

# Entanglement in the XX spin chain with an energy current

V. Eisler<sup>1,\*</sup> and Z. Zimborás<sup>2,†</sup>

<sup>1</sup>*Institute for Theoretical Physics, Eötvös University, 1117 Budapest, Pázmány sétány 1/a, Hungary*

<sup>2</sup>*Research Institute for Particle and Nuclear Physics, P.O. Box 49, H-1525 Budapest, Hungary*

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We consider the ground state of the XX chain that is constrained to carry a current of energy. The von Neumann entropy of a block of  $L$  neighboring spins, describing entanglement of the block with the rest of the chain, is computed. Recent calculations have revealed that the entropy in the XX model diverges logarithmically with the size of the subsystem. We show that the presence of the energy current increases the prefactor of the logarithmic growth. This result indicates that the emergence of the energy current gives rise to an increase of entanglement.

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## I. INTRODUCTION

Recently entanglement properties of various quantum systems have been the focus of numerous studies. Entanglement plays an essential role in several many-body quantum phenomena, such as superconductivity [1] and quantum phase transitions [2]. It is also regarded as an important resource in quantum computation and information processing [3]. Quantum spin chains offer an excellent theoretical framework for investigating entanglement properties, since several simple models can be solved analytically, and there also exist efficient numerical techniques. This motivated us to work with spin chains in order to investigate the effect of energy current on entanglement.

There are two widely used methods of characterising entanglement in spin chains. The first of these describes the entanglement between two spins in the chain with the quantity called concurrence [4, 5]. The other one measures entanglement of a block of spins with the rest of the chain with the von Neumann entropy, when the chain is in its ground state [6, 7, 8, 9, 10]. This latter method is especially useful when one tries to understand the role of entanglement in quantum phase transitions. These transitions manifest themselves in the appearance of gapless excitations, and are accompanied by a qualitative change in the correlations. In view of the connection between entanglement and quantum correlations, the motivation to characterize a critical system in terms of entanglement properties naturally emerges.

Vidal *et al.* [6] calculated the von Neumann entropy for a wide range of one-dimensional spin models and found that for critical (gapless) ground states the entropy of a block of spins diverges logarithmically with the size of the block, while for noncritical chains it converges to a finite value. The prefactor of the logarithm was argued to be one-third of the central charge of the underlying conformal field theory. These results were supported by

analytical calculations for the XX chain in [7] and for more general Hamiltonians in [9].

Spin chains are simple enough models to investigate also the nonequilibrium effects on entanglement. One can find states that are characterised by the presence of currents of some physical quantities such as energy or magnetization [11, 12]. An important effect of these currents is the rather drastic change in correlations. Therefore introduction of a current can be regarded as a quantum phase transition to a current-carrying phase. Consequently, it is interesting to find the entanglement properties of these current-carrying states.

In this paper we study an XX spin chain constrained to carry an energy current. We calculate the von Neumann entropy of a subsystem of  $L$  contiguous spins. The presence of the energy current maintains the logarithmic asymptotics of the entropy; however, the prefactor of the logarithm is increased from  $1/3$  to  $2/3$ , indicating a higher level of entanglement in the current-carrying states. We also show that at a special value of the current, where the symmetry of the state is enhanced, the asymptotics of the entropy is the same as in the XX chain without current. In the vicinity of these transition points the entropy is shown to display a special type of finite-size scaling.

## II. XX CHAIN WITH ENERGY CURRENT

The XX model is defined through the following Hamiltonian:

$$H^{XX} = - \sum_{l=1}^N (s_l^x s_{l+1}^x + s_l^y s_{l+1}^y) - h \sum_{l=1}^N s_l^z, \quad (1)$$

where  $s_l^\alpha$  ( $\alpha = x, y, z$ ) are the Pauli spin matrices at sites  $l = 1, 2, \dots, N$  of a periodic chain and  $h$  is the magnetic field. Our aim is to constrain the spin chain to carry a prescribed amount of energy current; therefore we use the technique introduced in [12]. Since the local energy satisfies a continuity equation with the local energy current, one can calculate the operator of the total energy

