On the boundary form factor program

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Abstract

Boundary form factor axioms are derived for the matrix elements of local boundary operators in integrable 1+1 dimensional boundary quantum field theories using the analyticity properties of correlators via the boundary reduction formula. Minimal solutions are determined for the integrable boundary perturbations of the free boson, free fermion (Ising), Lee-Yang and sinh-Gordon models and the two point functions calculated from them are checked against the exact solutions in the free cases and against the conformal data in the ultraviolet limit for the Lee-Yang model. In the case of the free boson/fermion the dimension of the solution space of the boundary form factor equation is shown to match the number of independent local operators. We obtain excellent agreement which proves not only the correctness of the solutions but also confirms the form factor axioms.

1 Introduction

The bootstrap program aims to classify and explicitly solve 1+1 dimensional integrable quantum field theories by constructing all of their Wightman functions. The first stage is the S-matrix bootstrap: the scattering matrix, connecting asymptotic *in* and *out* states, is determined from its properties such as factorizability, unitarity, crossing symmetry and the Yang-Baxter equation (YBE) supplemented by the maximal analyticity assumption. The result is the complete on-shell solution of the theory, i.e. the spectrum of excitations and their scattering amplitudes, which can be related to some independent definition of the model as a perturbed conformal field theory or a Lagrangian QFT (for reviews see [1, 2]). The second step is the form factor bootstrap, which allows one to determine matrix elements of local operators between asymptotic states using their analytic properties

originating from the already known S-matrix. Supposing maximal analyticity leads to a set of solutions each of which corresponds to a local operator of the theory. The form factors are then used to build the correlation (Wightman) functions via their spectral representations, yielding a complete off-shell description of the theory (see [3, 4] for reviews).

The first step of an analogous bootstrap program for 1+1 dimensional integrable *boundary* quantum field theories, the boundary R-matrix bootstrap, has been developed for several theories. In boundary theories the asymptotic states are connected by the reflection R-matrix, which satisfies unitarity and boundary crossing unitarity; for integrable boundary QFT, it also satisfies the boundary YBE (BYBE) and boundary bootstrap requirements. These equations supplemented by maximal analyticity assumptions make possible to determine the reflection matrices and provide the complete information about the theory on the mass shell [5].

For the second step matrix elements of local operators between asymptotic states have to be computed. In a boundary quantum field theory there are two types of operators, the bulk and the boundary operators, where their names indicate their localization point. Due to the broken translational invariance one point functions of bulk operators may acquire nontrivial space dependence behaving analogously to the two point functions in a bulk theory. Indeed this one point function can be calculated in the crossed channel, where the role of time and space is changed and the spatial boundary appears as a temporal one represented as an initial (boundary) state in the matrix element. Inserting a complete system of the bulk Hilbert space a spectral representation for the one point functions can be obtained in terms of the bulk form factors and the matrix element of the boundary state [6, 7]. Truncating this expansion at finite intermediate states provides a convergent large distance expansion. However, matrix elements of boundary operators cannot be computed in this way and the purpose of the present paper is to develop a technique to compute their correlation functions.

In this paper we initiate the second step of the boundary bootstrap program, namely the boundary form factor program for calculating the matrix elements of local boundary operators between asymptotic states. We derive their analytic structure from that of the R-matrix which, when supplemented by the assumption of maximal analyticity, leads to their determination. In the bulk case, it was shown in [8] that the solution space of the form factor equations can be brought into one-to-one correspondence with the operator content of the model. Based on this, we expect that the classification of the solutions of the boundary form factor axioms provides information on the boundary operator content of the theory, which in the ultraviolet limit is in a one-to-one correspondence with the Hilbert space of the model. Using the explicit form of the boundary form factors the spectral representation for the boundary correlation functions can be obtained.

The paper is organized as follows: first we define the boundary form factors by introducing asymptotic *in* and *out* multi-particle states, which are related by the multi-particle reflection matrix. Simple crossing relations are presented from which the form factor axioms follow easily, and then the axioms are verified by some consistency requirements. We outline a general strategy to solve theories with diagonal bulk scattering and boundary reflection amplitudes, and to compare the resulting two-point functions with their ultra-

violet limits. This idea is applied to integrable boundary perturbations of several models, such as the free boson model, free fermion (alias Ising) field theory, the scaling Lee-Yang model and sinh-Gordon theory. Appendix A contains a heuristic derivation of the crossing relations from the boundary reduction formula [9], while in Appendix B we present a formal derivation of the boundary form factor axioms from the boundary version of the Faddeev-Zamolodchikov algebra.

2 Boundary form factors

2.1 Definitions

The Hilbert space of a boundary quantum field theory consists of multi-particle states, which can be labeled by the particle species and the corresponding particle energies. To simplify the notations we restrict ourselves to theories containing only one particle type with a given mass m. In 1+1 dimensions it is convenient to work with the rapidity variable θ_i; the energy E_i of the particle can be written as $E_i = m\cosh\theta_i$, while the momentum is $p_i = m\sinh\theta_i$. Following the evolution of the multi-particle state in time to $t \to -\infty$ the particles get far away form each other and from the boundary, therefore forming an *in* state which is equivalent to a free multi-particle state and is denoted as[1]

$$|\theta_1, \theta_2, \ldots, \theta_n\rangle_{in} \quad ; \qquad \theta_1 > \theta_2 > \cdots > \theta_n > 0$$

Positivity of all incoming rapidities is a consequence of the assumption that the boundary is at the right end of the half line and it is a major difference from the bulk situation. This difference is essential because it influences the analyticity domain of matrix elements.

For $t \to +\infty$ all the scatterings and reflections are terminated, the particles are again far away from each other and from the boundary forming the *out* state,

$$|\theta_1^{'}, \theta_2^{'}, \ldots, \theta_m^{'}\rangle_{out} \quad ; \qquad \theta_1^{'} < \theta_2^{'} < \cdots < \theta_m^{'} < 0$$

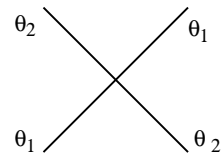
which is again equivalent to a free state. By the standard assumption of asymptotic completeness, the two sets of states form a complete basis separately and are connected by the reflection matrix, which is the boundary analogue of the S matrix. In an integrable theory, due to the infinite number of conserved charges, there is no particle creation ($n = m$), the set of rapidities changes only sign $\theta_i^{'} = -\theta_i$, and the reflection matrix factorizes into the product of pairwise bulk scatterings and individual reflections

$$|\theta_1, \theta_2, \ldots, \theta_n\rangle_{in} = \prod_{i<j} S(\theta_i - \theta_j) S(\theta_i + \theta_j) \prod_i R(\theta_i) | -\theta_1, -\theta_2, \ldots, -\theta_n\rangle_{out} \tag{1}$$

where $S(\theta_i - \theta_j)$ connects the two particle asymptotic *in* and *out* states in the bulk theory (without the boundary)

[1] In general, particles in an interacting two dimensional quantum field theory have an effective fermionic statistics with the sole exception of free bosonic theories, for which it is necessary to allow equality in the ordering of the particle rapidities.

$$|\theta_1,\theta_2\rangle_{in}^{bulk} = S(\theta_1-\theta_2)|\theta_2,\theta_1\rangle_{out}^{bulk} \qquad \text{depicted as}$$



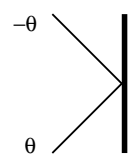
It is defined originally for $\theta_1 > \theta_2$ but can be analytically continued for complex rapidity parameters such that the extended function (denoted the same way) is meromorphic and satisfies unitarity and crossing symmetry

$$S(\theta)S(-\theta) = 1 \quad , \qquad S(i\pi-\theta) = S(\theta)$$

It might have poles on the imaginary axis at locations $\theta = iu_j$ with residue $-i\text{res}_{\theta=iu_j}S(\theta) = \Gamma_j^2$, some of which correspond to bound states.

The amplitude $R(\theta)$ connects the one particle asymptotic states in the boundary theory

$$|\theta\rangle_{in} = R(\theta)|-\theta\rangle_{out} \qquad \text{depicted as}$$



It can also be extended from the fundamental domain $\theta > 0$ to a meromorphic function on the whole complex θ plane satisfying unitarity and boundary crossing unitarity

$$R(\theta)R(-\theta) = 1 \quad , \qquad R(i\pi-\theta)S(2\theta) = R(\theta)$$

$R(\theta)$ may have poles at imaginary locations $\theta = iv_j$ $(0 < v_j < \pi/2)$, some corresponding to excited boundary states. If the interpolating field has a nontrivial vacuum expectation value then generally there is also a pole at $\theta = i\pi/2$ with residue

$$-i \operatorname*{Res}_{\theta=\frac{i\pi}{2}} R(\theta) = \frac{g^2}{2} \tag{2}$$

The boundary form factor is defined as the matrix element of some local boundary operator, $\mathcal{O}(t)$, between asymptotic states

$$_{out}\langle\theta_1',\theta_2',\ldots,\theta_m'|\mathcal{O}(t)|\theta_1,\theta_2,\ldots,\theta_n\rangle_{in} = F_{mn}^{\mathcal{O}}(\theta_1',\theta_2',\ldots,\theta_m';\theta_1,\theta_2,\ldots,\theta_n)e^{-imt(\sum\cosh\theta_i-\sum\cosh\theta_j')}$$

These form factors are defined only for $\theta_1 > \theta_2 > \cdots > \theta_n > 0$ and $\theta_1' < \theta_2' < \cdots < \theta_m' < 0$. We can introduce other form factors as

$$_{out}\langle\theta_1',\theta_2',\ldots,\theta_m'|\mathcal{O}(t)|-\theta_1,-\theta_2,\ldots,-\theta_n\rangle_{out} = F_{mn}^{\mathcal{O}}(\theta_1',\theta_2',\ldots,\theta_m';-\theta_1,-\theta_2,\ldots,-\theta_n)$$

and consider them as a continuation of the original ones in the rapidities. Expressing these form factors (via the boundary reduction formula [9]) in terms of correlation functions

an analytic continuation can be performed for any (even) complex values of the rapidity parameters. As a result the generalized form factors are meromorphic functions of the rapidity parameters, and we shall assume that their poles always have physical origins (maximal analyticity assumption). From the crossing formula

$$F^{\mathcal{O}}_{mn}(\theta_1^{'},\theta_2^{'},\ldots,\theta_m^{'};\theta_1,\theta_2,\ldots,\theta_n) = F^{\mathcal{O}}_{m-1n+1}(\theta_2^{'},\ldots,\theta_m^{'};\theta_1^{'}+i\pi,\theta_1,\theta_2,\ldots,\theta_n) + \text{disc} \quad (3)$$

derived in Appendix A, we can express all the form factors in terms of the elementary form factors

$${}_{out}\langle 0|\mathcal{O}(0)|\theta_1,\theta_2,\ldots,\theta_n\rangle_{in} = F^{\mathcal{O}}_n(\theta_1,\theta_2,\ldots,\theta_n)$$

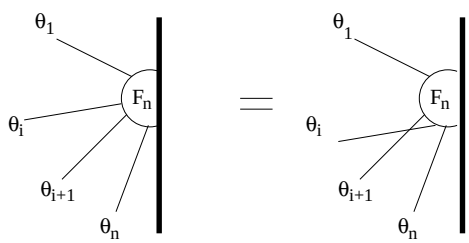
It is important to notice that the boundary form factors $F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_n)$, in contrast to the bulk case, do depend in general on all the rapidities θ_i, not just on their differences, since in the presence of a boundary Lorentz invariance is broken.

2.2 Axioms

In the Appendices we derive all the following properties of the matrix elements of local boundary operators valid in any integrable boundary quantum field theory. Following the general philosophy in the bulk case [3] we take them as axioms defining the local operators via their matrix elements.

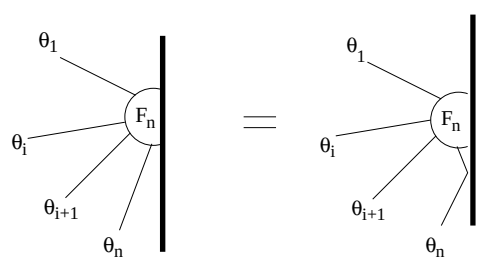
I. Permutation:

$$F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n) = S(\theta_i-\theta_{i+1})F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n)$$



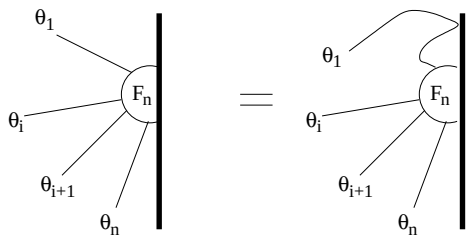
II. Reflection:

$$F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_{n-1},\theta_n) = R(\theta_n)F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_{n-1},-\theta_n)$$



III. Crossing reflection:

$$F_n^{\mathcal{O}}(\theta_1,\theta_2,\ldots,\theta_n) = R(i\pi-\theta_1)F_n^{\mathcal{O}}(2i\pi-\theta_1,\theta_2,\ldots,\theta_n)$$



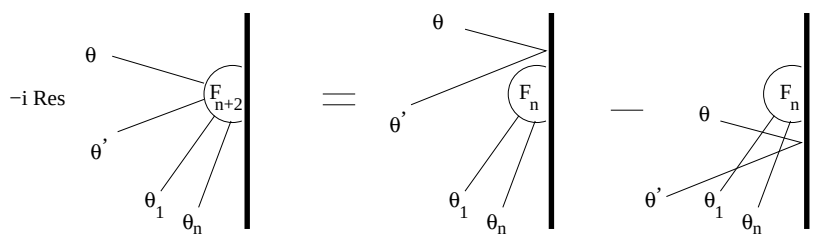
The singularity structure of the form factors is determined on physical grounds and can be axiomatized as follows:

