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Boundary finite size corrections for multiparticle states and planar AdS/CFT

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Abstract

We propose formulas for the Lüscher type finite size energy correction of multiparticle states on the interval and evaluate them for the simplest case in the AdS/CFT setting. By this we determine the leading wrapping correction to the anomalous dimension of the simplest determinant type operator, which corresponds to a one particle state on the $Y = 0$ brane.

1 Introduction

In the last few years much progress has been made in computing the spectrum of anomalous dimensions of planar $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory (SYM). The progress relied partly on the AdS/CFT correspondence [1, 2, 3] between this theory and the type IIB closed string theory on the $AdS_5 \times S^5$ background and partly on the integrability properties of both theories [4, 5, 6, 7, 8, 9, 10]. In particular it is accepted that the spectrum of anomalous dimensions of single trace operators containing an asymptotically large number of elementary SYM fields (which corresponds to the energy spectrum of strings moving freely on $AdS_5 \times S^5$ with large angular momentum) is fully determined by the system of asymptotic Bethe Ansatz equations (ABA) [11].

In extending these computations to operators with finite length "wrapping effects" [12], missing from the ABA, must be taken into account. On the string theory side they are related to vacuum polarization effects and, as was shown by Lüscher [13], can be described by the infinite volume scattering data. The Lüscher correction is just the leading term of a systematical expansion, which is summed up in the Thermodynamic Bethe Ansatz (TBA) program. The idea that the TBA approach can be applied to the superstring sigma model was advocated in [14]. The appropriate generalization of Lüscher's idea managed to describe the four [15] and five [16] loop corrections to the anomalous dimension of the Konishi operator in complete agreement with the direct gauge theory computations, whenever they were available [17, 18, 19]. This idea proved to be very useful to calculate the anomalous dimensions of various operators in the perturbative regime [20, 21, 22, 23, 24, 25]. A valid description to all couplings is based on the TBA description [26, 27, 28, 29, 30, 31, 32], which, as it is expected, in the weakly coupled regime reproduces the results of the generalized Lüscher correction [33, 34, 35, 36].

Parallel to these developments the study of finite size effects for determinant type operators or equivalently for open strings has also been started. After exhibiting the classical [37] and weak coupling [38] integrability of various open string models Hofman and Maldacena argued [39] that open strings on the $AdS_5 \times S^5$ background attached to the $Y = 0$ or to the $Z = 0$ giant graviton branes are integrable at all values of the coupling. The boundary version of the asymptotic BA have been worked out in [40, 41] for the $Y = 0$ brane, while in [42] for the $Z = 0$ brane.

Using the AdS/CFT generalization [43] of the boundary state formalism [44] the authors of [45] adopted the Lüscher type boundary finite size energy correction [46, 47] for the worldsheet QFT on a strip with width L to study the ground states of the $Y=0$ and $Z=0$ branes. As the $Y=0$ ground-state is left invariant by some supersymmetry transformation the finite size energy corrections vanish for all order of the perturbation theory, showing also that the corresponding determinant operator is protected. In contrast, in the $Z=0$ setting the vacuum transforms nontrivially under the symmetry leaving a room for finite size corrections. The authors of [45] managed to calculate the leading wrapping type boundary correction and also checked against their direct gauge theory results. The aim of the present paper is to extend their results for excited states.

In general we aim to derive the multiparticle generalization of Lüscher's formula for the boundary setting and apply them for magnons reflecting on the $Y=0$ brane. In the scattering description the $Y=0$ brane is much simpler than the $Z=0$ brane since the corresponding reflection factor is diagonal, opposed to the other case, where it is highly nondiagonal. In particular we would like to evaluate Lüscher's correction to the energy of a single magnon moving freely on a strip of width L reflecting on the boundaries. The Lüscher correction to the energy gives the leading exponential correction in L, and can naturally be converted into the leading wrapping correction to the anomalous dimension of the corresponding operator.

The paper is organized as follows: In the next Section we introduce the gauge invariant determinant-type operators, whose anomalous dimensions we are aiming to calculate. We will focus on a single impurity type operator which correspond to a one-particle state on the string side reflecting between two boundaries. We calculate its anomalous dimension in two different ways: from the spin-chain description originating from perturbative Feynman diagrams and from the integrable asymptotical boundary Bethe Ansatz. As the boundary BA is asymptotical we analyze next its Lüscher type correction. We propose expressions for the leading Lüscher-type energy corrections in Section 3 for both relativistic and non-relativistic theories. The non-relativistic expressions are then evaluated at leading order in Section 4 for the simplest nontrivial operator. Finally we conclude in Section 5 and outline some open problems. Some technical details on how we determined the boundary state is put into Appendix A, while in Appendix B we recall the explicit S-matrix elements we used to calculate the finite size corrections. Finally, for sake of completeness, we give the full weak coupling solution of the one particle boundary BY equation in Appendix C.

2 Determinant-type operators and the BBY equation

In this section we present the operator in the gauge theory description whose anomalous dimensions we are going to calculate. We calculate its perturbative anomalous dimension both from the dilatation operator and from the boundary Bethe-Yang equation of the string theory description.

2.1 Gauge theory description

In the gauge theory description the ground state of the $Y=0$ brane corresponds to the operator

$$\epsilon^{j_1\dots j_{N-1}j_N}_{i_1\dots i_{N-1}i_N} Y^{i_1}_{j_1}\dots Y^{i_{N-1}}_{j_{N-1}}(Z^J)^{i_N}_{j_N}$$

while the excitations we consider correspond to replacing one of the Z-s by an impurity χ:

$$\mathcal{O}_Y(Z^k\chi Z^{J-k-1}) = \epsilon^{j_1\dots j_{N-1}j_N}_{i_1\dots i_{N-1}i_N} Y^{i_1}_{j_1}\dots Y^{i_{N-1}}_{j_{N-1}}(Z^k\chi Z^{J-k-1})^{i_N}_{j_N} \tag{1}$$

The sets of fields inside $(\dots)^{i_N}_{j_N}$ constitute the states of the open spin chain. We keep the length of the chain (the number of fields inside $(\dots)^{i_N}_{j_N}$) finite.

The ground state for finite N is not supersymmetric [48]. In the planar limit ($N\to\infty$) however, what we are analyzing in this paper, integrability shows up and the supersymmetry of the ground-state seems to be restored. As a consequence, the anomalous dimension of the corresponding operator vanishes: indeed this was shown perturbatively up to two loops in [39] while at the level of the leading exponential corrections in [45].

We consider the $SU(2)$ sector first. In this case the impurity is given by Y. The fields at the two ends of the chain cannot be Y's since the operator then would factorize into a determinant and a single trace. Therefore Y can only occupy the 'internal' positions of the chain; to describe this we denote the first position of the chain by index 0 and the last one by $J-1$ when the Y can occupy $J-2$ different positions. Of course the total length of the chain is J. Furthermore we introduce the abbreviation $\mathcal{O}_Y(Y_j)$ for an operator of the form in (1) with Y standing at the j-th position in $(Z^{j-1}YZ^{J-j})^{i_N}_{j_N}$.

The final form of the integrable two loop Hamiltonian in the $SU(2)$ subsector of the $Y=0$ brane is given in [39]:

$$H=(2g^2-8g^4)\sum_{i=1}^{J-3}(\mathrm{I}-P_{i,i+1})+2g^4\sum_{i=1}^{J-4}(\mathrm{I}-P_{i,i+2})+(2g^2-4g^4)(q_1^Y+q_{J-2}^Y)+2g^4(q_2^Y+q_{J-3}^Y)$$

where $P_{i,k}$ is the permutation operator between sites i and k and q_i^Y acts as the identity if the field at the i-th position is Y and as zero if it is not.

The shortest conceivable string accommodating a one particle excitation is ZYZ having $J=3$. It is clear that only the third term of H has a non trivial action on $\mathcal{O}_Y(Y_1)$ and the corresponding energy eigenvalue is

$$\Delta_3=4g^2-8g^4$$

As will see later this corresponds to a particle of momentum $p=\frac{\pi}{2}$, see (2) below.

