Ergodic averages with prime divisor weights in L^1

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Abstract

We show that $\omega(n)$ and $\Omega(n)$, the number of distinct prime factors of n and the number of distinct prime factors of n counted according to multiplicity are good weighting functions for the pointwise ergodic theorem in L^1. That is, if g denotes one of these functions and $S_{g,K} = \sum_{n\leq K} g(n)$ then for every ergodic dynamical system $(X, \mathcal{A}, \mu, \tau)$ and every $f \in L^1(X)$

$$\lim_{K\to\infty} \frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n) f(\tau^n x) = \int_X f d\mu \text{ for } \mu \text{ a.e. } x \in X.$$

This answers a question raised by C. Cuny and M. Weber who showed this result for L^p, $p > 1$.

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1 Introduction

In [1] C. Cuny and M. Weber investigated whether some arthimetic weights are good weights for the pointwise ergodic theorem in L^p. In this paper we show that the prime divisor functions ω and Ω are both good weights for the L^1 pointwise ergodic theorem. The same fact for the spaces L^p, $p > 1$ was proved in [1] and our paper answers a question raised in that paper. Recall that if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ then $\omega(n) = k$ and $\Omega(n) = \alpha_1 + ... + \alpha_k$. We denote by g one of these functions. Given K we put

$$S_{g,K} = \sum_{n \le K} g(n).$$

We suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $\tau : X \to X$ is a measure preserving ergodic transformation. Given $f \in L^1(X)$ we consider the g-weighted ergodic averages

$$\mathcal{M}_{g,K} f(x) = \frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n) f(\tau^n x). \tag{1}$$

We show that for $g = \omega$, or Ω these averages μ a.e. converge to $\int_X f d\mu$, that is g is a good universal weight for the pointwise ergodic theorem in L^1. See Theorem 6.

For some similar ergodic theorems with other weights like the Möbius function, or its absolute value, or the Liouville function we refer to the papers of El Abdalaoui, Kułaga-Przymus, Lemańczyk and de la Rue, [3], and of Rosenblatt and Wierdl [8].

2 Preliminary results

We recall Theorem 430 from p. 72 of [5]

$$\sum_{n \le K} \omega(n) = K \log\log K + B_1 K + o(K) \text{ and} \tag{2}$$

$$\sum_{n \le K} \Omega(n) = K \log\log K + B_2 K + o(K). \tag{3}$$

Hence, for both cases we can assume that there exists a constant B (which depends on whether $g = \omega$, or $g = \Omega$) such that

$$\sum_{n \le K} g(n) = K \log\log K \Big(1 + \frac{B}{\log\log K} + \frac{o(K)}{K \log\log K}\Big). \tag{4}$$

From this it follows that there exists $C_g > 0$ such that for all $K \in \mathbb{N}$

$$\Big(\sum_{n \le K} g(n)\Big)^{\lfloor \log\log K \rfloor} = (S_{g,K})^{\lfloor \log\log K \rfloor} > C_g (K \lfloor \log\log K \rfloor)^{\lfloor \log\log K \rfloor}. \tag{5}$$

We need some information about the distribution of the functions ω and Ω. We use (3.9) from p. 689 of [6] by K. K. Norton which is based on a result of Halász [4] which is cited as (3.8) Lemma in [6]. Next we state (3.9) from [6] with $\delta = 0.1$ and $z = 2 - \delta = 1.9$.

Proposition 1. *There exists a constant $\widetilde{C}_H$ such that for every $K \ge 1$*

$$\sum_{n \le K} 1.9^{\omega(n)} \le \sum_{n \le K} 1.9^{\Omega(n)} \le \widetilde{C}_H K \exp(0.9 \cdot E(K)), \ \ \textit{where} \tag{6}$$

$E(K) = \sum_{p \le K} \frac{1}{p}$.