IV. Kinematical singularity

$$-i\operatorname*{Res}_{\theta=\theta'} F_{n+2}^{\mathcal{O}}(\theta+i\pi,\theta',\theta_1,\ldots,\theta_n) = \left(1-\prod_{i=1}^{n} S(\theta-\theta_i)S(\theta+\theta_i)\right) F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$

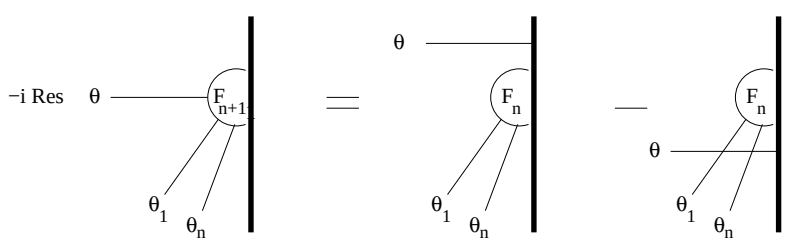
or equivalently described as

$$-i\operatorname*{Res}_{\theta=\theta'} F_{n+2}^{\mathcal{O}}(-\theta+i\pi,\theta',\theta_1,\ldots,\theta_n) = \left(R(\theta)-\prod_{i=1}^{n} S(\theta-\theta_i)R(\theta)S(\theta+\theta_i)\right) F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$



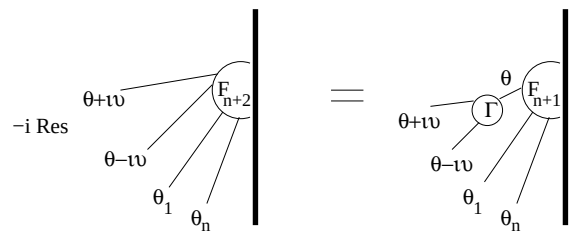
V. Boundary kinematical singularity

$$-i\operatorname*{Res}_{\theta=0} F_{n+1}^{\mathcal{O}}(\theta+\frac{i\pi}{2},\theta_1,\ldots,\theta_n) = \frac{g}{2}\Big(1-\prod_{i=1}^{n} S\big(\frac{i\pi}{2}-\theta_i\big)\Big)F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$



VI. Bulk dynamical singularity

$$-i \operatorname*{Res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(\theta+iu,\theta'-iu,\theta_1,\ldots,\theta_n) = \Gamma F^{\mathcal{O}}_{n+1}(\theta,\theta_1,\ldots,\theta_n)$$



VII. Boundary dynamical singularity

$$-i \operatorname*{Res}_{\theta=iv} F^{\mathcal{O}}_{n+1}(\theta_1,\ldots,\theta_n,\theta) = \tilde{g}\tilde{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$

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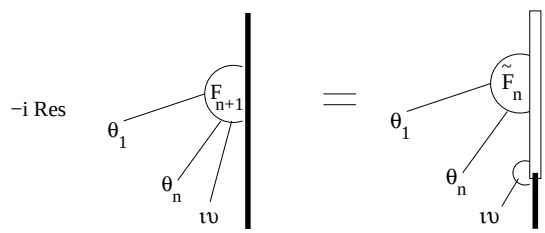


We note that equations similar to some of ours have been obtained earlier studying boundary form factors in specific spin chains. Using a concrete realization for the Hilbert space and the operators, these equations were extracted originally for the XXZ and XYZ models in [10] and extended for other spin chains in [11]. By extending the bulk free field representation for the boundary sine-Gordon model the analogues of XXZ equations were obtained in [12]. In all these approaches, however, there is no analogue of the axiom V, without which the equations do not determine completely the form factors as can be seen on the example of the sinh-Gordon model. In contrast, in our approach the form factor axioms are firmly established from first principles of local quantum field theory, thus they are valid in a general setting. As a further result of our systematic approach the axioms found form a complete system ready to be solved.

2.3 Consistency checks

Before proceeding to concrete examples we perform a few consistency checks of the axioms. First we note that they are self-consistent in the sense that for specific rapidities the $n+2$ particle form factor can be connected to the n particle form factor either by the kinematical singularity equations or by using twice the boundary kinematical equations, and the two

procedures give the same result. Indeed taking double residue in the first case, first at $\theta = \theta'$ and then at $\theta = i\frac{\pi}{2}$ gives

$$i \operatorname*{Res}_{\theta=\frac{i\pi}{2}} i \operatorname*{Res}_{\theta'=\theta} F^{\mathcal{O}}_{n+2}(-\theta + i\pi, \theta', \theta_1, \ldots, \theta_n) \quad = \\ \left(-i \operatorname*{Res}_{\theta=\frac{i\pi}{2}} R(\theta)\right) \left(1 - \prod_{i=1}^{n} S(\frac{i\pi}{2} - \theta_i) S(\frac{i\pi}{2} + \theta_i)\right) F^{\mathcal{O}}_n(\theta_1, \ldots, \theta_n)$$

Taking now the residue at $\theta = \frac{i\pi}{2}$ first then at $\theta' = \frac{i\pi}{2}$ and using that $S(0) = -1$ gives

$$i \operatorname*{Res}_{\theta=\frac{i\pi}{2}} i \operatorname*{Res}_{\theta'=\frac{i\pi}{2}} F^{\mathcal{O}}_{n+2}(-\theta + i\pi, \theta', \theta_1, \ldots, \theta_n) \quad = \\ \frac{g^2}{4} \left(1 + \prod_{i=1}^{n} S(\frac{i\pi}{2} - \theta_i)\right) \left(1 - \prod_{i=1}^{n} S(\frac{i\pi}{2} - \theta_i)\right) F^{\mathcal{O}}_n(\theta_1, \ldots, \theta_n)$$

The two different orders of taking the residues differ by a factor of 2 since in the first case after taking the residue at $\theta' = \theta$ we get a factor $f(2\theta - i\pi)$ which has a zero at $\theta = i\frac{\pi}{2}$ (due to $S(0) = -1$ the bulk minimal form factor vanishes at the threshold: $f(0) = 0$). In the second case after taking the first residue a factor $f(\theta - i\frac{\pi}{2})$ appears. When expanding around $\theta = i\frac{\pi}{2}$ to take the second residue there appears a factor 2 due to the difference in the arguments of this particular factor (all other terms are identical in the two cases). Combining the crossing symmetry of the S-matrix with the definition of g (2) the two expressions are easily seen to be equivalent.

It is worth emphasizing that in the boundary kinematical singularity axiom it is the particle-boundary coupling constant g which appears although the residue of the reflection factor determines only g^2. There are known examples where in two physically different situations the fundamental reflection amplitudes are the same and the two cases are distinguished only by the sign of g (e.g. the boundary Lee-Yang model with 1 boundary and with Φ boundary with a particular value of the boundary coupling [13] – see in more details in Sec. 3.3). Because of axiom V the solutions of the form factor axioms are different for the two cases, as shown in detail in Sec. 3.3.

As a second test we relate the two disconnected physical domains (*in/out*) of the definition of the form factor. By permuting successively each rapidity to the last position, applying a reflection and permuting back to their original position we obtain that

$$F^{\mathcal{O}}_n(\theta_1, \ldots, \theta_n) = \prod_{i<j} S(\theta_i + \theta_j) \prod_i R(\theta_i) \prod_{i<j} S(\theta_i - \theta_j) F^{\mathcal{O}}_n(-\theta_1, \ldots, -\theta_n)$$

The product appearing is nothing but the multi-particle R-matrix, (1), which connects the *in* and *out* states.

Finally we use the fact that the reflection matrix can be considered as a special form factor (of the identity operator Id) whose analytic properties are well known. By definition

$$F^{Id}_2(\theta' + i\pi, \theta) = {}_{out}\langle \theta' | Id | \theta \rangle_{in} = R(\theta)\, {}_{out}\langle \theta' | Id | -\theta \rangle_{out} = R(\theta)\delta(\theta + \theta')$$

Now using the permutation property and tricks as above we have

$$F_2^{Id}(\theta' + i\pi, \theta) = S(i\pi + \theta' - \theta) F_2^{Id}(\theta, \theta' + i\pi) = S(i\pi + \theta' - \theta) R(i\pi + \theta') \delta(\theta + \theta')$$

which, due to the boundary crossing unitarity, is equivalent to the previous expression.

2.4 General solution

In this section we describe the general procedure we use to obtain the solutions of the form factor equations in the various specific models. In doing so we emphasize the similarities and the differences between the boundary and bulk form factors and also separate the (boundary) operator dependent parts from the ones that depend on the specific field theory considered but are independent of the operators in question.

2.4.1 One particle form factors

In sharp contrast to the bulk case, in the boundary theory, the boundary operators in general may have non trivial one particle form factors (1PFF). Since the multi-particle form factors are recursively determined, the 1PFF-s are very important inputs to these recursions, and their determination is necessarily the first step. The equations for the 1PFF read:

$$F_1(\theta) = R(\theta) F_1(-\theta) \quad ; \quad F_1(i\pi + \theta) = R(-\theta) F_1(i\pi - \theta), \tag{4}$$

where the reflection amplitude $R(\theta)$ is analytic in the physical strip $0 \leq \Im m(\theta) \leq \pi/2$ (apart from the presence of finitely many discrete poles on the imaginary axis), and from general considerations using the reduction formulae we know that $F_1(\theta)$ is analytic for $0 \leq \Im m(\theta) \leq \pi$. Note that if $F_1(\theta)$ is a solution of (4) then $F_1(\theta)\Psi(\theta)$ is also a solution provided

$$\Psi(\theta) = \Psi(-\theta), \qquad \Psi(i\pi + \theta) = \Psi(i\pi - \theta),$$

i.e. if Ψ is even and $2\pi i$ periodic. Therefore one can take $\Psi(\theta) = \psi(y)$ with $y = e^{\theta} + e^{-\theta}$.

To construct solutions to (4) we reduce them to a problem already solved in the bulk form factor bootstrap. To this end we write $F_1(\theta) = g_1(\theta) g_2(i\pi - \theta)$ and suppose that

$$g_1(\theta) = R(\theta) g_1(-\theta) \quad ; \quad g_1(i\pi + \theta) = g_1(i\pi - \theta), \tag{5}$$

which are nothing else but the bulk two particle form factor equations [14], where the reflection amplitude, $R(\theta)$, plays the role of the S-matrix. Furthermore, plugging this product form F_1 into (4) reveals, that g_2 must also solve (5). Thus a solution to (4) can be constructed as

$$F_1(\theta) = g(\theta) g(i\pi - \theta),$$

where $g(\theta)$ is an appropriate solution of (5).

To obtain a solution of (5) we use the following theorem [15]. If the function $h(\theta)$ is meromorphic in the physical strip $0 \leq \Im m(\theta) < \pi$ with possible poles at $i\alpha_1, \ldots, i\alpha_l$

and zeros at $i\beta_1, \ldots, i\beta_k$ and grows as at most a polynomial in $\exp(|\theta|)$ for $|\Re e\,\theta| \to \infty$, furthermore it satisfies

$$h(\theta) = M(\theta)h(-\theta); \quad M(\theta) = \exp\left\{\int_0^\infty dt f(t) \sinh\left(\frac{t\theta}{i\pi}\right)\right\}; \quad h(i\pi - \theta) = h(i\pi + \theta);$$

then it is uniquely defined up to normalization as

$$h(\theta) = \frac{\prod_{j=1}^k \sinh\left(\frac{1}{2}(\theta - i\beta_j)\right) \sinh\left(\frac{1}{2}(\theta + i\beta_j)\right)}{\prod_{j=1}^l \sinh\left(\frac{1}{2}(\theta - i\alpha_j)\right) \sinh\left(\frac{1}{2}(\theta + i\alpha_j)\right)} \exp\left\{\int_0^\infty dt f(t) \frac{\sin^2\left(\frac{i\pi-\theta}{2\pi}t\right)}{\sinh x}\right\}.$$

Since the reflection amplitudes are usually expressed as products of the blocks (x_i)

$$R(\theta) = \prod_i (x_i); \qquad (x_i) = \frac{\sinh(\frac{\theta}{2} + i\frac{\pi x_i}{2})}{\sinh(\frac{\theta}{2} - i\frac{\pi x_i}{2})}, \tag{6}$$

to use this theorem we need the integral representation of one single block (x),:

$$-(x) = \exp\left\{2\int_0^\infty \frac{dt}{t} \frac{\sinh t(1-x)}{\sinh t} \sinh\left(\frac{t\theta}{i\pi}\right)\right\}.$$

Then, if $R(\theta)$ consists of an even number of blocks, the *minimal* solution (with *no zeroes and poles*) to eq.(4) can be written as

$$r^e_{\min}(\theta) = \exp\left\{2\int_0^\infty \frac{dt}{t} \frac{\sum_i \sinh t(1-x_i)}{\sinh^2 t}\left(1 - \cosh\frac{t}{2}\cos\frac{t}{\pi}\left(\frac{i\pi}{2} - \theta\right)\right)\right\}.$$

If $R(\theta)$ contains an extra minus sign, or is the product of an odd number of blocks, $R = -\prod(-(x_i))$, then the factor $g(\theta)$ necessarily contains a zero at the origin which is implemented by putting an extra $\sinh\frac{\theta}{2}$ into it; thus in this case $r^o_{\min}(\theta) = \sinh\theta\, r^e_{\min}(\theta)$. In the following an important role is played by the appropriate modification of the minimal 1PFF denoted by $r(\theta)$

$$r(\theta) = r_{\min}(\theta) \times \frac{\text{zeroes}}{\text{poles}}$$

where the last factor denotes an appropriate number of zeroes and poles at the right places (usually the same as in $R(\theta)$).