For $J=4$ one can look for the eigenstates of the Hamiltonian in the form $\psi(1)\mathcal{O}_Y(Y_1)+\psi(2)\mathcal{O}_Y(Y_2)$. In this case also the first and last terms of H give contributions and the eigenvalue equations have the form:

$$H\Psi=\Delta_4\Psi=\begin{pmatrix}4g^2-10g^4 & -2g^2+8g^4\\ -2g^2+8g^4 & 4g^2-10g^4\end{pmatrix}\begin{pmatrix}\psi(1)\\ \psi(2)\end{pmatrix}$$

Comparing the solution of the eigenvalue equation to [39] using $\psi(j)\sim\sin(jp)$ $j=1,2$ we see that one gets the first eigenvalue when $\psi(1)=\psi(2)$ i.e. when $p=\pi/3$ and then

$$\Delta_4^+=2g^2-2g^4=8g^2\sin^2(\pi/6)-32g^4\sin^4(\pi/6)$$

In a similar way the condition to get the second eigenvalue is $\psi(1)=-\psi(2)$ yielding $p=2\pi/3$ and also in this case

$$\Delta_4^-=6g^2-18g^4=8g^2\sin^2(\pi/3)-32g^4\sin^4(\pi/3)$$

2.2 Boundary Bethe Yang equations

Having only one impurity in the chain of Z-s corresponds to an excitation (magnon) moving freely between two boundaries and reflecting on them. The anomalous dimension of an operator of the form (1) is related to the bulk energy $E(p)=\sqrt{1+16g^2\sin^2(\frac{p}{2})}$ of the magnon as

$$\Delta_n=E(p_n)-1=8g^2\sin^2(\frac{p_n}{2})-32g^4\sin^4(\frac{p_n}{2})+\dots \tag{2}$$

where p_n are the discrete values of the momenta restricted by the BBY equation and the set of Δ_n should coincide with the (expansion of the) energy eigenvalues of the spin chain Hamiltonian with the two boundaries.

To describe the BBY equation we consider two boundaries labeled by α,β at a distance L and a particle (magnon) that propagates freely between them while undergoing nontrivial reflections at the two ends. The reflections of a magnon with momentum p, carrying the fundamental representation of $su(2|2)\otimes su(2|2)$, is described by the following matrices

$$\mathbb{R}_\alpha(-p)=\mathbb{R}_\beta(p)=\mathbb{R}(p)=R_0(p)\mathrm{diag}\left(-e^{i\frac{p}{2}},e^{-i\frac{p}{2}},1,1\right)\otimes\mathrm{diag}\left(-e^{i\frac{p}{2}},e^{-i\frac{p}{2}},1,1\right)$$

where the first two components correspond to the bosonic while the last two, (in which the reflection is trivial), to the fermionic components of the representation. The scalar factor can be determined from the boundary crossing unitarity equation [39, 49, 50] to be

$$R_0(p)=-e^{-ip}\sigma(p,-p) \tag{3}$$

where σ stands for the dressing phase [51], and e^{-ip} is a CDD factor which we fixed from the weak coupling limit.

The BBY equation encodes the periodicity of the particle's wavefunction[1]:

$$e^{-2ipL}\mathbb{R}_\alpha(-p)\mathbb{R}_\beta(p) \equiv e^{-2ip(L+1)}\sigma(p,-p)^2\mathrm{diag}(e^{ip},e^{-ip},1,1)\otimes\mathrm{diag}(e^{ip},e^{-ip},1,1) = 1 \quad (4)$$

Clearly the two pieces of the scalar factor effect the BBY equation and consequently the allowed momenta of the reflecting magnons in a different way: while up to g^6 one can forget about the dressing factor, the exponential factor effectively shifts the width of the strip by one unit. We analyze the general solution of the equation in Appendix C.

Here we analyze the weak coupling solutions of the BBY (4) up to the order of g^4. Since p must be in the range $0 < p < \pi$, its allowed values for a magnon with labels (11), which corresponds to the Y type impurity are as follows:

$$p_n = n\frac{\pi}{L}, \quad n = 1,\dots L-1$$

Now we can compare these momenta with the ones obtained from analysing the spin chain Hamiltonian. We can see that if $L = J-1$ then the two sets of energies are identical. (We also verified this for the $SU(3)$ subsector spanned by the three scalar fields W, Z and Y). This way we demonstrated that the weak coupling limit of the solutions of the BBY equations matches with the results of the spin chain calculations in both the $SU(2)$ and in the $SU(3)$ sectors.

2.3 The dressing factor and higher order weak coupling solutions

In the higher orders in g the dressing factor also effects the solutions of the BBY. Using the explicit form presented in [52] one finds

$$\sigma(p,-p) = e^{-ig^6 2^8\zeta(3)\sin^5\left(\frac{p}{2}\right)\cos\left(\frac{p}{2}\right)+\mathcal{O}(g^8)} = 1 - ig^6 2^8\zeta(3)\sin^5\left(\frac{p}{2}\right)\cos\left(\frac{p}{2}\right) + \mathcal{O}(g^8)$$

Writing the momentum of the magnon with labels (11) as

$$p = p_n + \delta p = n\frac{\pi}{L} + \delta p$$

in (4) yields

$$\delta p = -g^6\frac{2^8}{L}\zeta(3)\sin^5\left(\frac{p_n}{2}\right)\cos\left(\frac{p_n}{2}\right) + \mathcal{O}(g^8)$$

Since in the dispersion relation the momentum dependence is multiplied by g^2 , the shift in δp effects only the 8-th order term:

$$E(p_n) - 1 = 8g^2\sin^2\left(\frac{p_n}{2}\right) - 32g^4\sin^4\left(\frac{p_n}{2}\right) + 256g^6\sin^6\left(\frac{p_n}{2}\right)$$
$$-g^8\big(2560\sin^8\left(\frac{p_n}{2}\right) + \frac{2^{11}}{L}\zeta(3)\sin^6\left(\frac{p_n}{2}\right)\cos^2\left(\frac{p_n}{2}\right)\big) + \dots$$

Thus for a (11) magnon on the shortest possible strip - i.e. when $L = 2$ ($p_1 = \pi/2$) we find

$$E\left(\frac{\pi}{2}\right) - 1 = 4g^2 - 8g^4 + 32g^6 - g^8(160 + 64\zeta(3)) + \dots$$

The BBY equations (4) determine the *polynomial* finite size corrections only and next we consider the leading *exponential* corrections (known as Lüscher corrections). In particular we are interested in the corrections of the energy of a one particle state. The Lüscher corrections to the ground state energies for both the $Y = 0$ and the $Z = 0$ branes were determined already in [45]).

The analytic expression for the Lüscher correction to the energy of any excited state is not known in general even in the case of ordinary relativistic theories. Therefore, first we propose such a general expression which we extract/conjecture from studying certain excited state TBA and NLIE equations derived in some specific relativistic boundary theories. Then we generalize this to the AdS/CFT context on the $Y = 0$ brane.

[1] In the AdS conventions we use the inverse of the scattering matrix (compared to the relativistic case) in writing the ABA equations $e^{ip_jL} = \prod_k \mathbb{S}(p_j,p_k)$. To keep this convention we also write the boundary ABA into this convention.

3 Boundary finite size corrections for multiparticle states

In this Section we propose expressions for the leading finite size corrections to the BBY energy of multiparticle states on the strip. After reviewing the analog proposal for multiparticle states on the circle we formulate our conjecture in the relativistic boundary setting. Having confirmed the proposed expressions against exactly known integral equations we extend them to the non-relativistic realm valid also for the AdS/CFT correspondence.

3.1 Integrable systems on the circle

Suppose we analyze a system of particles interacting via a relativistically invariant integrable interaction.

