

Continuous horizontally rigid functions of two variables are affine

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Abstract

Cain, Clark and Rose defined a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ to be *vertically rigid* if $\text{graph}(cf)$ is isometric to $\text{graph}(f)$ for every $c \neq 0$. It is *horizontally rigid* if $\text{graph}(f(c\bar{x}))$ is isometric to $\text{graph}(f)$ for every $c \neq 0$ (see [1]).

In [2] the authors of the present paper settled Janković's conjecture by showing that a continuous function of one variable is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ ($a, b, k \in \mathbb{R}$). Later they

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proved in [4] that a continuous function of two variables is vertically rigid if and only if after a suitable rotation around the z -axis it is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ ($a, b, d, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \rightarrow \mathbb{R}$ continuous). The problem remained open in higher dimensions.

The characterization in the case of horizontal rigidity is surprisingly simpler. C. Richter proved in [3] that a continuous function of one variable is horizontally rigid if and only if it is of the form $a + bx$ ($a, b \in \mathbb{R}$). The goal of the present paper is to prove that a continuous function of two variables is horizontally rigid if and only if it is of the form $a + bx + dy$ ($a, b, d \in \mathbb{R}$). This problem also remains open in higher dimensions.

The main new ingredient of the present paper is the use of functional equations.

1 Introduction

Cain, Clark and Rose introduced the notion of vertical and horizontal rigidity in [1], which we now formulate for functions of several variables.

Definition 1.1. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *horizontally* or *vertically rigid*, if $\text{graph}(f(c \cdot))$ or $\text{graph}(cf)$ is isometric to $\text{graph}(f)$ for all $c \in (0, \infty)$, respectively. (Clearly, $c \in \mathbb{R} \setminus \{0\}$ would be the same.)

Definition 1.2. If C is a subset of $(0, \infty)$ and \mathcal{G} is a set of isometries of \mathbb{R}^{n+1} then we say that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is horizontally or vertically rigid *for a set* $C \subseteq (0, \infty)$ *via elements of* \mathcal{G} if for every $c \in C$ there exists a $\varphi \in \mathcal{G}$ such that $\varphi(\text{graph}(f)) = \text{graph}(f(c \cdot))$ or $\varphi(\text{graph}(f)) = \text{graph}(cf)$, respectively.

(If we do not mention C or \mathcal{G} then C is $(0, \infty)$ and \mathcal{G} is the set of all isometries.)

We characterized the continuous vertically rigid functions of one variable in [2], thus proving the conjecture of D. Janković.

Theorem 1.3 (Janković's Conjecture). *A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ ($a, b, k \in \mathbb{R}$, $k \neq 0$).*

C. Richter described the continuous horizontally rigid functions in dimension one in [3].

Theorem 1.4 (C. Richter). *A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is horizontally rigid if and only if it is of the form $a + bx$ ($a, b \in \mathbb{R}$). (It is actually enough to assume that f is horizontally rigid for a set $C \subseteq (0, \infty) \setminus \{1\}$ of at least 2 elements.)*

We characterized the continuous vertically rigid functions of two variables in [4].

Theorem 1.5. *A continuous function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is vertically rigid if and only if after a suitable rotation around the z -axis $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ ($a, b, d, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \rightarrow \mathbb{R}$ continuous).*

The goal of the present paper is to characterize the continuous horizontally rigid functions of two variables.

Theorem 1.6 (Main Theorem). *A continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is horizontally rigid if and only if it is of the form $a + bx + dy$ ($a, b, d \in \mathbb{R}$).*

In fact, we will prove a slightly stronger statement for which we introduce the following definition.

Definition 1.7. Let us say that a set $C \subseteq (0, \infty)$ *condensates to ∞* if for every $r \in \mathbb{R}$ the set $C \cap (r, \infty)$ is uncountable.

Theorem 1.8 (Main Theorem, technical form). *Let $C \subseteq (0, \infty)$ be a set condensating to ∞ . Then a continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is horizontally rigid for C if and only if it is of the form $a + bx + dy$ ($a, b, d \in \mathbb{R}$).*

The proof will very heavily rely on techniques developed in [4]. The key new ingredient will be that functional equations enter the picture. More precisely, we will study a specific system of functional equations in Section 2, which may be of interest in its own right, and then apply it in the course of the proof of the Main Theorem.