\*eisler@general.elte.hu

†cimbi@rmki.kfki.hu

current:

$$J^E = \sum_{l=1}^N [s_l^z (s_{l-1}^y s_{l+1}^x - s_{l-1}^x s_{l+1}^y) + h (s_l^x s_{l+1}^y - s_l^y s_{l+1}^x)] \quad (2)$$

In order to find the lowest-energy state among the states carrying a given current, one has to introduce a Lagrange multiplier  $\lambda$ , and diagonalize the following modified Hamiltonian:

$$H^E = H^{XX} - \lambda J^E. \quad (3)$$

The ground state of  $H^E$  can be considered as a current-carrying steady state of  $H^{XX}$  at zero temperature.

Since  $[H^{XX}, J^E] = 0$ , one can diagonalize  $H^E$  using the same methods which diagonalize  $H^{XX}$  [13], and the model can be transformed into a set of free fermions with the following spectrum:

$$\Lambda_k = (-\cos k - h)(1 - \lambda \sin k). \quad (4)$$

The ground state can be constructed by occupying all the modes with negative energy, and it remains the same as that of  $H^{XX}$  for  $\lambda \leq 1$ . If the driving field  $\lambda$  exceeds this critical value the energy current starts to flow, and the Fermi sea of the occupied modes splits into two parts. In order to illustrate the occupied regions it is useful to introduce the characteristic wavelengths  $k_h = \arcsin(h)$  and  $k_\lambda = \arccos(\lambda^{-1})$ . The ground state can be analyzed as a function of  $h$  and the expectation value of the current density  $j^E = \langle J^E/N \rangle$ , and the phase diagram shown on Fig.1 can be obtained. Three different phases can be

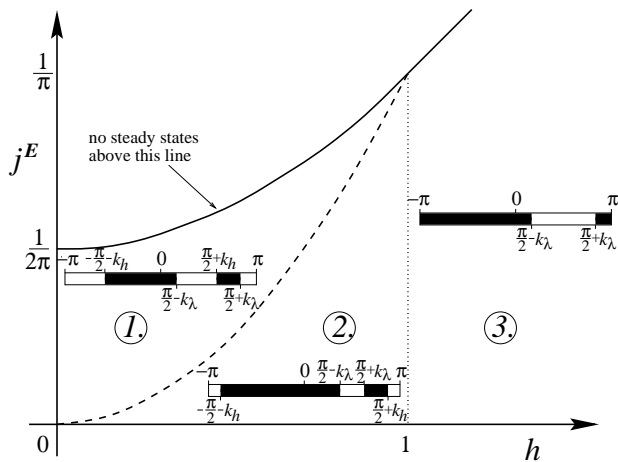


FIG. 1: Phase diagram of the XX model with energy current in the  $h - j^E$  plane, where  $h$  is the transverse field while  $j^E$  is the density of the flux of energy. The occupied fermionic modes are represented by the black parts of the rectangles [12].

distinguished. In phase ② and ③ only the magnetization energy part of the current is flowing, the current of interaction energy is zero, while the transverse magnetization,  $M^z$ , is nonzero. Entering phase ① the current of

interaction energy starts to flow, while  $M^z = 0$  throughout this region. On the line separating regions ① and ② ( $k_h = k_\lambda$ ) the symmetry of the ground state is enhanced, and it is characterised by a single Fermi sea. There are no states above the maximal current line, and in region ③ the ground state is the same along the  $j^E = \text{const.} \times h$  lines, thus it can be represented by the  $h = 1$  borderline, where the two Fermi seas merge. Details of the analysis of the phase space can be found in [12].

### III. ENTROPY OF A BLOCK OF SPINS

We are interested in the ground-state entanglement between a block of  $L$  contiguous spins and the rest of the chain. Following Bennett *et al.* [14] we use von Neumann entropy as a measure of entanglement. It is defined as

$$S_L = -\text{tr}(\rho_L \ln \rho_L), \quad (5)$$

where the reduced density matrix  $\rho_L = \text{tr}_{N-L} |\Psi_g\rangle \langle \Psi_g|$  of the block is obtained from the ground state  $|\Psi_g\rangle$  of the system by tracing out external degrees of freedom.