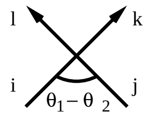
Infinite volume characteristics

In infinite volume the system is characterized by the dispersion relation

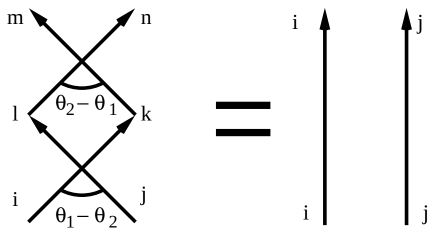
$$E(p_i) = \sqrt{m_i^2 + p_i^2} \qquad \leftrightarrow (E(\theta_i), p(\theta_i)) = m_i(\cosh\theta_i, \sinh\theta_i)$$

and the factorized scattering matrix

$$\mathbb{S}(\theta_1 - \theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)$$

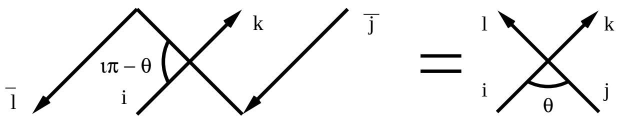


which satisfies unitarity



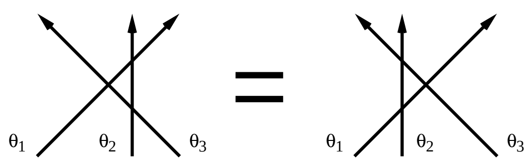
$$S_{ij}^{kl}(\theta_1 - \theta_2) S_{lk}^{nm}(\theta_2 - \theta_1) = \delta_i^m \delta_j^n$$

crossing symmetry



$$S_{\bar{l}i}^{\bar{j}k}(i\pi - \theta) = S_{ij}^{kl}(\theta)$$

and the Yang-Baxter equation



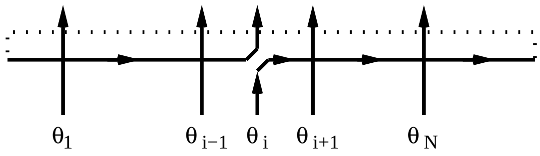
$$\mathbb{S}(\theta_2 - \theta_3)\mathbb{S}(\theta_1 - \theta_3)\mathbb{S}(\theta_1 - \theta_2) = \mathbb{S}(\theta_1 - \theta_2)\mathbb{S}(\theta_1 - \theta_3)\mathbb{S}(\theta_2 - \theta_3)$$

Asymptotically large volume spectrum

In finite but large volume, L, the energy level of an N particle state can be described up to exponentially small corrections as the sum of the individual energies

$$E(\theta_1,\dots,\theta_N)=\sum_i m_i\cosh\theta_i$$

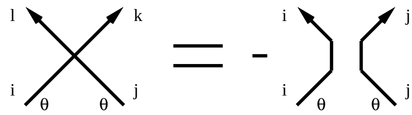
Here the rapidities are constrained by the Bethe-Yang (BY) equations



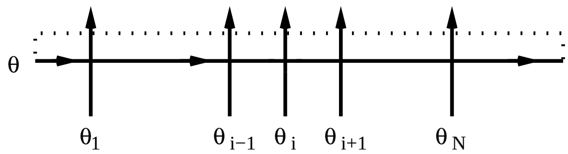
$$e^{ip(\theta_i)L}\mathbb{S}(\theta_i-\theta_{i+1})\dots\mathbb{S}(\theta_i-\theta_N)\mathbb{S}(\theta_i-\theta_1)\dots\mathbb{S}(\theta_i-\theta_{i-1})=\mathbb{I}\quad;\qquad i=1,\dots,N$$

where $\mathbb{S}$ denotes the full S-matrix and their product has to be diagonalized. If the S-matrix is regular, i.e. reduces to the permutation operator at vanishing arguments

$$\mathbb{S}(0)=-\mathbb{P},\quad S_{ij}^{kl}(0)=-\delta_i^l\delta_j^k$$



then the BY equation can be nicely formulated in terms of the asymptotic Y function which is defined by means of the transfer matrix



$$\mathbb{T}(\theta|\theta_1,\dots,\theta_N)=\mathrm{Tr}(\mathbb{S}(\theta-\theta_1)\dots\mathbb{S}(\theta-\theta_N))$$

If we denote the eigenvalue of the transfer matrix by $t(\theta|\theta_1,\dots,\theta_N)$ then the corresponding Y function can be written as

$$Y_{as}(\theta|\theta_1,\dots,\theta_N)=e^{ip(\theta)L}t(\theta|\theta_1,\dots,\theta_N)$$

and the BY equation takes a particularly simple form

$$Y_{as}(\theta_i|\theta_1,\dots,\theta_N)=-1$$

The asymptotic Y-function is also relevant in describing the leading exponentially small correction to the BY energies as we recall now.

Lüscher-type finite size corrections

The BY energies contain all polynomial finite size corrections in the inverse of the volume [13] and there is a systematic expansion for the additional exponentially small corrections which are organized according to their exponents. The leading corrections contain two terms: the so called integral or F-term and the residue or μ-term [53]. As the μ term is simply the residue of the integral term in the following we focus on the integral term only. In [15] a formula was proposed to describe the integral term of the leading exponential correction. It consists of two parts, the first directly changes the energy in the form

$$\Delta E = -\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\partial_\theta p(\theta)\, Y_{as}(\theta + i\frac{\pi}{2}|\theta_1,\dots,\theta_N)$$

and contains the vacuum polarization effects. The other one describes how the finite volume vacuum changes the momentum quantization (or other words the BY) equations:

$$\log Y_{as}(\theta_i|\theta_1,\dots,\theta_N) - \pi(2n+1) = -\partial_{\theta_i}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\, Y_{as}(\theta + i\frac{\pi}{2}|\theta_1,\dots,\theta_N)$$

These formulas have been tested against the available exact integral equations in diagonal theories like sinh-Gordon and Lee-Yang models in [15] and for non-diagonal theories in [54, 55, 35] and in both cases perfect agreement have been found.

Non-relativistic models

If the system is not relativistically invariant in infinite volume then the above formulas have to be modified. The main difference is that the scattering matrix no longer depends on the difference of the (generalized) rapidities u_i, rather, it depends individually on the two rapidities $\mathbb{S}(u_i, u_j)$. Still unitarity $\mathbb{S}(u_1, u_2) = \mathbb{S}(u_2, u_1)^{-1}$, crossing symmetry $\mathbb{S}^{c_1}(u_1, u_2) = \mathbb{S}(u_2, u_1 - \omega)$ (for some crossing parameter ω) and regularity ($\mathbb{S}(u, u) = -\mathbb{P}$) are supposed. Here $\mathbb{S}^{c_1}$ is charge conjugated in the first particle only and in the relativistic case $\omega = i\pi$.

In describing the finite volume spectrum similar formulas can be introduced as in the relativistic case. The transfer matrix is defined to be

$$\mathbb{T}(u|u_1,\dots,u_N) = \mathrm{Tr}(\mathbb{S}(u, u_1)\dots\mathbb{S}(u, u_N))$$

and with its eigenvalue $t(\theta|\theta_1,\dots,\theta_N)$ the corresponding asymptotic Y function can be written as

$$Y_{as}(u|u_1,\dots,u_N) = e^{ip(u)L}t(u|u_1,\dots,u_N)$$

From the regularity of the scattering matrix the BY equation follows

$$Y_{as}(u_i|u_1,\dots,u_N) = -1$$

Moreover, the direct finite size energy correction has a similar form as we have in the relativistic case:

$$\Delta E = -\int_{-\infty}^{\infty}\frac{du}{2\pi}\partial_u\tilde{p}(u)\, Y_{as}(u + \frac{\omega}{2}|u_1,\dots,u_N)$$

where the rapidity variable u has been analytically continued into its mirror domain: $u \to u + \frac{\omega}{2}$. The mirror theory can be obtained from the original theory as follows: first we define the Euclidean version of the model by analytically continuing in the time variable $t = iy$ and considering space x and imaginary time y on an equal footing. This Euclidean theory can be considered as an analytical continuation of another theory, in which x serves as the analytically continued time $x = i\tau$ and y is the space coordinate. The theory defined in terms of y, τ is called the mirror theory and its dispersion relation can be obtained by the same analytical continuation $E = i\tilde{p}$ and $p = i\tilde{E}$ see [56] in the AdS/CFT setting. In the general rapidity formulation we suppose the mirror theory can be described by the $u \to u + \frac{\omega}{2}$ shift: $\tilde{E}(u) = -ip(u + \frac{\omega}{2})$ and $\tilde{p}(u) = -iE(u + \frac{\omega}{2})$.

What is really difficult to figure out is the modification of the BA. In the paper [15] a special case was analyzed and the following form was proposed

$$\log Y_{as}(u_i|u_1,\dots,u_N) - \pi(2n+1) = \int_{-\infty}^{\infty}\frac{du}{2\pi}\, t^{'}(u + \frac{\omega}{2}|u_1,\dots,u_N)e^{ip(u+\frac{\omega}{2})L}$$

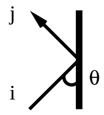
where $t^{'}(u|u_1,\dots,u_N)$ is the eigenvalue of $\mathrm{Tr}(\mathbb{S}(u, u_1)\dots(\partial_u\mathbb{S}(u, u_i))\dots\mathbb{S}(u, u_N))$ which was supposed to act diagonally on the multiparticle state whose energy correction we are calculating. These formulas[2] have been used in the AdS/CFT realm at five loop [16, 24] and compared to the TBA equations in [34, 35] where exact agreement have been found.

[2]To take into account the supersymmetric nature of the model the trace Tr()has to be replaced by the supertrace sTr().

