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A two-dimensional integrable axionic σ-model and T-duality

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Abstract

An S-matrix is proposed for the two dimensional O(3) σ-model with a dynamical θ-term (axion model). Exploiting an Abelian T-duality transformation connecting the axion model to an integrable SU(2)$\times$U(1) symmetric principal σ-model, strong evidence is presented for the correctness of the proposed S-matrix by comparing the perturbatively calculated free energies with the ones based on the Thermodynamical Bethe Ansatz. This T-duality transformation also leads to a new Lax-pair for both models. The quantum non-integrability of the O(3) σ-model with a *constant* θ-term, in contradistinction to the axion model, is illustrated by calculating the $2 \to 3$ particle production amplitude to lowest order in θ.

In this paper we shall study the following two dimensional σ-model described by the Lagrangian[1]:

$$\mathcal{L} = \frac{1}{2\tilde{\lambda}}\partial_\mu n^a \partial^\mu n^a + \frac{\tilde{\lambda}}{32\pi^2(1+\tilde{g})}\partial_\mu\theta\partial^\mu\theta + \frac{\theta}{8\pi}\epsilon^{\mu\nu}\epsilon^{abc}n^a\partial_\mu n^b\partial_\nu n^c\,, \tag{1}$$

where $n^a n^a = 1$. In fact (1) is an O(3) non-linear σ-model, coupled to a scalar field, θ, (whose normalization has been chosen for later convenience) through the Hopf term. This latter is proportional to the topological current of the O(3) model and with the normalization chosen in (1) its space-time integral (after a Wick rotation) yields the topological charge, which can take integer values only. This implies that the variable θ is actually an angle, taking its values between 0 and 2π. This observation will play an important role in all our considerations.

We shall refer to (1) as the 'axion model' since it can be thought of as the O(3) nonlinear σ-model with a dynamical θ-term which can be regarded as a two-dimensional analogue of its phenomenologically important four-dimensional counterpart [1]. The following (heuristic) consideration might be useful to gain some insight into the physics of the axion model. Let us integrate out the O(3) fields, n^a, in some generating functional of the theory (1). This way one would obtain a non-vanishing effective potential for the θ field. Because θ is 2π-periodic the effective potential must be also periodic. The effective theory of the θ field is therefore expected to be similar to the Sine-Gordon model, with a periodic potential and corresponding topological current $K_\mu = \epsilon_{\mu\nu}\partial^\nu\theta/2\pi$. Note that θ being an angular variable makes it difficult to integrate it out in the functional integral in spite of $\mathcal{L}$ being only quadractic in θ.

The axion model belongs to a family of (classically) integrable two-dimensional non-linear σ-models with an O(3) symmetry discovered in Ref. [2]. All of the models of Ref. [2] have target spaces with non-vanishing torsion (in addition to the metric tensor field) but the axion model is especially simple, as its torsion is constant.

It has already been observed in Ref. [3] that the axion model (1) can be mapped to a one parametric deformation of the SU(2) principal σ-model by an Abelian T-duality transformation. This latter (torsionless) model has an $\mathrm{SU_L(2)}\otimes\mathrm{U_R(1)}$ symmetry and recently there has been some revival of interest in it [4, 5]. Its Lagrangian can be written as:

$$\mathcal{L}_\Sigma = -\frac{1}{2\lambda}\Big\{L_\mu^a\,L^{a\mu} + gL_\mu^3\,L^{3\mu}\Big\}\,, \tag{2}$$

where

$$L_\mu = G^{-1}\partial_\mu G = \tau^a L_\mu^a\,, \tag{3}$$

[1] We use the following conventions. For a vector v in two-dimensional Minkowski space $v^\mu v_\mu = v_+ v_-$ where $v_\pm = v_0 \pm v_1$. The antisymmetric tensor is defined by $\epsilon^{01} = 1$, $\tau^a = \sigma^a/2$ with σ^a being the standard Pauli-matrices satisfying $\sigma^a\sigma^b = \delta^{ab} + i\epsilon^{abc}\sigma^c$.

and g is the parameter of deformation. The theory (2) is known to be integrable classically [6] and there is little doubt that it is also quantum-integrable [4, 5, 7]. Its spectrum contains two massive doublets (kinks) whose scattering is described by the tensor product of an SU(2)×U(1) symmetric solution of the bootstrap S-matrix equations:

$$S(\theta) = S^{(\infty)}(\theta) \otimes S^{(p)}(\theta)\,, \tag{4}$$

where $S^{(p)}(\theta)$ denotes the Sine-Gordon (SG) S-matrix. Depending on the value of the parameter, p, in addition to the kinks there are also some bound states (breathers) in the spectrum transforming as $3+1$ under SU(2).

