Application of the lattice Green's function for calculating the resistance of infinite networks of resistors

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We calculate the resistance between two arbitrary grid points of several infinite lattice structures of resistors by using lattice Green's functions. The resistance for d dimensional hypercubic, rectangular, triangular and honeycomb lattices of resistors is discussed in detail. We give recurrence formulas for the resistance between arbitrary lattice points of the square lattice. For large separation between nodes we calculate the asymptotic form of the resistance for a square lattice and the finite limiting value of the resistance for a simple cubic lattice. We point out the relation between the resistance of the lattice and the van Hove singularity of the tight-binding Hamiltonian. Our Green's function method can be applied in a straightforward manner to other types of lattice structures and can be useful didactically for introducing many concepts used in condensed matter physics.

I. INTRODUCTION

It is an old question to find the resistance between two adjacent grid points of an infinite square lattice in which all the edges represent identical resistances R. It is well known that the result is $R/2$, and an elegant and elementary solution of the problem is given by Aitchison[1]. The electric-circuit theory is discussed in detail in a classic text by van der Pol and Bremmer[2] and they derive the resistance between nearby points on the square lattice. In Doyle's and Snell's[3] book the connection between random walks and electric networks is presented, including many interesting results and useful references. Recently Venezian[4] and Atkinson et. al.[5] also studied the problem and in these papers the reader can find additional references. Venezian's method for finding the resistance between two arbitrary grid points of an infinite square lattice is based on the principle of the superposition of current distributions. This method was further utilized by Atkinson et. al.[5] and applied to two dimensional infinite triangular and hexagonal lattices as well as infinite cubic and hypercubic lattices. In this paper we present an alternative approach using lattice Green's functions. This Green's function method may have several advantages: (i) It can be used straightforwardly for more complicated lattice structures such as body and face centered cubic lattices. (ii) The results derived by this method reflect the symmetry of the lattice structures although they may not be suitable for numerical purposes. However, from these results one may derive other integral representations of the resistance between two nodes, which can be used for evaluating the integrals either algebraically or numerically with high precision. Later in this paper we shall give examples for this procedure. (iii) From the equation for the Green's function one can, in principle, derive some so-called recurrence formulas for the resistances between arbitrary grid points of an infinite lattice. In this paper such recurrence formulas are derived for an infinite square lattice[6] (for the first time to the best of the author's knowledge). (iv) In condensed matter physics the application of the lattice Green's function has become a very efficient tool. The analytical behavior of the lattice Green's function has been extensively studied over the past three decades for several lattice structures. We make use of the knowledge of this analytical behavior for an infinite cubic lattice and give the asymptotic value of the resistance as the separation of the two nodes tends to infinity. Throughout this paper we shall utilize the results known in the literature about the lattice Green's function and also give some important references. (v) Finally, our approach for networks of resistors may serve as a didactically good example for introducing the Green's function method as well as many basic concepts such as the Brillouin zone (BZ) used in solid state physics. We therefore feel that our Green's function method is of some physical interest.

The application of the Green's function proved to be a very effective method for studying the transport in inhomogeneous conductors and it has been used successfully by Kirkpatrick[7] for percolating networks of resistors. Our approach for obtaining the resistance of a lattice of resistors is closely related to that used by Kirkpatrick. Economou's book[8] gives an excellent introduction to the Green's function. A review of the lattice Green's function is given by Katsura et. al.[9] The lattice Green's function is also utilized in the theory of the Kosterlitz–Thouless–Berezinskii[10–12] phase transition of the screening of topological defects (vortices). A review of the latter problem is given by Chaikin et. al. in their book[13]. The phase transition in classical two-dimensional lattice Coulomb gases has been studied by Lee et. al.[14] also by using the lattice Green's function in their Monte Carlo simulations. The above examples of the applications of the lattice Green's function are just a selection of many problems known in the literature. A review of the Green's function of the so-called tight-binding Hamiltonian (TBH) used for describing the electronic band structures of crystal lattices is presented in Economou's book[8]. The lattice Green's function defined in this paper is related to the Green's function of the TBH. Below we shall point out that the resistance in a given lattice of resistors

is related to the Green's function of the TBH at the energy at which the density of states is singular. This singularity is one of the van Hove singularities of the density of states[15−17].

II. HYPERCUBIC LATTICE

Consider a d dimensional lattice which consists of all lattice points specified by position vectors $\mathbf{r}$ given in the form

$$\mathbf{r} = l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_d\mathbf{a}_d, \tag{1}$$

where $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_d$ are independent primitive translation vectors, and $l_1, l_2, \cdots, l_d$ range through all integer values (i.e. positive and negative integers, as well as zero). In the case of a d dimensional hypercube all the primitive translation vectors have the same magnitude, say a, i.e., $|\mathbf{a}_1| = |\mathbf{a}_2| = \cdots = |\mathbf{a}_d| = a$. Here a is the lattice constant of the d dimensional hypercube.

In the network of resistors we assume that the resistance of all the edges of the hypercube is the same, say R. We wish to find the resistance between the origin and a given lattice point $\mathbf{r}_0$ of the infinite hypercube. We denote the current that can enter at lattice point $\mathbf{r}$ by $I(\mathbf{r})$. However, for measuring the resistance between two sites the current has to be zero at all other sites. Similarly, the potential at site $\mathbf{r}$ will be denoted by $V(\mathbf{r})$. Then, at site $\mathbf{r}$, according to Ohm's and Kirchhoff's laws, we may write

$$I(\mathbf{r})R = \sum_{\mathbf{n}} \left[V(\mathbf{r}) - V(\mathbf{r} + \mathbf{n}) \right], \tag{2}$$

where the $\mathbf{n}$ are the vectors from site $\mathbf{r}$ to its nearest neighbors ($\mathbf{n} = \pm\mathbf{a}_i, i = 1, \cdots, d$). The right hand side of Eq. (2) may be expressed by the so-called *lattice Laplacian* defined on the hypercubic lattice:

$$\triangle_{(\mathbf{r})} f(\mathbf{r}) = \sum_{\mathbf{n}} \left[f(\mathbf{r} + \mathbf{n}) - f(\mathbf{r}) \right]. \tag{3}$$

The above defined lattice Laplacian corresponds to the finite-difference representation of the Laplace operator. The lattice Laplacian $1/a^2\triangle_{(\mathbf{r})} f(\mathbf{r})$ yields the correct form of the Laplacian in the continuum limit *i.e.* $a \to 0$. The lattice Laplacian is widely used to solve partial differential equations with the finite-difference method[18].

To measure the resistance between the origin and an arbitrary lattice point $\mathbf{r}_0$ we assume that a current I enters at the origin and exits at lattice point $\mathbf{r}_0$. Therefore, the current is zero at all the lattice points except for $\mathbf{r} = 0$ and $\mathbf{r}_0$, where it is I and $-I$, respectively. Thus, Eq. (2), with the lattice Laplacian, can be rewritten as

$$\triangle_{(\mathbf{r})} V(\mathbf{r}) = -I(\mathbf{r})R, \tag{4}$$

where current at lattice point $\mathbf{r}$ is

$$I(\mathbf{r}) = I\left(\delta_{\mathbf{r},0} - \delta_{\mathbf{r},\mathbf{r}_0} \right). \tag{5}$$

The resistance between the origin and $\mathbf{r}_0$ is

$$R(\mathbf{r}_0) = \frac{V(\mathbf{0}) - V(\mathbf{r}_0)}{I}. \tag{6}$$

To find the resistance we need to solve Eq. (4). This is a Poisson-like equation and may be solved by using the lattice Green's function:

$$V(\mathbf{r}) = R \sum_{\mathbf{r}'} G(\mathbf{r} - \mathbf{r}') I(\mathbf{r}'), \tag{7}$$

where the lattice Green's function is defined by

$$\triangle_{(\mathbf{r}')} G(\mathbf{r} - \mathbf{r}') = -\delta_{\mathbf{r},\mathbf{r}'}. \tag{8}$$

Finally, the resistance between the origin and $\mathbf{r}_0$ can be expressed by the lattice Green's function. Using Eq. (5) and (6) we obtain

$$R(\mathbf{r}_0) = 2R[G(\mathbf{0}) - G(\mathbf{r}_0)], \tag{9}$$

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where we have made use of the fact that the lattice Green's function is even, *i.e.* $G(\mathbf{r}) = G(-\mathbf{r})$. Equation (9) is our central result for the resistance.