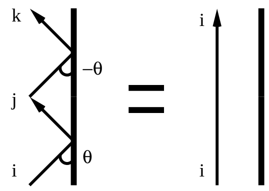
3.2 Integrable systems on the strip

A relativistic integrable *boundary* system in infinite volume is defined on the negative *half* line ($x \leq 0$) only and characterized, additionally to the dispersion relation and the two particle scattering matrix, by the one particle *reflection* matrix

$$\mathbb{R}(\theta) = R_i^j(\theta)$$

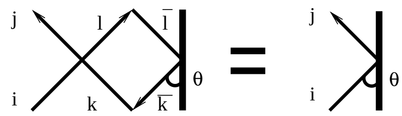


which satisfies unitarity



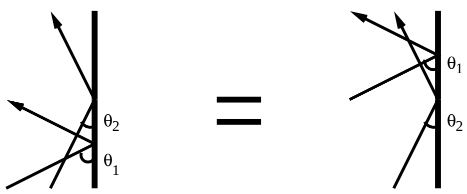
$$R_i^j(\theta)R_j^k(-\theta) = \delta_i^k \tag{5}$$

boundary crossing unitarity [44]



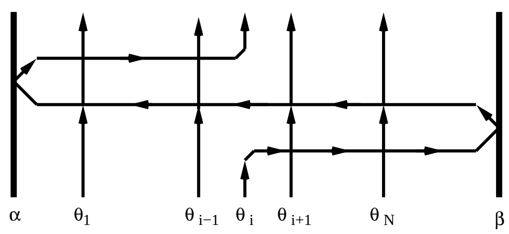
$$S_{ik}^{lj}(2\theta)R_{\bar{k}}^{\bar{l}}(i\pi - \theta) = R_i^j(\theta) \tag{6}$$

and the boundary Yang-Baxter equation.



$$\mathbb{S}(\theta_1 - \theta_2)\mathbb{R}(\theta_1)\mathbb{S}(\theta_1 + \theta_2)\mathbb{R}(\theta_2) = \mathbb{R}(\theta_2)\mathbb{S}(\theta_1 + \theta_2)\mathbb{R}(\theta_1)\mathbb{S}(\theta_1 - \theta_2) \tag{7}$$

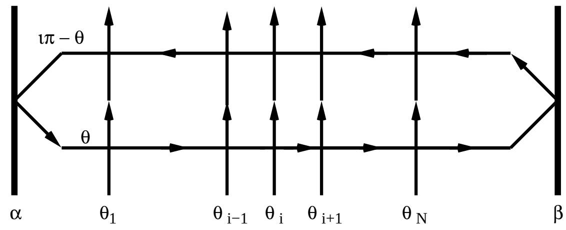
A finite volume boundary system of size L has two boundaries with left reflection factor $\mathbb{R}_\alpha$ and right reflection factor $\mathbb{R}_\beta$. When a particle with positive rapidity $\theta > 0$ reflects back from the right boundary with reflection factor $\mathbb{R}_\beta(\theta)$ it will reach the other boundary with $-\theta$. It is a standard convention to denote the left reflection factor of the particle with rapidity $-\theta$ by $\mathbb{R}_\alpha(\theta)$ as in this case $\mathbb{R}_\alpha(\theta)$ satisfies the same equations (5,6,7) as $\mathbb{R}_\beta(\theta)$ if the S-matrix is parity invariant, which is usually the case. The energy levels of a multiparticle state on the interval can be approximately described by the boundary BY (BBY) equations:



$$e^{2ip(\theta_i)L}\prod_{j=i+1}^{N}\mathbb{S}(\theta_i-\theta_j)\mathbb{R}_\beta(\theta_i)\prod_{j=N}^{1}\mathbb{S}(\theta_j+\theta_i)\mathbb{R}_\alpha(\theta_i)\prod_{j=1}^{i-1}\mathbb{S}(\theta_i-\theta_j)=\mathbb{I}\quad;\quad\theta_i>0\;;\;i=1,\dots,N$$

where similarly to the periodic case the product of reflection and scattering matrices have to be diagonalized for each i. The energy of the solution $E=\sum_{i=1}^{N}E(\theta_i)$ contains all the polynomial corrections in L^{-1}.

Just as in the periodic case this can be nicely derived by introducing the boundary analog of the transfer matrix, which is called the double-raw transfer matrix[3]



$$\mathbb{T}(\theta|\theta_1,\dots,\theta_N)=\mathrm{Tr}\left(\prod_{j=1}^{N}\mathbb{S}(\theta-\theta_j)\mathbb{R}_\beta(\theta)\prod_{j=N}^{1}\mathbb{S}(\theta_j+\theta)\mathbb{R}^c_\alpha(i\pi-\theta)\right)$$

where $\mathbb{R}^c$ is the charge conjugated reflection factor $\mathbb{R}^c=\mathbb{C}\mathbb{R}\mathbb{C}^{-1}$. Observe that $\mathbb{R}^c_\alpha(i\pi-\theta)$ satisfies the consistency equations (5,6,7), whenever $\mathbb{R}_\alpha(\theta)$ satisfies them. The reason why we have to use the reflection factor $\mathbb{R}^c_\alpha(i\pi-\theta)$ on the left boundary is that we would like to obtain the BBY equation. In specifying the spectral parameter to any of the particles' rapidity we have an extra scattering matrix $\mathbb{S}(2\theta_i)$ which should combine into $\mathbb{S}(2\theta_i)\mathbb{R}^c_\alpha(i\pi-\theta_i)=\mathbb{R}_\alpha(\theta)$.[4]

Let us also mention that using the crossing symmetry of the scattering matrix we might write the transfer matrix in an alternative way:

$$\mathbb{T}(\theta|\theta_1,\dots,\theta_N)=\mathrm{Tr}\left(\prod_{j=1}^{N}\mathbb{S}(\theta-\theta_j)\mathbb{R}_\beta(\theta)\prod_{j=N}^{1}\mathbb{S}^c(i\pi-\theta-\theta_j)\mathbb{R}^c_\alpha(i\pi-\theta)\right)$$

where $\mathbb{S}^c$ is charge conjugated in the auxiliary space only. This is the form of the double raw transfer matrix what is frequently used in boundary lattice models, see [59].

Due to the BYB and YB equations the transfer matrices commute for different θs [57]. The asymptotic Y-function is defined after the diagonalization of this family of double-raw transfer matrices. If the eigenvalue is denoted by $t(\theta|\theta_1,\dots,\theta_N)$ then the asymptotic Y-function is simply

$$Y_{as}(\theta|\theta_1,\dots,\theta_N)=e^{2ip(\theta)L}\,t(\theta|\theta_1,\dots,\theta_N)$$

One can check that, using the regularity of the scattering matrix, the boundary crossing unitarity of the reflection factor and the bulk YB equations, the BBY equation takes the same form as in the periodic case:

$$Y_{as}(\theta_i|\theta_1,\dots,\theta_N)=-1$$

What is really nice in this formulation is that the exponentially small finite size corrections can be described by exactly the same formulas we had in the periodic case. The vacuum polarization effects the energy of the N particle state as

$$\Delta E=-\int_0^\infty\frac{d\theta}{2\pi}\partial_\theta p(\theta)\,Y_{as}(\theta+i\frac{\pi}{2}|\theta_1,\dots,\theta_N)\tag{8}$$

[3]The concept of the double raw transfer matrix was introduced in integrable boundary spin chains in [57].

[4]A similar quantity was used in boundary kink theories to derive the BBY equation from a double raw transfer matrix in [58].

where due to the presence of the boundary we integrate only for positive momentum particles.

The leading finite size correction of the vacuum energy was derived in [46] and checked against exact integral equations. Evaluating Y_{as} for the vacuum state

$$Y_{as}^{vac}(\theta+\frac{i\pi}{2})=\mathrm{Tr}(\mathbb{R}_{\alpha}^{c}(\frac{i\pi}{2}-\theta)\mathbb{R}_{\beta}(\frac{i\pi}{2}+\theta))e^{-2mL\cosh\theta}$$

we can see that the generic formula (8) reproduces the vacuum result [46]. In the case of excited states the modification of the BBY equations are

$$\log(Y_{as}(\theta_i|\theta_1,\dots,\theta_N))-\pi(2n+1)=-\partial_{\theta_i}\int_0^{\infty}\frac{d\theta}{2\pi}\,Y_{as}(\theta+i\frac{\pi}{2}|\theta_1,\dots,\theta_N)$$

We have checked that these formulas correctly reproduce the leading exponential finite size corrections in the Lee-Yang and sinh-Gordon and for certain states in the sine-Gordon models with Dirichlet boundary conditions where exact integral equations were available [60, 46, 61, 62].