In the calculation of the entropy we use a similar approach that was successfully applied in case of the XX model [6]. The first step is to introduce the fermionic operators  $c_l$  and  $c_l^\dagger$  through the Jordan-Wigner transformation. Note, that due to the nonsymmetric spectrum, we have to use fermionic operators instead of the Majorana operators that were used earlier in case of the XX model. The ground state in our case can be completely characterised by the expectation values of the two-point correlations  $\langle c_m^\dagger c_n \rangle = G_{mn}$ ; any other expectation value can be expressed through Wick's theorem. The matrix  $G$  reads

$$G = \begin{bmatrix} g_0 & g_1 & \cdots & g_{N-1} \\ g_{-1} & g_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ g_{1-N} & \cdots & \cdots & g_0 \end{bmatrix}, \quad (6)$$

where the coefficients  $g_l$  for an infinite chain ( $N \rightarrow \infty$ ) are given by

$$g_l = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-il\theta} \frac{1}{2} \left( \frac{\Lambda_\theta}{|\Lambda_\theta|} + 1 \right), \quad (7)$$

where  $\Lambda_\theta$  is the spectrum defined in the previous section. Note that the integrand is just the characteristic function of the unoccupied fermionic modes.

From the correlation matrix  $G$  one can extract the entropy  $S_L$  of the block as follows. First, by eliminating from  $G$  the rows and columns corresponding to spins that do not belong to the block, we obtain the correlation matrix  $G_L$  of the state  $\rho_L$ :

$$G_L = \begin{bmatrix} g_0 & g_1 & \cdots & g_{L-1} \\ g_{-1} & g_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ g_{1-L} & \cdots & \cdots & g_0 \end{bmatrix}. \quad (8)$$

In principle one can reconstruct the reduced density matrix  $\rho_L$  using the matrix elements of  $G_L$ . However, the entropy of the block  $S_L$  can be computed in a more direct way from the correlation matrix. Let  $U \in \text{SU}(L)$  denote a unitary matrix that brings  $G_L$  into a diagonal form. This transformation defines a set of  $L$  fermionic operators  $b_m = \sum_{n=1}^L U_{nm} c_n$  which have a diagonal correlation matrix  $\tilde{G}_L = U^\dagger G_L U = \text{diag}(\lambda_1, \dots, \lambda_L)$ . The expectation values are thus

$$\langle b_m b_n \rangle = 0, \quad \langle b_m^\dagger b_n \rangle = \delta_{mn} \lambda_m, \quad (9)$$

that is, the above fermionic modes are *uncorrelated*. Therefore the reduced density matrix can be written as a product state

$$\rho_L = \rho_1 \otimes \dots \otimes \rho_L, \quad (10)$$

where  $\rho_n$  denotes the mixed state of mode  $n$ . Hence the entropy  $\rho_L$  is simply the sum of the entropy of each mode:

$$S_L = \sum_{n=1}^L [-\lambda_n \ln \lambda_n - (1 - \lambda_n) \ln(1 - \lambda_n)] \quad (11)$$

### A. Entropy asymptotics

It follows from Eq. (11) that in order to determine the entropy numerically, one only has to diagonalize the  $L \times L$  matrix  $G_L$ , instead of diagonalizing the original  $2^L \times 2^L$  reduced density matrix  $\rho_L$ . This method reduces considerably the computational effort, and the entropy can be obtained for relatively large block sizes. Fig.2 shows the results of the calculations.