In the following we shall show that the somewhat unexpected identification between the axion model and the deformed principal σ-model through a T-duality transformation allows us to learn more about both of them. Assuming the validity of the duality transformation at the *quantum level* between the two theories implies the absence of particle production in the axion model (1), and also that its factorized scattering theory is given by the S-matrix of Eq. (4).

The proposed quantum integrability of the axion model might seem somewhat surprising, as it is generally believed that the O(3) model with a *constant* θ-term is *not quantum integrable*, except for the special value $\theta = \pi$ [8] (despite the fact that the θ-term, being a total derivative, does not change the classical physics of the model). We now show that in the framework of the form-factor bootstrap approach the θ term mediates particle production in the O(3) σ-model, indeed. To lowest order in θ the $2 \to 3$ particle production amplitude can be written as

$$\begin{aligned} \langle p,b;p',b';p'',b''|q,a;q',a'\rangle_{(\theta)} &= (2\pi)^2\, i\,\theta\,\delta^{(2)}(p+p'+p''-q-q') \\ &\cdot \langle p,b;p',b';p'',b''|T(0)|q,a;q',a'\rangle_{(0)} + \mathcal{O}(\theta^2)\,, \end{aligned} \tag{5}$$

where in the first line the amplitude is in the O(3) model with a θ-term, while in the second line the matrix element of the topological charge density operator T is to be calculated in the original O(3) σ-model (with $\theta = 0$). In other words we simply apply perturbation theory in θ. Let us now consider the following simplified kinematical configuration: the incoming particles have momenta $q_1 = Q$ and $q_1' = -Q$, whereas the produced (outgoing) three particles have momenta $p_1 = Q'$, $p_1' = 0$ and $p_1'' = -Q'$ respectively. Here Q' can easily be expressed in terms of Q and the kink mass M using energy conservation. For large Q, using the results of Ref. [9], we find

$$\langle p,b;p',b';p'',b''|T(0)|q,a;q',a'\rangle_{(0)} \approx \pi^{\frac{5}{2}} \frac{Q^2}{\ln^3 Q/M} \left(\epsilon^{a'ba}\delta^{b'b''} - \epsilon^{b''ba}\delta^{b'a'}\right)\,. \tag{6}$$

Eq. (6) shows that already to first order in θ, the $2 \to 3$ particle production amplitude is different from zero. Thus at least for small values of θ, the introduction of this term destroys quantum integrability of the O(3) σ-model, indeed.

To exhibit now the classical T-duality transformation between the two models

[3], we introduce the parametrization

$$n^1 = \sin\vartheta \sin\varphi\,, \quad n^2 = \sin\vartheta\cos\varphi\,, \quad n^3 = \cos\vartheta\,, \qquad \theta = -\frac{4\pi}{\tilde{\lambda}}\sqrt{1+\tilde{g}}\,\chi\,, \tag{7}$$

in terms of which the Lagrangian (1) (after an integration by parts) becomes

$$\mathcal{L} = \frac{1}{2\tilde{\lambda}}\Big\{\partial_\mu\vartheta\partial^\mu\vartheta + \sin^2\vartheta\partial_\mu\varphi\partial^\mu\varphi + \partial_\mu\chi\partial^\mu\chi + 2\sqrt{1+\tilde{g}}\cos\vartheta\epsilon^{\mu\nu}\partial_\mu\chi\partial_\nu\varphi\Big\}\,. \tag{8}$$

We now perform an Abelian T-duality transformation [10] with respect to the variable χ, which corresponds to the canonical transformation [11]

$$\chi' = -\frac{\tilde{\lambda}}{\sqrt{1+\tilde{g}}}p_\alpha \qquad\qquad p_\chi = -\frac{\sqrt{1+\tilde{g}}}{\tilde{\lambda}}\alpha'\,, \tag{9}$$

where (and in the following) p_χ resp. p_α denote the canonical momenta conjugate to χ resp. to its 'dual' α. In terms of these new variables the dual Lagrangian turns out to be:

$$\mathcal{L}_\Sigma = \frac{1}{2\tilde{\lambda}}\Big\{\partial_\mu\vartheta\partial^\mu\vartheta + (1+\tilde{g}\cos^2\vartheta)\partial_\mu\varphi\partial^\mu\varphi + (1+\tilde{g})\left[\partial_\mu\alpha\partial^\mu\alpha + 2\cos\vartheta\partial_\mu\alpha\partial^\mu\varphi\right]\Big\}\,, \tag{10}$$

which is nothing but the Lagrangian (2), when parametrizing the SU(2) valued field, G, by the Euler angles