To find the lattice Green's function defined by Eq. (8) we take periodic boundary conditions at the edges of the hypercube. Consider a hypercube with L lattice points along each side. Thus the total number of sites in the d dimensional hypercube is L^d. Substituting the Fourier transform

$$G(\mathbf{r}) = \frac{1}{L^d} \sum_{\mathbf{k} \in \mathrm{BZ}} G(\mathbf{k}) e^{i\mathbf{k}\mathbf{r}} \qquad (10)$$

of the lattice Green's function into Eq. (8), we find

$$G(\mathbf{k}) = \frac{1}{\varepsilon(\mathbf{k})}, \qquad (11)$$

for the d dimensional hypercube where we have defined

$$\varepsilon(\mathbf{k}) = 2 \sum_{i=1}^{d} \left(1 - \cos \mathbf{k}\mathbf{a}_i\right). \qquad (12)$$

Owing to the periodic boundary conditions, the wave vector $\mathbf{k}$ in Eq. (11) is limited to the first Brillouin zone and is given by

$$\mathbf{k} = \frac{m_1}{L}\mathbf{b}_1 + \frac{m_2}{L}\mathbf{b}_2 + \cdots + \frac{m_d}{L}\mathbf{b}_d, \qquad (13)$$

where $m_1, m_2, \cdots m_d$ are integers such that $-L/2 \leq m_i \leq L/2$ for $i = 1, 2, \cdots, d$, and $\mathbf{b}_j$ are the reciprocal lattice vectors defined by $\mathbf{a}_i \mathbf{b}_j = 2\pi \delta_{ij}$, $i, j = 1, 2, \cdots, d$. Here we assumed that L is an even integer, which will be irrelevant in the limit $L \to \infty$. The mathematical description of the crystal lattice and the concept of the Brillouin zone can be found in many books on solid state physics[13,16,15,17].

Finally, the lattice Green's function takes the form

$$G(\mathbf{r}) = \frac{1}{L^d} \sum_{\mathbf{k} \in \mathrm{BZ}} \frac{e^{i\mathbf{k}\mathbf{r}}}{\varepsilon(\mathbf{k})}. \qquad (14)$$

If we take the limit $L \to \infty$ then the discrete summation over $\mathbf{k}$ can be substituted by an integral[16,13]:

$$\frac{1}{L^d} \sum_{\mathbf{k} \in \mathrm{BZ}} \to v_0 \int_{\mathbf{k} \in \mathrm{BZ}} \frac{d^d \mathbf{k}}{(2\pi)^d}, \qquad (15)$$

where $v_0 = a^d$ is the volume of the unit cell of the d dimensional hypercube. Thus the lattice Green's function is

$$G(\mathbf{r}) = v_0 \int_{\mathbf{k} \in \mathrm{BZ}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\mathbf{r}}}{\varepsilon(\mathbf{k})}. \qquad (16)$$

Using Eqs. (9) and (16) in d dimensions the resistance between the origin and lattice point $\mathbf{r}_0$ is

$$R(\mathbf{r}_0) = 2Rv_0 \int_{\mathbf{k} \in \mathrm{BZ}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1 - e^{i\mathbf{k}\mathbf{r}_0}}{\varepsilon(\mathbf{k})}. \qquad (17)$$

The above result can be simplified if we specify the lattice point as $\mathbf{r}_0 = l_1 \mathbf{a}_1 + l_2 \mathbf{a}_2 + \cdots + l_d \mathbf{a}_d$:

$$R(l_1, l_2, \cdots, l_d) = R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dx_d}{2\pi} \frac{1 - e^{i(l_1 x_1 + \cdots + l_d x_d)}}{\sum_{i=1}^{d}(1 - \cos x_i)}. \qquad (18)$$

From this final expression of the resistance one can see that the resistance does not depend on the angles between the unit vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_d$. Physically this means that the hypercube can be deformed without the change of the resistance between any two lattice points. The resistance in topologically equivalent lattices is the same. For further references we also give the lattice Green's function for a d dimensional hypercube:

$$G(l_1, l_2, \cdots, l_d) = \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dx_d}{2\pi} \frac{e^{i(l_1 x_1 + \cdots + l_d x_d)}}{2 \sum_{i=1}^{d}(1 - \cos x_i)}. \qquad (19)$$

A. Conducting medium, continuum limit

The resistance in an infinite conducting medium can be obtained by taking the limit as the lattice constant a tends to zero in Eq. (17). Denoting the electrical conductivity of the medium by σ, the resistance of a d dimensional hypercube with lattice constant a, according to Ohm's law, is given by

$$R = \frac{1}{\sigma a^{d-2}}. \tag{20}$$

Using the approximation $\varepsilon(\mathbf{k}) \approx \mathbf{k}^2 a^2$ for $|\mathbf{a}_i| = a \to 0$, Eq. (17) can easily be reduced to

$$R(\mathbf{r}_0) = \frac{2}{\sigma} \int_{\mathbf{k} \in \mathrm{BZ}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1 - e^{i\mathbf{k}\mathbf{r}_0}}{\mathbf{k}^2}. \tag{21}$$

The same result for the resistance of a conducting medium is given in Chaikin's book[13].

B. Linear chain, $d = 1$

Consider a linear chain of identical resistors R. The resistance between the origin and site l can be obtained by taking $d = 1$ in the general result given in Eq. (18):

$$R(l) = R \int_{-\pi}^{\pi} \frac{dx}{2\pi} \frac{1 - e^{ilx}}{1 - \cos x}. \tag{22}$$

The integral can be evaluated by the method of residues[19] and gives the following very simple result:

$$R(l) = Rl. \tag{23}$$

This can be interpreted as the resistance of l resistances R in series. The current flows only between the two sites separated by a finite distance. The two semi-infinite segments of the chain do not affect the resistance.

C. Square lattice, $d = 2$

Using Eq. (18), the resistance in two dimensions between the origin and $\mathbf{r}_0 = l_1 \mathbf{a}_1 + l_2 \mathbf{a}_2$ is

$$R(l_1, l_2) = R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \frac{1 - e^{il_1 x_1 + il_2 x_2}}{2 - \cos x_1 - \cos x_2} \tag{24}$$

$$= R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \frac{1 - \cos(l_1 x_1 + l_2 x_2)}{2 - \cos x_1 - \cos x_2}. \tag{25}$$

The resistance between two adjacent lattice sites can easily be obtained from the above expression without evaluating the integrals in Eq. (24) or (25). Note that interchanging l_1 and l_2 in Eq. (25) does not change the resistance, *i.e.*, $R(l_1, l_2) = R(l_2, l_1)$. This is consistent with the symmetry of the lattice. Then, from Eq. (25), the sum of $R(0, 1)$ and $R(1, 0)$ yields

$$R(0, 1) + R(1, 0) = R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} = R. \tag{26}$$

Thus $R(0, 1) = R(1, 0) = R/2$. The resistance between two adjacent lattice sites is $R/2$, which is a well known result[1,4].

In general the integrals in Eq. (24) have to be evaluated numerically. It is shown in Appendix A how one integral can be performed in Eq. (24). We found the same result as Venezian[4] and Atkinson[5]:

$$R(l_1, l_2) = R \int_0^{\pi} \frac{dy}{\pi} \frac{1 - e^{-|l_1|s} \cos l_2 y}{\sinh s}, \tag{27}$$

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where

$$\cosh s = 2 - \cos y. \tag{28}$$

It turns out that this expression is more stable numerically than Eq. (25). As an application of the above formula one can calculate the resistance between second nearest neighbors exactly. After some algebra one finds

$$R(1,1) = R \int_0^\pi \frac{dy}{\pi} \frac{(1 - \cos y)^2}{\sqrt{(2 - \cos y)^2 - 1}} = \frac{2}{\pi} R. \tag{29}$$

The energy dependent lattice Green's function of the tight-binding Hamiltonian for a square lattice is given by

$$G(E; l_1, l_2) = \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \frac{\cos(l_1 x_1 + l_2 x_2)}{E - \cos x_1 - \cos x_2} \tag{30}$$

