Universal threshold enhancement

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By assuming certain analytic properties of the propagator, it is shown that universal features of the spectral function including threshold enhancement arise if a pole describing a particle at high temperature approaches in the complex energy plane the threshold position of its two-body decay with the variation of T. The case is considered, when one can disregard any other decay processes. The quality of the proposed description is demonstrated by comparing it with the detailed large N solution of the linear σ model around the pole-threshold coincidence.

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The near-threshold two-body decay of a resonance is a well-studied problem in scattering theory [1]. In a medium with tunable parameters (e.g. temperature, density etc.) new features may arise [2]. It has been found [3, 4] that a hallmark of partial chiral symmetry restoration in a medium is the enhancement of the scalar-isoscalar spectral function near the two-pion threshold. In order to elucidate further the origins and the nature of this phenomenon, in this note we formulate a general approach to it. On the basis of this investigation some features of threshold enhancement appear to be *universal*, showing up whenever the mass of a stable particle can be driven by tuning some parameters of its environment to pass the threshold of a two-body decay into stable final particles. Following our general discussion we shall show evidence for this situation in the linear σ model by studying the propagator of the σ particle, neglecting the finite in-medium lifetime of the pions.

Let us start our general discussion by assuming the existence of a branch point in the complex $\sqrt{p^2}$ rest frame energy plane on the real energy axis for the propagator $G_\sigma(p^2,|\mathbf{p}|)$, $(p^2 = p_0^2 - \mathbf{p}^2)$, which corresponds to the $\sigma \to \alpha_1 + \alpha_2$ hypothetical decay. (In this part of our discussion σ refers to an arbitrary particle. The extra dependence on the 3-momentum is a medium effect.) We can restrict the discussion to the halfplane $\mathrm{Re}\sqrt{p^2} > 0$, due to the symmetric behavior of the propagator. At the branch point a square root singularity, proportional to $\delta \equiv \sqrt{1 - p^2/M^2}$, shows up in the self-energy contribution to the propagator, where M denotes the temperature dependent position of the branch point. The analytic (Riemann-sheet) structure of the $\sqrt{p^2}$-plane is specified by requiring that G_σ is analytically continued through the part of the real axis where $p^2 > M^2$, i.e. where the complex threshold factor δ is purely imaginary. Furthermore the plane we are working on is bounded by a cut along the section $p^2 < M^2$ of the real axis. Along the cut δ is real positive (negative) on the upper (lower) halfplane. Conventionally, one associates the upper halfplane with the first, the lower one with the second Riemann-sheet. We shall call the upper edge of the real axis reached from the first Riemann sheet as physical, and the lower edge of the cut as unphysical.

The goal of our analysis is to investigate the threshold behavior of the spectral function in the neighborhood of the branch point in a small interval around some temperature T^*, to be specified below. We first restrict the discussion to the case $|\mathbf{p}| = 0$, which makes our analysis easier to follow. There are now two small quantities characterizing this situation: $\epsilon \equiv (T - T^*)/T^*$ and δ. [It is important to note, that it is $M(T)$ and not $M(T^*)$, which appears in the definition of δ.] Our basic assumption is that the inverse propagator $G_\sigma^{-1}(p_0)$ can be expanded in terms of these two independent variables. Then to quadratic order one obtains

$$\begin{aligned} -G_\sigma^{-1}(p_0) &\approx a_1\epsilon + \tilde{a}_2\epsilon^2 + b_1\delta + b_2\delta^2 + c\delta\epsilon \\ &\approx \frac{1}{1 + c\epsilon/b_1}[a_1\epsilon + a_2\epsilon^2 + b_1\delta + b_2\delta^2], \end{aligned} \quad (1)$$

where $a_2 = \tilde{a}_2 + a_1 c/b_1$ was defined, after factoring out a temperature dependent finite constant.