$$G = e^{i\varphi\tau^3}\, e^{i\vartheta\tau^1}\, e^{i\alpha\tau^3}\,, \tag{11}$$

and taking into account the relations at the classical level between the couplings:

$$\tilde{\lambda} = \lambda\,, \qquad \tilde{g} = g\,. \tag{12}$$

The observation, that the axionic model is the T dual of $\mathcal{L}_\Sigma$ also explains why θ is an angular variable. Indeed it has been shown in Ref. [12] that in case of the principal σ-model ($g = 0$) the Abelian T duality (9) maps its target space (S^3) into $S^2 \times S^1$. The arguments of Ref. [12] can be easily applied to the present case with $g > -1$, and it is clear that in Eq. (1) n^a parametrize the S^2 and θ parametrizes the S^1.

In fact the equations of motion of both models (1) and (2) are known to admit a Lax representation indicating their (classical) integrability [2, 6]. Indeed, introducing the matrix valued current

$$I_\mu = \frac{\tilde{\lambda}}{8\pi}n\epsilon_{\mu\nu}\partial^\nu\theta - \frac{\sqrt{\tilde{g}}}{2}\epsilon_{\mu\nu}\partial^\nu n + \frac{1}{2}n\partial_\mu n\,, \tag{13}$$

where $n = in^a\sigma^a$, the equations of motion of (1) can be written as:

$$\partial^\mu I_\mu = 0\,, \qquad\qquad \partial_\mu I_\nu - \partial_\nu I_\mu = [I_\mu, I_\nu]\,. \tag{14}$$

The standard form (14) of the equations of motion allows for the introduction of a Lax pair

$$U_\pm = \frac{1}{1 \pm \omega}\, I_\pm \,, \tag{15}$$

satisfying the zero curvature equation

$$\partial_\mu U_\nu - \partial_\nu U_\mu = [U_\mu, U_\nu] \,, \tag{16}$$

for all values of the spectral parameter ω. The current, I_μ, is closely related to the matrix valued Noether current, $\mathcal{N}_\mu = -i\tau^a \mathcal{N}^a_\mu$, defined by $\delta\mathcal{L} = \partial^\mu \varepsilon^a \mathcal{N}^a_\mu$ corresponding to the symmetry transformation $\delta n^a = \epsilon^{abc}\varepsilon^b n^c$:

$$I_\mu = \tilde{\lambda}\mathcal{N}_\mu + \epsilon_{\mu\nu}\partial^\nu T \,, \qquad\qquad T = \Big(\frac{\tilde{\lambda}}{8\pi}\theta - \frac{\sqrt{g}}{2}\Big) n \,. \tag{17}$$

The fact that, apart from a trivially conserved piece, I_μ can be identified with the Noether current of the manifest O(3) symmetry of the Lagrangian explains only the first equation in Eq. (14). The trivially conserved part of I_μ is essential that the zero curvature equation be also satisfied.

The equations of motion of the deformed principal model (2) can be written entirely in terms of the current L_μ as

$$\partial^\mu L^3_\mu = 0 \,, \quad \partial^\mu L^1_\mu = -igL^{2\mu} L^3_\mu \,, \quad \partial^\mu L^2_\mu = igL^{1\mu} L^3_\mu \,. \tag{18}$$

It is known that this system can be put to the Lax form [6, 7], i.e. there is a spectral parameter dependent current, $V_\mu = \tau^a V^a_\mu$, satisfying the zero curvature equation (16). This current can be written as:

$$V^{1,2}_\pm = \alpha_\pm \, L^{1,2}_\pm \,, \qquad\qquad V^3_\pm = a_\pm \, L^3_\pm \,, \tag{19}$$

where

$$\alpha_\pm = -\frac{4 + g\omega^2}{4 - g\omega^2 \pm 4\omega} \,, \qquad a_\pm = -\frac{4 - g\omega^2 \mp 4g\omega}{4 - g\omega^2 \pm 4\omega} \,. \tag{20}$$

We can now use the classical T-duality transformation (9) to map the linear system of the axion model (14) to a new Lax pair for the deformed σ-model (2). It is given by Eq. (15), where the current, I_μ, has to be replaced by

$$\hat{I}_\mu = \partial_\mu G\, G^{-1} + g\Big(G\tau^3 G^{-1}\Big) L^3_\mu - i\sqrt{g}\epsilon_{\mu\nu}\partial^\nu \Big(G\tau^3 G^{-1}\Big) \,. \tag{21}$$