[see Eq. (5.31) in Economou's book[8]]. This is a generalization of our Green's function by introducing a new variable E instead of the value 2 in the denominator in Eq. (19) for $d = 2$. Note that a factor 2 appearing in the denominator of our Green's function in Eq. (19) is missing in Eq. (30). This is related to the fact that in the Schrödinger equation the Laplacian is multiplied by a factor of $1/2$ while in our case the Laplace equation is solved. Comparing Eqs. (19) (for $d = 2$) and (30) one can see that the resistance is

$$R(l_1, l_2) = R\left[G(2; 0, 0) - G(2; l_1, l_2) \right]. \tag{31}$$

Based on the equation for the Green's function Morita[20] derived the recurrence formulas for the Green's function $G(E; l_1, l_2)$ for an infinite square lattice (see Eqs. (3.8) and (4.2)-(4.4) in Morita's paper[20]). Applying Morita's results (with $E = 2$) to the resistance given in Eq. (31) we obtained the following recurrence formulas for the resistance:

$$
\begin{aligned}
R(m+1, m+1) &= \frac{4m}{2m+1} R(m,m) - \frac{2m-1}{2m+1} R(m-1, m-1), \\
R(m+1, m) &= 2R(m,m) - R(m, m-1), \\
R(m+1, 0) &= 4R(m,0) - R(m-1, 0) - 2R(m,1), \\
R(m+1, p) &= 4R(m,p) - R(m-1, p) - R(m, p+1) - R(m, p-1) \quad \text{if} \ \ 0 < p < m.
\end{aligned}
\tag{32}
$$

We have seen that $R(1,0) = R/2$ and $R(1,1) = 2R/\pi$. Since we know the exact values of $R(1,0)$ and $R(1,1)$ (obviously $R(0,0) = 0$) one can calculate the resistance exactly for arbitrary nodes by using the above given recurrence formulas. This way we obtained the same results as Atkinson et. al.[5] using Mathematica. The advantages of our recurrence relations are that they provide a new, very simple and effective tool to calculate the resistance between arbitrary nodes on a square lattice. We note that van der Pol and Bremmer[2] also gave the exact values of the resistance for nearby points in a square lattice using a different approach.

It is interesting to see the asymptotic form of the resistance for large values of l_1 or/and l_2. In Appendix B we derive the asymptotic form of the lattice Green's function for a square lattice [see Eq. (B9)]. Inserting Eq. (B9) into the general result of the resistance given in Eq. (9) the asymptotic form of the resistance is

$$R(l_1, l_2) = \frac{R}{\pi} \left(\ln \sqrt{l_1^2 + l_2^2} + \gamma + \frac{\ln 8}{2} \right), \tag{33}$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant[19]. The same result was obtained by Venezian[4] except that we got an exact value of the numerical constant in Eq. (33) whereas it was numerically approximated in Venezian's paper. The resistance is logarithmically divergent for large values of l_1 and l_2. A similar behavior was found for conducting medium by Chaikin et. al. in their book[13]. In the theory of the Kosterlitz–Thouless–Berezinskii[10−12] phase transition of the screening of topological defects (vortices) the same asymptotic form of the Green's function (as given in Eq. (B9)) has been used for a square lattice[11]. Finally, we note that Doyle and Snell showed in their book[3] (pp. 122–123) that the resistance goes to infinity as the separation between nodes tends to infinity but the asymptotic form was not derived.

D. Simple cubic lattice, $d = 3$

In three dimensions the resistance between the origin and a lattice point $\mathbf{r}_0 = l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + l_3\mathbf{a}_3$ can be obtained from Eq. (18):

$$R(l_1, l_2, l_3) = R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \int_{-\pi}^{\pi} \frac{dx_3}{2\pi} \frac{1 - e^{il_1x_1 + il_2x_2 + il_3x_3}}{3 - \cos x_1 - \cos x_2 - \cos x_3} \tag{34}$$

$$= R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \int_{-\pi}^{\pi} \frac{dx_3}{2\pi} \frac{1 - \cos(l_1x_1 + l_2x_2 + l_3x_3)}{3 - \cos x_1 - \cos x_2 - \cos x_3}. \tag{35}$$

Similarly to the case of a square lattice, the exact value of the resistance between two adjacent lattice sites can be obtained from the above expression. Clearly, from Eq. (35) (and for symmetry reasons, too) $R(1, 0, 0) = R(0, 1, 0) = R(0, 0, 1)$ and

$$R(1, 0, 0) + R(0, 1, 0) + R(0, 0, 1) = R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \int_{-\pi}^{\pi} \frac{dx_3}{2\pi} = R. \tag{36}$$

Therefore, the resistance between adjacent sites is $R/3$. It is easy to show much in the same way that in a d dimensional hypercube the resistance between adjacent sites is R/d.

Unlike in a square lattice, in a simple cubic lattice the resistance does not diverge as the separation of the entering and exiting sites increases, but tends to a finite value. One can write for the resistance $R(l_1, l_2, l_3) = 2[G(0, 0, 0) - G(l_1, l_2, l_3)]$ where $G(l_1, l_2, l_3)$ is given in Eq. (19) for $d = 3$. It is well known from the theory of Fourier series (Riemann's lemma) that $\lim_{p \to \infty} \int_a^b dx\, \varphi(x) \cos px \to 0$ for any integrable function $\varphi(x)$. Hence, $G(l_1, l_2, l_3) \to 0$ (which indeed corresponds to the boundary condition of the Green's function at infinity) and thus, $R(l_1, l_2, l_3) \to 2G(0, 0, 0)$ when any of the $l_1, l_2, l_3 \to \infty$. The value of $G(0, 0, 0)$ was evaluated for the first time by Watson[21] and subsequently by Joyce[22] in a closed form in terms of elliptic integrals. The following exact result was found

$$2G(0, 0, 0) = \left(\frac{2}{\pi}\right)^2 \left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}\right) [\mathbf{K}(k_0)]^2 = 0.505462\ldots, \tag{37}$$

where $k_0 = (2 - \sqrt{3})(\sqrt{3} - \sqrt{2})$ and

$$\mathbf{K}(k) = \int_0^{\pi/2} d\theta \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{38}$$

is the complete elliptic integral of the first kind. It is worth mentioning that a simpler result was obtained by Glasser et. al.[23] (see also Doyle's and Snell's book[3]) who calculated the integrals in terms of gamma functions:

$$2G(0, 0, 0) = \frac{\sqrt{3} - 1}{96\pi^3} \Gamma^2(1/24)\Gamma^2(11/24). \tag{39}$$

Thus, the resistance in units of R for a simple cubic lattice tends to the finite value $0.505462\ldots$ when the separation between the entering and exiting sites tends to infinity.

For finite separations the formula for the resistance given in Eq. (35) is not suitable for numerical purposes since the integrals converge slowly with increasing the number of mesh points. Similarly to the case of a square lattice, we can use the energy dependent lattice Green's function of the tight-binding Hamiltonian defined for a simple cubic lattice as

$$G(E; l_1, l_2, l_3) = \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \int_{-\pi}^{\pi} \frac{dx_3}{2\pi} \frac{\cos(l_1x_1 + l_2x_2 + l_3x_3)}{E - \cos x_1 - \cos x_2 - \cos x_3} \tag{40}$$

[see Eq. (5.49) in Economou's book[8]]. This is a generalization of our Green's function by introducing a new variable E instead of the value 3 in the denominator in Eq. (19) for $d = 3$. Note that the reason for the missing factor of 2 in Eq. (40) is the same as that explained in the previous section. Comparison of Eqs. (19) (for $d = 3$) and (40) yields

$$R(l_1, l_2, l_3) = R\left[G(3; 0, 0, 0) - G(3; l_1, l_2, l_3)\right]. \tag{41}$$