We investigate the behavior of the spectral function, which can be derived from (1) by studying the pole structure starting at high temperature. In the concrete case of the σ-model the tendency to restore the chiral symmetry with increasing temperature reduces the σ-mass below $2m_\pi(T)$. For such temperature σ is necessarily stable if the pions are assumed to be stable.

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When decreasing the temperature the location of the σ-pole moves along the physical real axis to the two-body decay threshold from below. Let us assume that this pole reaches the branch point at $T = T^*$. This is the temperature T^* around which the behavior of the spectral function will be investigated.

The stability of the $O(N)$ multiplet enforced by the chiral symmetry is now postulated, without reference to any particular symmetry, for the general discussion. We assume that for $\epsilon > 0$ in addition to the approximate stability of the particles α_1 and α_2, σ is stable. This requires a single pole, δ_+, for G_σ on the physical real axis below the threshold ($1 > \delta > 0$), corresponding to a stable state. The requirement, that no other positive root should be present is fulfilled only if $a_1 < 0, b_1 > 0, b_2 > 0$. This choice implies the existence of a root $\tilde{\delta} < 0$ on the unphysical real axis for both signs of ϵ. This pole stays away from the threshold even when $\epsilon \to 0$, see Fig. 1.

When the temperature is decreased to the region $\epsilon < 0$, G_σ^{-1} has no roots on the physical real axis. The pole, which did correspond to the stable particle moves continuously over the threshold to the unphysical real axis. In this temperature range this solution is denoted by δ_-. This means that for $\epsilon < 0$ one has two poles ($\delta_-, \tilde{\delta}$) on the unphysical real axis for sufficiently small $|\epsilon|$.

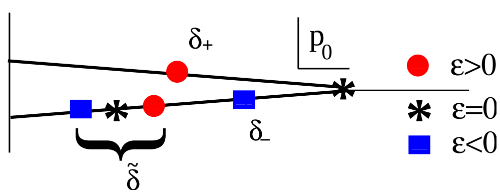

FIG. 1: Schematic presentation of the pole evolution.

In this approximation the spectral function $\rho(\epsilon, \delta) = -\mathrm{Im}G_\sigma(p_0)/\pi$ has the following parameterization above the threshold $p_0^2 > M^2$:

$$\rho(p_0, \epsilon) = -\frac{\left[1 + \frac{c}{b_1}\epsilon\right](b_2\pi)^{-1}\sqrt{\frac{p_0^2}{M^2} - 1}(\tilde{\delta} + \delta_\pm)}{(\frac{p_0^2}{M^2} - 1 - \delta_\pm\tilde{\delta})^2 + (\frac{p_0^2}{M^2} - 1)(\tilde{\delta} + \delta_\pm)^2} \quad (2)$$

(the expression depends on ϵ also indirectly, through the $\tilde{\delta} \cdot \delta_\pm$ term). The pole terms associated with δ_+ and δ_- in the respective temperature ranges $\epsilon > 0$ and $\epsilon < 0$ can be uniquely separated in G_σ. The approximate spectral function based exclusively on this singular "pole" contribution, above the threshold M reads:

$$\rho_{\rm sing} = \frac{(1 + c\epsilon/b_1)}{b_2(\delta_\pm - \tilde{\delta})\pi} \times \frac{|\delta|}{|\delta|^2 + \delta_\pm^2}. \quad (3)$$

The maximal threshold enhancement clearly occurs for $\epsilon = 0$, when $\rho_{\rm sing} \sim 1/|\delta|$, which has been found also in Ref. [5]. Since $\delta_+(\epsilon)$ and $-\delta_-(\epsilon)$ in the respective temperature ranges are equal to linear order in ϵ, the mirror variation to $\mathcal{O}(\epsilon)$ of $\rho_{\rm sing}$ around T^* is deformed only by the temperature dependent first factor on the right hand side of Eq. (3).