Recall that by Theorem 427 in [5]

$$E(K) = \sum_{p \le K} \frac{1}{p} = \log\log K + B_1 + o(1). \tag{7}$$

The constant B_1 is the same which appears in (2). The way we will use this is the following: there exists a constant C_P such that for $K > 3$

$$E(K) = \sum_{p \le K} \frac{1}{p} < C_P \log\log K. \tag{8}$$

Combining this with (6) we obtain that for $g = \omega$, or Ω we have for $K > 3$

$$\sum_{n \le K} 1.9^{g(n)} < \widetilde{C}_H \cdot K \cdot \exp(0.9 \cdot C_P \log\log K) \le C_H \cdot K \exp(0.9 \cdot C_P \lfloor \log\log K \rfloor), \tag{9}$$

with a suitable constant C_H not depending on K.

In [1] a result of Delange [2] was used to deduce Theorem 2.7 in [1]. The result of Delange is the following

Theorem 2. *For every $m \geq 1$ we have*

$$\sum_{n \leq K} g(n)^m = K(\log\log K)^m + O(K(\log\log K)^{m-1}).$$

We were unable to use this result since the constant in $O(K(\log\log K)^{m-1})$ cannot be chosen not depending on $m \geq 1$.

Hence we use (9) in the proof of the following lemma.

Lemma 3. *There exists a constant $C_{\Omega,max}$ such that for all $K \geq 16$*

$$\sum_{n \leq K} \omega(n)^{\lfloor \log\log K \rfloor} \leq \sum_{n \leq K} \Omega(n)^{\lfloor \log\log K \rfloor} < K(C_{\Omega,max} \lfloor \log\log K \rfloor)^{\lfloor \log\log K \rfloor}. \tag{10}$$

We remark that the assumption $K \geq 16$ implies that $\log\log K > 1.01 > 1$.

Proof. Since $\omega(n) \leq \Omega(n)$ the first inequality is obvious in (10),

We assume that $K \geq 16$ is fixed and for ease of notation we put $\nu = \lfloor \log\log K \rfloor$. Set

$$N_{l,K} = \{n \leq K : 2^l \nu \leq \Omega(n) < 2^{l+1}\nu\}. \tag{11}$$

By (9) $N_{l,K} \cdot 1.9^{2^l \nu} < C_H K \exp(0.9 \cdot C_P \nu)$. This implies that

$$N_{l,K} < C_H K \cdot \exp((0.9 \cdot C_P - 2^l \log 1.9)\nu). \tag{12}$$

Since $\log 1.9 > 0.6$ we can choose l_0 such that for $l \geq l_0$

$$0.9 \cdot C_P - 2^l \log 1.9 + (l+1)\log 2 < -0.5 \cdot 2^l = -2^{l-1}. \tag{13}$$

From (12) and (13) we infer

$$\sum_{n \leq K} \Omega(n)^\nu < K \cdot (2\nu)^\nu + \sum_{l=1}^{\infty} N_{l,K}(2^{l+1}\nu)^\nu \leq \tag{14}$$

$$K \cdot (2\nu)^\nu + \sum_{l=1}^{l_0-1} K(2^{l+1}\nu)^\nu + \sum_{l=l_0}^{\infty} C_H K \nu^\nu \exp(((\log 2^{l+1}) + 0.9C_P - 2^l \log 1.9)\nu) <$$

(using (13) with a suitable constant $C_{\Omega,1} > 2$ we obtain)

$$C_{\Omega,1}^{\nu} K\nu^{\nu} + \sum_{l=l_0}^{\infty} C_H K\nu^{\nu} \exp(-2^{l-1}\nu) <$$

(recalling that $\nu = \lfloor \log\log K \rfloor \geq \lfloor \log\log 16 \rfloor = 1$, with a suitable constant $C_{\Omega,max}$ we have)

$$K\nu^{\nu}\Big(C_{\Omega,1}^{\nu} + C_H \sum_{l=l_0}^{\infty} \exp(-2^{l-1})\Big) < C_{\Omega,max}^{\nu} K\nu^{\nu} =$$

$$K(C_{\Omega,max}\lfloor \log\log K \rfloor)^{\lfloor \log\log K \rfloor}.$$

□

We need the following (probably well-known) elementary inequality to which we could not find a reference and hence provided the short proof.