The analytic behavior of the lattice Green's function $G(E; l_1, l_2, l_3)$ has been extensively studied over the past three decades. Numerical values of $G(E; l_1, l_2, l_3)$ were given by Koster et. al.,[24] and Wolfram et. al.,[25] by using the following representation of the Green's function for a simple cubic lattice:

$$G(E \geq 3; l, m, n) = \int_0^\infty dt\, e^{-Et} I_l(t) I_m(t) I_n(t), \tag{42}$$

where I_l is the modified Bessel function of order l. However, it turns out that another representation of the lattice Green's functions is more suitable for numerical calculations. For $3 \leq E$ the following form of the Green's function along an axis was given in terms of complete elliptic integrals of the first kind by Horiguchi[26]

$$G(E; l, 0, 0) = \frac{1}{\pi^2} \int_0^\pi dx\, \mathbf{K}(k) k \cos lx, \tag{43}$$

where

$$k = \frac{2}{E - \cos x} \tag{44}$$

and $\mathbf{K}(k)$ is defined in Eq. (38). In Fig. 1 we plotted the numerical values of the resistance using Eq. (43) for $E = 3$ and $1 \leq l \leq 100$.
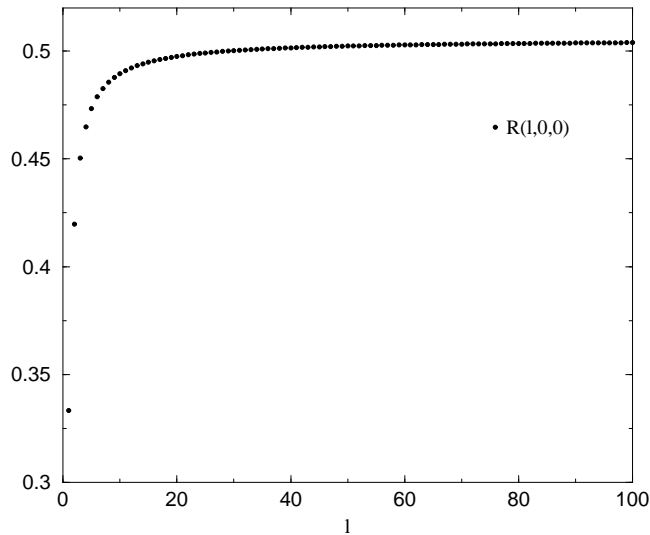

FIG. 1. The resistance $R(l, 0, 0)$ in units of R along an axis for a simple cubic lattice.

It is seen from the figure that the resistance tends rapidly to its asymptotic value given in Eq. (37). We would mention that Glasser et. al.[31] gave other useful integral representations of the lattice Green's function for the hypercubic lattice for arbitrary dimension d.

It was shown by Joyce[22] that the function $G(E; 0, 0, 0)$ can be expressed in the form of a product of two complete elliptic integrals of the first kind. Horiguchi[26] and later on Morita[27] obtained recurrence relations for the function $G(E; l_1, l_2, l_3)$ of the simple cubic lattice for an arbitrary site l_1, l_2, l_3 in terms only of $G(E; 0, 0, 0)$, $G(E; 2, 0, 0)$ and $G(E; 3, 0, 0)$. The last two Green's functions were expressed by Horiguchi and Morita[28] in closed form in terms of complete elliptic integrals. From these results one can calculate the Green's function for an arbitrary lattice point. As an example, one of the recurrence formulas we obtained from the recurrence formula of the Green's function (at $E = 3$) given by Horiguchi[26] is

$$R(m, 1, 0) = -\frac{1}{4} R(m-1, 0, 0) + \frac{3}{2} R(m, 0, 0) - \frac{1}{4} R(m+1, 0, 0). \tag{45}$$

Together with other recurrence formulas given in Horiguchi's and Morita's paper one can find $R(l_1, l_2, l_3)$ for arbitrary values of l_1, l_2, l_3. The numerical values of the resistance for small l_1, l_2, l_3 are listed in Table I.

l_1, l_2, l_3	$R(l_1, l_2, l_3)/R$
1,0,0	1/3
1,1,0	0.395079
1,1,1	0.418305
2,0,0	0.419683
2,1,0	0.433599
2,1,1	0.441531
2,2,0	0.449352
3,0,0	0.450372
3,1,0	0.454416
4,0,0	0.464884
∞	0.505462[21,22]

TABLE I. Numerical values of $R(l_1, l_2, l_3)$ in units of R for a simple cubic lattice.

Our results are in agreement with those given by Atkinson[5].

It is worth mentioning some useful references for the lattice Green's function of other three dimensional lattices, such as body centered and face centered. The exact values of $G(1; 0, 0, 0)$ as well as other exact results can be found in Joyce's paper[29] for a body centered cubic lattice. Morita[27] gave the recurrence relations of the lattice Green's function for a body centered cubic lattice. Inoue[30] derived the recurrence relations for a face centered cubic lattice and obtained the exact values of the lattice Green's function at some lattice points. Based on the results given in these papers one can calculate the resistance between arbitrary nodes for cubic lattices.

Finally we would like to point out that in Eq. (41) the resistance is related to the energy dependent Green's function at energy $E = 3$. However, it is known that the density of states at this energy corresponds to one type of the van Hove singularities[15,16]. Therefore, the resistance for a simple cubic lattice is related to the Green's function of the tight-binding Hamiltonian at the value of the energy at which the density of states has a van Hove type singularity. In the general case of a d dimensional hypercube the Green's function of the tight-binding Hamiltonian is infinite at $E = d$ for $d = 2$ (logarithmically divergent) while it has a finite value for $d = 3$ as it was shown above. In the latter case the derivative of the Green's function with respect to E is singular.

III. RECTANGULAR LATTICE

In this section we shall calculate the resistance of a rectangular lattice in which the resistance of each edge is proportional to its length. Consider a rectangular lattice with unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ and introduce parameter $p = |\mathbf{a}_1| / |\mathbf{a}_2|$. Let R be the resistance of the edge along the direction of $\mathbf{a}_2$. To find the resistance between the origin and site $\mathbf{r}_0$ assume a current I enters at the origin and exits at site $\mathbf{r}_0$. We denote the potential at lattice point $\mathbf{r}$ by $V(\mathbf{r})$. Then, according to Ohm's and Kirchhoff's laws, we may write

$$\triangle_{\mathrm{r}}^{\mathrm{rect}} V(\mathbf{r}) = -I \left(\delta_{\mathbf{r},0} - \delta_{\mathbf{r},\mathbf{r}_0} \right) R, \tag{46}$$

where the 'rectangular' Laplacian is defined by

$$\triangle_{\mathrm{r}}^{\mathrm{rect}} V(\mathbf{r}) = \frac{V(\mathbf{r} + \mathbf{a}_1) - V(\mathbf{r})}{p} + \frac{V(\mathbf{r} - \mathbf{a}_1) - V(\mathbf{r})}{p} + V(\mathbf{r} + \mathbf{a}_2) - V(\mathbf{r}) + V(\mathbf{r} - \mathbf{a}_2) - V(\mathbf{r}). \tag{47}$$

Then the equation for the lattice Green's function corresponding to the 'rectangular' Laplace equation is

$$\triangle_{\mathrm{r}'}^{\mathrm{rect}} G(\mathbf{r} - \mathbf{r}') = -\delta_{\mathbf{r},\mathbf{r}'}. \tag{48}$$

The Green's function can be calculated in a way similar to the hypercubic case in Sec. II. We have

$$G(\mathbf{r}) = v_0 \int_{\mathbf{k} \in \mathrm{BZ}} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\mathbf{r}}}{\varepsilon(\mathbf{k})}, \tag{49}$$

where $v_0 = pa^2$ is the area of the unit cell, and the Brillouin zone is a rectangle with sides $2\pi / |\mathbf{a}_1|$ and $2\pi / |\mathbf{a}_2|$ along the directions of $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, and

$$\varepsilon(\mathbf{k}) = 2 \left[\frac{1}{p} (1 - \cos \mathbf{k}\mathbf{a}_1) + 1 - \cos \mathbf{k}\mathbf{a}_2 \right]. \tag{50}$$

Using Eq. (9), the resistance between the origin and lattice point $\mathbf{r}_0 = l_1 \mathbf{a}_1 + l_2 \mathbf{a}_2$ for a given p is

$$R(p; l_1, l_2) = R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \frac{1 - e^{il_1 x_1 + il_2 x_2}}{\frac{1}{p} (1 - \cos x_1) + 1 - \cos x_2} \tag{51}$$

$$= R \int_{-\pi}^{\pi} \frac{dx_1}{2\pi} \int_{-\pi}^{\pi} \frac{dx_2}{2\pi} \frac{1 - \cos (l_1 x_1 + l_2 x_2)}{\frac{1}{p} (1 - \cos x_1) + 1 - \cos x_2}. \tag{52}$$

The integral over x_2 can be evaluated similarly to a square lattice (see Appendix A). We find

$$R(p; l_1, l_2) = R \int_0^{\pi} \frac{dx}{\pi} \frac{1 - e^{-|l_2|s} \cos l_1 x}{\sinh s}, \tag{53}$$

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where

$$\cosh s = 1 + \frac{1}{p} - \frac{1}{p} \cos x. \tag{54}$$

Note that l_1 and l_2 are interchanged here compared to Eq. (27).