In a nonrelativistic boundary theory these formulas have to be modified. As the scattering matrix depends individually on its arguments the boundary crossing equation is modified:

$$\mathbb{R}(u)=\mathbb{S}(u,-u)\mathbb{R}^{c}(\omega-u)$$

where $\mathbb{R}^c=\mathbb{C}\mathbb{R}\mathbb{C}^{-1}$. The BYBE takes the form

$$\mathbb{S}(u_1,u_2)\mathbb{R}(u_1)\mathbb{S}(u_2,-u_1)\mathbb{R}(u_2)=\mathbb{R}(u_2)\mathbb{S}(u_2,-u_1)\mathbb{R}(u_1)\mathbb{S}(u_1,u_2)$$

Still one can define the double-raw transfer matrix

$$\mathbb{T}(u|u_1,\dots,u_N)=\mathrm{Tr}\left(\prod_{j=1}^{N}\mathbb{S}(u,u_j)\mathbb{R}_{\beta}(u)\prod_{j=N}^{1}\mathbb{S}(u_j,-u)\mathbb{R}_{\alpha}^{c}(\omega-u)\right)$$

which commutes for different spectral parameters u. The crossing symmetry of the bulk scattering matrix provides an equivalent formula

$$\mathbb{T}(u|u_1,\dots,u_N)=\mathrm{Tr}\left(\prod_{j=1}^{N}\mathbb{S}(u,u_j)\mathbb{R}_{\beta}(u)\prod_{j=N}^{1}\mathbb{S}^{c}(\omega-u,u_j)\mathbb{R}_{\alpha}^{c}(\omega-u)\right)$$

Using the eigenvalue of the transfer matrix one can define the asymptotic Y- function

$$Y_{as}(u|u_1,\dots,u_N)=e^{2ip(u)L}t(u|u_1,\dots,u_N)$$

which can be used to describe the BBY equations

$$Y_{as}(u_i|u_1,\dots,u_N)=-1$$

The energy correction is expected to be

$$\Delta E=-\int_0^{\infty}\frac{du}{2\pi}\partial_u\tilde{p}(u)\,Y_{as}(u+\frac{\omega}{2}|u_1,\dots,u_N) \tag{9}$$

For the vacuum state the correction reduces to $Y_{as}^{vac}(u+\frac{\omega}{2})=\mathrm{Tr}(\mathbb{R}_{\alpha}^{c}(\frac{\omega}{2}-u)\mathbb{R}_{\beta}(\frac{\omega}{2}+u))e^{2iLp(u+\frac{\omega}{2})}$. In this case there is no BBA equation to be modified. For a general multiparticle state the modification of the BBA is conjectured to be

$$\log Y_{as}(u_i|u_1,\dots,u_N)-\pi(2n+1)=\int_0^{\infty}\frac{du}{2\pi}\,t^{'}(u+\frac{\omega}{2}|u_1,\dots,u_N)e^{2ip(u+\frac{\omega}{2})L}$$

where $t^{'}(u|u_1,\dots,u_N)$ is nothing but the eigenvalue of the operator

$$\mathrm{Tr}(\mathbb{S}(u,u_1)\dots(\partial_u\mathbb{S}(u,u_i))\dots\mathbb{S}(u,u_N)\mathbb{R}_{\beta}(u)\prod_{j=N}^{1}\mathbb{S}(u_j,-u)\mathbb{R}_{\alpha}^{c}(\omega-u))$$

on the state under investigation and we supposed, similarly to the periodic case, that the eigenstate of $\mathbb{T}$ is also an eigenstate of this operator.

4 Lüscher-type correction in AdS/CFT

In this section we elaborate the previously conjectured finite size energy correction formulas for a one particle state in the AdS/CFT setting. In particular we focus on the $Y = 0$ brane.

Lüscher-type correction

In order to make connection to the general description we recall that the dispersion relation of the Q magnon bound states

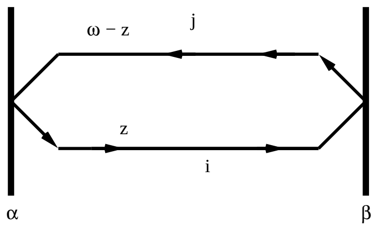
$$E^2 - 16g^2 \sin^2 \frac{p}{2} = Q^2$$

can be uniformized on the torus with parameter z in terms of the Jacobi amplitudes as [63, 56]:

$$p = 2 \ \mathrm{am}(z,k) \quad ; \qquad E = Q\,\mathrm{dn}(z,k) \quad ; \qquad k = -16\frac{g^2}{Q^2}$$

The real period of the torus is $2\omega_1 = 4K(k)$ and the imaginary period is $2\omega_2 = 4iK(1-k) - 4K(k)$. The crossing parameter in this theory is the half of the imaginary period $\omega = \omega_2$ and z plays the role of the generalized rapidity, in terms of which the S-matrix satisfies, unitarity $\mathbb{S}(z_1,z_2)\mathbb{S}(z_2,z_1) = \mathbb{I}$, crossing symmetry $\mathbb{S}^{c_1}(z_1,z_2) = \mathbb{S}(z_2, z_1+\omega_2)$ and the YBE [64]. As we already mentioned, in the AdS literature we use a different convention for the scattering matrix, in which the ABA takes the form $e^{-ip_jL}\prod_k \mathbb{S}(p_j,p_k) = 1$. This means that instead of continuing to $u \to u + \frac{\omega}{2}$ we have to continue the result to $z \to z - \frac{\omega_2}{2}$.

Taking this into account as a first application we take formula (9) and evaluate for the $\omega \to -\omega_2$ continuation and describe the energy correction of the vacuum



$$\Delta E(L) = -\sum_Q \int_0^{\frac{\omega_1}{2}} \frac{dz}{2\pi}(\partial_z \tilde{p}_Q(z))\mathbb{R}_i^j(-\frac{\omega_2}{2}+z)\mathbb{C}_{j\bar{j}}\mathbb{R}_{\bar{i}}^{\bar{j}}(-\frac{\omega_2}{2}-z)\mathbb{C}^{i\bar{i}}e^{-2\tilde{\epsilon}_Q L} \qquad (10)$$

where we have to sum over the full infinite spectrum of the mirror theory. This expression was evaluated in [45]. Let us recall their results: first the reflection factors of the mirror boundstates have to determined. According to the tensor product nature of the mirror bound-states the reflection factor can be factorized as

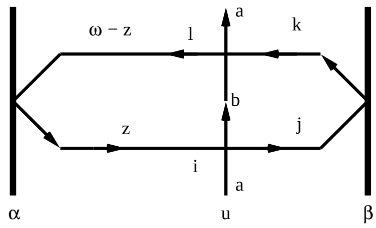
$$\mathbb{R}(z) = R_0(z)R(z) \otimes R(z)$$

Each $su(2|2)$ factor can be further decomposed with respect to the unbroken $su(2)$ symmetry as $(Q+1; Q-1; Q; Q)$ where Q is the charge of the bound-state. Interestingly the unbroken $su(2|1)$ symmetry turns out to be restrictive enough to fix the matrix part of the reflection factor completely:

$$R(z) = \mathrm{diag}(\mathbb{I}_{Q+1}; -\mathbb{I}_{Q-1}; -e^{i\frac{p}{2}}\mathbb{I}_Q, e^{-i\frac{p}{2}}\mathbb{I}_Q)$$

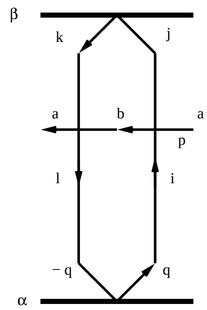
This is in stark contrast to the case of physical bound-states which transform under the totally symmetric representations. Indeed there the unbroken $su(2|1)$ does not fix completely the matrix part and higher symmetries as the Yangian have to be used. See [65, 66] for an exhaustive analysis of the $Q = 2$ case. The scalar part of the bound-state reflection factor R_0 can be fixed from the fusion principle and will be evaluated below. The nonzero matrix elements of the fundamental charge conjugation matrix are $\mathbb{C}_{12} = -\mathbb{C}_{21} = -i$ and $\mathbb{C}_{34} = -\mathbb{C}_{43} = 1$ from which it can easily be extended to any representation. Evaluating the Lüscher correction (10) with the analytically continued bound-state reflection factors gives vanishing result (independently of R_0) which is consistent with the unbroken supersymmetry of the vacuum.