Lemma 4. *Suppose $K, \nu \in \mathbb{N}$, $b_1, ..., b_K$ are nonnegative numbers and we have permutations $\pi_j : \{1, ..., K\} \to \{1, ..., K\}$, $j = 1, ..., \nu$. Then*

$$b_{\pi_1(1)} \cdots b_{\pi_\nu(1)} + ... + b_{\pi_1(K)} \cdots b_{\pi_\nu(K)} \leq b_1^{\nu} + ... + b_K^{\nu}. \tag{15}$$

Proof. Without limiting generality we can suppose that $0 \leq b_1 \leq ... \leq b_K$. First observe that if $A > B \geq 0$ and $C > D \geq 0$ then

$$\text{from } (A-B)(C-D) \geq 0 \text{ it follows that } AC + BD \geq AD + BC. \tag{16}$$

Set $\pi_{j,1}(k) = \pi_j(k)$ for $j = 1, ..., \nu$ and $k = 1, ..., K$. If $\pi_{j,l}$ is defined for an $l \in \mathbb{N}$ then set

$$\mathfrak{M}_l^* = \max_k b_{\pi_{1,l}(k)} \cdots b_{\pi_{\nu,l}(k)}.$$

We want to define a sequence of permutations such that for every l

$$b_{\pi_{1,l-1}(1)} \cdots b_{\pi_{\nu,l-1}(1)} + ... + b_{\pi_{1,l-1}(K)} \cdots b_{\pi_{\nu,l-1}(K)} \leq \tag{17}$$

$$b_{\pi_{1,l}(1)} \cdots b_{\pi_{\nu,l}(1)} + ... + b_{\pi_{1,l}(K)} \cdots b_{\pi_{\nu,l}(K)}.$$

Suppose that $\mathfrak{M}_l^* < b_K^{\nu}$. Select k^* such that $\mathfrak{M}_l^* = b_{\pi_{1,l}(k^*)} \cdots b_{\pi_{\nu,l}(k^*)}$. Then we can select j^* such that $b_{\pi_{j^*,l}(k^*)} < b_K$ and k^{**} such that $b_{\pi_{j^*,l}(k^{**})} = b_K$. Set $A = b_K = b_{\pi_{j^*,l}(k^{**})}$, $B = b_{\pi_{j^*,l}(k^*)}$, $C = b_{\pi_{1,l}(k^*)} \cdots b_{\pi_{\nu,l}(k^*)}/B = \mathfrak{M}_l^*/B$

and $D = b_{\pi_{1,l}(k^{**})} \cdots b_{\pi_{\nu,l}(k^{**})}/A$. Then $A > B \geq 0$ and $C > D \geq 0$. Set $\pi_{j^*,l+1}(k^{**}) = \pi_{j^*,l}(k^*)$, $\pi_{j^*,l+1}(k^*) = \pi_{j^*,l}(k^{**})$, and for any other j and k set $\pi_{j,l+1}(k) = \pi_{j,l}(k)$. From (16) it follows that (17) holds with l replaced by $l+1$ and $\mathfrak{M}^*_{l+1} > \mathfrak{M}^*_l$. Hence in finitely many steps there is l_1 such that $\mathfrak{M}^*_{l_1} = b^\nu_K$.