Now it is interesting to see how the resistance between adjacent sites varies with $p = |\mathbf{a}_1| / |\mathbf{a}_2|$. The resistance between nearest neighbors is different along the x and the y axis. Thus, unlike in the case of a square lattice, $R(p; 1, 0) \neq R(p; 0, 1)$ for a rectangular lattice except for the trivial case $p = 1$ (square lattice). No sum rule exists for $R(p; 1, 0) + R(p; 0, 1)$ as in the case of a square lattice. In Fig. 2 the resistances $R(p; 1, 0)$ and $R(p; 0, 1)$ are plotted as functions of p.
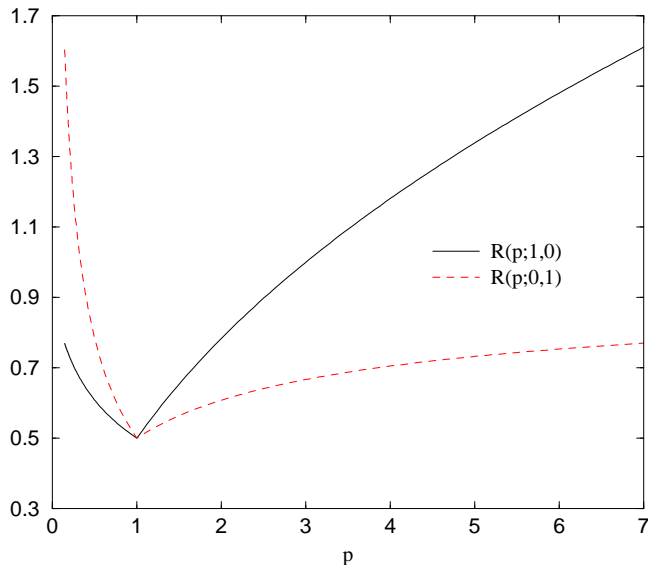

FIG. 2. The resistances $R(p; 1, 0)$ and $R(p; 0, 1)$ in units of R as functions of p for a rectangular lattice with $p = |\mathbf{a}_1| / |\mathbf{a}_2|$.

From Fig. 2 one can see that resistance $R(p; 1, 0)$ increases with increasing p. It can be shown that $R(p; 1, 0) \to \infty$ as $p \to \infty$. This is physically clear since the lattice constant along the x axis increases resulting in an increasing resistance of each of the segments in this direction. On the other hand along the y axis a saturation of $R(p; 0, 1)$ can be seen, which is not obvious at all. Expanding the integral in the expression of $R(p; 0, 1)$ in powers of p, for large p we get $R(p; 0, 1) \approx R(1 - 2/\pi p^{-1/2} + O(p^{-3/2}))$. Thus, $R(p; 0, 1) \to R$ when $p \to \infty$. Some additional results are presented for the energy dependent lattice Green's function for a rectangular lattice in the papers of Morita et. al.,[32] and Katsura et. al[33].

IV. TRIANGULAR LATTICE

In this section we calculate the resistance in a triangular lattice in which the resistance of each edge is identical, say R. First we consider a triangular lattice with unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, and with a lattice constant $a = |\mathbf{a}_1| = |\mathbf{a}_2|$. We choose $\mathbf{a}_1$ and $\mathbf{a}_2$ such that the angle between them is $120°$. We introduce a third vector by $\mathbf{a}_3 = -(\mathbf{a}_1 + \mathbf{a}_2)$. The vectors drawn from each lattice point to its 6 nearest neighbors are $\pm \mathbf{a}_1, \pm \mathbf{a}_2, \pm \mathbf{a}_3$. Assume that a current I enters at the origin and exits at site $\mathbf{r}_0$, and that the potential at site $\mathbf{r}$ is $V(\mathbf{r})$. Again, from Ohm's and Kirchhoff's laws, we find

$$\triangle_{\mathbf{r}}^{\text{triang}} V(\mathbf{r}) = -I \left(\delta_{\mathbf{r},0} - \delta_{\mathbf{r},\mathbf{r}_0} \right) R, \tag{55}$$

where the 'triangular' Laplacian is defined by

$$\triangle_{\mathbf{r}}^{\text{triang}} V(\mathbf{r}) = \sum_{i=1}^{3} \left[V(\mathbf{r} - \mathbf{a}_i) - 2V(\mathbf{r}) + V(\mathbf{r} + \mathbf{a}_i) \right]. \tag{56}$$

It is important to note that the triangular Laplacian $\triangle_r^{\text{triang}}$ defined above is $2/(3a^2)$ times that used for solving the Laplace equation with the finite-difference method on a triangular lattice. The factor $2/(3a^2)$ ensures that the lattice Laplacian in the finite-difference method yields the correct form of the Laplacian in the continuum limit ($a \to 0$). However, in our case this limit does not exist since there are also connections (resistors) between adjacent lattice points along the direction of the vector $\mathbf{a}_3$.

The equation for the lattice Green's function corresponding to the 'triangular' Laplace equation is given by

$$\triangle_{r'}^{\text{triang}} G(\mathbf{r} - \mathbf{r}') = -\delta_{\mathbf{r},\mathbf{r}'}. \tag{57}$$

The Green's function can be calculated in the same way as in the hypercubic case. We have

$$G(\mathbf{r}) = v_0 \int_{\mathbf{k} \in \text{BZ}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\mathbf{r}}}{2 \sum_{i=1}^3 (1 - \cos \mathbf{k}\mathbf{a}_i)}, \tag{58}$$

where $v_0 = \sqrt{3}/2a^2$ is the area of the unit cell, and the vector $\mathbf{k}$ given in Eq. (13) runs over the Brillouin zone of the triangular lattice, which is a regular hexagon[13].

The resistance between the origin and site $\mathbf{r}_0$ can be obtained from Eq. (9) with the lattice Green's function for a triangular lattice and it yields

$$R(\mathbf{r}_0) = Rv_0 \int_{\mathbf{k} \in \text{BZ}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1 - e^{i\mathbf{k}\mathbf{r}_0}}{\sum_{i=1}^3 (1 - \cos \mathbf{k}\mathbf{a}_i)} = Rv_0 \int_{\mathbf{k} \in \text{BZ}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1 - \cos \mathbf{k}\mathbf{r}_0}{\sum_{i=1}^3 (1 - \cos \mathbf{k}\mathbf{a}_i)}. \tag{59}$$

Note that the factor 2 dropped out from the denominator.

Using Eq. (59) we can easily find the resistance between two adjacent lattice sites. For symmetry reasons it is clear that $R(\mathbf{a}_1) = R(\mathbf{a}_2) = R(\mathbf{a}_3)$. On the other hand

$$\sum_{i=1}^3 R(\mathbf{a}_i) = Rv_0 \int_{\mathbf{k} \in \text{BZ}} \frac{d^2\mathbf{k}}{(2\pi)^2} = R. \tag{60}$$

In the last step we have made use of the fact that the volume of the Brillouin zone is $1/v_0$. Hence, the resistance between two adjacent lattice sites is $R(\mathbf{a}_1) = R/3$.

It is worth mentioning that without changing the resistance between two arbitrary lattice points one can transform the triangular lattice to a square lattice in which there are also resistors between the end points of one of the diagonals of each square. This topologically equivalent lattice is more suitable for evaluating the necessary integrals over the Brillouin zone since the Brillouin zone becomes a square. The same transformation was used by Atkinson et. al. in their paper[5]. If we choose $\mathbf{a}_1 = (1,0)$ and $\mathbf{a}_1 = (0,1)$, then the resistance between the origin and lattice point $\mathbf{r}_0 = n\mathbf{a}_1 + m\mathbf{a}_2$ is given by

$$R(n,m) = R \int_{-\pi}^{\pi} \frac{dx}{2\pi} \int_{-\pi}^{\pi} \frac{dy}{2\pi} \frac{1 - e^{inx+imy}}{3 - \cos x - \cos y - \cos(x+y)}. \tag{61}$$

In Appendix C we perform one integral in Eq. (61) using the method of residues[19] in a similar way to Appendix A. After some algebra it is easy to see that the result given in Eq. (C4) is in agreement with Atkinson's result[5]:

$$R(n,m) = R \int_0^{\pi/2} \frac{dx}{\pi} \frac{1 - e^{-|n-m|s} \cos(n+m)x}{\sinh s \cos x}, \tag{62}$$

where $\cosh s = 2 \sec x - \cos x$.