Thus the general solution of eq.(4) can be written as

$$F_1(\theta) = r(\theta)Q_1(y), \qquad y = e^\theta + e^{-\theta},$$

where the choice of $Q_1(y)$ is restricted by the analyticity and the possible asymptotics of F_1. It is the $Q_1(y)$ in the 1PFF that carries the dependence on the boundary operator $\mathcal{O}$. Note in particular that if $Q_1(y)$ corresponds to the operator $\mathcal{O}$ then $\tilde{Q}_1(y) \sim y^N Q_1(y)$ with N integer $N \geq 1$, describes the 1PFF of the operator $\partial_\tau^N \mathcal{O}$.

2.4.2 Two-particle form factors

The next step is to investigate the two-particle form factors (2PFF). The novel feature compared to the 1PFF is that their equations contain also the bulk S-matrix. It is worthwhile to go through the analysis in some detail since it is straightforward to write down the general form of the n-particle form factors once that of the 2PFF-s is obtained. The equations for the 2PFF-s have the form

$$\begin{aligned} F_2(\theta_1,\theta_2) &= S(\theta_1-\theta_2)F_2(\theta_2,\theta_1), \quad (a) \qquad F_2(\theta_1,\theta_2) = R(\theta_2)F_2(\theta_1,-\theta_2) \quad (b) \\ F_2(i\pi+\theta_1,\theta_2) &= R(-\theta_1)F_2(i\pi-\theta_1,\theta_2). \quad (c) \end{aligned} \tag{7}$$

Note that if $F_2(\theta_1,\theta_2)$ is a solution to these equations then so is $F_2(\theta_1,\theta_2)H(\theta_1,\theta_2)$ provided H is a symmetric, even and $2i\pi$ periodic function.

To construct solutions to eq.(7) we write

$$F_2(\theta_1,\theta_2) = f(\theta_1-\theta_2)\Psi(\theta_1,\theta_2)$$

where $f(\theta)$ is the minimal bulk two particle form factor [14], i.e. the minimal solution of

$$f(\theta) = S(\theta)f(-\theta), \qquad f(i\pi+\theta) = f(i\pi-\theta).$$

Plugging this F_2 into (7a) reveals that Ψ must be symmetric $\Psi(\theta_1,\theta_2) = \Psi(\theta_2,\theta_1)$. The most convenient way to satisfy (7b) is that Ψ has the form

$$\Psi(\theta_1,\theta_2) = f(\theta_1+\theta_2)r(\theta_1)r(\theta_2)\Phi(\theta_1,\theta_2)$$

where Φ is symmetric and even $\Phi(\theta_1,\theta_2) = \Phi(\theta_1,-\theta_2)$. Finally this F_2 satisfies eq.(7c) also if $\Phi(i\pi-\theta_1,\theta_2) = \Phi(i\pi+\theta_1,\theta_2)$. The conditions on Φ can be satisfied simply by writing $\Phi(\theta_1,\theta_2) = \phi(y_1,y_2)$ where ϕ is a symmetric function of the y_i-s ($y_i = e^{\theta_i}+e^{-\theta_i}$, $i=1,2$). Thus the general form of the 2PFF, compatible with eq.(7) is

$$F_2(\theta_1,\theta_2) = r(\theta_1)r(\theta_2)f(\theta_1-\theta_2)f(\theta_1+\theta_2)\phi(y_1,y_2), \qquad \phi(y_1,y_2) = \phi(y_2,y_1).$$

Different choices of the boundary operator $\mathcal{O}$ correspond to different functions $\phi(y_1,y_2)$ in this expression.

2.4.3 Multi-particle form factors

From the explicit form of the 2PFF it is clear that the general form of the multi-particle form factors can be written in the following form:

$$F_n(\theta_1,\theta_2,\ldots,\theta_n) = G_n(\theta_1,\theta_2,\ldots,\theta_n)\prod_{i=1}^{n} r(\theta_i)\prod_{i<j} f(\theta_i-\theta_j)f(\theta_i+\theta_j), \tag{8}$$

where $f(\theta)$ is the minimal bulk two particle form factor. As a consequence of the form factor equations G_n is a $2\pi i$ periodic, symmetric and even function of the rapidities: θ_i,

i.e. it is symmetric in the variable $y_i = 2\cosh\theta_i$. When the bulk S-matrix is nontrivial, the bulk kinematical singularity equations

$$-i \operatorname*{Res}_{\theta'=\theta} F_{n+2}(\theta' + i\pi, \theta, \theta_1, \ldots, \theta_n) = (1 - \prod_{i=1}^{n} S(\theta - \theta_i) S(\theta + \theta_j)) F_n(\theta_1, \ldots, \theta_n) \qquad (9)$$

give recursive relations linking G_n to G_{n+2}. (Note that these singularities are *absent* in the two particle case). The advantage of using the y_i-s becomes clear if one tries to describe the bulk kinematical singularities: since $y(i\pi + \theta) = -y(\theta)$, thus including a (symmetric) factor $y_i + y_j$ in the denominator automatically accounts for the pole. Therefore in the following we put

$$G_n(\theta_1, \theta_2, \ldots, \theta_n) = \frac{Q_n(y_1, y_2 \ldots, y_n)}{\prod\limits_{i<j}(y_i + y_j)},$$

(with Q_n being a symmetric function of $y_1, \ldots, y_n$) and then eq.(9) give recursive relations between the functions Q_n. Clearly the actual form of these recursive relations varies from model to model since they depend on the bulk S-matrix. The form of the recursions depends also on the choice of the 1PFF $r(\theta)$; it is useful to choose an $r(\theta)$ which gives the simplest possible recursion. Writing the 2PFF in the same form as the n-particle one

$$\phi(y_1, y_2) = \frac{Q_2(y_1, y_2)}{y_1 + y_2}$$

then the absence of kinematical singularities requires $Q_2(y, -y) = 0$.

If the bulk S-matrix is nontrivial and the reflection factor has a pole at $\frac{i\pi}{2}$ then the form factors with odd and even particle number are connected by the boundary kinematical singularity equation:

$$-i \operatorname*{Res}_{\theta=0} F^{\mathcal{O}}_{n+1}(\theta + \frac{i\pi}{2}, \theta_1, \ldots, \theta_n) = \frac{g}{2}\Big(1 - \prod_{i=1}^{n} S\big(\frac{i\pi}{2} - \theta_i\big)\Big) F^{\mathcal{O}}_n(\theta_1, \ldots, \theta_n)$$

The corresponding pole in the n particle form factor can be included as

$$G_n(\theta_1, \theta_2, \ldots, \theta_n) = \frac{Q_n(y_1, y_2 \ldots, y_n)}{\prod_i y_i \prod\limits_{i<j}(y_i + y_j)},$$

and the boundary kinematical singularity equation relates Q_n to Q_{n+1}.

The even and the odd particle form factors are also related if the bulk S-matrix has a "self fusing" pole describing the 2 particle $\to$ 1 particle process, which parallels the bulk situation (this happens e.g. in the Lee-Yang model). (In this case it is customary to include this pole also in $f(\theta)$). Since the fusing angle in this process is necessarily $2\pi/3$, one finds from bootstrap that in this case the dynamical singularities imply

$$-i \operatorname*{Res}_{\theta'=\theta} F_{n+2}(\theta' + \frac{i\pi}{3}, \theta - \frac{i\pi}{3}, \theta_1, \ldots, \theta_n) = \Gamma F_{n+1}(\theta, \theta_1, \ldots, \theta_n), \qquad (10)$$

where Γ is related to the residue of the S-matrix at the self fusing pole: $-i\mathrm{res}_{\theta=\frac{2\pi i}{3}}S(\theta)=\Gamma^2$.

An important restriction follows on the form factor functions from requiring a power law bounded ultraviolet behaviour for the two point correlator of the boundary operators $\langle 0|\mathcal{O}(\tau)\mathcal{O}(0)|0\rangle$: the growth of the function $F_n(\theta_1,\ldots,\theta_n)$ must be bounded by some exponential of the rapidity as $\theta\to\infty$ (i.e. the form factors only grow polynomially with particle energy). This can be shown using an argument identical to that in the bulk case [16]. If $r(\theta)$ and $f(\theta)$ are specified in a way to include all poles induced by the dynamics of the model, then it follows that the functions Q_n must be *polynomials* of the y_i. Therefore in the following we only look for explicit *polynomial* solutions of the various recursion equations. This is a posteriori confirmed since we find as many polynomial solution of the boundary form factor equation as many independent local operator exist in the theories.

2.5 Two-point function

Once an appropriate solution of the form factor axioms is found it can be used to describe correlators of boundary operators. The two-point function of the boundary operator $\mathcal{O}$ can be computed by inserting a complete set of states

$$\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle=\sum_{n=0}^{\infty}\frac{1}{(2\pi)^n}\int_{\theta_1>\theta_2>\cdots>\theta_n>0}d\theta_1 d\theta_2\ldots d\theta_n e^{-imt\sum_i\cosh\theta_i}F_nF_n^+ \tag{11}$$

where time translation invariance was used and the form factors are

$$F_n=\langle 0|\mathcal{O}(0)|\theta_1,\theta_2,\ldots,\theta_n\rangle_{in}=F_n^{\mathcal{O}}(\theta_1,\theta_2,\ldots,\theta_n)$$

and

$$F_n^+={}_{in}\langle\theta_1,\theta_2,\ldots,\theta_n|\mathcal{O}(0)|0\rangle=F_n^{\mathcal{O}}(i\pi+\theta_n,i\pi+\theta_{n-1},\ldots,i\pi+\theta_1)$$

which, for unitary theories, is the complex conjugate of the previous one: $F_n^+=F_n^*$. In the Euclidean ($\tau=it$) version of the theories the form factor expansion of the correlator for large separations converges rapidly since multi-particle terms are exponentially suppressed.

The identification between solutions of the form factor equations and operators of the theory is a central issue. One possible way is to analyze the behaviour of the boundary correlators for short distances. Although on general grounds one may expect the form factor expansion to converge rapidly only in the infrared (large volume) regime, the examples from the various bulk theories, where the form factor expansion converges even in the UV domain (see e.g. [17]), suggest that similar behaviour may happen in the boundary setting as well. If the theory can be described as a relevant perturbation of a conformal field theory, then in the UV domain the two-point function must follow a behaviour dictated by this limiting theory. The short distance singularity exponent is related to the scaling dimension of the operator $\mathcal{O}$ and can be calculated from the asymptotic growth of the form factors.

3 Model studies

In this section we carry out a detailed investigation of the solutions of the form factor equations in four different models. The first two models (the massive scalar field with linear boundary interaction and the Ising model interacting with a boundary magnetic field [5]) are free in the bulk and the correlation functions are known explicitly, thus the form factors obtained from the explicit field theoretic solutions can be compared directly to the solutions of the form factor equations. In both cases we find that the space of appropriate *polynomial* solutions of the FF equations can be identified with the space of local boundary operators obtained from the explicit construction. In the case of the Ising model we also show how the conformal dimensions of the various operators of the ultraviolet limiting BCFT can be obtained from the solutions of the FF equations. The third and fourth models, namely the scaling Lee-Yang and the sinh-Gordon models with integrability preserving boundaries are among the simplest boundary integrable theories. In contrast to the previous cases they cannot be solved directly so one has to rely upon the solution of the form factor equations. Since these models contain nontrivial bulk interactions the recursion relations connecting the multi-particle form factors are no longer trivial, and in these cases we investigate their solutions in detail.

3.1 Massive scalar with linear boundary interaction

3.1.1 Direct calculation

The free massive scalar field $\Phi(x,t)$ restricted to the negative half-line $x \leq 0$ subject to linear boundary condition

$$\partial_x \Phi(x,t)|_{x=0} = -\lambda(\Phi(0,t) - \Phi_0). \tag{12}$$

can be described by the following Lagrangian:

$$\mathcal{L} = \Theta(-x)\left(\frac{1}{2}(\partial_t\Phi)^2 - \frac{1}{2}(\partial_x\Phi)^2 - \frac{m^2}{2}\Phi^2\right) - \delta(x)\frac{\lambda}{2}(\Phi - \Phi_0)^2,$$

This one parameter family of linear boundary condition interpolates between Neumann $\partial_x\Phi|_{x=0} = 0$ (for $\lambda = 0$) and Dirichlet $\Phi|_{x=0} = \Phi_0$ (for $\lambda \to \infty$) boundary conditions. Since for any λ we are dealing with a free theory it can be solved explicitly. The mode decomposition of the field is

$$\Phi(x,t) = Ae^{mx} + \int_0^\infty \frac{dk}{\omega(k)}\Big\{a(k)e^{-i\omega(k)t}\big(e^{ikx} + R(k)e^{-ikx}\big) + a^+(k)e^{i\omega(k)t}\big(e^{-ikx} + R(-k)e^{ikx}\big)\Big\}$$

where $A = \frac{\lambda}{m+\lambda}\Phi_0$ and

$$R(k) = \frac{k - i\lambda}{k + i\lambda}$$

is the reflection factor on the boundary at $x = 0$. The creation/annihilation operators are normalized as

$$[a(k), a^+(k')] = 2\pi\omega(k)\delta(k - k') \quad , \quad k\,,\, k' > 0$$

The boundary two-point function can be calculated easily

$$\langle 0|\Phi(0,t)\Phi(0,t')|0\rangle = A^2 + \int_0^\infty \frac{dk}{2\pi\omega(k)} e^{-i\omega(k)(t-t')} \left(1 + R(k)\right)\left(1 + R(-k)\right)$$