After step l_1 arguing as above we can still define the permutations $\pi_{j,l}$ so that (17) holds at each step and can reach a step l_2 such that $\mathfrak{M}^*_{l_2} = b^\nu_K$ and the second largest term among $b_{\pi_{1,l_2}(k)} \cdots b_{\pi_{\nu,l_2}(k)}$, $k = 1, ..., K$ equals b^ν_{K-1}. Repeating this procedure one can obtain (15). □

We will use the transference principle and hence we need to consider functions on the integers. Suppose $\varphi : \mathbb{Z} \to [0, +\infty)$ is a function on the integers with compact/bounded support. Again g will denote ω, or Ω. Put

$$M_{g,K}\varphi(j) = \frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n)\varphi(j+n) \text{ for } j \in \mathbb{Z}.$$

First we prove a "localized" maximal inequality.

Lemma 5. *There exists a constant $C_{g,max} > 0$ such that for any $\varphi : \mathbb{Z} \to [0, +\infty)$, $K \geq 16$ and $k \in \mathbb{Z}$*

$$\sum_{j=1}^{K} (M_{g,K}\varphi(k+j))^{\lfloor \log\log K \rfloor} \leq \Big(\sum_{j=2}^{2K} \varphi(k+j) \Big) \Big(\frac{C_{g,max}}{K} \sum_{j=2}^{2K} \varphi(k+j) \Big)^{\lfloor \log\log K \rfloor - 1}. \tag{18}$$

Proof. Without limiting generality we can suppose that $k = 0$ and $K \geq 16$ is fixed. We use again the notation $\nu = \nu_K = \lfloor \log\log K \rfloor$. We put

$$\widetilde{g}(n) = \widetilde{g}_K(n) = \begin{cases} g(n) & \text{if } 1 \leq n \leq K \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

We need to estimate

$$\sum_{j=1}^{K} \Big(\frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n)\varphi(j+n) \Big)^{\nu} =$$

$$\frac{1}{S^\nu_{g,K}} \sum_{j=1}^{K} \sum_{n_1=1}^{K} ... \sum_{n_\nu=1}^{K} g(n_1) \cdots g(n_\nu) \cdot \varphi(j+n_1) \cdots \varphi(j+n_\nu) =$$

$$\frac{1}{S^{\nu}_{g,K}}\sum_{n'=1}^{K}\sum_{j_1=2}^{2K}...\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\widetilde{g}(n')\widetilde{g}(n'+j_2-j_1)\cdots\widetilde{g}(n'+j_\nu-j_1)=$$

$$\frac{1}{S^{\nu}_{g,K}}\sum_{j_1=2}^{2K}...\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\sum_{n'=1}^{K}\widetilde{g}(n')\widetilde{g}(n'+j_2-j_1)\cdots\widetilde{g}(n'+j_\nu-j_1)\leq$$

(using Lemma 4 and (19))

$$\frac{1}{S^{\nu}_{g,K}}\sum_{j_1=2}^{2K}...\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\sum_{n'=-K+2}^{2K-1}(\widetilde{g}(n'))^{\nu}=$$

$$\frac{1}{S^{\nu}_{g,K}}\sum_{j_1=2}^{2K}...\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\sum_{n'=1}^{K}(g(n'))^{\nu}\leq$$

(by using Lemma 3)

$$K\cdot C^{\nu}_{\Omega,max}\nu^{\nu}\frac{1}{S^{\nu}_{g,K}}\Big(\sum_{j=2}^{2K}\varphi(j)\Big)^{\nu}<$$

(by (5))

$$K\cdot C^{\nu}_{\Omega,max}\nu^{\nu}\frac{1}{C_g(K\nu)^{\nu}}\Big(\sum_{j=2}^{2K}\varphi(j)\Big)^{\nu}<$$

(with a suitable constant $C_{g,max}>0$)

$$<\Big(\sum_{j=2}^{2K}\varphi(j)\Big)\cdot\Big(C_{g,max}\frac{1}{K}\sum_{j=2}^{2K}\varphi(j)\Big)^{\nu-1}.$$