The threshold enhancement sets in in a universal way, reflecting the motion of $\delta_+(\delta_-)$ along the physical (unphysical) real axis towards the position of the threshold. Concerning universality based on the expansion given in Eq. (1) three further remarks are in order: i) the linear dependence of $\delta_\pm$ on ϵ is universal, which determines the leading term of the enhancement according to the second factor in Eq. (3), ii) when looking at the first correction to this leading behavior the one-pole description is not sufficient, $\tilde{\delta}$ plays an important role (if the expansion is pushed to higher orders, further roots would appear), iii) it is obvious that the expansion given in Eq. (1) is independent of the low temperature quasiparticle spectra.

Another characteristic temperature value may be defined, corresponding to the temperature where δ_- and $\tilde{\delta}$ become degenerate on the unphysical real axis, when the temperature is gradually decreased further. It is determined by the smaller negative root of the equation $b_1^2 - 4b_2(a_1\epsilon_0 + a_2\epsilon_0^2) = 0$. For smaller temperature values ($\epsilon < \epsilon_0$) the two poles become complex. One of the poles moves into the second Riemann sheet and can be interpreted as a resonance of finite lifetime if the temperature becomes sufficiently low. The other pole is its "mirror" located on the continuation of the second Riemann sheet through the unphysical real axis into the positive half-plane, and does not have any physical meaning. This picture fully agrees with the qualitative pole trajectory found in [6]. It could happen that the validity of the quadratic expansion of G_σ^{-1} breaks down already before the degeneracy of the two poles on the unphysical real axis occurs and δ_- may stay on this axis until the temperature is decreased to zero. This alternative seems to be realized in the calculation presented in [7]. It would be very interesting to see the presence of such a pole in the low temperature representation of the pion-pion scattering amplitude based on chiral perturbation theory [8].

The extension of the investigation to the case of nonzero spatial momentum $|\mathbf{p}|/M$, regarded as a new small parameter can be readily done, by assuming the analyticity of G_σ also on this quantity. Now the threshold parameter δ takes its original Lorentz invariant form: $\delta = \sqrt{1 - p^2/M^2}$, where p^2 is the squared four-momentum. At finite temperature, in the propagator also a separate dependence on the spatial momentum appears. Therefore the expansion Eq. (1) will contain also terms of the form $(\mathbf{p}^2/M^2)^l\epsilon^k\delta^m$ (note, that ϵ remains as it was introduced for $|\mathbf{p}| = 0$!). To lowest order (single pole approximation) one keeps terms linear in ϵ and δ and the coefficient d of a new term $d \times |\mathbf{p}|^2/M^2$ has to be computed.

The coefficient d determines the shift in the value of the temperature for which the maximal threshold enhancement of ρ_{sing} occurs at small finite $|\mathbf{p}|$: $T^*(|\mathbf{p}|) = T^*[1 + \frac{d}{4|a_1|}\frac{\mathbf{p}^2}{m_\pi^2}]$. Since the term proportional to $\mathbf{p}^2$ is real even after the continuation above the threshold, $|\mathbf{p}|$ and ϵ play similar roles in the spectral function. For fixed T, it is d, which controls the location and height of the maximum of ρ_{sing} when $|\mathbf{p}|$ is increasing. For instance, one finds $\delta_{max} \approx |a_1\epsilon + d|\mathbf{p}|^2/M^2|/b_1$.

Below, we illustrate the validity of the simplified representation discussed above for the scalar-isoscalar spectral function of the linear σ model, obtained to leading order in an expansion with respect to the inverse of the number of the Goldstone bosons (large N approximation). For this approximate solution one checks explicitly the existence of the power series of G_σ^{-1} in δ, $|\mathbf{p}|$ and ϵ. Furthermore, the ranges of validity in $|\epsilon|$ of the single and the two pole descriptions of the spectral function can be analyzed by comparing Eq. (2) and Eq. (3) with the complete expression of the spectral function in the model.