Building heavily on [4], the structure of the proof will be as follows. In Section 2 we prove a theorem about the solutions of a specific system of functional equations, which will be applied at the end of the proof. In Section 3 we cite a theorem of C. Richter which shows that if all the isometries are translations then the continuous horizontally rigid function f of arbitrary variables is constant. We also characterize the horizontally rigid functions via translations in arbitrary dimensions. In Section 4 we consider the set S_f of directions of segments connecting pairs of points on $\text{graph}(f)$ (see Definition 4.1). We cite a theorem from [4] to determine the possible S_f 's. In Section 5 we complete the proof by handling these cases using various methods. Finally, in Section 6 we collect the open questions.

2 A system of functional equations

In this section we explicitly solve a system of functional equations which may be interesting in its own right. Our motivation is of course that we will apply this result for the proof of our Main Theorem.

Theorem 2.1. *Suppose that $C \subseteq (0, \infty)$ is an uncountable set and $u_c, v_c, h_c \in \mathbb{R}$ for all $c \in C$. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous solution to the following system of functional equations.*

$$g(x) = h_c g(cx + u_c) + v_c. \quad (x \in \mathbb{R}, c \in C) \quad (2.1)$$

Then either $h_c = c^{-s}$ ($s \in \mathbb{R}, s > 0$) for all $c \in C$ and $g(x) = a + b_1(d - x)^s$ for $x < d$ and $g(x) = a + b_2(x - d)^s$ for $x \geq d$ ($a, b_1, b_2, d \in \mathbb{R}, (b_1, b_2) \neq (0, 0)$) or g is constant.

Proof. Let us introduce the function $h(c) = h_c$ for the sake of notational simplicity.

If g is constant then the proof is complete, so we may assume that this is not the case. If there exists a $c_0 \in C$ such that $h(c_0) = 0$ then (2.1) implies $g(x) \equiv v_{c_0}$, but this is impossible. Therefore $h(c) \neq 0$ for all $c \in C$. Applying (2.1) twice for $c_1, c_2 \in C$ we obtain the following.

$$\begin{aligned} g(x) &= h(c_1)g(c_1x + u_{c_1}) + v_{c_1} \\ &= h(c_1)(h(c_2)g(c_2(c_1x + u_{c_1}) + u_{c_2}) + v_{c_2}) + v_{c_1} \\ &= h(c_1)h(c_2)g(c_1c_2x + c_2u_{c_1} + u_{c_2}) + h(c_1)v_{c_2} + v_{c_1}. \end{aligned}$$

Interchanging c_1 and c_2 follows

$$\begin{aligned} g(x) &= h(c_1)h(c_2)g(c_1c_2x + c_2u_{c_1} + u_{c_2}) + h(c_1)v_{c_2} + v_{c_1} \\ &= h(c_1)h(c_2)g(c_1c_2x + c_1u_{c_2} + u_{c_1}) + h(c_2)v_{c_1} + v_{c_2}. \end{aligned}$$

Therefore

$$\begin{aligned} &g(c_1c_2x + c_2u_{c_1} + u_{c_2}) + \frac{h(c_1)v_{c_2} + v_{c_1}}{h(c_1)h(c_2)} \\ &= g(c_1c_2x + c_1u_{c_2} + u_{c_1}) + \frac{h(c_2)v_{c_1} + v_{c_2}}{h(c_1)h(c_2)}. \end{aligned} \quad (2.2)$$

Let $y = c_1c_2x + c_1u_{c_2} + u_{c_1}$ be the argument of g in the second line, and $u_{c_1, c_2} = u_{c_1}(c_2 - 1) - u_{c_2}(c_1 - 1)$, then the argument of g in the first line becomes $c_1c_2x + c_2u_{c_1} + u_{c_2} = y + u_{c_1, c_2}$.