By comparing this expression to the form factor expansion of the two-point function (11), the form factor of the elementary field can be extracted:

$$\langle 0|\Phi(0,t)|\theta\rangle = e^{-i\omega(k)t}\left(1 + R(k)\right)$$

The same result can be obtained by taking the general (space-dependent) two point function

$$\begin{aligned}\langle 0|T\left(\Phi(x,t)\Phi(x',t')\right)|0\rangle &= A^2 e^{m(x+x')} \\ &\quad + \int \frac{d^2k}{(2\pi)^2}\frac{i}{k^2 - m^2 + i\epsilon} e^{-ik_0(t-t')}\left(e^{-ik_1(x-x')} + R(k)e^{+ik_1(x+x')}\right)\end{aligned}$$

and using the boundary reduction formula [9]

$$\begin{aligned}\langle 0|\Phi(x,t)|\theta\rangle &= 2i\int_{-\infty}^0 dx' \int_{-\infty}^\infty dt'\; e^{i\omega(\theta)t'} \cos(p(\theta)x')\left\{\partial_{t'}^2 - \partial_{x'}^2 + m^2 + \delta(x')\partial_{x'}\right\} \\ &\qquad \langle 0|T\left(\Phi(x,t)\Phi(x',t')\right)|0\rangle\end{aligned}$$

where $\omega(\theta) = m\cosh(\theta)$ and $p(\theta) = m\sinh(\theta)$. Performing explicitly the calculation yields

$$\langle 0|\Phi(x,t)|\theta\rangle = e^{-i\omega(\theta)t}(e^{ip(\theta)x} + R(\theta)e^{-ip(\theta)x}) \tag{13}$$

which for the form factor of the operator $\Phi(x = 0, t)$ reads as

$$\langle 0|\Phi(0,t)|\theta\rangle = e^{-i\omega(\theta)t}\left(1 + R(\theta)\right)$$

Introducing $\tau = -it$ one also gets

$$\langle 0|\partial_\tau^n \Phi(0,0)|\theta\rangle = \omega(\theta)^n\left(1 + R(\theta)\right), \quad n > 0.$$

Clearly these operators have no multi-particle matrix elements. It is important to realize that $\partial_x\Phi(0,0)$ is not an independent operator since the boundary condition eq.(12) relates it to $\Phi(0,0)$, thus the set of independent boundary operators having only one particle matrix elements is given by $\partial_\tau^n\Phi(0,0)$. To obtain multi-particle matrix elements one has to consider

$$\langle 0| : \Phi(x_1,t_1)\ldots\Phi(x_k,t_k) : |\theta_1\ldots\theta_k\rangle.$$

Using the analogous boundary reduction formula and the Wick theorem we obtain

$$\langle 0| : \Phi(x_1,t_1)\dots\Phi(x_k,t_k) : |\theta_1\dots\theta_k\rangle = \tag{14}$$
$$\left\{e^{-i\omega(\theta_1)t_1}\left(e^{ip(\theta_1)x_1}+R(\theta_1)e^{-ip(\theta_1)x_1}\right)\right\}\dots\left\{e^{-i\omega(\theta_k)t_k}\left(e^{ip(\theta_k)x_k}+R(\theta_k)e^{-ip(\theta_k)x_k}\right)\right\}+\dots,$$

where the ellipses at the end represent additional terms which make it completely symmetric in all coordinates. From this expression one can extract the form factor of the most general boundary operator of the theory

$$\langle 0| : \partial_{\tau_1}^{n_1}\Phi(0,0)\dots\partial_{\tau_k}^{n_k}\Phi(0,0) : |\theta_1\dots\theta_k\rangle = \omega(\theta_1)^{n_1}\left(1+R(\theta_1)\right)\dots\omega(\theta_k)^{n_k}\left(1+R(\theta_k)\right)+\dots$$

where again a complete symmetrization in the θ_i rapidities is understood. Checking the leading asymptotic behaviour of these form factors gives that for all $\theta_i \sim \theta$ large they grow as $e^{N\theta}$, where $N = n_1+\dots+n_k$ is the total number of derivatives in the expression. We note that we have as many operators for a given N as many partition N has into the numbers $1,2,\dots,k$. This can be seen by writing $N = N_1+2N_2+\dots+kN_k$ and associating to it the operator with $n_1 = N_1+N_2\dots+N_k$, $n_2 = N_2+\dots+N_k$ ... $n_k = N_k$ derivatives.

The Dirichlet boundary condition ($R=-1$) can be obtained in the $\lambda\to\infty$ limit. Clearly $\Phi|_{x=0} = \Phi_0$ is a c-number and the Dirichlet boundary condition does not connect the operator $\partial_x\Phi|_{x=0}$ to $\Phi|_{x=0}$. We can extract, however, the form factors of the operator $\partial_x\Phi(0,t)$ from that of $\Phi(0,t)$ by taking the $\lambda\to\infty$ limit carefully in (13):

$$\langle 0|\partial_x\Phi(0,t)|\theta\rangle = e^{-i\omega(\theta)t}2ip(\theta)$$

and for its derivatives

$$\langle 0|\partial_\tau^n\partial_x\Phi(0,0)|\theta\rangle = \omega(\theta)^n 2ip(\theta), \quad n>0.$$

This can be extended similarly to the most general operator as

$$\langle 0| : \partial_\tau^{n_1}\partial_x\Phi(0,0)\dots\partial_\tau^{n_k}\partial_x\Phi(0,0) : |\theta_1\dots\theta_k\rangle = \omega(\theta_1)^{n_1}2ip(\theta_1)\dots\omega(\theta_k)^{n_k}2ip(\theta_k)+\dots$$

where again a complete symmetrization in the θ_i rapidities is understood. Checking the leading asymptotic behaviour of these form factors gives that for all $\theta_i\sim\theta$ large they grow as $e^{N\theta}$, where $N = k+n_1+\dots+n_k$ is the total number of derivatives in the expression.

3.1.2 Solving the form factor equations

The bulk S-matrix of the theory together with the reflection factor are

$$S(\theta)=1, \quad R(\theta) = \frac{\sinh\theta - i\frac{\lambda}{m}}{\sinh\theta + i\frac{\lambda}{m}} = -\left(1+\frac{B}{2}\right)\left(-\frac{B}{2}\right), \quad \frac{\lambda}{m} = \sin\frac{\pi B}{2},$$

where the block notation (6) is used to express $R(\theta)$. As a consequence of $S=1$ the minimal bulk form factor is trivial: $f(\theta)=1$. To determine the 1PFF we note that this reflection factor is identical to the two particle S-matrix of the sinh-Gordon model if the

above identification of parameters is done. Therefore eq.(5) in this case is identical to the equation for the minimal bulk form factor of the sinh-Gordon model. Choosing for $g(\theta)$ the solution given in [14] (described in detail in the sinh-Gordon section) gives

$$r(\theta) = 2g(\theta)g(i\pi - \theta) = \frac{2\sinh\theta}{\sinh\theta + i\frac{\lambda}{m}} = 1 + R(\theta)$$

Clearly this corresponds to the form factor of the operator $\Phi(0,0) - A$.

Now we demonstrate that the number of independent solutions of the form factor equations coincides with the number of local boundary operators. In this case the general Ansatz (8) takes the following form

$$F_n(\theta_1, \ldots, \theta_n) = P_n(y_1, \ldots, y_n) \prod_i r(\theta_i)$$

where $y_i = 2\cosh\theta_i$ as before. Since the bulk S-matrix is trivial there are no bulk/boundary kinematical singularities and P_n is a completely symmetric polynomial in the y_i-s. One can count the independent solutions of the BFF equations by counting the possible solutions for P_n. If P_n has degree N then the solutions are given by the partitions of N into the numbers $1, 2, \ldots n$ in the following way. Since the completely symmetric polynomials of n variable are generated by the σ_i-s (elementary symmetric polynomials of degree i) one can write:

$$\prod_{i=1}^{n}(x + x_i) = \sum_{i=1}^{n} \sigma_i\, x^{n-i}; \quad P_n \propto \sigma_1^{k_1}\sigma_2^{k_2}\ldots\sigma_n^{k_n}; \quad N = \sum ik_i$$

It is clear that this space has the same dimension as the space of boundary operators having only n particle matrix elements with asymptotic growth $e^{N\theta}$.

The Dirichlet ($\lambda \to \infty$) limit for the operator $\partial_x\Phi(0,0)$ can be obtained using its relation to $\Phi(0,0)$ via the boundary condition (12) as we did in the Lagrangian framework.

3.2 Ising model with boundary magnetic field

3.2.1 Direct calculation

The Ising model with a boundary magnetic field can be described by a free massive Majorana fermion perturbed at the boundary [5]. In Minkowskian formalism the Dirac equation can be obtained form the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

The gamma matrices are chosen as

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

in order for the Dirac equation to be real:

$$\begin{pmatrix} -m & \partial_x + \partial_t \\ \partial_x - \partial_t & -m \end{pmatrix} \Psi = 0$$

Thus the Majorana condition corresponds to taking Ψ real. Using the component notation

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

the Lagrangian of the boundary field theory takes the form

$$\mathcal{L} = -i\Theta(-x)\left(\frac{1}{2}\psi_+(\partial_t - \partial_x)\psi_+ - \frac{1}{2}\psi_-(\partial_t + \partial_x)\psi_- - m\psi_+\psi_-\right) - i\delta(x)U_B$$

where

$$U_B = \frac{1}{2}\psi_+\psi_- + \frac{1}{2}a\dot{a} + \frac{1}{2}ha(\psi_+ + \psi_-)$$

The operator a is a boundary fermion $a^2 = 1$, which implements the boundary condition

$$\partial_t(\psi_+ - \psi_-) = \frac{h^2}{2}(\psi_+ + \psi_-)$$

Since the theory is free we can solve it explicitly. The mode expansion of the fermionic fields are

$$\begin{aligned} \psi_\pm(x,t) \quad = \quad & \int_0^\infty \frac{d\theta}{2\pi}\Bigg\{ b(\theta)e^{-i\omega(\theta)t}\left(u_\pm(\theta)e^{ik(\theta)x} + R(\theta)u_\pm(-\theta)e^{-ik(\theta)x}\right) \\ & + b^+(\theta)e^{i\omega(\theta)t}\left(v_\pm(\theta)e^{-ik(\theta)x} + R(-\theta)v_\pm(-\theta)e^{ik(\theta)x}\right)\Bigg\} \end{aligned}$$

where $u_\pm(\theta) = v_\pm^*(\theta) = \sqrt{m}e^{\mp\frac{i\pi+2\theta}{4}}$ are the spinor amplitudes, $R(\theta)$ is nothing else but the one-particle reflection factor

$$R(\theta) = i\tanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right)\frac{\sinh\theta + i\kappa}{\sinh\theta - i\kappa} \qquad , \quad \kappa = 1 - \frac{h^2}{2m}$$

and the creation/annihilation operators are normalized as

$$\{b(\theta), b^+(\theta')\} = 2\pi\delta(\theta - \theta')$$

The boundary two point function can be calculated explicitly:

$$\langle 0|\psi_\pm(0,t)\psi_+(0,t')|0\rangle = \int_0^\infty \frac{d\theta}{2\pi}e^{-i\omega(\theta)(t-t')}\left(u_\pm(\theta) + R(\theta)u_\pm(-\theta)\right)\left(v_+(\theta) + R(-\theta)v_+(-\theta)\right) \tag{15}$$

which, as compared to the form factor expansion (11), gives

$$\langle 0|\psi_\pm(0,t)|\theta\rangle = e^{-i\omega(\theta)t}\left(u_\pm(\theta) + R(\theta)u_\pm(-\theta)\right) \tag{16}$$

These two operators are not independent since they are related by the boundary condition and so there is only one boundary fermion field, say ψ_+. As a result, the algebraically independent operators at the boundary are the fermion field and its derivatives $\partial_\tau^n\psi_+$. Note that $\partial_x\psi_+|_{x=0}$ is not an independent field, as it is determined by the Dirac equation in terms of $\partial_\tau\psi_+|_{x=0}$ and $\psi_-|_{x=0}$. As a consequence of the fermionic nature of the field the most general boundary operator has the form $\partial_\tau^{n_1}\psi_+\partial_\tau^{n_2}\psi_+\ldots\partial_\tau^{n_k}\psi_+$ where $n_1 > n_2 > \cdots > n_k$ (the inequalities are strict, in contrast to the bosonic case discussed earlier). $N = n_1 + n_2 + \cdots + n_k$ is called the level of the operator, and operators at level N can be brought in one-to-one correspondence with partitions of N into the numbers $1, 2, \ldots, k$. For a partition

$$N = kN_k + (k-1)N_{k-1} + \cdots + 2N_2 + N_1$$

we associate the operator above with $n_k = N_k$; $n_{k-1} = N_k + N_{k-1}; \ldots;$ $n_1 = N_k + N_{k-1} + \cdots + N_1$.

(Had we included also the operator a we would have had to perform a GSO type projection, leaving only non-fermionic operators. This would amount to keeping all operators with an even number of fermion factors plus all odd ones multiplied with a factor a, but this would lead to the same number of operators.)