Eq. (21) is obtained from (13) by the T-duality transformation (9). $\hat{I}_\mu$ is related to the Noether current $\hat{\mathcal{N}}_\mu$, corresponding to the manifest symmetry $\delta G = -i\varepsilon^a \tau^a G$ of (2) and can be written analogously to $\mathcal{N}_\mu$:

$$\hat{I}_\mu = \lambda\hat{\mathcal{N}}_\mu + \epsilon_{\mu\nu}\partial^\nu \hat{T} \,, \qquad\qquad \hat{T} = -i\sqrt{g}\, G\tau^3 G^{-1} \,. \tag{22}$$

It is clear that the new Lax pair (15) and the 'old' one, (19), cannot be related by a gauge transformation since they have different pole structures as functions of the spectral variable, ω. In the $g \to -1$ limit, the axion model reduces to the original O(3) σ-model (decoupled from the θ field), and the Lax pair (15) becomes equivalent to that of Ref. [13], where it has been pointed out that the corresponding $\hat{I}_\mu$'s are *ultralocal* currents. We note that the Lax pairs (15) and (19) correspond to (different) deformations of the usual Lax pairs of the principal chiral σ-model, linear in $\partial_\mu G G^{-1}$ respectively $G^{-1}\partial_\mu G$.

Next we carry out a standard test on the proposed S-matrix (4) for the axion model by comparing its (zero temperature) free energy obtained from the Thermodynamical Bethe Ansatz (TBA) and in weak coupling perturbation theory (PT) [14]. For the deformed σ-model (2) this comparison has been done in Ref. [5] where complete consistency has been found between the results of PT and of the TBA. For the axion model it is sufficient to compute the free energy in PT as the results of the TBA can be literally taken over from Ref. [5]. In the present case one obtains as a bonus, a further nontrivial check on the quantum equivalence between the axion and the deformed σ-model, hence also on the validity of the T-duality transformation at the quantum level. Up to now when quantum equivalence between dually related models has been tested, mostly β-functions have been compared. The fact that the higher coefficients of the β-functions are scheme dependent makes such a comparison more difficult and less conclusive.

The equivalence of the β-functions is certainly a necessary condition for the validity of quantum T-duality. At one loop order the β-functions of the couplings, β_λ, β_g and $\beta_{\tilde\lambda}$, $\beta_{\tilde g}$ are simply obtained from each other by the classical relation (12). At two loops, however, it has been found in [3] that using the background field method and dimensional regularization the following perturbative redefinition of the couplings

$$\tilde\lambda = \lambda + \frac{\lambda^2}{4\pi}(1+g)\,, \quad \tilde g = g + \frac{\lambda}{4\pi}(1+g)^2\,, \tag{23}$$

(i.e. a change of scheme) is induced by the T-duality transformation. Taking into account Eqs. (23) the two loop β-functions of the two models turn out to be equivalent. Alternatively, introducing a renormalization group (RG) invariant combination of the two couplings:

$$p = 2\pi \lim_{t\to\infty} \frac{1+g(t)}{\lambda(t)}\,, \qquad \tilde p = 2\pi \lim_{t\to\infty} \frac{1+\tilde g(t)}{\tilde\lambda(t)}\,, \tag{24}$$

where $t \propto \ln h$, one finds $p = \tilde p$ up to two loops [3]. It is important to note that the RG invariant quantity (24) can be consistently *identified* with the parameter p in the S matrix (4). Let us introduce an effective β-function for $\lambda(t)$ by $\beta_{\rm eff}(\lambda, p) = \beta_\lambda(\lambda, \Gamma(\lambda, p))$, expressing $g(t)$, in terms of the running coupling, $\lambda(t)$, and p as

$g(t) = \Gamma(\lambda(t)\,,p)$. Using the perturbative result for $\Gamma(\lambda(t),p)$ [5] one finds

$$\beta_{\text{eff}}(\lambda,p) = \beta_{\text{eff}}(\tilde{\lambda},\tilde{p}) = -\frac{\lambda^2}{2\pi} + \frac{p-2}{8\pi^2}\lambda^3 + \cdots\,. \tag{25}$$

Thus as far as coupling constant renormalization is concerned, the two models are equivalent, both are asymptotically free, and the actual value of p effects only the two loop coefficient.

The classical free energy density is obtained by minimizing the Legendre transform of the Hamiltonian density coupled to some conserved currents

$$\hat{\mathcal{H}} = \mathcal{H}_0 - h_i J_0^i\,, \qquad \hat{H} = \int dx \hat{\mathcal{H}} = H - h_i Q_i\,. \tag{26}$$

Since the axion field, θ, is actually an angle, its winding number (topological charge) can be non trivial. Therefore we present here the Legendre transformation of the modified Hamiltonian (26) for a rather general case.