Moreover, set $v_{c_1, c_2} = \frac{v_{c_1}(h(c_2)-1) - v_{c_2}(h(c_1)-1)}{h(c_1)h(c_2)}$, then (2.2) attains the form

$$g(y + u_{c_1, c_2}) = g(y) + v_{c_1, c_2}.$$

Since y can also be an arbitrary real number, we can replace it by x and hence we obtain that for every $x \in \mathbb{R}$ and $c_1, c_2 \in C$

$$g(x + u_{c_1, c_2}) = g(x) + v_{c_1, c_2}. \quad (2.3)$$

Define

$$U = \{u_{c_1, c_2} : c_1, c_2 \in C\}.$$

First assume that U is uncountable. Let us denote by \widehat{U} the additive group generated by U , and let $G: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $G(x) = g(x) - g(0)$. Clearly, G is continuous. If $n \in \mathbb{Z}$ then applying equation (2.3) $|n|$ times follows that for all $c_1, c_2 \in C$

$$\begin{aligned} G(x + nu_{c_1, c_2}) &= g(x + nu_{c_1, c_2}) - g(0) = g(x) + nv_{c_1, c_2} - g(0) \\ &= G(x) + nv_{c_1, c_2}. \end{aligned} \quad (2.4)$$

Every $\hat{u} \in \widehat{U}$ can be written as $\hat{u} = \sum_{i=1}^k n_i u_{c_i, c'_i}$ ($k \in \mathbb{N}$, $n_i \in \mathbb{Z}$, $c_i, c'_i \in C$, $i \in \{1, \dots, k\}$). Then (2.4) and $G(0) = 0$ imply

$$G\left(\sum_{i=1}^k n_i u_{c_i, c'_i}\right) = \sum_{i=1}^k n_i v_{c_i, c'_i}.$$

Thus G is additive on \widehat{U} . Since U is uncountable, \widehat{U} is dense in \mathbb{R} . Therefore the continuity of G follows that G is additive. Since G is an additive and continuous function, it is of the form $G(x) = bx$ ($b \in \mathbb{R}$). Hence the non-constant function g is of the form $g(x) = a + bx$, where $b \in \mathbb{R} \setminus \{0\}$ and $a = g(0) \in \mathbb{R}$. If $c \in C$ is fixed then (2.1) implies $a + bx = h(c)(a + bcx + u_c) + v_c$. Comparing the coefficients of x on the two sides of the equation yields $h(c) = c^{-1}$. Therefore g and h are of the required form.

Suppose next that U is countable and $1 \notin C$. Let $c_1, c_2 \in C$ be arbitrarily fixed. The map $F: C \rightarrow U \times U$, $F(c) = (u_{c_1, c}, u_{c_2, c})$ has an uncountable domain and a countable range, so it is not one-to-one. Hence there exist $c_3, c_4 \in C$, $c_3 \neq c_4$ such that $F(c_3) = F(c_4)$. Therefore $u_{c_1, c_3} = u_{c_1, c_4}$ and $u_{c_2, c_3} = u_{c_2, c_4}$, so by the definition of u_{c_1, c_2}

$$\begin{aligned} u_{c_1}(c_3 - 1) - u_{c_3}(c_1 - 1) &= u_{c_1}(c_4 - 1) - u_{c_4}(c_1 - 1), \\ u_{c_2}(c_3 - 1) - u_{c_3}(c_2 - 1) &= u_{c_2}(c_4 - 1) - u_{c_4}(c_2 - 1). \end{aligned}$$

Thus

$$\begin{aligned} u_{c_1}(c_3 - c_4) &= (u_{c_3} - u_{c_4})(c_1 - 1), \\ u_{c_2}(c_3 - c_4) &= (u_{c_3} - u_{c_4})(c_2 - 1), \end{aligned}$$

and dividing by $(c_3 - c_4)(c_1 - 1)$ or $(c_3 - c_4)(c_2 - 1)$, respectively infers

$$\frac{u_{c_1}}{c_1 - 1} = \frac{u_{c_3} - u_{c_4}}{c_3 - c_4} = \frac{u_{c_2}}{c_2 - 1}.$$

As $c_1, c_2 \in C$ were arbitrary, there exists $d \in \mathbb{R}$ such that $\frac{u_c}{c-1} = -d$ for all $c \in C$, that is, $u_c = d(1 - c)$.

Substituting this into (2.1) we obtain $g(x) = h(c)g(cx + d(1 - c)) + v_c$. Let $g_d(x) = g(x + d)$, so we have $g_d(x - d) = h(c)g_d(c(x - d)) + v_c$. Putting x for $x - d$ implies

$$g_d(x) = h(c)g_d(cx) + v_c \quad (x \in \mathbb{R}, c \in C). \quad (2.5)$$

Plugging in $x = 0$ into this equation we obtain $v_c = g_d(0)(1 - h(c))$, and then writing this back to (2.5) yields $g_d(x) - g_d(0) = h(c)(g_d(cx) - g_d(0))$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = g_d(x) - g_d(0)$. Clearly, f is continuous, and for all $c \in C$ and $x \in \mathbb{R}$

$$f(x) = h(c)f(cx). \quad (2.6)$$

If $n \in \mathbb{Z}$ then applying (2.6) $|n|$ times follows that for all $c \in C$ and $x \in \mathbb{R}$ we have

$$f(x) = h^n(c)f(c^n x). \quad (2.7)$$

Let us now examine $g|_{(-\infty, d)}$.