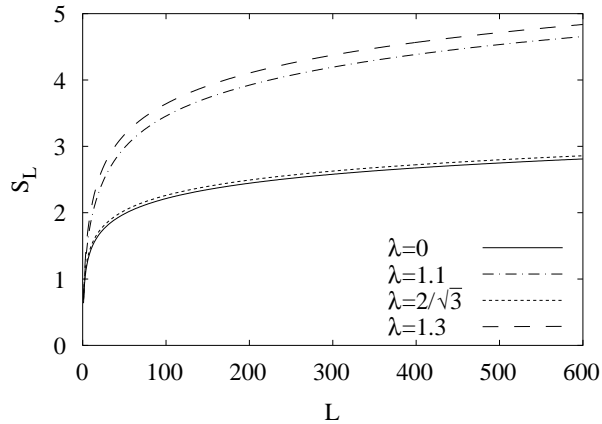


FIG. 2: Entropy calculated from the reduced density matrix as a function of the block size  $L$ . The magnetic field is set to  $h = 0.5$ ; the curves correspond to different values of the current driving field  $\lambda$ . The entropy grows as  $(2/3) \ln L$  in the current-carrying phases ① ( $\lambda = 1.1$ ) and ② ( $\lambda = 1.3$ ) except at the borderline ( $\lambda = 2/\sqrt{3}$ ), where the asymptotics is  $(1/3) \ln L$ , just as in the case of the XX model ( $\lambda = 0$ ).

The ground state entropy of the XX model was first investigated in [6, 7] and for  $h < 1$  it was found to grow

asymptotically as  $\frac{1}{3} \ln L$  with the block size. As one starts to increase the value of the driving field  $\lambda$ , the ground state (and the entropy as well) remains the same up to the critical field  $\lambda_c = 1$ . Further increasing  $\lambda$  one enters the current-carrying phase ② (see Fig.1), where the asymptotics of the entropy changes to  $\frac{2}{3} \ln L$ , and the same behaviour can be observed in phase ①. The only exception is the borderline of these phases (which is characterised by the condition  $k_h = k_\lambda$ ), where the entropy growth is again  $\frac{1}{3} \ln L$ .

For values  $h \geq 1$  of the magnetic field, all the spins are aligned in the ground state of the XX model, thus the entropy of a block vanishes. Switching on the current one observes a  $\frac{1}{3} \ln L$  entropy asymptotics in phase ③.

Summarizing the above results, one concludes that the introduction of an energy current may lead to a more rapid entropy growth, indicating a higher level of ground-state entanglement. The von Neumann entropy can be given for large block sizes ( $L \rightarrow \infty$ ) as

$$S_L = \frac{R}{3} \ln L + S_0, \quad (12)$$

where  $R$  is the number of Fermi seas in the spectrum and  $S_0$  is a function of the parameters  $h$  and  $\lambda$ , independent of  $L$ .

The above result was obtained analytically by Keating and Mezzadri [9] for general quadratic Hamiltonians that have a correlation matrix of Toeplitz type with symmetric fermionic spectrum. In our case the correlation matrix is also of Toeplitz type, but the presence of the current breaks the left-right symmetry, resulting in a nonsymmetric spectrum. Nevertheless, the numerical results indicate that the above asymptotic form (12) of the entropy seems to hold also in this more general case.

The next to leading term  $S_0$  in the entropy is also given in a closed form in [9] for symmetric spectra. Although the spectrum is not symmetric in our model, there are special cases when it can be transformed to a symmetric form. First we note that shifting the wave numbers by  $\varphi$  in the spectrum is equivalent to a unitary transformation  $V^\dagger G V$  of the correlation matrix, where  $V = \text{diag}(1, e^{i\varphi}, e^{2i\varphi}, \dots, e^{(N-1)i\varphi})$ . Since this transformation is diagonal, it leaves the eigenvalues of the reduced density matrix  $G_L$  and thus the entropy invariant.