For $n = m$ in Eq. (62) we have

$$R(n,n) = 2R \int_0^{\pi/2} \frac{dx}{\pi} \frac{1 - \cos 2nx}{\sqrt{(3 - \cos 2x)^2 - 4\cos^2 x}}. \tag{63}$$

Evaluating this expression by means of the program Maple we obtained the same results for $n = m$ as Atkinson[5]. Like in the case of a square lattice we believe that similar recurrence formulas exist for a triangular lattice but further work is necessary along this line. Similarly, it would be interesting to find the asymptotic form of the resistance as the separation between the two nodes tends to infinity. According to some preliminary work the resistance is again logarithmically divergent.

V. HONEYCOMB LATTICE

In this section we calculate the resistances in an infinite honeycomb lattice of resistors. Atkinson and Steenwijk[5] have studied this lattice structure exploiting the fact that the hexagonal lattice can be constructed from the triangular lattice by the application of the so-called $\Delta - Y$ transformation[34]. One advantage of our Green's function method is that we do not use such a $\Delta - Y$ transformation, specific only for the honeycomb lattice, therefore our method can also be used in a straightforward manner for other lattice structures. As we shall see later, in the honeycomb lattice each unit cell contains *two* lattice points, which is the main structural difference from the triangular lattice. Including more than one lattice point in the unit cell, the method outlined in this section can be viewed as a generalization of the Green's function method discussed in the previous sections.

We assume that all the resistors have the same resistance R. The lattice structure and the unit cell are shown in Fig. 3. The angle between $\mathbf{a}_1$ and $\mathbf{a}_2$ is $120°$, and $|\mathbf{a}_1| = |\mathbf{a}_2| = \sqrt{3}a$, where a is the length of the edges of the hexagons. There are two types of lattice points in each unit cell denoted by A and B in Fig. 3.
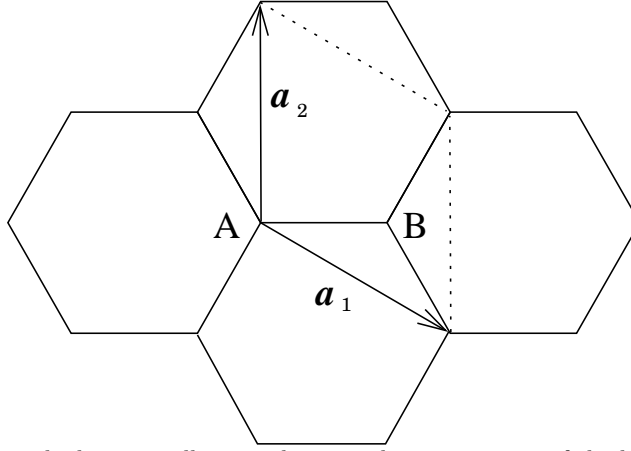

FIG. 3. The honeycomb lattice with the unit cell. $\mathbf{a}_1$ and $\mathbf{a}_2$ are the unit vectors of the lattice. Each unit cell contains two types of lattice points, A and B.

From now on the position of the lattice points A and B will be given by the position of the unit cell in which they are located. Assume the origin is at one of the lattice points A, and then the position of a unit cell can be specified by the position vector $\mathbf{r} = n\mathbf{a}_1 + m\mathbf{a}_2$, where n, m are arbitrary integers. We denote the potential and the current in one of the unit cells by $V_A(\mathbf{r})$ and $V_B(\mathbf{r})$, and $I_A(\mathbf{r})$ and $I_B(\mathbf{r})$, respectively, where subscripts A and B refer to the corresponding lattice points. Owing to Ohm's and Kirchhoff's laws the currents $I_A(\mathbf{r})$ and $I_B(\mathbf{r})$ in the unit cell specified by $\mathbf{r}$ satisfy the following equations:

$$I_A(\mathbf{r}) = \frac{V_A(\mathbf{r}) - V_B(\mathbf{r})}{R} + \frac{V_A(\mathbf{r}) - V_B(\mathbf{r} - \mathbf{a}_1)}{R} + \frac{V_A(\mathbf{r}) - V_B(\mathbf{r} - \mathbf{a}_1 - \mathbf{a}_2)}{R},$$

$$I_B(\mathbf{r}) = \frac{V_B(\mathbf{r}) - V_A(\mathbf{r})}{R} + \frac{V_B(\mathbf{r}) - V_A(\mathbf{r} + \mathbf{a}_1)}{R} + \frac{V_B(\mathbf{r}) - V_A(\mathbf{r} + \mathbf{a}_1 + \mathbf{a}_2)}{R}. \tag{64}$$

Assuming periodic boundary conditions again the potential $V_A(\mathbf{r})$ can be given by its Fourier transform

$$V_A(\mathbf{r}) = \frac{1}{N} \sum_{\mathbf{k} \in \text{BZ}} V_A(\mathbf{k}) \, e^{i\mathbf{k}\mathbf{r}}, \tag{65}$$

where N is the number of unit cells, and analogous expressions are valid for $V_B(\mathbf{r}), I_A(\mathbf{r})$ and $I_B(\mathbf{r})$. Here $\mathbf{k}$ is in the Brillouin zone[16,15,17]. Thus, we may rewrite Eq. (64) as

$$\mathbf{L}(\mathbf{k}) \begin{bmatrix} V_A(\mathbf{k}) \\ V_B(\mathbf{k}) \end{bmatrix} = -R \begin{bmatrix} I_A(\mathbf{k}) \\ I_A(\mathbf{k}) \end{bmatrix}, \tag{66}$$

where

$$\mathbf{L}(\mathbf{k}) = \begin{pmatrix} -3 & h^* \\ h & -3 \end{pmatrix}, \tag{67}$$

and

$$h = 1 + e^{i\mathbf{k}\mathbf{a}_1} + e^{i\mathbf{k}(\mathbf{a}_1 + \mathbf{a}_2)}. \tag{68}$$

Equation (66) is indeed the Fourier transform of the Poisson-like equation that determines the potentials for a given current distribution. However, in this case the Laplace operator is a 2x2 matrix. In $\mathbf{r}$-represantion the analogous equation for a hypercubic lattice was given in Eq. (4).

The equation for the Fourier transform of the Green's function (which is a 2x2 matrix for a honeycomb lattice, too) can be defined analogously to a hypercubic lattice in Eq. (8):

$$\mathbf{L}(\mathbf{k})\mathbf{G}(\mathbf{k}) = -1. \tag{69}$$

The solution of Eq. (69) for the Green's function is

$$\mathbf{G}(\mathbf{k}) = \frac{1}{9 - |h|^2} \begin{pmatrix} 3 & h^* \\ h & 3 \end{pmatrix}, \tag{70}$$

where $9 - |h|^2 = 2\left(3 - \cos\mathbf{k}\mathbf{a}_1 - \cos\mathbf{k}\mathbf{a}_2 - \cos\mathbf{k}\mathbf{a}_3\right)$ and we have introduced a third vector, $\mathbf{a}_3 = -(\mathbf{a}_1 + \mathbf{a}_2)$.

There are four types of resistance. We denote the resistance between a lattice point A as the origin and site $\mathbf{r}_0 = n\mathbf{a}_1 + m\mathbf{a}_2$ (which is an A-type site) by $R_{AA}(\mathbf{r}_0)$, while the resistance between the origin and site $\mathbf{r}_0 + (2\mathbf{a}_1 + \mathbf{a}_2)/3$ (which is a B-type site in the unit cell at $\mathbf{r}_0$) is denoted by $R_{AB}(\mathbf{r}_0)$. For symmetry reasons, it follows that for the other two types of resistance: $R_{BB}(\mathbf{r}_0) = R_{AA}(\mathbf{r}_0)$ and $R_{BA}(\mathbf{r}_0) = R_{AB}(\mathbf{r}_0)$. To measure $R_{AA}(\mathbf{r}_0)$ the current at sites A and B in the unit cell at $\mathbf{r}$ are

$$I_A(\mathbf{r}) = I\left(\delta_{\mathbf{r},0} - \delta_{\mathbf{r},\mathbf{r}_0}\right) \quad \text{and} \quad I_B(\mathbf{r}) = 0, \tag{71}$$

while for measuring $R_{AB}(\mathbf{r}_0)$ we have

$$I_A(\mathbf{r}) = I\,\delta_{\mathbf{r},0} \quad \text{and} \quad I_B(\mathbf{r}) = -I\,\delta_{\mathbf{r},\mathbf{r}_0}. \tag{72}$$

