g^2+g^3z^6+z^6+2\,\rho^2z^4+\rho^4z^2) \tag{75}$$

For $g=0$ these expressions simplify and become identical to the well known expressions [9],[20],[32] for the non Abelian dual of the $SU(2)$ principal model. For $g=-1$ $\hat{R}_{12}$, $\hat{R}_{22}$, and $\hat{R}_{23}$ vanish identically; this is in accord with the decoupling of the α field.

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