3.2.2 Solution of the FF bootstrap

Using again the block notation (6) the S-matrix of the theory and the reflection factor are [5]

$$S(\theta) = -1 \quad , \qquad R_x(\theta) = [x]\left(-\frac{1}{2}\right) \quad , \qquad [x] = (x)(1-x)$$

where x is related to the boundary magnetic field as

$$\sin \pi x = 1 - \frac{h^2}{2m} = \kappa$$

For $h = 0$ we recover the free boundary condition with

$$R_{\text{free}}(\theta) = \left(\frac{1}{2}\right)$$

which has a pole at $i\frac{\pi}{2}$ corresponding to the fact that the ground state is doubly degenerate. In contrast to the generic situation the pole at $i\frac{\pi}{2}$ is a *dynamical* pole and not a kinematical one (since the field has no vacuum expectation value at all). The $h \to \infty$ limit corresponds to the fixed boundary condition (when the Ising spin takes a fixed value at the boundary), and the reflection factor is

$$R_{\text{fixed}}(\theta) = \left(-\frac{1}{2}\right)$$

which has no pole in the physical strip at all. The minimal one particle form factor for the fixed case can be calculated directly using the recipe in Section 2

$$r_{\text{fixed}}(\theta) = \frac{\sinh\theta}{\sinh(\frac{\theta}{2} + i\frac{\pi}{4})}$$

For the free case we include the dynamical singularity into the 1PFF

$$r_{\text{free}}(\theta) = -2i\frac{\sinh\theta}{\cosh\theta}\sinh\left(\frac{\theta}{2} + i\frac{\pi}{4}\right)$$

The simplest solution which interpolates between these two cases and has a pole exactly at the location of the boundary dynamical singularity of the reflection factor is

$$r(\theta) = \frac{\sinh\theta}{\sinh(\frac{\theta}{2} + i\frac{\pi}{4})}\frac{\cosh\theta + i(1-\kappa)}{\sinh\theta - i\kappa}. \tag{17}$$

This expression is the same we obtained from the exact solution of the model (16). The minimal bulk two particle form factor is simply

$$f(\theta) = \sinh\frac{\theta}{2}.$$

Since $R(\theta)$ has no kinematical pole at $i\frac{\pi}{2}$, boundary kinematical singularities are absent, and since the bulk S-matrix is -1 there are no bulk kinematical singularities either. Thus we look for the n particle form factors in the form (8)

$$F_n(\theta_1, \ldots, \theta_n) = P_n(y_1, \ldots, y_n)\prod_i r(\theta_i)\prod_{i<j} f(\theta_i - \theta_j)f(\theta_i + \theta_j),$$

where $y_i = 2\cosh\theta_i$ and P_n is a completely symmetric polynomial in the y_i-s. Taking into account the special form of $f(\theta)$ the form factor simplifies to

$$F_n(\theta_1, \ldots, \theta_n) = P_n(y_1, \ldots, y_n)\prod_i r(\theta_i)\prod_{i<j}(y_i - y_j) \tag{18}$$

The independent solutions are counted in the same way as in the bosonic case, i.e. by the partitions of N into the numbers $1, 2, \ldots, n$ and are generated by the σ_i-s. It is clear that the dimension of the space they span is the same as the one of the boundary operators obtained from the direct calculation.

Since the UV limit of this theory is a boundary conformal field theory one can go further than in the bosonic case and calculate the UV dimension of the various boundary operators. As the form factor equations are not coupled we can choose a basis among operators consisting of those having matrix elements only with a certain fixed number of particles.

If the operator has only one-particle matrix element then its correlator is

$$\langle 0|\mathcal{O}(\tau)\mathcal{O}(0)|0\rangle = \int_0^\infty \frac{d\theta}{2\pi}|F_1^{\mathcal{O}}(\theta)|^2 e^{-m\cosh\theta\tau} \tag{19}$$

where $F_1^{\mathcal{O}}(\theta) = r(\theta)P_1(y)$. Plugging (17) into (19) we obtain the exact correlator (15). If the operator $\mathcal{O}$ goes to a scaling operator in the UV limit ($\tau \to 0$) then the exact correlator has the short distance asymptotics $\tau^{-2\Delta}$, where Δ is the appropriate scaling dimension in the ultraviolet BCFT. In (19) the singularity comes from the large θ asymptotics of the form factor. If $|F_1^{\mathcal{O}}(\theta)|^2$ diverges as y^n here, then the corresponding weight is $\Delta = \frac{n}{2}$. Taking the simplest solution $P_1(y) = 1$ the weight is $\Delta = \frac{1}{2}$ which corresponds to the boundary fermion field. Choosing $P_1(y) = \sigma_1^n(y)$ corresponds to the n-th derivative of this operator which has weight $n + \frac{1}{2}$.

Similarly we can analyze an operator having n-particle matrix element only. The corresponding correlator is

$$\langle 0|\mathcal{O}_{(n)}(\tau)\mathcal{O}_{(n)}(0)|0\rangle = \int_0^\infty \frac{d\theta_1}{2\pi}\cdots\int_0^\infty \frac{d\theta_n}{2\pi}\frac{1}{n!}|F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)|^2 e^{-m(\cosh\theta_1+\cdots+\cosh\theta_n)\tau}$$

The operator which has the mildest UV behaviour corresponds to $P_n = 1$. The corresponding form factor square for large θ-s behaves as

$$|F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)|^2 \propto \exp(\theta(n + n(n-1)) = e^{\theta n^2},$$

thus the UV dimension of $\mathcal{O}_{(n)}$ is $\Delta = \frac{n^2}{2}$. The explicit boundary operator which has nonzero matrix elements only with n particle states and has the mildest UV behaviour is

$$\psi_+\partial_\tau\psi_+\ldots\partial_\tau^{n-1}\psi_+$$

with dimension $\Delta = \frac{n}{2} + \frac{n(n-1)}{2}$; therefore it can be associated to $\mathcal{O}_{(n)}$.

To match the descendent operators, note that to any partition of $N = k_1 + 2k_2 + \cdots + n\cdot k_n$ there exists a solution of the form factor equations with $P_n^N = \sigma_1^{k_1}\ldots\sigma_n^{k_n}$. The number of such polynomials is the same as the number of descendants of $\mathcal{O}_{(n)}$ at level N: to the given partition we can associate the operator

$$\partial_\tau^{k_n}\psi_+\partial_\tau^{1+k_{n-1}+k_n}\psi_+\ldots\partial_\tau^{n-1+k_1+\cdots+k_n}\psi_+$$

Conversely, given an operator of the form

$$\partial_\tau^{p_1}\psi_+\ldots\partial_\tau^{p_n}\psi_+ \qquad , \qquad 0 \le p_1 < p_2 \cdots < p_n$$

of weight $N + \frac{n^2}{2}$, one can define a partition as $k_n = p_1$, $k_{n-1} = p_2 - k_n - 1$, ... and thus associate a polynomial solution of the form factor equations with appropriate asymptotic behaviour. It is important to emphasize that we do not claim that the form factor related to P_n^N belongs to the operator above, what we have shown is only that the dimension of the space of operators with certain scaling dimension is the same as the dimension of the solution of the form factor equations.

3.3 The boundary scaling Lee-Yang model

The scaling Lee-Yang model with boundary is a combined bulk and boundary perturbation of the boundary version of the $\mathcal{M}(2/5)$ minimal model [7, 18]. In the bulk the perturbation is given by the unique relevant spinless field ϕ, at the boundary the perturbation depends on which of the two possible conformal boundary conditions is present in the unperturbed model. One, denoted by 1 in [18], does not have any relevant boundary fields - thus can have no boundary perturbation either -, while the other, denoted Φ in [18], has a single relevant boundary field φ with scaling dimension $-1/5$. In this latter case the general perturbed action is

$$\mathcal{A}_{\lambda,\Phi(h)} = \mathcal{A}_\Phi + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \phi(x,y) + h \int_{-\infty}^{\infty} dy \varphi(y),$$

where $\mathcal{A}_\Phi$ denotes the action for $\mathcal{M}(2/5)$ with the Φ boundary condition imposed at $x = 0$, and λ and h denote the bulk and boundary couplings respectively. The action of $\mathcal{A}_{\lambda,1}$ is similar, but the last term on the right hand side is missing. If $\lambda > 0$ then in all cases the bulk behaviour is described by an integrable massive theory having only a single particle type with the following S matrix [19]:

$$S(\theta) = -\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = -\left[\frac{1}{3}\right] \quad ; \quad (x) = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi x}{2}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi x}{2}\right)}.$$

The pole at $\theta = \frac{2\pi i}{3}$ corresponds to the "φ^3 property", i.e. the particle appears as a bound state of itself. The minimal bulk two particle form factor which has only a zero at $\theta = 0$ and a pole at $\theta = \frac{2\pi i}{3}$ in the strip $0 \leq \Im m(\theta) < \pi$ has the form [17]:

$$f(\theta) = \frac{y-2}{y+1} v(i\pi - \theta) v(-i\pi + \theta) \quad , \quad y = e^\theta + e^{-\theta},$$

where

$$v(\theta) = \exp\left\{2\int_0^\infty \frac{dt}{t} e^{i\frac{\theta t}{\pi}} \frac{\sinh\frac{t}{2}\sinh\frac{t}{3}\sinh\frac{t}{6}}{\sinh^2 t}\right\}.$$

In the boundary theory with the perturbed Φ boundary, the reflection amplitude of the particle depends on the strength of the coupling constant of the boundary field as [18]

$$R(\theta)_\Phi = R_0(\theta) R(b,\theta) = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(-\frac{2}{3}\right)\left[\frac{b+1}{6}\right]\left[\frac{b-1}{6}\right],$$

where the dimensionless parameter b is related to the dimensionful h as

$$h(b) = \sin\left(\left(b + \frac{1}{2}\right)\frac{\pi}{5}\right) m(\lambda)^{6/5} h_{crit}, \qquad h_{crit} = -\pi^{\frac{3}{5}} 2^{\frac{4}{5}} 5^{\frac{1}{4}} \frac{\sin\frac{2\pi}{5}}{\sqrt{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}} \left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{3}{5})}\right)^{\frac{6}{5}},$$

and $m(\lambda)$ is the mass of the particle giving the overall scale in the infrared description. In the case of the 1 boundary the reflection amplitude is the parameter independent expression

$$R(\theta)_1 = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(-\frac{2}{3}\right).$$

Note that $R(\theta)_1$ is identical to $R(\theta)_\Phi$ at $b=0$ and so both have a pole at $\theta = i\pi/2$ coming from the $\left(\frac{1}{2}\right)$ block but their g factors differ in a sign [13].

3.3.1 Lee-Yang model with perturbed Φ boundary

We consider first the Lee-Yang model with perturbed Φ boundary. The 1PFF corresponding to $R(\theta)_\Phi$ is chosen as

$$r(\theta) = \frac{i\sinh\theta}{(\sinh\theta - i\sin\frac{\pi(b+1)}{6})(\sinh\theta - i\sin\frac{\pi(b-1)}{6})} u(\theta),$$

where

$$u(\theta) = \exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{1}{\sinh\frac{t}{2}} - 2\cosh\frac{t}{2}\cos\left[\left(\frac{i\pi}{2}-\theta\right)\frac{t}{\pi}\right]\frac{\sinh\frac{5t}{6}+\sinh\frac{t}{2}-\sinh\frac{t}{3}}{\sinh^2 t}\right]\right\}.$$

Note that $r \sim 1$ at $y \to \infty$, and $r(\theta)$ satisfies the $r(\theta + i\pi) = r(\theta)^*$ reality condition for real θ. The general n-particle form factors have the form

$$F_n(\theta_1,\ldots,\theta_n) = H_n Q_n(y_1,\ldots,y_n)\prod_i \frac{r(\theta_i)}{y_i}\prod_{i<j}\frac{f(\theta_i-\theta_j)f(\theta_i+\theta_j)}{y_i+y_j}, \tag{20}$$

where we separated a normalizing factor H_n from the polynomials Q_n. The various F_n-s are related to each other by both the kinematical and the dynamical singularity equations, since the S-matrix is nontrivial and also has a φ^3 pole with $\Gamma = i2^{\frac{1}{2}}3^{\frac{1}{4}}$. In addition, these F_n-s also have to satisfy the equation coming from the residue of the pole at $\theta = i\pi/2$:

$$-i \operatorname*{Res}_{\theta=i\pi/2} F_{n+1}(\theta,\theta_1,\ldots,\theta_n) = \frac{g_\Phi}{2}\left(1-\prod_{j=1}^n S(i\frac{\pi}{2}-\theta_j)\right)F_n(\theta_1,\ldots,\theta_n). \tag{21}$$

Our strategy is to solve the recursion equations coming from the first two conditions first and check whether the solutions also satisfy the third requirement (21). By choosing the normalizing factors H_n and introducing the useful quantities β_k

$$H_n = N\left(\frac{i3^{\frac{1}{4}}}{2^{\frac{1}{2}}v(0)}\right)^n \qquad \beta_k(b) = 2\cos\frac{\pi}{6}(b+k), \quad k\in \mathrm{Z}, \tag{22}$$

the overall normalization N drops out and the recursion equations coming from the dynamical (resp. kinematical) singularities read

$$\begin{aligned} Q_2(y_+,y_-) &= (y^2-\beta_{-3}^2)Q_1(y), \\ Q_{n+2}(y_+,y_-,y_1,\ldots,y_n) &= Q_{n+1}(y,y_1,\ldots,y_n)\,(y^2-\beta_{-3}^2)\prod_{i=1}^n(y+y_i), \quad n>0; \end{aligned} \tag{23}$$

$$Q_{n+2}(-y,y,y_1,\ldots,y_n) = Q_n(y_1,\ldots,y_n)\,(y^2-\beta_{-1}^2)(y^2-\beta_1^2)\,P_n, \tag{24}$$

where

$$P_n = \frac{1}{2(y_+-y_-)}\left[\prod_{i=1}^{n}(y_i-y_-)(y_i+y_+) - \prod_{i=1}^{n}(y_i+y_-)(y_i-y_+)\right], \tag{25}$$

and

$$y_+ = \omega x + \omega^{-1}x^{-1}; \quad y_- = \omega^{-1}x + \omega x^{-1}; \quad x = e^{\theta}; \quad \omega = e^{i\frac{\pi}{3}}, \quad y = x + x^{-1}. \tag{26}$$

Next we present the minimal solution of these recursion equations up to $n = 3$. The solution is called minimal, if the leading overall degrees of the F_n-s in all of the y variables are the smallest possible ones. Of course the solution also depends on the input function $Q_1(y_1)$. Since F_1 can have no pole at $\theta = i\pi/2$ while $r(\theta)/y$ has one, Q_1 must be chosen to cancel this pole; the choice with the minimal degree is $Q_1(y_1) = y_1 = \sigma_1(y_1)$. Using this as input, we find from (23 ,24) the *unique* solution

$$\begin{aligned} Q_1(y_1) &= \sigma_1, \qquad\qquad Q_2(y_1,y_2) = \sigma_1(\sigma_2+3-\beta_{-3}^2), \\ Q_3(y_1,y_2,y_3) &= \sigma_1\left[\sigma_1(\sigma_2+\beta_{-1}^2)(\sigma_2+\beta_1^2) - (\sigma_2+3)(\sigma_1\sigma_2-\sigma_3)\right]. \end{aligned} \tag{27}$$

The remarkable property of this solution is that it contains no free parameters. A simple counting of the various powers shows that the leading overall degree of F_1, F_2, and F_3 vanish.