Now, if  $h\lambda = 1$  ( $k_h + k_\lambda = \frac{\pi}{2}$ ) then the two intervals of the vacant fermionic modes (white parts of the rectangles on Fig.1) have equal lengths, and a shift of the wave numbers by  $\frac{\pi}{4}$  symmetrizes the spectrum. In this case the constant term  $S_0$  can be expressed as follows:

$$S_0 = \begin{cases} \frac{2}{3} \left( \ln \sqrt{4(1 - \lambda^{-2})(2\lambda^{-2} - 1)} + C \right), & k_\lambda < k_h \\ \frac{2}{3} \left( \ln \sqrt{\frac{1 - 2\lambda^{-2}}{1 - \lambda^{-2}}} + C \right), & k_\lambda > k_h \end{cases}, \quad (13)$$

where  $C = 1 + \gamma_E - 6I \ln 2$  is a constant defined through the Euler constant  $\gamma_E$  and  $I \approx 0.0221603$  is a numerically evaluated integral expression [7]. Thus one can see that the entropy can be written with a scaling variable as

$S_L = \frac{2}{3}(\ln \mathcal{L} + C)$ , where

$$\mathcal{L} = \begin{cases} L\sqrt{4(1-\lambda^{-2})(2\lambda^{-2}-1)}, & k_\lambda < k_h \\ L\sqrt{\frac{1-2\lambda^{-2}}{1-\lambda^{-2}}}, & k_\lambda > k_h \end{cases}. \quad (14)$$

Fig.3 shows the numerically calculated entropy with the logarithmic part subtracted for different block sizes. The points perfectly fit the analytically calculated curve (13) except near  $\lambda^{-1} = \frac{1}{\sqrt{2}}$  which corresponds to the line separating the current-carrying phases ① and ②. Therefore, formula (13) is applicable when  $\mathcal{L} \gg 1$ .

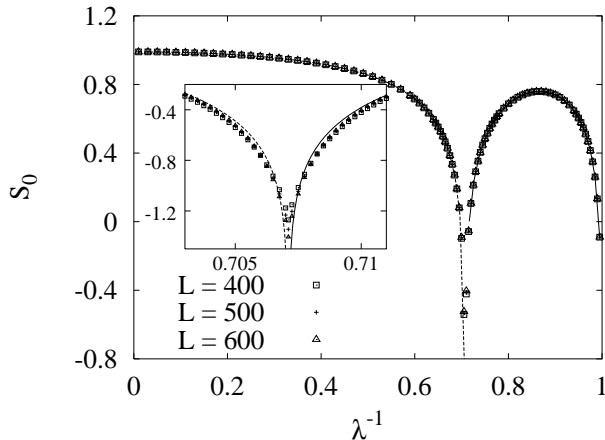


FIG. 3: The next to leading order part of the entropy along the  $h\lambda = 1$  line for different block sizes as a function of  $\lambda^{-1}$ . The lines are analytically calculated from the form of the spectrum. The points perfectly fit the calculated curve except near the vicinity of  $\lambda^{-1} = 1/\sqrt{2}$  (see inset) and  $\lambda^{-1} = 1$ , which corresponds to the boundary of the current-carrying phases.

For nonsymmetric spectra the calculation of the next to leading order term in the entropy is mathematically more involved; hence we were only able to treat the problem numerically. The results reveal that approaching the lines characterised by  $k_h = k_\lambda$ ,  $k_h = 0$  or  $k_\lambda = 0$ ,  $S_0$  seems to diverge. This divergence is a consequence of the changing of the amplitude of the leading term: at the boundaries of the different phases the entropy grows as  $\frac{1}{3} \ln L$ . Thus approaching the boundaries, the  $\frac{2}{3} \ln L$  asymptotics has to be compensated with a negative logarithmic divergence in  $S_0$ .

### B. Finite-size scaling

Obviously, for a fixed  $L$ , there is a finite neighbourhood around the transition lines, where Eq.(12) cannot be used. In the case of a symmetric spectrum it was seen that it holds only when  $\mathcal{L} \gg 1$ . If we would like to characterise the behaviour of the entropy near these transition lines, we have to note that we can associate a diverging length scale, or alternatively a vanishing characteristic wave number to each of the lines. Similarly

to finite-size scaling one writes the entropy near phase transition points as the sum of the “critical” entropy and a term depending only on the product of the block size and the characteristic wave number. For example near the high-symmetry transition line ( $k_h = k_\lambda$ ) it can be written as:

$$S_L(k_h, k_\lambda) = S_L^c + S(L|k_h - k_\lambda|), \quad (15)$$

where  $S_L^c$  is the value of the entropy on the transition line. The numerical calculations support the above type of scaling. Fig.4 shows the numerical results for the scaling function near the high-symmetry line. Similarly, near the other transition lines ( $k_\lambda = 0$  or  $k_h = 0$ ) this type of scaling is valid, but with an other scaling function.