The first goal in the second part is to sketch the way the σ propagator given by Eq. (19) of Ref. [6] can be cast for $p_0 < M \equiv 2m_\pi(T)$ and $|\mathbf{p}| = 0$ in the form presented in Eq. (1) of the present paper. Employing an obvious notation, the renormalized couplings (m_R^2, λ_R, h) of this model were fixed at $T = 0$ at values which ensure $m_\pi(0) = 140$ MeV, $f_\pi = \sqrt{N}\Phi(T = 0) = 93$ MeV, $m_\sigma = 3.95 f_\pi$, $m_\sigma/\Gamma_\sigma \sim 1.4$. The temperature dependence of the vacuum expectation value of the $O(N)$ field, $\Phi(T)$, was determined by the equation of state (its renormalized form appears in Eq. (11) of Ref. [6]):

$$m^2 + \frac{\lambda}{6}\left[\Phi^2(T) + \int_k \left[n(\omega_k) + \frac{1}{2}\right]\frac{1}{\omega_k}\right] - \frac{h}{\Phi(T)} = 0, \quad (4)$$

where n is the Bose-Einstein factor and $\omega_k^2 = \mathbf{k}^2 + m_\pi^2$. The expansion of $m_\pi(T)$ to quadratic order in ϵ is provided through the combination of the relation $m_\pi^2(T) = h/\Phi(T)$ with the expansion of $\Phi(T)$ around $\Phi(T^*)$. $\Phi(T^*)$ is found from Eq. (31) of Ref. [6].

The selfenergy of σ is determined to leading large N order by a sum of the chain of pion bubbles, characterized by the function $b^>(p_0)$ as shown by Eq. (5) of Ref. [6] (the index $>$ refers to the first Riemann sheet). The expression of the bubble above the threshold given in Ref. [6] is written, after continuing to the physical real axis below the threshold and some convenient integral transformations, in a form where the expansion in powers of δ and ϵ actually starts from:

$$4\pi^2 b^>(p_0) = \frac{1}{4}\ln\frac{m_\pi^2(T)}{M_0^2} + \frac{\delta\arccos(\delta)}{2\sqrt{1-\delta^2}} - \int_\xi^\infty dx \frac{(x^2-\xi^2)^{-\frac{1}{2}}}{e^x - 1} + \delta\int_0^\infty dx \frac{[(1+x^2)\sqrt{1+\delta^2x^2}]^{-1}}{e^{\xi\sqrt{1+\delta^2x^2}} - 1}, \quad (5)$$

where $m_\pi(T)$ and $\xi = m_\pi(T)/T$ carry the explicit T dependence (M_0 is the normalization scale of the theory defined at $T = 0$). One performs first the expansion in powers of δ to quadratic order. For the second integral in Eq. (5) one finds to linear order $(\frac{\pi}{2} - \delta)n(m_\pi(T))$. Next, the remaining explicit pion mass dependence (also in ξ!) is expanded around T^* with the procedure sketched above. With these hints one easily establishes the approximate expression Eq. (1), obtaining numerically the following values for the coefficients: $a_1 = -18.32$, $a_2 = -23.14$, $b_1 = 8.59$, $b_2 = 14.48$, $c = 9.67$.