To check eq.(21) we need the following relations following from the explicit solution (27) and from the various identities among the β_k-s:

$$\begin{aligned} Q_2(0,y_2) &= \sigma_1(y_2)(3-\beta_{-3}^2), \\ Q_3(0,y_2,y_3) &= \beta_1\beta_{-1}\sigma_1(y_2,y_3)Q_2(y_2,y_3) = (3-\beta_{-3}^2)\sigma_1(y_2,y_3)Q_2(y_2,y_3). \end{aligned}$$

Indeed using them in eq.(21) leads to a consistency condition on the ratio of the H_n-s:

$$\frac{H_{n+1}}{H_n}r(i\frac{\pi}{2})(3-\beta_{-3}^2) = -2i\sqrt{3}g_\Phi, \qquad n = 1,2.$$

Since

$$r(i\frac{\pi}{2}) = \frac{4u(i\frac{\pi}{2})}{i(\sqrt{3}-\beta_{-3})^2}, \quad \text{and} \quad g_\Phi = i2(3)^{1/4}(2-\sqrt{3})^{1/2}\frac{\sqrt{3}+\beta_{-3}}{\sqrt{3}-\beta_{-3}},$$

the b dependence cancels from the consistency condition and using the actual form of the H_n-s leads to

$$\frac{u(i\frac{\pi}{2})}{\sqrt{2}v(0)} = \sqrt{3}\,(2-\sqrt{3})^{1/2},$$

which we checked numerically up to 7 digits.

To test these form factors numerically against the predictions of conformal field theory, we take the spectral representation of the boundary two-point function

$$\begin{aligned}\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle &= \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n)^{+}F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n)e^{-mt\sum_{i=1}^n\cosh\theta_i}\\ &= \sum_{n=0}^{\infty}(-)^n\int_0^{\infty}\frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}|F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n)|^2e^{-mt\sum_{i=1}^n\cosh\theta_i}\end{aligned}$$

which we truncate to the first few terms in the boundary form factor expansion. Since the minimal solution of the form factor problem has the mildest UV behaviour it is natural to assume, that in the UV it corresponds to the boundary field φ. Therefore $\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle$ obtained from the FF expansion must be compared to the short distance expansion:

$$\langle 0|m^{\frac{1}{5}}\varphi(t)m^{\frac{1}{5}}\varphi(0)|0\rangle = -(mt)^{\frac{2}{5}} + (mt)^{\frac{1}{5}}C^{\varphi}_{\varphi\varphi}\langle m^{\frac{1}{5}}\varphi\rangle + \dots$$

where appropriate powers of m were inserted to make the expression dimensionless and the fusion coefficient is

$$C^{\varphi}_{\varphi\varphi} = -\sqrt{\frac{1+\sqrt{5}}{2}}\sqrt{\frac{\Gamma(\frac{1}{5})\Gamma(\frac{6}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}}$$

while the (b-dependent, dimensionless) vacuum expectation value

$$\langle m^{\frac{1}{5}}\varphi\rangle = -\frac{5}{6h_{crit}}\frac{\cos(\frac{\pi b}{6})}{\cos(\frac{\pi}{10}(2b+1))}$$

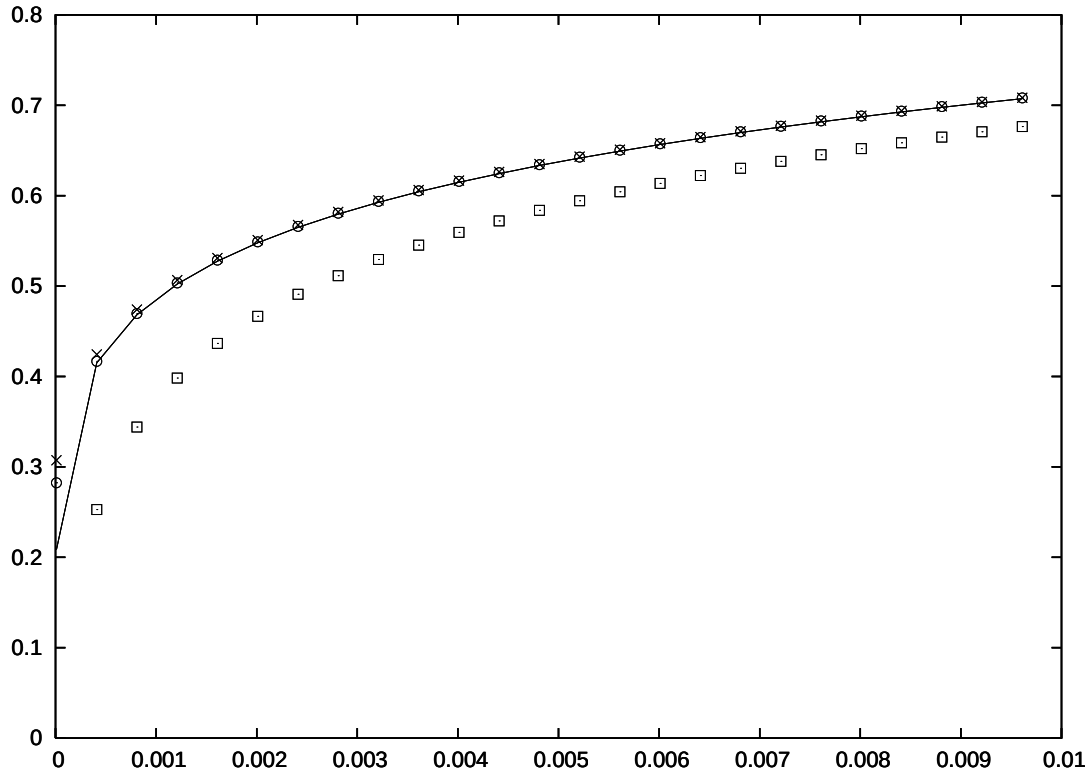
is given explicitly in [18]. In analogy with the bulk case [7] we choose the normalization factor N in (22) as the vacuum expectation value of the boundary field

$$N = \langle m^{\frac{1}{5}}\varphi\rangle \tag{28}$$

With this choice the boundary form factor expansion gives

$$\begin{aligned}\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle &= |F_0^{\mathcal{O}}|^2 - \int_0^{\infty}\frac{d\theta}{2\pi}|F_1^{\mathcal{O}}|^2e^{-mt\cosh\theta}\\ &+ \int_0^{\infty}\frac{d\theta_1 d\theta_2}{2(2\pi)^2}|F_2^{\mathcal{O}}(\theta_1,\theta_2)|^2e^{-mt(\cosh\theta_1+\cosh\theta_2)}\\ &- \int_0^{\infty}\frac{d\theta_1 d\theta_2 d\theta_3}{6(2\pi)^3}|F_3^{\mathcal{O}}(\theta_1,\theta_2,\theta_3)|^2e^{-mt(\cosh\theta_1+\cosh\theta_2+\cosh\theta_3)} + \dots\end{aligned}$$

The two expansions are compared on the next figure



where $b = -1.05$ and the dimensionless correlation function is plotted against mt. The predicted UV behaviour is given by the continuous line and the numerically determined form factor expansion truncated at 1, 2 and 3 particle intermediate states is denoted by the symbols $\square$, $\times$ and $\circ$, respectively.

We checked that the agreement between the form factor expansion truncated at three particles and the UV CFT prediction holds for various values of the parameter b: indeed as we change b the two curves move together. The agreement above also confirms our choice (28) for the normalization of the form factors. Based on all these we associate the boundary operator corresponding to the minimal solution of the form factor axioms to the one, that in the UV limit becomes the φ field of the boundary Lee-Yang model.

3.3.2 Lee-Yang model with the 1 boundary

The 1PFF corresponding to the parameter free $R(\theta)_1$ is chosen as

$$r_1(\theta) = i \sinh\theta \; u(\theta),$$

where $u(\theta)$ is the same as in the previous subsection. Note that r_1 also satisfies $r_1(\theta + i\pi) = r_1(\theta)^*$ but its asymptotic behaviour $r_1 \sim y^2$ at $y \to \infty$ is different from that of the r in the previous case. Since $R(\theta)_1$ also has a pole at $\theta = i\pi/2$ we introduce a similar Ansatz as in (20)

$$F_n(\theta_1, \ldots, \theta_n) = H_n^1 Q_n(y_1, \ldots, y_n) \prod_i \frac{r_1(\theta_i)}{y_i} \prod_{i<j} \frac{f(\theta_i - \theta_j) f(\theta_i + \theta_j)}{y_i + y_j}, \tag{29}$$

with $H_n^1 = 4^n H_n$ where H_n is the same as in (22). Then one finds the following recursion equations for the Q_n-s from the dynamical (resp. kinematical) singularity equations:

$$\begin{aligned} Q_2(y_+, y_-) &= Q_1(y), \\ Q_{n+2}(y_+, y_-, y_1, \ldots, y_n) &= Q_{n+1}(y, y_1, \ldots, y_n) \prod_{i=1}^{n} (y + y_i), \quad n > 0; \end{aligned} \tag{30}$$

$$Q_{n+2}(-y, y, y_1, \ldots, y_n) = Q_n(y_1, \ldots, y_n)\, P_n, \tag{31}$$

where the various symbols are the same as in eq.(23,24). Up to $n = 4$ the *unique* minimal solution of these recursion equations with the input $Q_1(y_1) = \sigma_1$ is

$$Q_1(y_1) = \sigma_1, \quad Q_2(y_1, y_2) \sim \sigma_1, \quad Q_3(y_1, y_2, y_3) \sim \sigma_1^2, \quad Q_4(y_1, y_2, y_3, y_4) \sim \sigma_1^2(\sigma_2 + 3).$$

It is easy to show that the leading overall degree of the first four form factors $F_1, \ldots, F_4$ is two. This indicates that the operator that corresponds to this set is different from the one encountered in the case of the perturbed Φ boundary. Therefore in the conformal limit this operator is different from the φ field and this is in accord with the fact that only the identity operator and its descendents can live on the conformal boundary condition 1. Based on the asymptotics of the form factors for large θ the corresponding operator has ultraviolet dimension 2 and can be identified with the unique such operator in the conformal vacuum module which is the L_{-2} descendent of the identity. This identification is further confirmed by comparing the numerically obtained truncated form factor expansion to the conformal two-point function.

Since

$$r_1(i\frac{\pi}{2}) = -u(i\frac{\pi}{2}), \quad \text{and} \quad g_1 = -i2(3)^{1/4}(2 - \sqrt{3})^{1/2},$$

(see also [13]) one can readily show that these four form factors also satisfy the equation coming from the residue of the pole at $\theta = i\pi/2$.

3.4 The boundary sinh-Gordon model

The sinh-Gordon theory in the bulk is defined by the Lagrangian[2]:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{m^2}{b^2}(\cosh b\Phi - 1)$$

It can be considered as the analytic continuation of the sine-Gordon model for imaginary coupling $\beta = ib$. The S-matrix of the model is

$$S(\theta) = -\left(1 + \frac{B}{2}\right)\left(-\frac{B}{2}\right) = -\left[-\frac{B}{2}\right] \qquad ; \quad B = \frac{2b^2}{8\pi + b^2}$$

[2]Note that the parameter b is used here with a different meaning compared to the former case of the boundary Lee-Yang model.

The minimal bulk two particle form factor belonging to this S-matrix is [14]

$$f(\theta) = \mathcal{N} \exp\left[8 \int_0^\infty \frac{dx}{x} \sin^2\left(\frac{x(i\pi - \theta)}{2\pi}\right) \frac{\sinh\frac{xB}{4} \sinh(1-\frac{B}{2})\frac{x}{2} \sinh\frac{x}{2}}{\sinh^2 x} \right],$$

and it satisfies

$$f(\theta) f(\theta + i\pi) = \frac{\sinh\theta}{\sinh\theta + i \sin\frac{\pi B}{2}} \ . \tag{32}$$

The sinh-Gordon theory can be restricted to the negative half-line, but the integrability is maintained only by imposing either the Dirichlet

$$\Phi(0,t) = \Phi_0^D$$

or the two parameter family of perturbed Neumann

$$V_B(\Phi(0,t)) = M_0 \cosh\left(\frac{b}{2}(\Phi(0,t) - \Phi_0)\right) - M_0$$

boundary conditions. The latter interpolates between the Neumann and the Dirichlet boundary conditions, since for $M_0 = 0$ we recover the Neumann, while for $M_0 \to \infty$ the Dirichlet boundary condition with $\Phi_0^D = \Phi_0$. The reflection factor which depends on two continuous parameters can be written as

$$R(\theta) = R_0(\theta) R(E,F,\theta) = \left(\frac{1}{2}\right)\left(\frac{1}{2} + \frac{B}{4}\right)\left(-1 - \frac{B}{4}\right)\left[\frac{E-1}{2}\right]\left[\frac{F-1}{2}\right]$$

in terms of the parameterization used in [20]. They are related to the parameters of the Lagrangian as

$$\begin{aligned} \cos\frac{E}{16}(b^2+8\pi)\cos\frac{F}{16}(b^2+8\pi) &= \frac{M_0}{M_{crit}}\cosh\frac{b\Phi_0}{2} \\ \sin\frac{E}{16}(b^2+8\pi)\sin\frac{F}{16}(b^2+8\pi) &= -\frac{M_0}{M_{crit}}\sinh\frac{b\Phi_0}{2} \end{aligned}$$

where $M_{crit} = m\sqrt{\frac{2}{b^2\sinh(b^2/8)}}$. Note that for generic values of the parameters ($E \neq 0$, $F \neq 0$) this reflection factor has a pole at $\theta = i\pi/2$ coming from the $\left(\frac{1}{2}\right)$ factor. Imposing Dirichlet boundary condition instead of the general one corresponds to removing the F dependent factor from $R(\theta)$. Then the remaining parameter E is related to the Φ_0 boundary value of the sinh-Gordon field as $E = i8b\Phi_0/(b^2+8\pi)$.