Let us consider a general sigma model with torsion

$$\mathcal{L}_0 = \frac{1}{2} g_{AB}\partial^\mu X^A \partial_\mu X^B + \frac{1}{2} b_{AB}\epsilon^{\mu\nu}\partial_\mu X^A \partial_\nu X^B, \tag{27}$$

and the following Ansatz for a set of conserved currents

$$J_\mu^i = \mathcal{C}_A^i(X)\partial_\mu X^A + \epsilon_\mu^{\ \nu}\mathcal{B}_A^i(X)\partial_\nu X^A\,, \tag{28}$$

sufficiently general to include topological currents. The Legendre transformation of (26) yields the Lagrangian of the modified model which can be written as

$$\hat{\mathcal{L}} = \mathcal{L}_0 + h^i J_0^i + \frac{1}{2} h^i h^j \mathcal{C}_A^i \mathcal{C}^{Aj}\,. \tag{29}$$

In fact $\hat{\mathcal{L}}$ can be obtained by gauging $\mathcal{L}_0$ i.e. by the substitution

$$\partial_\mu X^A \to \partial_\mu X^A + h^i \delta_{\mu 0}\mathcal{C}^{iA} \tag{30}$$

when the antisymmetric field b_{AB} is invariant (without compensating gauge transformation) under the symmetry transformation generated by the conserved currents (28).

Below we also give a class of classical ground states (around which $\hat{\mathcal{L}}$ is to be expanded) assuming that the metric, the antisymmetric tensor field and the quantities characterizing the currents are independent of a set of coordinates, θ^α, corresponding to the splitting $X^A = (y^k, \theta^\alpha)$:

$$g_{AB} = g_{AB}(y), \quad b_{AB} = b_{AB}(y), \quad \mathcal{C}_A^i = \mathcal{C}_A^i(y), \quad \mathcal{B}_A^i = \mathcal{B}_A^i(y)\,. \tag{31}$$

In this case the ground state is characterized by constant y^k-s and constant θ'^α-s $y^k \equiv y_0^k$, $\theta'^\alpha \equiv \theta'^\alpha_0$, where the y_0^k-s stand for the extrema of

$$H_{\rm eff} = -\frac{1}{2} h^i h^j (g_{AB}\mathcal{C}^{Ai}\mathcal{C}^{Bj} + \mathcal{B}^i_\alpha \mathcal{B}^j_\beta (\gamma^{-1})^{\alpha\beta})\,, \tag{32}$$

and

$$\theta'^\alpha_0 = (\gamma^{-1})^{\alpha\beta}\mathcal{B}^i_\beta(y_0)h^i. \tag{33}$$

In (32-33) $\gamma_{\alpha\beta}$ denotes the restriction of g_{AB} to the submanifold coordinatized by θ^α.

The axion model has a 'manifest' (i.e. up to a total derivative) SU(2)×U$_\theta$(1) symmetry, where the U$_\theta$(1) subgroup is generated by the shift $\theta \to \theta + {\rm const}$. It is very important to note that although the SU(2) symmetry of the Lagrangian (1) corresponds to that of the S-matrix, the 'manifest' U$_\theta$(1) symmetry *cannot be identified* with the corresponding one of the S-matrix (4). Here the duality transformation provides the clue; the corresponding $\tilde{\rm U}_{\rm R}(1)$ symmetry of the axion model is actually the image of the manifest ${\rm U}_{\rm R}(1)$ symmetry of (2) under the duality transformation, thus it is generated by the *topological current* of the axion field.

Corresponding to the ${\rm SU}_{\rm L}(2)\otimes{\rm U}_{\rm R}(1)$ symmetry of the deformed σ-model there are two Noether charges, $Q_{\rm L}$ resp. $Q_{\rm R}$, associated to the ${\rm U}_{\rm L}(1)$ resp. ${\rm U}_{\rm R}(1)$ subgroups. Introducing two chemical potentials coupled to the $Q_{\rm L}$ resp. $Q_{\rm R}$, charges the Hamiltonian (26) takes the form: $H = H_\Sigma - h_{\rm L}Q_{\rm L} - h_{\rm R}Q_{\rm R}$. Then one can distinguish between three different types of finite density ground states: LEFT with $h_{\rm L} > 0$, and $h_{\rm R} = 0$, RIGHT with $h_{\rm L} = 0$, and $h_{\rm R} > 0$, and DIAG where $h_{\rm L}, h_{\rm R} > 0$. As found in [5] the RIGHT case is obtained from DIAG by letting $h_{\rm L} = 0$ in the *final* results.