Case I. $f|_{(-\infty, 0)} \equiv 0$.

This holds iff $g_d|_{(-\infty, 0)} \equiv g_d(0)$ iff $g|_{(-\infty, d)} \equiv g(d)$ iff $g|_{(-\infty, d)}$ is constant (note that g is continuous at d). Hence $g|_{(-\infty, d)}$ is of the required form with $a = g(d), b_1 = 0$ and $s \in \mathbb{R}$ arbitrary. This clearly extends to $x = d$ as well.

Case II. There is an $x_0 < 0$ such that $f(x_0) \neq 0$, that is, $g|_{(-\infty, d)}$ is not constant.

Let us denote by \tilde{C} the multiplicative group generated by C , and let $H: (0, \infty) \rightarrow \mathbb{R}$ be defined by $H(y) = \frac{f(yx_0)}{f(x_0)}$. Then H is continuous and (2.6) follows that $H(c) = \frac{1}{h(c)}$ for all $c \in C$. Every $\tilde{c} \in \tilde{C}$ can be written as $\tilde{c} = \prod_{i=1}^k c_i^{n_i}$ ($k \in \mathbb{N}, n_i \in \mathbb{Z}, c_i \in C, i \in \{1, \dots, k\}$). Then (2.7) implies

$$\begin{aligned} H\left(\prod_{i=1}^k c_i^{n_i}\right) &= \frac{f(c_1^{n_1} \cdots c_k^{n_k} x_0)}{f(x_0)} = \prod_{i=1}^k \frac{f(c_1^{n_1} \cdots c_i^{n_i} x_0)}{f(c_1^{n_1} \cdots c_{i-1}^{n_{i-1}} x_0)} \\ &= \prod_{i=1}^k h^{-n_i}(c_i), \end{aligned}$$

thus H is multiplicative on \tilde{C} . Since C is uncountable, \tilde{C} is dense in $(0, \infty)$. Therefore the continuity of H follows that H is multiplicative. Since H is a non-zero continuous multiplicative function, it is of the form $H(y) = y^s$ ($s \in \mathbb{R}$). Then, on the one hand, $h(c) = \frac{1}{H(c)} = c^{-s}$ for all $c \in C$, so h is of the required form (later we will prove $s > 0$). On the other hand, for an arbitrary $x < 0$ letting $y = \frac{x}{x_0} > 0$ yields

$$\begin{aligned} g_d(x) - g_d(0) &= f(x) = f(yx_0) = f(x_0)H(y) \\ &= f(x_0)y^s = f(x_0)\left(\frac{x}{x_0}\right)^s = f(x_0)\frac{|x|^s}{|x_0|^s}. \end{aligned}$$

Hence, applying the definition of g_d and writing $x - d$ for x gives that for some $s \in \mathbb{R}$

$$g(x) = g(d) + \frac{f(x_0)}{|x_0|^s} |x - d|^s \quad (x < d), \quad (2.8)$$

which is the required form. The continuity of g at d implies that $s \geq 0$. Since $g|_{(-\infty, d)}$ is not constant, we also have $b_1 \neq 0$ and $s > 0$. Then (2.8) holds for $x = d$ as well.

An analogous argument works for $x \geq d$. Since g is not constant, Case II holds either for $x \leq d$ or for $x \geq d$. Therefore there exists an $s > 0$ such that $h(c) = c^{-s}$ for every $c \in C$, and g is of the required form with $(b_1, b_2) \neq (0, 0)$. (Note that s is determined by h , so it does not depend on $x \leq d$ or $x \geq d$, and similarly for a , since $a = g(d)$.) This finishes the proof for U countable and $1 \in C$.