□

3 Main result

Theorem 6. *For every ergodic dynamical system $(X,\mathcal{A},\mu,\tau)$ and every $f\in L^1(X)$*

$$\lim_{K\to\infty}\mathcal{M}_{g,K}f(x)=\int_X f d\mu \text{ for } \mu \text{ a.e. } x\in X. \tag{20}$$

Proof. By Theorem 2.5 and Remark 2.6 of [1] we know that ω and Ω are good weights for the pointwise ergodic theorem in L^p for $p > 1$. This means that we have a dense set of functions in L^1 for which the pointwise ergodic theorem holds. In Theorem 2.5 of [1] it is not stated explicitely that the limit function of the averages $\mathcal{M}_{g,K}f$ is $\int_X f d\mu$, but from the proof of this theorem it is clear that $\mathcal{M}_{g,K}f$ not only converges a.e., but its limit is indeed $\int_X f d\mu$ (at least for $f \in L^\infty(\mu)$). Indeed, from (2.2) in [1] it follows that $\mathcal{M}_{g,K}f$ can be written as the sum of an ordinary Birkhoff-average of f and an error term which tends to zero as $K \to \infty$.

Hence by standard application of Banach's principle (see for example [7] p. 91) the following weak L^1-maximal inequality proves Theorem 6.

Proposition 7. *There exists a constant C_{max} such that for every ergodic dynamical system $(X, \mathcal{A}, \mu, \tau)$ for every $f \in L^1(\mu)$ and $\lambda \geq 0$*

$$\mu\{x : \sup_{K \geq 1} \mathcal{M}_{g,K}f(x) > \lambda\} \leq C_{max} \frac{||f||_1}{\lambda}. \tag{21}$$

□

Proof of Proposition 7. By standard transference arguments, see for example [8] Chapter III, it is sufficient to establish a corresponding weak maximal inequality on the integers with $\lambda = 1$ for nonnegative functions with compact support. Hence, this proof will be completed by Proposition 8 below. □

Thus we need to state and prove the following maximal inequality:

Proposition 8. *There exists a constant C_{max} such that for every $\varphi : \mathbb{Z} \to [0, \infty)$ with compact support*

$$\#\{j : \sup_{K \in \mathbb{N}} M_{g,K}\varphi(j) > 1\} \leq C_{max} ||\varphi||_{\ell_1}.$$

Proposition 8 can also be reduced further to the following Claim. Set $M_l = M_{g,2^l}$.

Claim 9. *There exists a constant C'_{max} such that for every $\varphi : \mathbb{Z} \to [0, +\infty)$ with compact support*

$$\#\{j : \sup_{l \in \mathbb{N}} M_l\varphi(j) > 1\} \leq C'_{max} ||\varphi||_{\ell_1}. \tag{22}$$

Proof of Proposition 8 based on Claim 9. Given $K \in \mathbb{N}$ choose $l_K \in \mathbb{N}$ such that $2^{l_K-1} < K \leq 2^{l_K}$. By (2), or (3) there exists a constant $C_R > 0$ not depending on K such that $S_{g,2^{l_K}} \leq C_R S_{g,K}$. We have

$$1 < M_{g,K}\varphi(j) = \frac{1}{S_{g,K}} \sum_{j=1}^{K} g(n)\varphi(j+n) \leq$$

$$\frac{S_{g,2^{l_K}}}{S_{g,K}} \cdot \frac{1}{S_{g,2^{l_K}}} \sum_{n=1}^{2^{l_K}} g(n)\varphi(j+n) \leq C_R M_{g,2^{l_K}}\varphi(j).$$

Hence, $1 < M_{g,K}\varphi(j)$ implies $\frac{1}{C_R} < M_{g,2^{l_K}}\varphi(j) = M_{l_K}\varphi(j)$.