We compute below the corresponding ground state energies to one loop order in the axion model (1), starting with the LEFT case first. With the Euler angle parametrization of G (11) the ${\rm U}_{\rm L}(1)$ transformation, $G \mapsto e^{i\kappa\tau^3}G$ of the deformed σ-model (2) acts as a simple shift, $\varphi(x) \mapsto \varphi(x) + \kappa$. The corresponding Noether charge, $Q_{\rm L}$, and its image under the T-duality transformation, $\tilde{Q}_{\rm L}$, are simply

$$Q_{\rm L} = \int dx p_\varphi\,, \quad \tilde{Q}_{\rm L} = \int dx \tilde{p}_\varphi\,, \quad \text{where} \quad p_\varphi = \frac{\partial \mathcal{L}_\Sigma}{\partial\dot\varphi}\,,\ \tilde{p}_\varphi = \frac{\partial\mathcal{L}}{\partial\dot\varphi}\,,$$

since the canonical transformation implementing the T-duality mapping (9) effects only p_α, χ', α' and p_χ, leaving the other fields, φ, ϑ, p_φ, p_ϑ, unchanged.

Since in the LEFT case the b_{AB} field in Eq. (1) is invariant, one can simply 'gauge' the Lagrangian of the axion model in an external ($h_{\rm L}$) field (see Eq. (30). The classical ground state is found to be $\varphi \equiv \chi \equiv 0$, $\vartheta \equiv \pi/2$. (The corresponding solution of the deformed σ-model is given by $\varphi \equiv \alpha \equiv 0$, $\vartheta \equiv \pi/2$.)

Expanding the (Euclidean) Lagrangian (after suitable rescalings, etc.) we obtain

$$\overline{\mathcal{L}} = -\frac{2h_{\rm L}^2}{\tilde{\lambda}_0} + \frac{1}{2} m\mathcal{M}m^T + \mathsf{o}(\tilde{\lambda}), \tag{34}$$

where

$$\mathcal{M} = \begin{pmatrix} -\partial^2 + 4h_{\rm L}^2 & 0 & 2h_{\rm L}\sqrt{1+\tilde{g}_0}\,\epsilon_{\mu 2}\partial_\mu \\ 0 & -\partial^2 & 0 \\ -2h_{\rm L}\sqrt{1+\tilde{g}_0}\,\epsilon_{\mu 2}\partial_\mu & 0 & -\partial^2 \end{pmatrix}, \tag{35}$$

and $m = (\vartheta, \varphi, \chi)$. ($\tilde{\lambda}_0$, $\tilde{g}_0$ denote the bare coupling and parameter of the axion/dual model). In Eq. (35) we kept the ϵ tensor explicitly, as it requires a careful definition in $n = 2 - \epsilon$ dimensions which we use to regularize the momentum integrals. We adopt the definiton of [15], where this antisymmetric tensor corresponds to an almost complex structure: $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, $\epsilon_{\mu\nu}\epsilon_{\mu\sigma} = \delta_{\nu\sigma}$. The one loop quantum corrections to the classical ground state (the first term in Eq. (34)) require the calculation of a functional determinant, leading to

$$\mathcal{F}(h) = \frac{4h_{\rm L}^2}{n} \int \frac{d^n p}{(2\pi)^n} \frac{\tilde{p}_1^2 - \tilde{g}_0 \tilde{p}_2^2}{\tilde{p}^4 + 4h_{\rm L}^2(\tilde{p}_1^2 - \tilde{g}_0\tilde{p}_2^2)}\,, \quad \tilde{p}_\mu = \epsilon_{\mu\nu} p_\nu\,. \tag{36}$$

To evaluate (36) we apply the modified dimensional regularization of Ref. [5] as $\tilde{p}_2 = \epsilon_{2\nu} p_\nu$ plays here a distinguished role and it is kept as a one dimensional variable. In fact for our purposes it is sufficient to calculate the *difference* $\mathcal{F}(h) - \mathcal{F}_\Sigma(h)$, where $\mathcal{F}_\Sigma(h)$ is the corresponding determinant in the deformed σ-model (Eq. (3.12) in Ref. [5]). Since both $\mathcal{F}(h)$ and $\mathcal{F}_\Sigma(h)$ are already the first quantum corrections to the classical expressions we may set $\tilde{g} = g$ (and make no distinction between bare and renormalized g's) when computing their difference to lowest order and we end up with

$$\mathcal{F}(h) - \mathcal{F}_\Sigma(h) = \frac{(2h_{\rm L})^n}{n}(1+g) \int \frac{d^n q}{(2\pi)^n} \frac{(q_1^2 - q_2^2)q^4}{N_1 N_2} = \frac{(2h_{\rm L})^n}{n}(1+g)w(g), \tag{37}$$