Finally, if U is countable and $1 \in C$ then the above argument for $C \setminus \{1\}$ implies that $g, h|_{C \setminus \{1\}}$ are of the required forms, so it is enough to prove that

$h(1) = 1$. Since $(b_1, b_2) \neq (0, 0)$, we may assume by the symmetry that $b_2 \neq 0$. Applying (2.1) for $c = 1$ and $x > \max\{d, d - u_1\}$ we obtain $a + b_2(x - d)^s = h(1)(a + b_2(x + u_1 - d)^s) + v_1$. That is $h(1) = \lim_{x \rightarrow \infty} \frac{a - v_1 + b_2(x - d)^s}{a + b_2(x + u_1 - d)^s} = 1$. The proof is complete. \square

Remark 2.2. From the above theorem it is easy to get all the solutions of the system of functional equations in question, but we did not write it out that way, since it would have made it very difficult to read.

3 Horizontal rigidity via translations

Theorem 3.1 (C. Richter). *Let $c \in (0, \infty) \setminus \{1\}$ be arbitrary, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function that is horizontally rigid for c via a translation. Then f is constant.*

Proof. The proof of [3, Prop. 3] works with the obvious modifications, just replace x and u by vectors. \square

The following theorem generalizes the one-dimensional result [2, Thm. 6.2]. It will not be used in the sequel, but it is interesting in its own right. Note that we do not assume continuity.

Theorem 3.2. *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is horizontally rigid via translations if and only if there exists a $\vec{r} \in \mathbb{R}^n$ so that f is constant on every open halfline starting from \vec{r} .*

Sketch of the proof. If $\vec{r} \in \mathbb{R}^n$ and f is constant on every open halfline starting from \vec{r} then f is clearly horizontally rigid via translations.

In order to prove the nontrivial direction, we use induction on n . For $n = 1$ the statement is [2, Thm. 6.2]. Suppose that the statement is true for $n - 1$.

First, assume that f is not periodic. Then the proof of [2, Thm. 6.2] applies with the obvious modifications, we just replace x, u, p and r by vectors.

Finally, assume that f is periodic modulo $\vec{p} \in \mathbb{R}^n$ for some $\vec{p} \neq \vec{0}$. As in the proof of [2, Thm. 6.2], we obtain that f is also periodic modulo $c\vec{p}$ for every $c \in (0, \infty)$.

Let φ be an isometry of \mathbb{R}^n . It is not hard to see that if f is horizontally rigid via translations then so is $f \circ \varphi$. Similarly, if f is constant on every open halfline starting from some \vec{r} then so is $f \circ \varphi$. Therefore we may assume that f is periodic modulo every vector of the form $(0, \dots, 0, x_{n+1})$. In other words, f is of the form $g \circ \text{pr}$, where $g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a function and $\text{pr}((x_1, \dots, x_n)) = (x_1, \dots, x_{n-1})$ is the projection of \mathbb{R}^n onto the first $n - 1$ coordinates. Since f is horizontally rigid via translations, for all $c \in (0, \infty)$ there are vectors $\vec{u}_c, \vec{v}_c \in \mathbb{R}^n$ such that $f(c\vec{x}) = f(\vec{x} + \vec{u}_c) + \vec{v}_c$. Then clearly for all $\vec{y} \in \mathbb{R}^{n-1}$ we have $g(c\vec{y}) = g(\vec{y} + \text{pr}(\vec{u}_c)) + \text{pr}(\vec{v}_c)$, so g is also horizontally rigid via translations. By the inductual hypothesis g is of the required form. Thus $f = g \circ \text{pr}$ is also of the required form, and the proof is complete. \square

4 Determining the possible set of directions

Now we start working on the case of general isometries. We follow [4].

Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ denote the unit sphere. For a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ let S_f be the set of directions between pairs of points on the graph of f , that is,

Definition 4.1.

$$S_f = \left\{ \frac{p - q}{|p - q|} \in \mathbb{S}^2 : p, q \in \text{graph}(f), p \neq q \right\}.$$

Recall that a *great circle* is a circle line in \mathbb{R}^3 of radius 1 centered at the origin. We call it *vertical* if it passes through the points $(0, 0, \pm 1)$.

Definition 4.2. Let $h: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be defined as follows. Every $\vec{x} \in \mathbb{S}^1$ is in a unique half great circle connecting $(0, 0, 1)$ and $(0, 0, -1)$. It is not hard to see using the continuity of f that the intersection of S_f with this great circle is an arc. Define $h(\vec{x})$ as the top endpoint of this arc.