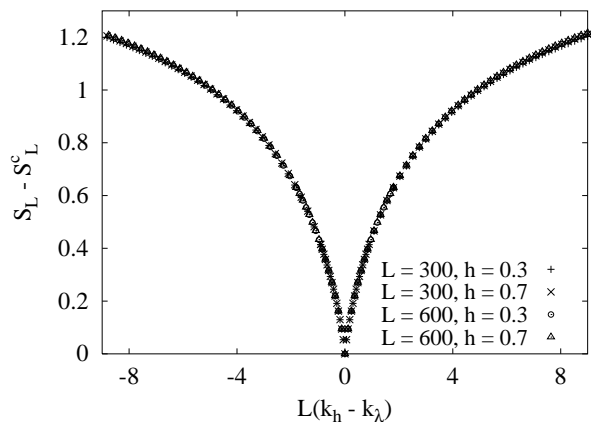


FIG. 4: Scaling function of the entropy near the  $k_h = k_\lambda$  transition line. The “critical” entropy is subtracted from the total entropy and plotted against the scaling variable  $L(k_h - k_\lambda)$ . The curves are calculated for different lattice sizes ( $L = 300$  and  $600$ ) and for different values of the magnetic field ( $h = 0.3$  and  $0.7$ ). The points all fit the same scaling function.

## IV. FINAL REMARKS

We should note that analogously to the energy current, it is possible to introduce a current of magnetization. The resulting spectrum can be written in the same form as that of the XX chain, however with a shift in the wave numbers and a decreased effective magnetic field [12]. Hence, the asymptotics of the entropy will be the same; only the  $L$ -independent constant increases. Thus, interestingly, the magnetization current has a much smaller effect on entanglement.

In summary, we have shown on the example of the XX chain that the introduction of an energy current results a more entangled state. It would be worth considering current-carrying steady states in other spin models, to check whether the increase of entanglement is a general consequence of currents.

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- [1] M. Tinkham, *Introduction to Superconductivity*, 2nd ed. (McGraw Hill, New York, 1996).
  - [2] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, U.K., 2001).
  - [3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Communication* (Cambridge University Press, Cambridge, U.K., 2000).
  - [4] A. Osterloh, L. Amico, G. Falci and R. Fazio, *Nature* (London) **416**, 608 (2002).
  - [5] T. J. Osborne and M. A. Nielsen, *Phys. Rev. A* **66**, 032110 (2002).
  - [6] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, *Phys. Rev. Lett.* **90**, 227902 (2003); J. I. Latorre, E. Rico and G. Vidal, *Quantum Inf. Comput.* **4**, 48 (2004).
  - [7] B. Q. Jin and V. E. Korepin, *J. Stat. Phys.* **116**, 79 (2004).
  - [8] M. Fannes, B. Haegemann and M. Mosonyi, *J. Math. Phys.* **44**, 6005 (2003).
  - [9] J. P. Keating and F. Mezzadri, *Commun. Math. Phys.* **252**, 543 (2004).
  - [10] V. Popkov and M. Salerno, *Phys. Rev. A* **71**, 012301 (2005).
  - [11] T. Antal, Z. Rácz and L. Sasvári, *Phys. Rev. Lett.* **78**, 167 (1997).
  - [12] T. Antal, Z. Rácz, A. Rákos and G.M. Schütz, *Phys. Rev. E* **57**, 5184 (1998).
  - [13] E. Lieb, T. Schultz and D. Mattis, *Ann. Phys. (N.Y.)* **16** (1961) 403.
  - [14] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, *Phys. Rev. A*, **53**, 2046 (1996).