Let us turn to the description of the finite size energy correction of a one particle state. As the dispersion relation contain a factor g^2 in front of $\sin^2\left(\frac{p}{2}\right)$, the weak coupling expansion of the finite size corrections appearing in the momentum quantization $p \to p + \delta p$ will be suppressed by one order compared to the direct energy corrections. Consequently the leading wrapping correction according to (8) turns out to be



$$\Delta E_a(L) = -\sum_Q \int_0^{\frac{\omega_1}{2}} \frac{dz}{2\pi} (\partial_z \tilde{p}_Q(z)) \mathbb{S}_{ia}^{jb}(\frac{\omega}{2}+z,u)\mathbb{R}_j^k(\frac{\omega}{2}+z)\mathbb{S}_{lb}^{ka}(\frac{\omega}{2}-z,u)\mathbb{C}^{l\bar{l}}\mathbb{R}_{\bar{l}}^{\bar{i}}(\frac{\omega}{2}-z)\mathbb{C}_{\bar{i}i}e^{-2\tilde{\epsilon}_Q L} \quad (11)$$

where as we mentioned $\omega = -\omega_2$. An alternative formula can be obtained by working directly in the mirror theory where the same contribution can be depicted as



$$\Delta E_a(L) = -\sum_Q \int_0^\infty \frac{dq}{2\pi} \mathbb{K}^{\bar{l}i}(q)\mathbb{S}_{ia}^{jb}(q,p)\bar{\mathbb{K}}_{j\bar{k}}(q)\mathbb{S}_{\bar{l}b}^{\bar{k}a}(-q,p)e^{-2\tilde{\epsilon}_Q L} \quad (12)$$

and a refers to the particle type whose energy correction we are calculating. This expression contains the boundary state amplitudes, $\mathbb{K}^{ij}(q)$ for each bound-state, which are related to the reflection factors by analytical continuation $\mathbb{K}^{ij}(z) = \mathbb{C}^{i\bar{i}}\mathbb{R}_{\bar{i}}^j(\frac{\omega}{2} - z)$. For convenience we write the formula in terms of the momentum of the mirror particles $q = \tilde{p}_Q$.

Now we turn to the evaluation of the Lüscher formulas. Both the reflection factors (boundary state amplitudes) and the scattering matrices can be factorized according to the two to identical $su(2|2)$ "color" factors:

$$\mathbb{S}(q,p) = S_0(q,p)\,S(q,p) \otimes S(q,p) \quad ; \quad \mathbb{K}(q) = K_0(q)\,K(q) \otimes K(q)$$

In this decomposition the energy correction can be written as

$$\Delta E(L) = -\sum_Q \int_0^\infty \frac{dq}{2\pi} K_0(q)S_0(q,p)\bar{K}_0(q)S_0(-q,p)\left[\mathrm{Tr}\left(\bar{K}(q)S(q,p)K(q)S(-q,p)\right)\right]^2 e^{-2\tilde{\epsilon}_Q L} \quad (13)$$

where we calculate the Lüscher correction for a particle state labeled by $a = (11)$ in (12). This means that the operator $\mathrm{Tr}(\bar{K}(q)S(q,p)K(q)S(-q,p))$ acts diagonally on the one particle states and we simply take its eigenvalue on the state labeled by a. We also note, that due to the particular definition of $\mathbb{K}$ and the charge conjugated $\bar{\mathbb{K}}$ (or the definition the double raw transfer matrix containing $\mathbb{R}_\alpha^c(\omega - u)$) we do not have to use supertrace. Alternatively, one can avoid the charge conjugation on the right boundary and use the graded double raw transfer matrix, see [67] for a discussion of this issue.

As we are calculating the leading wrapping correction we expand all functions in leading order in g. The expression $\tilde{\epsilon}_Q$ denotes the mirror energy of the charge Q bound-state which in the rapidity parametrization has the following weak coupling expansion:

$$e^{-\tilde{\epsilon}_Q} = \frac{4g^2}{q^2+Q^2} + O(g^4)$$

In the formula (12) we have to sum also over all polarization of the mirror bound-states. Let us focus on one copy of the two $su(2|2)$ factors. The bound-states can be labeled as follows: we decompose the $4Q$ dimensional completely antisymmetric representation space as $4Q = (Q+1)+(Q-1)+Q+Q$ and parametrize the sub-spaces in the superfield formalism [56, 15] by

$$\begin{aligned}
Q+1 \longrightarrow |j\rangle^1 &= \frac{1}{\sqrt{(Q-j)!j!}} w_3^{Q-j} w_4^j && ; \quad j = 0,1,\ldots,Q \\
Q-1 \longrightarrow |j\rangle^2 &= \frac{1}{\sqrt{(Q-2-j)!j!}} w_3^{Q-2-j} w_4^j \theta_1\theta_2 && ; \quad j = 0,1,\ldots,Q-2 \\
Q \longrightarrow |j\rangle^3 &= \frac{1}{\sqrt{(Q-1-j)!j!}} w_3^{Q-1-j} w_4^j \theta_1 && ; \quad j = 0,1,\ldots,Q-1 \\
Q \longrightarrow |j\rangle^4 &= \frac{1}{\sqrt{(Q-1-j)!j!}} w_3^{Q-1-j} w_4^j \theta_2 && ; \quad j = 0,1,\ldots,Q-1
\end{aligned}$$

where we payed attention to the proper normalization of the states. In this basis the boundary state amplitudes have the following nonzero matrix elements

$$K^{11}_{j,Q-j} = (-1)^j \,; \quad K^{22}_{j,Q-2-j} = -(-1)^j \,; \quad K^{34}_{j,Q-1-j} = i(-1)^j e^{\frac{\tilde{\epsilon}_Q}{2}} \,; \quad K^{43}_{j,Q-1-j} = i(-1)^j e^{-\frac{\tilde{\epsilon}_Q}{2}}$$

where the upper index refers to the subspace, while the lower labels the state within. This form of the boundary state amplitudes follows from requiring its vanishing under the unbroken $su(1|2)$ symmetry, as we explicitly show in Appendix A. The conjugated boundary states read as

$$\bar{K}^{11}_{j,Q-j} = (-1)^j \,; \quad \bar{K}^{22}_{j,Q-2-j} = -(-1)^j \,; \quad \bar{K}^{43}_{j,Q-1-j} = i(-1)^j e^{\frac{\tilde{\epsilon}_Q}{2}} \,; \quad \bar{K}^{34}_{j,Q-1-j} = i(-1)^j e^{-\frac{\tilde{\epsilon}_Q}{2}}$$

Thus basically a $3 \leftrightarrow 4$ change is made.

The energy correction (12,13) contains also the scattering of the mirror bound-states with the fundamental physical particle. To describe this scattering matrix we collect the relevant coefficient from [15] in Appendix B. Using the non-vanishing S-matrix elements from the Appendix of [15] the contributions of the various subspaces can be written as:

- the $Q+1$ dimensional contribution

$$\sum_{j=0}^{Q} \bar{K}^{11}_{j,Q-j} \left[a_5^5(p,-q) K^{11}_{j,Q-j} a_5^5(p,q) - \frac{1}{2} a_5^6(p,-q) K^{34}_{j,Q-1-j} a_2^3(p,q) \right]$$

- the $Q-1$ dimensional subspace contributes as

$$\sum_{j=0}^{Q-2} \bar{K}^{22}_{j,Q-2-j} \left[2a_8^8(p,-q) K^{22}_{j,Q-2-j} 2a_8^8(p,q) + \frac{Q}{Q-1} a_8^7(p,-q) K^{34}_{j,Q-1-j} a_4^3(p,q) \right]$$