3.4.1 Sinh-Gordon model with $\Phi_0 = 0$ Dirichlet b.c.

This case is interesting because $E = 0$ implies that the pole at $\theta = i\pi/2$ is *absent* in this case. Therefore the equation coming from the residue of this pole is also absent and the

form factors are less restricted. The 1PFF corresponding to the reflection amplitude on the $E=0$ Dirichlet boundary is

$$r_0(\theta) = \frac{\sinh\theta}{\sinh\theta + i} u(\theta, B),$$

where

$$u(\theta, B) = \exp\left[-2\int_0^\infty \frac{dx}{x}\left[\cos(\frac{i\pi}{2}-\theta)\frac{x}{\pi} - 1\right]\frac{\cosh\frac{x}{2}}{\sinh^2 x}\left(\sinh\frac{xB}{4} + \sinh(1-\frac{B}{2})\frac{x}{2} + \sinh\frac{x}{2}\right)\right].$$

Note that $r_0 \sim y$ at $y \to \infty$ and has no pole at $\theta = i\pi/2$. At $B=0$ - which corresponds to a free theory - $u(\theta, 0)$ can be integrated explicitly yielding $r_0(\theta)|_{B=0} = (-i\sinh\theta)/2$; and this, apart from a trivial normalizational phase coincides with the 1PFF for a free scalar with Dirichlet b.c. (As discussed in Section 3.1.1 in this case $\partial_x\Phi(0,0)$ is the operator having one particle matrix element only).

We write the n particle form factors in the general form:

$$F_n(\theta_1, \ldots, \theta_n) = H_n Q_n(y_1, \ldots, y_n) \prod_i r_0(\theta_i) \prod_{i<j} \frac{f(\theta_i - \theta_j) f(\theta_i + \theta_j)}{y_i + y_j}.$$

Since there is no self fusing pole in the S matrix of the sinh-Gordon model, the F_n-s are related only by the kinematical singularity equations. Choosing the ratio of the H_n-s appropriately the recursion equations originating from here take the form:

$$Q_{n+2}(-y, y, y_1, \ldots, y_n) = -Q_n(y_1, \ldots, y_n)\, P_n,$$

where P_n is given by eq.(25,26) with $\omega = e^{i\pi\frac{B}{2}}$.

As r_0 has no pole at $\theta = i\pi/2$, one can have a minimal solution of this recursion equations starting with $Q_1 = 1$ which has non vanishing form factors for *odd* particle numbers. We calculated up to $n=5$ and found that the solution is uniquely given by

$$Q_1(y_1) = 1, \qquad Q_3(y_1, y_2, y_3) = -\sigma_1,$$

$$Q_5(y_1, \ldots, y_5) = \sigma_1[\sigma_3\sigma_2 - (\omega+\omega^{-1})^2\sigma_5 + (\omega-\omega^{-1})^4\sigma_1 - (\omega-\omega^{-1})^2(\sigma_3 + \sigma_1\sigma_2)],$$

with all the F_1, F_3 and F_5 form factors having leading overall degree one. There is a unique local operator with this property, namely $\partial_x\Phi$.

Of course one can also find non vanishing solutions with *even* particle numbers also starting with a non trivial Q_2. Since F_2 can have no kinematical singularity, the minimal choice is $Q_2(y_1, y_2) = \sigma_1$. With this input we obtained again a unique solution

$$Q_2(y_1, y_2) = \sigma_1, \qquad Q_4(y_1, \ldots, y_4) = \sigma_1^2(\sigma_2 - (\omega-\omega^{-1})^2),$$

where both the F_2 and the F_4 have leading overall degree two.

For $\Phi_0 = 0$ ($E=0$) the $\Phi \to -\Phi$ reflection symmetry of the bulk sinh-Gordon model survives also in the boundary theory. Therefore the boundary operators can be classified as even or odd ones, having only non-vanishing even or odd particle form factors, respectively. Thus the second form factor family can be identified with the operator $(\partial_x\Phi)^2$.

3.4.2 Sinh-Gordon model with $\Phi_0 \neq 0$ Dirichlet b.c.

For $\Phi_0 \neq 0$ ($E \neq 0$) the reflection factor acquires a pole at $\theta = i\pi/2$ due to the fact that the field has a nontrivial vacuum configuration. At the same time the reflection symmetry of the bulk sinh-Gordon model is violated in the boundary theory. Therefore the boundary operators cannot be classified into representations of this symmetry, and the equation coming from the residue of the pole at $\theta = i\pi/2$ connects the form factors with even and odd particle numbers. Note that now this equation plays an essential role as it is the only one that relates these two sets of form factors to each other.

The 1PFF corresponding to the reflection amplitude on the $E \neq 0$ Dirichlet boundary is

$$r_E(\theta) = \frac{\sinh\theta}{\sinh\theta - i\sin\gamma} u(\theta, B) \quad , \qquad \gamma = \frac{\pi}{2}(E-1),$$

where $u(\theta, B)$ is the same as in the previous case. Note that $r_E \sim y$ at $y \to \infty$.

Writing the n particle form factors in the general form

$$F_n(\theta_1, \ldots, \theta_n) = H_n Q_n(y_1, \ldots, y_n) \prod_i \frac{r_E(\theta_i)}{y_i} \prod_{i<j} \frac{f(\theta_i - \theta_j) f(\theta_i + \theta_j)}{y_i + y_j},$$

and choosing the ratio of the H_n-s appropriately the recursion equations originating from the kinematical singularity equation take the form:

$$Q_{n+2}(-y, y, y_1, \ldots, y_n) = (y^2 - 4\cos^2\gamma) Q_n(y_1, \ldots, y_n)\, P_n, \tag{33}$$

where P_n is given by eq.(25,26) with $\omega = e^{i\pi\frac{B}{2}}$. Next we show how the equation coming from the residue of the pole at $\theta = i\pi/2$:

$$-i \operatorname*{Res}_{\theta = i\pi/2} F_{n+1}(\theta, \theta_1, \ldots, \theta_n) = \frac{g_E}{2} \left(1 - \prod_{j=1}^{n} S(i\frac{\pi}{2} - \theta_j) \right) F_n(\theta_1, \ldots, \theta_n), \tag{34}$$

helps to eliminate the arbitrariness in the minimal solution of the recursion equations.

$Q_1(y_1) = \sigma_1$ is the minimal choice that guarantees that F_1 has no pole at $\theta = i\pi/2$. Using this in the recursion equation (33) gives that the most general Q_3 has the form:

$$Q_3(y_1, y_2, y_3) = -\sigma_1^2(\sigma_2 + 4\cos^2\gamma) + (A + B\sigma_1)(\sigma_1\sigma_2 - \sigma_3),$$

where A and B are arbitrary constants. Eq.(34) leads to the following relation between Q_3 and Q_2:

$$H_3 r_E(i\frac{\pi}{2}) Q_3(0, y_2, y_3) = g_E 2\sin\frac{B\pi}{2} H_2 \sigma_1(y_2, y_3) Q_2(y_2, y_3).$$

Since

$$Q_3(0, y_2, y_3) = \sigma_1 \left\{ -\sigma_1(\sigma_2 + 4\cos^2\gamma) + (A + B\sigma_1)\sigma_2 \right\},$$

the expression in the curly bracket should be proportional to Q_2. This observation fixes the values of A and B: Q_2 has to be proportional to σ_1 to guarantee that F_2 has no kinematical

singularity and this requirement is met only if $A = 0$, while F_2 has a leading degree not exceeding that of F_1 and F_3 if $B = 1$. Thus with these two requirements one obtains a parameter free solution starting with $Q_1 = \sigma_1$; up to $n = 4$ it has the form:

$$Q_2(y_1, y_2) \sim -4\cos^2\gamma\sigma_1, \quad Q_3(y_1, y_2, y_3) = -\sigma_1\sigma_3 - 4\cos^2\gamma\sigma_1^2,$$

$$Q_4(y_1, \ldots, y_4) \sim -4\cos^2\gamma(\sigma_1\sigma_3 + 4\cos^2\gamma\sigma_1^2)(\sigma_2 + 4\sin^2\frac{\pi B}{2}).$$

Note that both Q_2 and Q_4 vanish for $\gamma = -\pi/2$ ($E = 0$). Furthermore for $E = 0$ one also has $Q_1/y_1 = 1$, and $Q_3/(y_1y_2y_3) = -\sigma_1$, thus the solution goes over smoothly into the one with $E = 0$. Since

$$r_E(i\frac{\pi}{2}) = \frac{1}{1-\sin\gamma}, \qquad \text{and} \qquad g_E = \frac{2(1+\cos\frac{\pi B}{4}+\sin\frac{\pi B}{4})}{\sqrt{\sin\frac{\pi B}{2}}}\frac{\cos\gamma}{1-\sin\gamma},$$

the γ dependence cancels from the ratios of H_3/H_1 and H_4/H_2 when we use eq.(34) for $n = 1, 2, 3$:

$$-i\frac{H_4}{H_2} = -i\frac{H_3}{H_1} = \left(1+\cos\frac{\pi B}{4}+\sin\frac{\pi B}{4}\right)^2 4\sin\frac{\pi B}{2}.$$

In the $b \to 0$ limit these ratios vanish, therefore the higher form factors decouple in accord with the fact that the kinematical singularity axiom becomes trivial for the free field theory.

4 Conclusion

In this paper we treated the form factor bootstrap for boundary operators in integrable boundary quantum field theory. Although there have been earlier treatment of form factors for specific (mainly lattice) models [10, 11, 12], none of these has actually given a complete formulation similar to the axiomatic approach by Smirnov for the bulk case [3]. The present work initiates an extension of this axiomatic program to boundary fields.

We have given a complete axiomatization of the properties of boundary form factors, derived from first principles of quantum field theory (unitarity and the boundary extension of the LSZ reduction formulae). In particular, the axiom describing boundary kinematic singularities is an entirely new result of this paper, as this has never been treated before in any previous study. We have shown that these axioms are consistent with many known aspects of integrable boundary field theory. In particular, the relation between the residue of the reflection factor at $i\pi/2$ and the one-particle contribution to the boundary state, noted previously in the context of finite size effects, was confirmed once more as a necessary condition for the consistency between the boundary and the bulk kinematical axiom (the only exception to this relation is when the bulk is free, but then the two axioms are trivial). Therefore it seems that this particular relation is a consequence of integrability and the existence of a nontrivial bulk scattering matrix.

We then proceeded to give a systematic method to solve the boundary form factor axioms for the case of diagonal scattering. The solution is a natural generalization of the

bulk case, but necessitates the introduction of a minimal boundary form factor function in addition to the already known minimal bulk form factor. The periodicity, permutation and reflection axioms can then be solved by a general Ansatz, and the residue axioms can be recast as recursion relations for certain *polynomial* functions which characterize the form factor solution completely.

In particular, we treated the case of the free boson and the free fermion (noncritical Ising model with boundary magnetic field), where the polynomial solutions of the form factor axioms were shown to be identical to the explicit solutions obtained from the field theory, and it was also shown that the polynomial solutions of the form factor bootstrap match the full boundary operator content expected from the Lagrangian approach.

As example for the interacting case, we first treated the Lee-Yang model, where the boundary kinematical singularity axiom makes its first appearance, and it is very important in order to distinguish between boundary conditions that have different conformal limits. We have also computed the spectral expansion of the two-point correlation function for the operator with the lowest conformal dimension and have shown that it matches perfectly with the ultraviolet expansion of the same correlation function obtained from boundary conformal field theory.

Our second interacting example is the sinh-Gordon model, with Dirichlet boundary condition (an extension to the general case is in principle straightforward, but we decided to treat only the Dirichlet case to keep it short and simple). The boundary conditions with zero and with nonzero value of the field on the boundary are differentiated again by the boundary kinematical axiom, and we have shown that the results of the boundary form factor bootstrap fit perfectly well with expectations from the Lagrangian approach.

An open question is to find and classify non-minimal solutions of the form factor equations and interpret them in terms of the local boundary operator algebra of the underlying field theory, extending the method presented for the bulk sinh-Gordon model in [21]. In particular it is interesting to find out whether the counting of operators in the conformal limit can be matched with the full set of solutions in the interacting case.

It is obvious that the results presented in this paper can be applied directly to any integrable boundary quantum field theory for which the factorized scattering theory is known, and that they formulate a well-defined program to determine form factors and correlation functions of boundary operators, similar to the approach used in the bulk case. We have also shown how to solve the axioms for theories with diagonal bulk and boundary scattering.

It is an interesting problem to extend these results to the case of nondiagonal scattering (with boundary sine-Gordon theory as the most prominent example). The extension of the axioms is straightforward: they must be decorated by multiplet indices, just like in the bulk case, although here we avoided to give this extension explicitly to keep the exposition simple. However, solving them will probably encounter much more difficulties, and just as in the bulk, new methods must be devised for the task, like the boundary extension of the Lukyanov free field representation in [12].

The comparison to the Lagrangian and perturbed conformal field theory description would be greatly facilitated by establishing sum rules for the spectral representation of

the boundary correlators, similar to the c-theorem [22] and Δ-theorem [23] in the bulk case, and is one of the most important problems left open by the present work. Another promising open direction is to consider possible applications of boundary form factors and correlation functions in the area of boundary quantum field theory and condensed matter.