For any $\widetilde{\varphi} : \mathbb{Z} \to [0, +\infty)$ with compact support taking $\varphi = C_R\widetilde{\varphi}$ by Claim 9 we obtain

$$\#\{j : \sup_{K\in\mathbb{N}} M_{g,K}\widetilde{\varphi}(j) > 1\} \leq \#\{j : \sup_{l\in\mathbb{N}} M_l\varphi(j) > 1\} \leq$$

$$C'_{max}||\varphi||_{\ell_1} = C'_{max}C_R||\widetilde{\varphi}||_{\ell_1}.$$

□

Proof of Claim 9. If $1 \leq l \leq 4$ then consider the set $E_l = \{j : M_l\varphi(j) > 1\}$ and the system of intervals $\mathcal{I}_l = \{[j+1, j+2^l] \cap \mathbb{Z} : j \in E_l\}$. Then $E_l + 1 \subset \cup_{I\in\mathcal{I}_l} I$ and hence $\#E_l \leq \# \cup_{I\in\mathcal{I}_l} I$. We can select a subsystem $\mathcal{I}'_l \subset \mathcal{I}_l$ such that no point of $\mathbb{Z}$ is covered by more than two intervals belonging to $\mathcal{I}'_l$ and $\cup_{I\in\mathcal{I}'_l} I = \cup_{I\in\mathcal{I}_l} I$.

Suppose $I = [j+1, j+2^l] \cap \mathbb{Z} \in \mathcal{I}'_l \subset \mathcal{I}_l$. Then $M_l\varphi(j) > 1$ implies that

$$1 < \frac{1}{S_{g,2^l}} \sum_{n=1}^{2^l} g(n)\varphi(j+n),$$

that is

$$S_{g,2^l} \leq \sum_{n=1}^{2^l} g(n)\varphi(j+n) = \sum_{k\in I} g(k-j)\varphi(k).$$

Thus

$$1 \leq \frac{S_{g,2^l}}{\max_{k\leq 2^l} g(k)} \leq \sum_{k\in I} \varphi(k).$$

If $l \le 4$ then we have $\#I/16 \le 1 \le \sum_{k\in I}\varphi(k)$. Since no point is covered by more than two intervals $I \in \mathcal{I}_l'$, that is, $\sum_{I\in\mathcal{I}_l'}\chi_I(j) \le 2$, $(j \in \mathbb{Z})$ we obtain that for $l \le 4$

$$\#E_l \le \# \cup_{I\in\mathcal{I}_l'} I \le 32||\varphi||_{\ell_1}$$

and hence

$$\#\{j : \sup_{1\le l\le 4} M_l\varphi(j) > 1\} \le 128||\varphi||_{\ell_1}. \tag{23}$$

Next suppose that $l > 4$. We consider the dyadic intervals $(r2^l, (r+1)2^l] \cap \mathbb{Z}$, $r \in \mathbb{Z}$. We say that $r \in R_{l,+}$ if

$$\frac{1}{2^l}\sum_{j=r2^l+1}^{r2^l+2\cdot 2^l}\varphi(j) > \frac{1}{100\cdot C_{g,max}}. \tag{24}$$

Otherwise, if $r \notin R_{l,+}$ we say that $r \in R_{l,-}$.

For $r \in R_{l,-}$ we use Lemma 5 and the negation of (24) to deduce that for $l > 4$

$$\sum_{j=1}^{2^l}(M_l\varphi(r2^l+j))^{\lfloor \log\log 2^l\rfloor} < \Big(\sum_{j=2}^{2\cdot 2^l}\varphi(r2^l+j)\Big)\cdot\Big(\frac{1}{100}\Big)^{\lfloor \log\log 2^l\rfloor - 1} \le \tag{25}$$

$$100^2\Big(\sum_{j=2}^{2\cdot 2^l}\varphi(r2^l+j)\Big)\cdot\Big(\frac{1}{100}\Big)^{\log\log 2^l} \le$$

$$100^2\Big(\sum_{j=2}^{2\cdot 2^l}\varphi(r2^l+j)\Big)\cdot\exp(-(\log 100)\cdot\log\log 2^l) \le$$

$$100^2\Big(\sum_{j=2}^{2\cdot 2^l}\varphi(r2^l+j)\Big)\cdot\frac{6}{l^2}, \text{ where we used that}$$