where $N_1 = q^4 + q_1^2 - g q_2^2$, $N_2 = q^4 + q_2^2 - g q_1^2$. Although the integrand yielding $w(g)$ is antisymmetric under $q_1 \leftrightarrow q_2$, the integral is divergent by power counting for $n = 2$, i.e. it must be computed in $n = 2 - \epsilon$ dimensions. Its derivative, $w'(g)$, is, however, *convergent* by power counting and it has also an antisymmetric integrand, therefore this latter may be evaluated in $n = 2$ dimensions giving $w'(g) \equiv 0$. Then to compute $w(g)$ one may choose e.g. the point $g = -1$:

$$w(-1) = \int \frac{d^n q}{(2\pi)^n} \frac{q_1^2 - q_2^2}{(q^2+1)^2} = \frac{n-1-1}{n} \int \frac{d^n q}{(2\pi)^n} \frac{q^2}{(q^2+1)^2} = -\frac{1}{4\pi}, \tag{38}$$

where writing the second equality, we used that q_1 is $n-1$ dimensional, while q_2 is a 1 dimensional variable. From (38) one finds that after taking into account the change of the renormalization scheme (23), in PT the free energy densities of the two models (1) and (2) do indeed coincide for the LEFT case. Recently this calculation has been performed also at the two-loop level [16].

To discuss the RIGHT and DIAG cases we find it more convenient to use the parametrization of [5] for the SU(2) valued field, G:

$$G = \frac{i\sigma^2}{\sqrt{1+|\Psi|^2}} \begin{pmatrix} 1 & -\Psi^* \\ \Psi & 1 \end{pmatrix} \begin{pmatrix} e^{-i\Phi} & 0 \\ 0 & e^{i\Phi} \end{pmatrix}, \tag{39}$$

where Ψ resp. Φ is a complex resp. a real scalar field. Now $U_{\rm R}(1)$ acts as a shift, $\Phi \mapsto \Phi + \kappa$, and then the corresponding Noether charge of the deformed σ-model is simply $Q_{\rm R} = \int dx p_\Phi$. $Q_{\rm L}$ is slightly more complicated when expressed in terms of the canonical momenta (as $e^{i\kappa\sigma^3} i\sigma^2 = i\sigma^2 e^{-i\kappa\sigma^3}$): $Q_{\rm L} = \int dx[p_\Phi + 2i(p_\Psi \Psi - p_{\Psi^*}\Psi^*)]$.

Using Buscher's rule [10], the Lagrangian of the axion (dual) model now takes the form:

$$\mathcal{L}^d = \frac{\tilde\lambda}{8(1+\tilde g)}(\partial_\mu f)^2 + \frac{2}{\tilde\lambda}\frac{\partial_\mu\Psi\partial^\mu\Psi^*}{N^2} + \frac{1+\tilde g}{\tilde\lambda}\hat{\mathcal{A}}_\mu\hat{\mathcal{A}}^\mu - \frac{i}{2}\epsilon^{01}(\dot f\hat{\mathcal{A}}_1 - f'\hat{\mathcal{A}}_0), \tag{40}$$

where f is the dual to Φ, and

$$\hat{\mathcal{A}}_\mu = \mathcal{A}_{\Psi^*}\partial_\mu\Psi^* - \mathcal{A}_\Psi\partial_\mu\Psi = \frac{1}{N}(\Psi\partial_\mu\Psi^* - \Psi^*\partial_\mu\Psi), \quad N = 1+|\Psi|^2. \tag{41}$$

The canonical transformation connecting $\mathcal{L}_\Sigma$ and $\mathcal{L}^d$, maps $Q_{\rm R}$ and $Q_{\rm L}$ to

$$\tilde Q_{\rm R} = -\int dx f', \quad \tilde Q_{\rm L} = \int dx\,[-f' + 2i(\tilde p_\Psi\Psi - \tilde p_{\Psi^*}\Psi^*)], \tag{42}$$

i.e. $\tilde Q_{\rm R}$ and $\tilde Q_{\rm L}$ do indeed contain the topological charge of the axion field (proportional to f).