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A Heuristic derivation of the FF axioms

We present some heuristic arguments, along the lines of [3], for the derivation of boundary form factor axioms using the boundary reduction formula [9].

We analyze the analyticity properties of the form factor

$$F_n^{\mathcal{O}} := F_n^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n) = \langle 0|\mathcal{O}(0)|\theta_1, \theta_2, \ldots, \theta_n\rangle_{in}$$

as a function of the variable θ_1.

We follow the notations of [9]: The asymptotic creation/annihilation operators can be expressed in terms of the free asymptotic fields as

$$\begin{aligned} a_{in}(\theta) &= 2i \int_{-\infty}^{0} dx \cos(k(\theta)x) e^{i\omega(\theta)t} \overleftrightarrow{\partial}_t \Phi_{in}(x,t) \\ a_{in}^{+}(\theta) &= -2i \int_{-\infty}^{0} dx \cos(k(\theta)x) e^{-i\omega(\theta)t} \overleftrightarrow{\partial}_t \Phi_{in}(x,t) \ . \end{aligned} \tag{35}$$

where $k(\theta) = m \sinh\theta$ and $\omega(\theta) = m\cosh\theta$. The in state is a free state and we have

$$\langle 0|\mathcal{O}(0)|\theta_1, \theta_2, \ldots, \theta_n\rangle_{in} = \langle 0|\mathcal{O}(0) a_{in}^{+}(\theta_1)|\theta_2, \ldots, \theta_n\rangle_{in} \tag{36}$$

We use (35) together with

$$\mathcal{O}(0)\Phi_{in}(x,t) = [\mathcal{O}(0), \Phi_{in}(x,t)] + \Phi_{in}(x,t)\mathcal{O}(0)$$

to obtain

$$F_n^{\mathcal{O}} = \text{disc.} - 2i \int_{-\infty}^{0} dx \cos(k(\theta_1)x) e^{-i\omega(\theta_1)t} \overleftrightarrow{\partial}_t \langle 0|[\mathcal{O}(0), \Phi_{in}(x,t)]|\theta_2, \ldots, \theta_n\rangle_{in} \ . \tag{37}$$

where the disconnected part is

$$\text{disc.} = \langle 0|a_{in}^{+}(\theta_1)\mathcal{O}(0)|\theta_2, \ldots, \theta_n\rangle_{in} = \langle 0|a_{in}^{+}(\theta_1)|0\rangle F_{n-1}^{\mathcal{O}}(\theta_2, \ldots, \theta_n)$$

Note that in theories with nonzero vacuum expectation values of the field Φ the matrix element $\langle 0|a^+_{in}(\theta_1)|0\rangle$ is nonzero and can be written as

$$\langle 0|a^+_{in}(\theta_1)|0\rangle = \frac{g}{2}2\pi\delta(\theta_1 - \frac{i\pi}{2})$$

which corresponds to the one particle term in the boundary state in the crossed channel [5]. It was conjectured in [7] and later confirmed using TBA arguments [24] that the one particle contribution to the boundary state has a coefficient equal to $\frac{g}{2}$ rather than g as suggested in [5]. In the channel we use here this translates directly into the equation above.

Supposing that the *in* field can be expressed in terms of the interacting field as $\Phi(x,t) \to Z^{1/2}\Phi_{in}(x,t)$ for $t \to -\infty$ and that $[\mathcal{O}(0), \Phi(x,0)] = 0$, the connected part can be written in the form

$$\text{conn.} = iZ^{-1/2}2\int_{-\infty}^{0} dx \int_{-\infty}^{0} dt\partial_t \left\{ \cos(k(\theta_1)x)e^{-i\omega(\theta_1)t}\overleftrightarrow{\partial}_t \langle 0|[\mathcal{O}(0), \Phi(x,t)]|\theta_2, \ldots, \theta_n\rangle_{in} \right\}$$

Performing the usual partial integration while taking care of the surface term we obtain

$$\text{conn.} = iZ^{-1/2}2\int d^2x e^{-i\omega(\theta_1)t}\cos(k(\theta_1)x)\Theta(-t)\langle 0|[\mathcal{O}(0), J(x,t)]|\theta_2, \ldots, \theta_n\rangle_{in} \qquad (38)$$

where $J(x,t) = \{\Box + m^2 + \delta(x)\partial_x\}\Phi(x,t)$ and the integration goes over the entire spacetime. The range of the integration is the interior of the past light cone due to the presence of $\Theta(-t)$ and of the vanishing of $[\mathcal{O}(0), J(x,t)]$ on space-like intervals. The analytic properties of the integral are determined by the exponent for large negative times. The exponent decreases if $\Im m(\omega(\theta_1)) > 0$ thus the *in* form factor ($\theta_1 > \theta_2 > \cdots > \theta_n > 0$) allows an analytical continuation into the domain:

$$0 < \Im m(\theta_1) < \pi$$

Repeating the same procedure for the *out* matrix elements

$$F^{\mathcal{O}}_n(-\theta_1, -\theta_2, \ldots, -\theta_n) = \langle 0|\mathcal{O}(0)| - \theta_1, -\theta_2, \ldots, -\theta_n\rangle_{out}$$

we obtain the domain of analytical continuation: $0 < \Im m(-\theta_1) < \pi$.

To derive the crossing relation we consider the following matrix element

$$F^{\mathcal{O}}_{1n-1} := F^{\mathcal{O}}_{1n-1}(\theta_1|\theta_2, \ldots, \theta_n) = {}_{in}\langle\theta_1|\mathcal{O}(0)|\theta_2, \ldots, \theta_n\rangle_{in}$$

Applying the reduction formula to the particle with rapidity θ_1 (35) we obtain

$$F^{\mathcal{O}}_{1n-1} = \text{disc} - 2i\int_{-\infty}^{0} dx \cos(k(\theta_1)x)e^{i\omega(\theta_1)t}\overleftrightarrow{\partial}_t \langle 0|[\mathcal{O}(0), \Phi_{in}(x,t)]|\theta_2, \ldots, \theta_n\rangle_{in}$$

where the disconnected part (supposing $\theta_1 \geq \theta_2$) is

$$\text{disc.} = \langle 0|\mathcal{O}(0)a^+_{in}(\theta_1)|\theta_2, \ldots, \theta_n\rangle_{in} = {}_{in}\langle\theta_1|\theta_2\rangle_{in}F^{\mathcal{O}}_{n-2}(\theta_3, \ldots, \theta_n)$$

Performing a partial integration as before the result for the connected component is

$$\text{conn.} = iZ^{-1/2} 2 \int d^2x e^{i\omega(\theta_1)t} \cos(k(\theta_1)x)\Theta(-t)\langle 0|[\mathcal{O}(0), J(x,t)]|\theta_2, \ldots, \theta_n\rangle_{in} \tag{39}$$

which has an analytic continuation for

$$-\pi < \Im m(\theta_1) < 0$$

Comparing (38) with (39) and using that $m\cosh(\theta_1 + i\pi) = -m\cosh\theta_1$ the crossing relation (3) is proved. Similar result can be obtained for an *out* state and the $-\theta_1 < -\theta_2 < \cdots < -\theta_n < 0$ range of the parameters.

The reflection property (Axiom II) can be shown by considering

$$F_n^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n) = \langle 0|\mathcal{O}(0)|\theta_1, \theta_2, \ldots, \theta_n\rangle_{in}$$

and crossing all particles except the one with rapidity θ_n to the left. Now inserting a complete set of *out* states we have

$${}_{in}\langle \ldots |\mathcal{O}(0)|\theta_n\rangle_{in} = \sum_{out} {}_{in}\langle \ldots |A(0)|n\rangle_{out\ out}\langle n|\theta_n\rangle_{in}$$

where only the first two terms are nonzero:

$${}_{in}\langle \ldots |\mathcal{O}(0)|\theta_n\rangle_{in} = {}_{in}\langle \ldots |A(0)|0\rangle\langle 0|\theta_n\rangle_{in} + \sum_{\theta} \langle \ldots |A(0)|\theta\rangle_{out\ out}\langle \theta|\theta_n\rangle_{in}$$

The connected part gives the required R factor while the disconnected one combined with the disconnected part in (37) and the permutation property gives the boundary kinematical singularity.

The permutation property in the bulk case is usually derived from very similar argumentation we used above for showing the reflection property. Note, however, that the same result can be obtained from the analysis of the singularity structure of the Green functions: the part, which is responsible for the discontinuity in the form factor by changing two neighboring rapidities, is related to the bulk S-matrix. The permutation property in the boundary case (Axiom I) can be derived only from the second approach, namely from a detailed investigation of the singularity structure of the Green functions. By extending the result on the two point function in [9] one can show that multi point functions have momentum preserving parts identical to their bulk counterparts and exactly these parts contribute only, when two neighboring (both positive) rapidities are changed, and cause the same discontinuity in the form factor we met in the bulk case.

The kinematical singularity equation (Axiom IV) can be obtained (using the permutation and reflection axioms) from the analysis of the disconnected components in the crossing relations as obtained for the *in* and for the *out* states:

$$F_{1n-1}^{\mathcal{O}}(\pm\theta_1| \pm \theta_2, \ldots, \pm\theta_n) = F_n^{\mathcal{O}}(i\pi \pm \theta_1, \pm\theta_2, \ldots, \pm\theta_n) + 2\pi\delta(\theta_1 - \theta_2)F_{n-2}^{\mathcal{O}}(\pm\theta_3, \ldots, \pm\theta_n)$$

Although our derivation of the boundary form factor axioms is heuristic to some extent we expect that the formulation of the same ideas in a rigorously defined quantum field theoretical framework would lead to the proper and mathematically founded derivation (but note that this has not been performed yet in the bulk case either).

B Formal derivation of the FF axioms

Here we show how the boundary form factor axioms can be formally derived from a boundary analogue of the Faddeev-Zamolodchikov algebra.

In the bulk case one introduces creation $Z^*(\theta)$ and annihilation $Z(\theta)$ operators corresponding to asymptotic states. They are defined for real rapidities $\theta \in R$ and satisfy the following defining relations

$$\begin{aligned} Z^*(\theta_1)Z^*(\theta_2) &= S(\theta_1-\theta_2)Z^*(\theta_2)Z^*(\theta_1) \\ Z(\theta_1)Z(\theta_2) &= S(\theta_1-\theta_2)Z(\theta_2)Z(\theta_1) \\ Z(\theta_1)Z^*(\theta_2) &= S(\theta_2-\theta_1)Z^*(\theta_2)Z(\theta_1) + 2\pi\delta(\theta_1-\theta_2) \end{aligned} \tag{40}$$

Extending Z, Z^* to imaginary rapidities (treating θ as a $2\pi i$ periodic complex variable) we encounter singularities in their products at $\theta_1 = \theta_2 \pm i\pi$ with residues

$$-i \operatorname*{Res}_{\theta_1=\theta_2+i\pi} Z^*(\theta_1)Z^*(\theta_2) = 1$$

and

$$-i \operatorname*{Res}_{\theta_1=\theta_2+i\pi} Z(\theta_1)Z(\theta_2) = 1$$

These can be formulated by postulating the crossing property

$$Z(\theta) = Z^*(\theta + i\pi)$$

and taking into account the defining relations (40). We note that using this identification all of the defining relations (40) can be combined into a single one

$$Z^*(\theta_1)Z^*(\theta_2) = S(\theta_1-\theta_2)Z^*(\theta_2)Z^*(\theta_1) + 2\pi\delta(\theta_1-\theta_2-i\pi)$$

In the boundary case the generators Z, Z^* are defined only for positive values of the rapidity arguments and additionally two new formal generators are introduced creating the boundary vacuum as follows

$$|0\rangle_B = B^*|0\rangle \quad , \qquad {}_B\langle 0| = \langle 0|B$$

We introduce two new relations

$$Z^*(\theta)B^* = R(\theta)Z^*(-\theta)B^*$$

and

$$BZ(\theta) = BZ(-\theta)R(-\theta)$$

which describe how we can extend the generators for negative rapidities. By analytically continuing in the rapidity again we have singularities in the operator products

$$-i \operatorname*{Res}_{\theta=i\frac{\pi}{2}} Z^*(\theta)B^* = \frac{g}{2}B^*$$

and

$$i \operatorname*{Res}_{\theta=-i\frac{\pi}{2}} BZ(\theta) = \frac{g}{2}B$$

They would correspond to particles with real rapidity in the crossed channel (exchanging time and space coordinates). These new relations can again be summarized in a single one

$$Z^*(\theta)B^* = R(\theta)Z^*(-\theta)B^* + 2\pi\delta(\theta - \frac{i\pi}{2})\frac{g}{2}B^*$$

together with its formal conjugate

$$BZ(\theta) = BZ(-\theta)R(-\theta) + 2\pi\delta(\theta + \frac{i\pi}{2})\frac{g}{2}B$$

We claim that representing the form factor of the boundary operator $\mathcal{O}(0)$ as

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) = \langle 0|B\,\mathcal{O}(0)\,Z^*(\theta_1)\ldots Z^*(\theta_n)B^*|0\rangle$$

and supposing locality

$$[\mathcal{O}(0), Z^*(\theta)] = 0$$

we can recover all the non-singularity type form factor axioms immediately. For deriving the singularity axioms we have to observe that singularity appears not only from a single term. E.g. in the boundary kinematical singularity axiom, the form factor $F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n)$ exhibits a singularity in θ_1 at $i\frac{\pi}{2}$ coming from two places: the operator product of both B and B^* with $Z^*(\theta_1)$ is singular. Supposing that they appear in additive terms of the form factor we can obtain the desired formula.

Finally, we note that formulating the boundary FZ algebra in the spirit of [25] might lead to a more rigorous derivation of our axioms.

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