Thus

$$R_{AA}(\mathbf{r}_0) = \frac{V_A(\mathbf{r} = \mathbf{0}) - V_A(\mathbf{r}_0)}{I}. \tag{73}$$

$$R_{AB}(\mathbf{r}_0) = \frac{V_A(\mathbf{r} = \mathbf{0}) - V_B(\mathbf{r}_0)}{I}. \tag{74}$$

First, consider the resistance $R_{AA}(\mathbf{r}_0)$. From Eqs. (66), (69)–(71) we obtain

$$V_A(\mathbf{k}) = IR\,G_{11}(\mathbf{k})\left(1 - e^{-i\mathbf{k}\mathbf{r}_0}\right). \tag{75}$$

Hence, using Eq. (65) we can obtain the $\mathbf{r}$-dependence of the potential $V_A(\mathbf{r})$ and then Eq. (73) yields

$$R_{AA}(\mathbf{r}_0) = R\frac{2}{N}\sum_{\mathbf{k}\in\text{BZ}} G_{11}(\mathbf{k})\left(1 - \cos\mathbf{k}\mathbf{r}_0\right) = R\frac{3}{N}\sum_{\mathbf{k}\in\text{BZ}} \frac{1 - \cos\mathbf{k}\mathbf{r}_0}{3 - \cos\mathbf{k}\mathbf{a}_1 - \cos\mathbf{k}\mathbf{a}_2 - \cos\mathbf{k}\mathbf{a}_3}. \tag{76}$$

Using Eq. (15), the summation over $\mathbf{k}$ can be substituted by an integral and we obtain

$$R_{AA}(\mathbf{r}_0) = 3Rv_0 \int_{\mathbf{k}\in\text{BZ}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1 - \cos\mathbf{k}\mathbf{r}_0}{3 - \cos\mathbf{k}\mathbf{a}_1 - \cos\mathbf{k}\mathbf{a}_2 - \cos\mathbf{k}\mathbf{a}_3}, \tag{77}$$

where $v_0 = 3\sqrt{3}a^2$ is the area of the unit cell. One can see that the same expression was found for a triangular lattice except for the factor $3v_0$ here [see Eq. (59)]. Note that although the area of the unit cell v_0 is 6 times bigger than that for the triangular lattice, the area of the Brillouin zone is 6 times less and thus, the ratio of the resistances for honeycomb and triangular lattices is 3. Therefore, the resistance between two A-type nodes is three times the corresponding resistance in the triangular lattice. This again agrees with Atkinson's result.

Similarly, using Eqs. (66), (69), (70) and (72) we can obtain $V_A(\mathbf{k})$ and $V_B(\mathbf{k})$. From Eq. (65) the potential $V_A(\mathbf{r})$ can be determined, and analogously $V_B(\mathbf{r})$. Finally, Eq. (74) leads to

$$R_{AB}(\mathbf{r}_0) = R\frac{1}{N}\sum_{\mathbf{k}\in\mathrm{BZ}} \left(G_{11}(\mathbf{k}) + G_{22}(\mathbf{k}) - G_{12}(\mathbf{k})\,e^{-i\mathbf{k}\mathbf{r}_0} - G_{21}(\mathbf{k})\,e^{i\mathbf{k}\mathbf{r}_0}\right)$$

$$= Rv_0 \int_{\mathbf{k}\in\mathrm{BZ}} \frac{d^2\mathbf{k}}{(2\pi)^2}\, \frac{3 - \cos\mathbf{k}\mathbf{r}_0 - \cos\mathbf{k}\,(\mathbf{r}_0 + \mathbf{a}_1) - \cos\mathbf{k}\,(\mathbf{r}_0 + \mathbf{a}_1 + \mathbf{a}_2)}{3 - \cos\mathbf{k}\mathbf{a}_1 - \cos\mathbf{k}\mathbf{a}_2 - \cos\mathbf{k}\mathbf{a}_3}$$

$$= \frac{1}{3}\left[R_{AA}(\mathbf{r}_0) + R_{AA}(\mathbf{r}_0 + \mathbf{a}_1) + R_{AA}(\mathbf{r}_0 + \mathbf{a}_1 + \mathbf{a}_2)\right]. \tag{78}$$

The same result was found by Atkinson and Steenwijk.

The resistance between second nearest neighbor lattice sites may be found from Eq. (77). For symmetry reasons $R_{AA}(\mathbf{a}_1) = R_{AA}(\mathbf{a}_2) = R_{AA}(\mathbf{a}_3)$ and from Eq. (77) we have $\sum_{i=1}^{3} R_{AA}(\mathbf{a}_i) = 3R$. Thus, the resistance between second nearest neighbor lattice sites is R. Using Eq. (78), the resistance for adjacent lattice sites is $R_{AB}(\mathbf{0}) = 1/3\left[R_{AA}(\mathbf{0}) + R_{AA}(\mathbf{a}_1) + R_{AA}(\mathbf{a}_1 + \mathbf{a}_2)\right] = 2R/3$, since obviously, $R_{AA}(\mathbf{0}) = 0$. We would just mention that the limiting value of the resistance is again infinite as the separation between nodes tends to infinity (see p. 139 of Doyle's and Snell's book[3]).

We note that one can transform the honeycomb lattice to a topologically equivalent brick-type lattice shown in Fig. 4 without changing the resistance between two arbitrary lattice points.
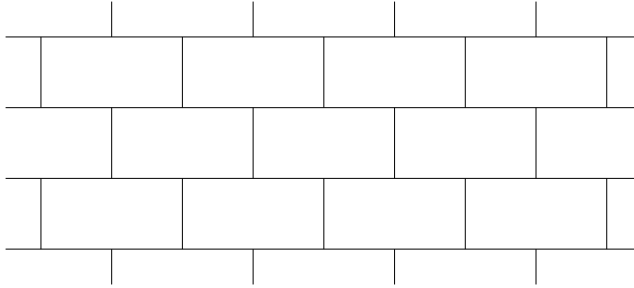

FIG. 4. The honeycomb lattice can be transformed into a topologically equivalent brick-type lattice.

This fact has been utilized in the case of the triangular lattice in the previous section.

As it can be seen from the above results, the resistance between arbitrary lattice points in a honeycomb lattice can be related to the corresponding triangular lattice. This kind of relation is the consequence of the so-called *duality transformation* in which the variables are transformed to Fourier transform variables (for more details see Chaikin's book[13] pp. 578–584). The dual lattice of a triangular lattice is a honeycomb lattice. However, if the unit cell contains more than two non-equivalent lattice points then such a connection might not be used. On the other hand, the Green's function method outlined in the example of the honeycomb lattice can still be applied straightforwardly.

ACKNOWLEDGMENTS

APPENDIX A: INTEGRATION IN THE EXPRESSION OF THE RESISTANCE FOR A SQUARE LATTICE

Starting from Eq. (24) for the resistance $R(n, m)$ we have

$$R(n, m) = R\int_{-\pi}^{\pi} \frac{dy}{2\pi}\, I(y), \tag{A1}$$

where

$$I(y) = \int_{-\pi}^{\pi} \frac{dx}{2\pi}\, \frac{1 - e^{inx}e^{imy}}{2 - \cos x - \cos y}. \tag{A2}$$

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Since $R(n,m) = R(-n,m)$, we take $n > 0$. To proceed further, we perform the integral in $I(y)$ using the method of residues. Introducing the complex variable $z = e^{ix}$ the integral $I(y)$ reads

$$I(y) = -2i \oint \frac{dz}{2\pi} \frac{1 - z^n e^{imy}}{2z(2 - \cos y) - z^2 - 1}, \tag{A3}$$

where the path of integration is the unit circle. The denominator has roots at $z_+ = e^{i\alpha_+}$ and $z_- = e^{i\alpha_-}$, where α_+ and α_- satisfy the equation $\cos \alpha = 2 - \cos y$ and $\alpha_- = -\alpha_+$. It is clear that for $-\pi < y < \pi$ we have $2 - \cos y > 1$, therefore the solution for α is purely imaginary. Thus we introduce s with $\alpha_+ = -\alpha_- = is$, where s satisfies the equation

$$\cosh s = 2 - \cos y. \tag{A4}$$

For $-\pi < y < \pi$ it is true that $s > 0$, so the two poles of the integrand in $I(y)$ are real numbers and satisfy the following inequalities: $z_+ = e^{-s} < 1$ and $z_- = e^s > 1$. Thus the pole z_+ is within the unit circle, while z_- is outside. According to the residue theorem[19] $I(y) = -2i2\pi i \sum \text{residues}$ within the unit circle. Using Eq. (A4) the residue of the integrand of $I(y)$ at z_+ is

$$\frac{1}{2\pi} \frac{1 - e^{-ns} e^{imy}}{2(2 - \cos y) - 2z_+} = \frac{1}{2\pi} \frac{1 - e^{-ns} e^{imy}}{2\sinh s}. \tag{A5}$$

We obtain

$$I(y) = -2i2\pi i \frac{1}{2\pi} \frac{1 - e^{-ns} e^{imy}}{2\sinh s} = \frac{1 - e^{-ns} e^{imy}}{\sinh s}, \tag{A6}$$

where s satisfies Eq. (A4). Finally, the resistance $R(n,m)$ for arbitrary integers n, m becomes

$$R(n,m) = R \int_{-\pi}^{\pi} \frac{dy}{2\pi} \frac{1 - e^{-|n|s} e^{imy}}{\sinh s} = R \int_{0}^{\pi} \frac{dy}{\pi} \frac{1 - e^{-|n|s} \cos my}{\sinh s}. \tag{A7}$$

The same result was found by Venezian[4].