Applying now the general framework, Eqs. (32-33) to the present cases; $i =$(L,R), $X^A = (f,\Psi,\Psi^*)$, with $\theta^\alpha = (f)$, $y^k = (\Psi,\Psi^*)$. Using the explicit form of $\tilde p_\Psi$ and $\tilde p_{\Psi^*}$ one finds for $\tilde J_\mu^{\rm R,\,L}$:

$$\mathcal{C}^{\rm R}_A \equiv 0, \qquad \mathcal{B}^{\rm R}_A = \begin{cases} -1, & A = f \\ 0, & A = \Psi,\ \Psi^*, \end{cases} \tag{43}$$

$$\mathcal{C}^{\rm L}_A = \begin{cases} 0, & A = f \\ -\mathcal{N}\mathcal{A}_\Psi/\tilde\lambda, & A = \Psi \\ \mathcal{N}\mathcal{A}_{\Psi^*}/\tilde\lambda, & A = \Psi^* \end{cases} \qquad \mathcal{B}^{\rm L}_A = \begin{cases} -(1-2|\Psi|^2/N), & A = f \\ 0, & A = \Psi,\ \Psi^*, \end{cases} \tag{44}$$

where $\mathcal{N} = 4i(1-2(1+\tilde g)|\Psi|^2)/N$. Substituting these $\mathcal{C}^i_A$ and $\mathcal{B}^i_A$ into Eq. (32) reveals that $H_{\rm eff}$ depends only on $|\Psi|^2$ and that its extremum is at $\Psi = 0 = \Psi^*$. In the DIAG case the actual value of the ground state energy density at this extremum is given by:

$$\hat H|_{\rm min} = -\frac{2(1+\tilde g)}{\tilde\lambda}(h_{\rm R} + h_{\rm L})^2, \tag{45}$$

and the expectation value of f' is: $f'_0 = -4(1+\tilde{g})(h_{\rm R}+h_{\rm L})/\tilde{\lambda}$. We note that $\hat{H}|_{\rm min}$ agrees (as it should) with the corresponding result of the deformed σ-model with $\tilde{\lambda} \mapsto \lambda$, $\tilde{g} \mapsto g$ (Eq. (3.20) in [5]). For the RIGHT case the analogous expressions of the axion model are simply obtained from (45) by setting $h_{\rm L} = 0$.

At this point we recall the somewhat unusual feature of the axion model once more, i.e. that the U(1) symmetry of the S-matrix (4) is realized through a topological current analogously to the Sine-Gordon theory. To emphasize this we quote here the value of the classical free energy density corresponding to the Noether charge of the 'manifest' $\mathrm{U}_\theta(1)$ symmetry of the Lagrangian (1): $\hat{H}|^{(\theta)}_{\rm min} = -2h_{\rm R}^2/\tilde{\lambda}$, quite different from Eq. (45) with $h_{\rm L} = 0$.

To obtain the one loop correction to the free energy, one has to expand $\hat{\mathcal{L}}^d$ around the minimum, Eq. (45). Writing $f = xf'_0 + \hat{f}$, where the expectation value of $\hat{f}$ vanishes, one finds that the quadratic terms containing $\hat{f}$ are independent of $h_{\rm L}$, $h_{\rm R}$, so the quadratic pieces of $\hat{\mathcal{L}}^d$ (hence the one loop correction) are effectively determined by $\Psi = \sqrt{\tilde{\lambda}/2}\,\psi$ only:

$$\mathcal{L}_2 = \partial_\mu\psi\partial^\mu\psi^* - m^2|\psi|^2 + iq(\psi\dot{\psi}^* - \psi^*\dot{\psi})\,, \tag{46}$$

where $m^2 = 4h_{\rm L}(h_{\rm L}\tilde{g}_0 + h_{\rm R}(1+\tilde{g}_0))$ and $q = h_{\rm L}(1-\tilde{g}_0) - h_{\rm R}(1+\tilde{g}_0)$. After continuation to Euclidean space and writing $\psi = (\phi_1 + i\phi_2)/\sqrt{2}$, Eq. (46) becomes identical to the corresponding pieces for the deformed σ-model, Eq. (C.11) in [5] (with $\tilde{\lambda}_0 \mapsto \lambda_0$, $\tilde{g}_0 \mapsto g_0$). From this it follows that the free energy densities fully agree also for the DIAG and RIGHT cases in both models.

Now the results of the comparison between the free energies computed by the TBA (based on the proposed S matrix (4)) and in PT for the deformed σ-model (2) in Ref. [5] can be simply taken over for the axion model. The conclusion is that there is complete consistency between the TBA and the perturbative calculations, providing good evidence for the validity of the proposed S matrix (4) for the axion model. Since the effective coupling (25) is identical in the two models the $m/\Lambda_{\overline{\rm MS}}$ ratio found in [5] stays unchanged.

Finally we would like to point out that it would be interesting to study the axion model by lattice Monte-Carlo simulations. This would provide us with a completely non-perturbative way of testing quantum T-duality. Technical difficulties arising from the non-reality of the Euclidean action in the context of the lattice Monte-Carlo study of the $O(3)$ model with a constant θ term and a suggestion how to circumvent them is discussed in [17].

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