Clearly, the bottom endpoint of this arc is $-h(-\vec{x})$, so the ‘top function bounding the strip S_f is $h(\vec{x})$ and the bottom function is $-h(-\vec{x})$ ’. The coordinate functions of h are denoted by (h_1, h_2, h_3) , where $h_3: \mathbb{S}^1 \rightarrow [-1, 1]$ encodes all information about h .

Definition 4.3. For $c > 0$ let $\psi_c: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ denote the map that ‘deforms S_f according to the map $c \mapsto f(c \cdot)$ ’, that is,

$$\psi_c((x, y, z)) = \frac{\left(\frac{x}{c}, \frac{y}{c}, z\right)}{\left|\left(\frac{x}{c}, \frac{y}{c}, z\right)\right|} = \frac{(x, y, cz)}{|(x, y, cz)|} \quad ((x, y, z) \in \mathbb{S}^2).$$

The above equation shows that the map that ‘deforms S_f according to the map $c \mapsto cf$ ’ is also ψ_c . This is the key connection between vertical and horizontal rigidity. This explains why we can replace vertically rigid functions by horizontally rigid functions in [4, Thm. 5.3] and obtain the following theorem by the same proof.

Theorem 4.4. *Let $C \subseteq (0, \infty)$ be a set condensating to ∞ , and let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function horizontally rigid for C . Then one of the following holds.*

- **Case A.** *There is a vertical great circle that intersects S_f in only two points.*
- **Case B.** $S_f = \mathbb{S}^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$.
- **Case C.** *There exists an $\vec{x}_0 \in \mathbb{S}^1$ such that $h_3(\vec{x}_0) = 0$ and $h_3(\vec{x}) = 1$ for every $\vec{x} \neq \vec{x}_0$, that is, S_f is ‘ \mathbb{S}^2 minus two quarters of a great circle’.*
- **Case D.** *There exists a closed interval I in \mathbb{S}^1 with $0 < \text{length}(I) < \pi$ such that $h_3(\vec{x}) = 0$ if $\vec{x} \in I$, and $h_3(\vec{x}) = 1$ if $\vec{x} \notin I$, that is, S_f is ‘ \mathbb{S}^2 minus two spherical triangles’.*

5 The end of the proof

This section builds heavily on the corresponding section of [4].

The following lemma will be useful, the easy proof is left to the reader.

Lemma 5.1. *Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be horizontally rigid for c_0 via φ_0 and for c via φ . Then $f(c_0 \cdot)$ is horizontally rigid for $\frac{c}{c_0}$ via $\varphi \circ \varphi_0^{-1}$.*

The following technical lemma is [4, Lemma 2.1].

Lemma 5.2. *Let $f(x, y) = g(x) + dy$, where $d > 0$ and let $c > 0$. If we rotate $\text{graph}(f)$ around the x -axis by the angle $\alpha_c = \arctan(cd) - \arctan(d)$ then the intersection of this rotated graph with the xy -plane is the graph of a function of the form $y = -w_{c,d}g(x)$, where $w_{c,d} = \frac{1}{cd} \sqrt{\frac{(cd)^2 + 1}{d^2 + 1}}$.*

Remark 5.3. Let φ_c be the isometry mapping $\text{graph}(f)$ onto $\text{graph}(f(c \cdot))$. Every isometry φ is of the form $\varphi^{\text{trans}} \circ \varphi^{\text{ort}}$, where φ^{ort} is an orthogonal transformation and φ^{trans} is a translation. Moreover, if φ is orientation-preserving then φ^{ort} is a rotation around a line passing through the origin. A key observation is the following: The horizontal rigidity of f for C implies that $\psi_c(S_f) = \varphi_c^{\text{ort}}(S_f)$ for every $c \in C$.

Now we complete the proof of the technical form of the Main Theorem. We repeat the statement here.

Theorem 5.4 (Main Theorem, technical form). *Let $C \subseteq (0, \infty)$ be a set condensing to ∞ . Then a continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is horizontally rigid for C if and only if it is of the form $a + bx + dy$ ($a, b, d \in \mathbb{R}$).*

Proof. By Theorem 4.4 it suffices to consider Cases A-D.

Case A. There is a vertical great circle that intersects S_f in only two points.

We may assume using a suitable rotation around the z -axis that the vertical great circle is in the yz -plane, hence $f(x, y)$ is of the form $g(x) + dy$. (Note that both the class of affine functions and the class of functions that are horizontally rigid for C are closed under rotations around the z -axis.) The continuity of f implies that g is also continuous.