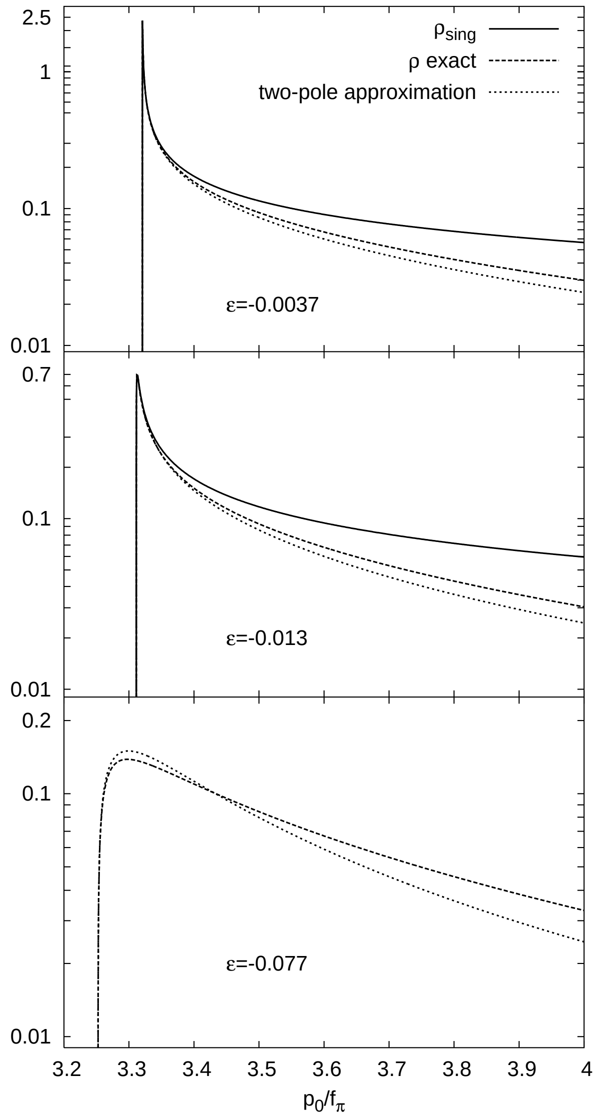

FIG. 2: The spectral function $\rho(p_0, \epsilon) f_\pi^2$ in various approximations as a function of p_0 at different reduced temperature ϵ, and $|\mathbf{p}| = 0$.

In Fig. 2 the spectral function and its universal approximations are shown for three different temperatures. The curves labeled "exact" are calculated from the complete expression derived in Ref. [6]. They are compared with the two approximate forms described above. All temperatures were chosen below T^* (above it the situation is qualitatively similar). For the highest value of ϵ the pole δ_- lies almost on the threshold, for its smallest value ($\epsilon = \epsilon_0 \approx -0.077$) one has $\delta_- = \tilde{\delta}$. When calculated from the exact expression one obtains: $\epsilon_0 = -0.13$. According to these curves the region where the one-pole description gives a good approximation is rather narrow in ϵ and in particular in δ. The correction due to the second pole contribution is quite important for increasing frequency values. For $\epsilon = \epsilon_0$ the line corresponding to

ρ_{sing} was omitted, since its expression becomes meaningless. It is, however, remarkable how good approximation can be constructed using Eq. (1). The range of validity of the quadratic expansion of G_σ^{-1} in ϵ includes also the degeneracy temperature of the poles, persisting even for somewhat lower temperatures when both poles are complex and lie equidistantly from the threshold. The specific form of the spectral function, $\rho \sim |\delta|/(|\delta|^2+const)^2$, characteristic at the degeneracy temperature is expected to occur even in the exact solution of the $N=4$ linear σ model with physically relevant couplings.

For the case of the $\sigma-\pi$ system, the coefficient d, which determines the influence of the explicit $|\mathbf{p}|$-dependence, comes uniquely from the temperature dependent part of the pion bubble, $b_T^> \equiv b_T^>(p_0,|\mathbf{p}|)$. Since the $T=0$ part of the expression of the bubble is Lorentz invariant, it remains unchanged when expressed with the complete expression of δ. The expression of $b_T^>$ for $\mathbf{p}\neq 0$, see e.g. Refs. [7, 9] can be cast in the form

$$b_T^> = \int_1^\infty \frac{dt}{8\pi^2}\frac{n(tm_\pi)}{|\mathbf{p}|}\ln\left|\frac{\left[\frac{p^2}{2}+|\mathbf{p}|\sqrt{t^2-1}\right]^2-p_0^2t^2}{\left[\frac{p^2}{2}-|\mathbf{p}|\sqrt{t^2-1}\right]^2-p_0^2t^2}\right|. \tag{6}$$