$4.61 \ge \log 100 \ge 4.60517$ and $\log\log 2 > -0.37$ implies that

$$\exp(-(\log 100)\cdot\log\log 2^l) = \exp(-(\log 100)((\log l) + \log\log 2)) =$$

$$\exp(-(\log 100)\log\log 2)\cdot\exp(-(\log 100)\log l) < \frac{6}{l^2}.$$

Set $\mathcal{M}_l^* = \{j : M_l\varphi(j) > 1\}$ and $\mathcal{M}^* = \cup_l\mathcal{M}_l^*$.

If $r \in R_{l,-}$ then by (25)

$$\#(\mathcal{M}_l^* \cap (r2^l, (r+1)2^l]) \leq \sum_{j=1}^{2^l} (M_l \varphi(r2^l + j))^{\lfloor \log\log 2^l \rfloor} \leq$$

$$6 \cdot 100^2 \cdot \frac{1}{l^2} \Big(\sum_{j=2}^{2 \cdot 2^l} \varphi(r2^l + j) \Big).$$

Hence

$$\#(\mathcal{M}_l^* \cap \bigcup_{r \in R_{l,-}} (r2^l, (r+1)2^l]) \leq$$

$$12 \cdot 100^2 \frac{1}{l^2} ||\varphi_{\ell_1}||$$

and

$$\#\Big(\bigcup_l (\mathcal{M}_l^* \cap \cup_{r \in R_{l,-}} (r2^l, (r+1)2^l]) \Big) \leq 12 \cdot 100^2 \frac{\pi^2}{6} ||\varphi_{\ell_1}||. \qquad (26)$$

On the other hand,

$$\cup_{l>4} \cup_{r \in R_{l,+}} (r2^l, (r+1)2^l] \cap \mathbb{Z} \subset \cup_{l>4} \cup_{r \in R_{l,+}} [r2^l, (r+2)2^l] \cap \mathbb{Z}. \qquad (27)$$

We can again select a subsystem $\mathcal{I}_+^*$ of the intervals $\mathcal{I}_+ = \{[r2^l, (r+2)2^l] : l > 4,\ r \in R_{l,+}\}$ such that

$$\sum_{I \in \mathcal{I}_+^*} \chi_I(j) \leq 2 \text{ for all } j \in \mathbb{Z} \text{ and } \cup_{I \in \mathcal{I}_+} I = \cup_{I \in \mathcal{I}_+^*} I. \qquad (28)$$

From (24) it follows that if $[r2^l, (r+2)2^l] = I \in \mathcal{I}_+^*$ then

$$C_{g,max} \cdot 400 \sum_{j \in I} \varphi(j) > 4 \cdot 2^l > \#(I \cap \mathbb{Z}).$$

Thus, by (28)

$$\#(\cup_{I \in \mathcal{I}_+} I \cap \mathbb{Z}) = \#(\cup_{I \in \mathcal{I}_+^*} I \cap \mathbb{Z}) < C_{g,max} \cdot 800 ||\varphi||_{\ell_1}.$$

Hence, by (27)

$$\#\Bigg(\cup_{l>4} \cup_{r \in R_{l,+}} (r2^l, (r+1)2^l] \cap \mathbb{Z} \Bigg) \leq C_{g,max} \cdot 800 ||\varphi||_{\ell_1}.$$

From this, (23) and (26) it follows that

$$\#\mathcal{M}^* \leq (128 + 12 \cdot 100^2 \frac{\pi^2}{6} + 800 C_{g,max}) ||\varphi||_{\ell_1} = C'_{max} ||\varphi||_{\ell_1}.$$

□

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