- the Q dimensional subspace with index 3 as

$$\sum_{j=0}^{Q-1} \bar{K}^{34}_{j,Q-1-j} \left[a_9^9(p,-q) K^{34}_{j,Q-1-j} \frac{1}{2} (a_9^9(p,q) + a_3^3(p,q)) \right]$$

finally the Q dimensional subspace with index 4 as

$$\sum_{j=0}^{Q-1}\bar{K}^{43}_{j,Q-1-j}\Big[\frac{1}{2}(a_9^9(p,-q)+a_3^3(p,-q))K^{43}_{j,Q-1-j}a_9^9(p,q)$$
$$+\frac{1}{2}(a_9^9(p,-q)-a_3^3(p,-q))K^{34}_{j,Q-1-j}\frac{1}{2}(a_9^9(p,q)-a_3^3(p,q))$$
$$-\frac{Q+1}{2Q}a_2^3(p,-q)K^{11}_{j,Q-j}a_5^6(p,q)+a_4^3(p,-q)K^{22}_{j,Q-2-j}a_8^7(p,q)\Big]$$

where the S^{kl}_{ij} S-matrix elements are obtained by multiplying the a^n_m projector coefficients of [15] by $(-1)^{\epsilon_i\epsilon_j}$. We use the explicit expressions of $a^j_i(p,q)$ from Appendix B together with the parametrization:

$$x^{\pm}(u)=\frac{2u\pm i}{4g}\left[1+\sqrt{1-\frac{16g^2}{(2u\pm i)^2}}\right]\quad;\qquad z^{\pm}(q)=\frac{q+iQ}{4g}\left[\sqrt{1+\frac{16g^2}{Q^2+q^2}}\pm 1\right]$$

and make the weak coupling expansion of each term. Here u parametrizes the physical momentum $p(u)$, and stands for the usual rapidity variable of AdS/CFT and not for the generalized one z (for which the crossing equation is valid). The leading order contribution vanishes and for the first non-vanishing contribution we obtain

$$\mathrm{Tr}\left(\bar{K}(q)S(q,u)K(q)S(-q,u)\right)=-\frac{4Q(q^2+Q^2+1+4u^2)(q^2+Q^2-1-4u^2)}{(q^2+(-1+Q-2iu)^2)(2u+i)^3(2u-i)}$$

The scalar part of the scattering matrix between the charge Q mirror bound-state and the fundamental physical particle according to [51] can be written as

$$S_0(q,u)=\frac{(x^+-z^-)^2(-1+x^-z^-)}{(x^--z^+)(x^+-z^+)(-1+x^+z^-)}\tilde{\Sigma}_{Q1}(q,u)$$

where $\tilde{\Sigma}$ contains the dressing phase and its weak coupling expansion starts as $\tilde{\Sigma}_{Q1}(q,u)=1+g^2(\dots)$. Thus the leading order expansion of the S-matrix scalar factor is

$$S_0(q,u)=\frac{(2u+i)^2(2u-q+i(Q-1))}{(2u-q-i(Q+1))(2u-q-i(Q-1))(2u-q+i(Q+1))}+O(g^2)$$

The scalar part of the boundary state amplitude for a charged Q mirror boundstate can be analytically continued from the boundstate reflection factor, whose scalar part can be calculated by the bootstrap principle [68]. The charge Q bound-state composed of elementary magnons as $x=(x_1,\dots,x_Q)$, such that $x^-=x_1^-$ and $x_Q^+=x^+$and the bound-state condition is also satisfied $x_i^+=x_{i+1}^-$. Thus the full scalar factor as the product of the elementary scalar factors turns out to be :

$$R_0^Q(x)=\prod_{i=1}^{Q}R_0^1(x_i^{\pm})\prod_{i<j}S_0^{11}(x_i^{\pm},-x_j^{\mp})\tag{14}$$

where $R_0^1(x^{\pm})$ denotes the scalar factor of a fundamental particle, (3), while $S_0^{11}(x_1,x_2)$ denotes the scalar factor of their scattering matrices. The combination appearing in $K_0\bar{K}_0$ can be easily calculated following [45] and one finds that

$$\bar{K}_0(q)K_0(q)=\frac{4(1+z^+z^-)^2}{(z^++\frac{1}{z^+})(z^-+\frac{1}{z^-})(z^-+z^+)^2}=\frac{256q^2g^4}{(q^2+Q^2)^3}+\dots$$

Putting together in eq. (13) the contributions of the matrix part and the scalar parts of the S-matrices and the boundary states one obtains that the leading correction is proportional to $g^{4(L+1)}$ in case of a strip with width L. Evaluating the correction (13) we must set the momentum (or rapidity) of the fundamental particle equal to the value(s) allowed by the BBY equation for the particular L. Thus for $J = 3$ ($L = 2$) we have to use $p = \frac{\pi}{2}$. Evaluating the formula we obtain

$$\Delta E = 192g^{12}(4\zeta(5) - 7\zeta(9))$$

The integrand satisfied a very nontrivial consistency relation, namely the contribution of the dynamical poles vanished when we summed them over the boundstate spectrum (Q). The transcendentality property of ΔE is what is expected from a six loop gauge theory calculation: the maximal transcendentality is $2(\#\text{loops}) - 3$, as it happened for the Konishi operator at four and five loops. It is interesting to compare our result to the analogous result for the anomalous dimension of the Konishi operator in the periodic theory: first the wrapping correction in the strip geometry consists of the linear combinations of ζ functions only without the additional integer term, and second it seems to appear in relatively higher orders in g. This is a generic feature of the boundary finite size corrections as they start at e^{-2mL} compared to the periodic case which starts at e^{-mL}. There is one exceptional case, namely when the boundary reflection factor admits a kinematical pole at $q = 0$, since then the boundary correction starts also at e^{-mL}. This phenomena does not appear for the $Y = 0$ brane, but it indeed happens for $Z = 0$ [45].

5 Conclusions

In this paper we have proposed formulas to describe the leading Lüscher-type finite size energy correction for multiparticle states on the strip. By this we generalized two results in the literature. On one hand we generalized the multiparticle finite size energy corrections from the periodic (cylinder) [15] to the boundary (strip) setting. On the other we extended the boundary finite size correction from the vacuum state [46, 47, 45] to excited multiparticle states. We then evaluated the proposed formulas for a single particle excitation reflecting diagonally on the $Y = 0$ brane and determined the wrapping contribution to the anomalous dimension of the simplest determinant operator of the form $O_Y(ZYZ^J)$.

The calculation, after internal consistency checks, resulted in sums of ζ-functions, which are also expected from a gauge theory calculation. Nevertheless a stringent consistency check could be obtained by calculating the wrapping contribution directly from Feynman diagrams on the gauge theory side. This can be done after identifying the wrapping type diagrams, analogously to the case what was developed for single trace operators [69].

In this paper we calculated the finite size correction to the energy of the simplest one particle state. Clearly the approach is quite general and can be applied, in principle, to any multiparticle state over the $Y = 0$ ground state, although it can be cumbersome to collect the various S-matrix elements. A bit more sophisticated approach can be based on the Y-system. In [26] Y-system type functional relations was proposed to describe the spectrum of planar AdS/CFT and, in the same time, its asymptotic solution was expressed in terms of the transfer matrix eigenvalues of the ABA. We expect that the same Y-system describes the boundary AdS/CFT, too, and that the corresponding asymptotical solution can be expressed also in terms of the eigenvalues of the double raw transfer matrices. A work is in progress in this direction.

Although the conjectured Y-system could describe all the excited states, it is useful only if its analytical structure is completely understood. A derivation based on the boundary Thermodynamical Bethe Ansatz would provide not only the rigorous establishment of the Y-system, but also present the needed analytical structure.

Recently there is a growing interest in the less supersymmetric, β-deformed, version of the AdS/CFT correspondence [70, 71, 36, 72, 73]. We believe that our boundary formulation can be extended to this realm, too.

Acknowledgments

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A Mirror boundary state

In this Appendix we show that the matrix structure of the boundary state amplitudes for each Q (mirror) bound states can be obtained by requiring the boundary state to be a singlet under the unbroken $su(1|2)$ symmetry. The boundary state can be written as

$$|B\rangle = \sum_{Q=1}^{\infty} |B_Q\rangle, \qquad |B_Q\rangle = \sum_{a,b=1}^{4} \sum_{ij} K_{ij}^{ab} |i\rangle_{(-q)}^{a} \otimes |j\rangle_{(q)}^{b}$$

where a, b are running over the the four subspaces of the $4Q$ dimensional representation space of the mirror Q bound state and $|j\rangle_{(q)}^{b}$ represent (the superfields description of) the states belonging to them as described in the main body of the paper. (The subscript on these symbols indicate the momentum of the bound state). The bosonic generators of the unbroken symmetry can be described in terms of the fermionic θ_1 θ_2 and bosonic w_3, w_4 parameters as

$$L_1^1 = \frac{1}{2}(\theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}), \quad R_{\alpha}^{\beta} = w_{\alpha} \frac{\partial}{\partial w_{\beta}} - \frac{1}{2} \delta_{\alpha}^{\beta} w_{\gamma} \frac{\partial}{\partial w_{\gamma}}, \quad \alpha,\, \beta = 3,\, 4$$