For the threshold behavior of the spectral function to the lowest (linear) order in $(\epsilon,\delta,\mathbf{p}^2)$, one can use $\delta=0$ in the integral, that is for the quantities scaled by $m_\pi(T)$ one can put: $p^2=4$, $p_0^2=4+\mathbf{p}^2$. With a suitable transformation of the variable of integration we can ensure the smoothness of the integrand, obtaining:

$$b_T^> = \int_0^\infty \frac{dt}{4\pi^2\xi}\ln\frac{1-\exp\left[-\xi\sqrt{1+\left(\frac{|\mathbf{p}|}{2}\tanh\frac{|\mathbf{p}|}{2}t\right)^2}\right]}{1-\exp\left[-\xi\sqrt{1+\left(\frac{|\mathbf{p}|}{2}\coth\frac{|\mathbf{p}|}{2}t\right)^2}\right]}. \tag{7}$$

Performing a partial integration one obtains the first two terms of the asymptotic expansion of the integral valid for small $|\mathbf{p}|$, relevant for our analysis. Next to the unchanged $|\mathbf{p}|$-independent term with the complete δ one can calculate the coefficient of $\mathbf{p}^2$ by performing the t-integral with the value of ξ determined at T^*. For the actual couplings $d=1.65$ is found.

The assumed expansion of the inverse propagator around the threshold in powers of ϵ and δ resembles the starting point of the Landau theory of second order phase transitions. Like in the Landau theory, our purpose has not been to prove the analytic property of the relevant quantity, but to explore the consequences of its postulation. We conjecture the universality of the threshold behavior in any channel where a particle pole describing at high temperature a stable particle might pass through the threshold of its lowest two-body decay. Our discussion implies that for $T\lesssim T^*$ the pole dominating analytically the spectral function cannot correspond to any particle. Whether this remains valid down to $T=0$ is model dependent.

A possible generalization of Eq. (2), in which only the quadratic dependence of the inverse propagator on δ is assumed, can be easily written down:

$$\rho(\epsilon,\mathbf{p},p^2)=\frac{R_1(\epsilon,\mathbf{p})\sqrt{\frac{p^2}{M^2}-1}}{[\frac{p^2}{M^2}-1+R_2(\epsilon,\mathbf{p})]^2+R_3(\epsilon,\mathbf{p})[\frac{p^2}{M^2}-1]}. \tag{8}$$

The functions R_i are real, at $T=0$ they assume constant values. ϵ is measured from the value T^* determined by the condition $R_2(\epsilon=0,0)=0$. A sequence of reduced temperatures $\epsilon(\mathbf{p})\equiv(T^*(\mathbf{p})-T^*)/T^*$ can be defined as the solution of the equation $R_2(\epsilon(\mathbf{p}),\mathbf{p})=0$. Obviously $T^*(0)=T^*$. Asymptotically one expects the "scaling" behavior $R_2\sim[\epsilon-\epsilon(\mathbf{p})]^\alpha, \alpha>0$. The treatment based on Eq. (1) leads to $\alpha=1$, but in principle a nontrivial non-integer scaling exponent can be envisaged. Note, that when there is no explicit symmetry breaking the branch point tends to the origin if $|\mathbf{p}|$ goes to zero and $T^*=T_c$ independently of $|\mathbf{p}|$ [10].

The next step will be to discuss the effect of the finite pion lifetime. It shifts the branch-point into the lower halfplane. Around it there is still an enhancement (similar to the undamped case) but in the spectral function varying along the real axis the effect is smeared out partially, see [11]. Either a complete two-loop (improved) perturbative analysis or the evaluation of the next-to-leading large N correction to G_σ is needed to reliably assess the importance of the finite pion width.

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