APPENDIX B: THE ASYMPTOTIC FORM OF THE LATTICE GREEN'S FUNCTION FOR A SQUARE LATTICE

In this Appendix we derive the asymptotic form of the lattice Green's function for a square lattice. The lattice Green's function at site $\mathbf{r} = 0$ is divergent since $\varepsilon(\mathbf{k}) = 0$ for $\mathbf{k} = 0$. Therefore we calculate the asymptotic form of $G(\mathbf{0}) - G(\mathbf{r})$. Starting from Eq. (19) the lattice Green's function for site $\mathbf{r} = n\mathbf{a}_1 + m\mathbf{a}_2$ in a square lattice becomes

$$G(\mathbf{0}) - G(n,m) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dy}{2\pi} \int_{-\pi}^{\pi} \frac{dx}{2\pi} \frac{1 - e^{inx} e^{imy}}{2 - \cos x - \cos y}. \tag{B1}$$

The integral over x is the same as $I(y)$ in Eq. (A2), so we can use the result obtained in Eq. (A6):

$$G(\mathbf{0}) - G(n,m) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dy}{2\pi} \frac{1 - e^{-|n|s} e^{imy}}{\sinh s} = \int_{0}^{\pi} \frac{dy}{2\pi} \frac{1 - e^{-|n|s} \cos my}{\sinh s}. \tag{B2}$$

We follow the same method for deriving the asymptotic form of the lattice Green's function for large m and n as Venezian[4]. A similar method was used in Chaikin's book[13] (see pp. 295–296) in the case of a continuous medium in two dimensions. We break the integral in Eq. (B2) into three parts:

$$G(\mathbf{0}) - G(n,m) = I_1 + I_2 + I_3, \tag{B3}$$

where

$$I_1 = \int_{0}^{\pi} \frac{dy}{2\pi} \frac{1 - e^{-|n|y} \cos my}{y},$$

$$I_2 = \int_{0}^{\pi} \frac{dy}{2\pi} \left(\frac{1}{\sinh s} - \frac{1}{y} \right),$$

$$I_3 = \int_{0}^{\pi} \frac{dy}{2\pi} \left(\frac{e^{-|n|y} \cos my}{y} - \frac{e^{-|n|s} \cos my}{\sinh s} \right), \tag{B4}$$

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and s satisfies Eq. (A4). The first integral I_1 can be expressed by the integral exponential $\text{Ein}(z)$[35]:

$$I_1 = \frac{1}{2\pi}\text{Re}\left\{\int_0^\pi dy\,\frac{1-e^{(|n|-im)y}}{y}\right\} = \frac{1}{2\pi}\text{Re}\left\{\text{Ein}\left[\pi\left(|n|-im\right)\right]\right\}, \tag{B5}$$

where $\text{Ein}(z)$ is defined by

$$\text{Ein}(z) = \int_0^z dt\,\frac{1-e^{-t}}{t}. \tag{B6}$$

For large values of its argument, $\text{Ein}(z) \approx \ln z + \gamma$, where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. Thus, for large n and m I_1 can be approximated by

$$I_1 \approx \frac{1}{2\pi}\left(\ln|\pi\left(|n|-im\right)| + \gamma\right) = \frac{1}{2\pi}\left(\ln\sqrt{n^2+m^2} + \gamma + \ln\pi\right). \tag{B7}$$

Using Eq. (A4) the integral I_2 can be evaluated exactly:

$$I_2 = \int_0^\pi \frac{dy}{2\pi}\left(\frac{1}{\sqrt{\left(2-\cos y\right)^2-1}} - \frac{1}{y}\right) = \frac{1}{2\pi}\left(\frac{\ln 8}{2} - \ln\pi\right). \tag{B8}$$

In the integral I_3 the integrand is close to zero for small values of y and s since $s \approx \sinh s \approx y$, while for larger values of y and s the exponentials are negligible, therefore $I_3 \approx 0$.

Finally, we find that the lattice Green's function for large arguments, *i.e.*, $|\mathbf{r}| = a\sqrt{n^2+m^2} \to \infty$ becomes

$$G(\mathbf{r}) = G(\mathbf{0}) - \frac{1}{2\pi}\left(\ln\frac{|\mathbf{r}|}{a} + \gamma + \frac{\ln 8}{2}\right). \tag{B9}$$

The same result is quoted on page 296 of Chaikin's book[13]. In the theory of the Kosterlitz–Thouless–Berezinkii phase transition[10–12] Kosterlitz used the same asymptotic form of the Green's function for a square lattice[11].

APPENDIX C: INTEGRATION IN THE EXPRESSION OF THE RESISTANCE FOR A TRIANGULAR LATTICE

Introducing the coordinate transformations $x' = (x+y)/2$ and $y' = (x-y)/2$, and using the complex variable $z = e^{iy'/2}$ we have $R(n,m) = 1/2\,R\int_{-\pi}^\pi dx'/2\pi\,I(x')$ from Eq. (61), where

$$I(x') = 2i\oint \frac{dz}{2\pi}\,\frac{1 - z^{(n-m)}\,e^{i(n+m)\frac{x'}{2}}}{z^2\cos\frac{x'}{2} - z\left(3-\cos x'\right) + \cos\frac{x'}{2}}, \tag{C1}$$

and the path of integration is the unit circle. The factor $1/2$ in front of $R(n,m)$ is the Jacobian corresponding to the transformations of variables. Since $R(n\mathbf{a}_1 - m\mathbf{a}_2) = R(m\mathbf{a}_1 - n\mathbf{a}_2)$, we take $n-m > 0$. The denominator has roots in $z_+ = e^{i\alpha_+/2}$ and $z_- = e^{i\alpha_-/2}$ and it is easy to see that they satisfy equations $\alpha_+ = -\alpha_-$ and $2\cos(\alpha_+/2) = (3-\cos x')/\cos(x'/2)$. For $-\pi < x' < \pi$ it is clear that α_+ is purely imaginary. If we choose α_+ such that $\text{Re}\,\alpha_+ > 0$, then $z_+ < 1$ and $z_- > 1$. Thus, with $\alpha_+ = is$, the residue of the integrand of $I(x')$ at z_+ (which is inside the unit circle, *i.e.*, $s > 0$) is

$$\frac{1}{2\pi}\frac{1 - e^{-(n-m)\frac{s}{2}}\,e^{i(n+m)\frac{x'}{2}}}{2z_+\cos\frac{x'}{2} - \left(3-\cos x'\right)} = -\frac{1}{2\pi}\frac{1 - e^{-(n-m)\frac{s}{2}}\,e^{i(n+m)\frac{x'}{2}}}{2\sinh\frac{s}{2}\,\cos\frac{x'}{2}}, \tag{C2}$$

where s satisfies the equation

$$2\cosh\frac{s}{2} = \frac{3-\cos x'}{\cos\frac{x'}{2}}. \tag{C3}$$

Finally, the resistance for arbitrary integers n, m is

$$R(n,m) = \frac{R}{2}\int_{-\pi}^\pi \frac{dx'}{2\pi}\,\frac{1 - e^{-|n-m|\frac{s}{2}}\,e^{i(n+m)\frac{x'}{2}}}{\sinh\frac{s}{2}\,\cos\frac{x'}{2}}. \tag{C4}$$

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