The requirements $L_1^1 |B_Q\rangle = 0$, $R_{\alpha}^{\beta} |B_Q\rangle = 0$ restrict the form of $|B_Q\rangle$ as

$$|B_Q\rangle = K^{11} \sum_{j=0}^{Q} (-1)^j |j\rangle_{(-q)}^{1} \otimes |Q-j\rangle_{(q)}^{1} + K^{22} \sum_{j=0}^{Q-2} (-1)^j |j\rangle_{(-q)}^{2} \otimes |Q-2-j\rangle_{(q)}^{2}$$

$$+ K^{34} \sum_{j=0}^{Q-1} (-1)^j |j\rangle_{(-q)}^{3} \otimes |Q-1-j\rangle_{(q)}^{4} + K^{43} \sum_{j=0}^{Q-1} (-1)^j |j\rangle_{(-q)}^{4} \otimes |Q-1-j\rangle_{(q)}^{3}$$

To impose also $Q_{\alpha}^1 |B_Q\rangle = 0$ we need not only the explicit form of the supersymmetry generators in the superfield formalism

$$Q_{\alpha}^{1} = a w_{\alpha} \frac{\partial}{\partial \theta_1} + b \epsilon^{12} \epsilon_{\alpha\beta} \theta_2 \frac{\partial}{\partial w_{\beta}}$$

but also the fact, that they have a non trivial coproduct [56]:

$$Q_{\alpha}^{1} (|j\rangle_{(-q)}^{b} \otimes |l\rangle_{(q)}^{b}) = e^{\epsilon_Q(q)/4} (Q_{\alpha}^{1} |j\rangle_{(-q)}^{b}) \otimes |l\rangle_{(q)}^{b} + e^{-\epsilon_Q(q)/4} |j\rangle_{(-q)}^{b} \otimes (Q_{\alpha}^{1} |l\rangle_{(q)}^{b})$$

Here a, b are the q dependent coefficients, obtained by the $x^{\pm}(p) \mapsto z^{\pm}(q)$ analytic continuation from the magnon channel

$$a(q) = \sqrt{\frac{g}{Q}} \eta(q), \quad b(q) = \sqrt{\frac{g}{Q}} \frac{i}{\eta(q)} (\frac{z^+}{z^-} - 1), \quad \eta(q) = e^{\epsilon_Q(q)/4} \sqrt{i(z^- - z^+)}$$

and we also exploited that $\epsilon_Q(q) = \epsilon_Q(-q)$. The requirement $Q_{\alpha}^1 |B_Q\rangle = 0$ leads to a (compatible) homogeneous linear system of equations for the remaining K^{ab}. The solution can be written in terms of the undetermined K^{11} as

$$K^{22} = -K^{11}, \quad K^{34} = -i K^{11} e^{-\epsilon_Q(q)/2}, \quad K^{43} = -i K^{11} e^{\epsilon_Q(q)/2}$$

If we, instead of demanding the conservation of Q_{α}^1, impose the $Q_{\alpha}^2 |B_Q\rangle = 0$ requirement the analogous calculation yields

$$K^{22} = -K^{11}, \quad K^{34} = i K^{11} e^{\epsilon_Q(q)/2}, \quad K^{43} = i K^{11} e^{-\epsilon_Q(q)/2}$$

This solution is equivalent to the one obtained by analytical continuation from the diagonal reflection matrix $R(z)$, therefore we use this boundary state to calculate the Lüscher correction.

B Scattering matrix coefficients

Here we collect the explicit form of the scattering matrix elements we used to calculate the finite size corrections.

$$a_5^5 = \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-}\frac{\tilde{\eta}_1}{\eta_1}$$

$$a_6^5 = \sqrt{Q}\frac{(x_2^+ - x_2^-)}{(x_1^+ - x_2^-)}\frac{\tilde{\eta}_1}{\eta_2}$$

$$a_9^9 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{\tilde{\eta}_1}{\eta_1}\frac{\tilde{\eta}_2}{\eta_2}$$

$$a_8^8 = \frac{1}{2}\frac{x_1^-(x_1^- - x_2^+)(1 - x_1^+ x_2^-)}{x_1^+(x_1^+ - x_2^-)(1 - x_1^- x_2^-)}\frac{\tilde{\eta}_1}{\eta_1}\frac{\tilde{\eta}_2^2}{\eta_2^2}$$

$$a_7^8 = -\frac{i}{\sqrt{Q}}\frac{x_1^- x_2^-(x_1^- - x_2^+)}{x_1^+ x_2^+(x_1^+ - x_2^-)(1 - x_1^- x_2^-)}\frac{\tilde{\eta}_1\tilde{\eta}_2^2}{\eta_2}$$

$$a_3^2 = -\frac{2i}{\sqrt{Q}}\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^+ - x_2^-)(1 - x_1^- x_2^-)\eta_1\eta_2}$$

$$a_4^3 = \frac{Q-1}{\sqrt{Q}}\frac{x_1^-(x_2^+ - x_2^-)(1 - x_1^+ x_2^-)}{x_1^+(x_1^+ - x_2^-)(1 - x_1^- x_2^-)}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_2^2}$$

$$a_3^3 = -\frac{(-x_1^- x_1^+(1 + x_1^- x_2^- - 2x_1^+ x_2^-) - (x_1^+ + x_1^-(-2 + x_1^+ x_2^-))x_2^+)}{x_1^+(x_1^+ - x_2^-)(1 - x_1^- x_2^-)}\frac{\tilde{\eta}_1}{\eta_1}\frac{\tilde{\eta}_2}{\eta_2}$$

where the following phase factors have been chosen:

$$\tilde{\eta}_1 = e^{i\frac{p_1}{4}}\sqrt{i(x_1^- - x_1^+)} \quad ; \quad \eta_1 = e^{i\frac{p_2}{2}}\tilde{\eta}_1 \quad ; \quad \eta_2 = e^{i\frac{p_2}{4}}\sqrt{i(x_2^- - x_2^+)} \quad ; \quad \tilde{\eta}_2 = e^{i\frac{p_1}{2}}\eta_2$$

C Complete solution of the BBY

In this appendix we present the complete weak coupling solution of the BBY (4) up to the order of g^4. To describe them we introduce $J = L + 1$. Since p must be in the range $0 < p < \pi$, its allowed values are as follows:

magnon labels	allowed p-s	nature
(33) (34) (43) (44) (12) (21)	$p_n = n\frac{\pi}{J}, \quad n = 1, \dots J-1$	bosonic
(11)	$p_n = n\frac{\pi}{J-1}, \quad n = 1, \dots J-2$	bosonic
(22)	$p_n = n\frac{\pi}{J+1}, \quad n = 1, \dots J$	bosonic
(13) (14) (31) (41)	$p_n = \frac{2n\pi}{2J-1}, \quad n = 1, \dots J-1$	fermionic
(23) (24) (32) (42)	$p_n = \frac{2n\pi}{2J+1}, \quad n = 1, \dots J$	fermionic

(In writing the entries of the table we also exploited that J is an integer).

The different allowed momenta for various different magnon labels indicate that the presence of the two boundaries splits the 16-fold degeneracy of the bulk magnon states. A particularly interesting aspect of these allowed p values is to consider the difference between the sum of bosonic and fermionic energies $E_B - E_F$ where

$$E_B = 6\sum_{n=1}^{J-1}\sqrt{1 + 16g^2\sin^2(\frac{n\pi}{2J})} + \sum_{n=1}^{J-2}\sqrt{1 + 16g^2\sin^2(\frac{n\pi}{2(J-1)})} + \sum_{n=1}^{J}\sqrt{1 + 16g^2\sin^2(\frac{n\pi}{2(J+1)})}$$

and

$$E_F = 4(\sum_{n=1}^{J-1}\sqrt{1 + 16g^2\sin^2(\frac{n\pi}{2J-1})} + \sum_{n=1}^{J}\sqrt{1 + 16g^2\sin^2(\frac{n\pi}{2J+1})})$$

since this difference may be thought of as the sum of the (single magnon) vacuum fluctuations (zero mode sum) around the ground state (i.e. two boundaries with no particle between them). The interesting observation is that adding to this half of the energies of the $p = \pi$ bosonic modes $E_Z/2 = 4\sqrt{1+16g^2}$ we get precisely zero

$$E_B + E_Z/2 - E_F = 0.$$

We checked this analytically up to g^8 for a number of (integer) J-s and also numerically for some randomly chosen g-s. The vanishing of this sum may be consistent with the ground state preserving one supersymmetry. It would be interesting to perform an analogous calculation for the $Q = 2$ bound-state particles [65].

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