Subcase A1. $d = 0$.

Let $c \in C$ be fixed, and let φ_c be the corresponding isometry. The graph of $f(c \cdot)$ is invariant under translations parallel to the y -axis. As the same holds for f , by rigidity, $f(c \cdot)$ is also invariant under translations parallel to the φ_c -image of the y -axis. If these two directions are nonparallel, then $\text{graph}(f(c \cdot))$ is a plane, and hence so is $\text{graph}(f)$. Therefore we may assume that all lines parallel to the y -axis are taken to lines parallel to the y -axis, but then all planes parallel to the xz -plane are taken to planes parallel to the xz -plane. But this shows (by considering the intersections of the graphs with the xz -plane) that g is horizontally rigid for c , hence $g(x)$ is of the form $a + bx$ ($a, b \in \mathbb{R}$) by Theorem 1.4, and we are done.

Subcase A2. $d \neq 0$.

We may assume that $d > 0$, since otherwise we may consider $-f$.

For every $c \in C$ let φ_c be the corresponding isometry. We claim that we may assume that all these are orientation-preserving. First, if $\{c \in C : \varphi_c \text{ is orientation-preserving}\}$ condensates to ∞ then we are done by shrinking C . Otherwise, we may assume that they are all orientation-reversing (note that if we split C into two pieces then at least one of them still condensates to ∞). Let us fix a $c_0 \in C$ and consider $c_0 f$ instead of f . By Lemma 5.1 this function is rigid for an uncountable set with all isometries orientation-preserving, and if it is of the desired form then so is f , so we are done.

We may assume $1 \notin C$. Let us fix a $c \in C$. Similarly as in the previous subcase, we may assume that lines parallel to $(0, 1, d)$ are taken to lines parallel to $(0, 1, cd)$ as follows. The special form of f implies that $\text{graph}(f)$ is invariant under translations in the $(0, 1, d)$ -direction, hence $\text{graph}(f(c\cdot))$ is invariant under translations in the $(0, 1, cd)$ -direction, moreover, by rigidity, $\text{graph}(f(c\cdot))$ is also invariant under translations parallel to the φ_c -image of the lines of direction $(0, 1, d)$. If these two latter directions do not coincide then $\text{graph}(f(c\cdot))$ is a plane, and we are done.

Therefore the image of every line parallel to $(0, 1, d)$ is a line parallel to $(0, 1, cd)$ under the orientation-preserving isometry φ_c . As in Remark 5.3, write $\varphi_c = \varphi_c^{\text{trans}} \circ \varphi_c^{\text{ort}}$, where φ_c^{ort} is a rotation about a line containing the origin and φ_c^{trans} is a translation. Since the translation does not affect directions, the rotation φ_c^{ort} takes the direction $(0, 1, d)$ to the nonparallel direction $(0, 1, cd)$ ($d \neq 0$), therefore the axis of the rotation has to be orthogonal to the plane spanned by these two directions. Hence the axis has to be the x -axis. Moreover, the angle of the rotation is easily seen to be $\arctan(cd) - \arctan(d)$.

We now show that we may assume that φ_c^{trans} is a horizontal translation. Decompose the translation as $\varphi_c^{\text{trans}} = \varphi_c^{\vec{u}} \circ \varphi_c^{\vec{v}}$, where $\varphi_c^{\vec{v}}$ is a horizontal translation and $\varphi_c^{\vec{u}}$ is a translation in the $(0, 1, cd)$ -direction. Since $\varphi_c^{\text{ort}}(\text{graph}(f))$ is invariant under translations in the $(0, 1, cd)$ -direction, so is $\varphi_c^{\vec{v}} \circ \varphi_c^{\text{ort}}(\text{graph}(f))$, hence

$$\begin{aligned} \varphi_c^{\vec{v}} \circ \varphi_c^{\text{ort}}(\text{graph}(f)) &= \varphi_c^{\vec{u}} \circ \varphi_c^{\vec{v}} \circ \varphi_c^{\text{ort}}(\text{graph}(f)) \\ &= \varphi_c(\text{graph}(f)) = \text{graph}(f(c\cdot)), \end{aligned}$$

so we can assume $\varphi_c = \varphi_c^{\vec{v}} \circ \varphi_c^{\text{ort}}$, and we are done.

Let us denote the xy -plane by $\{z = 0\}$ and consider the intersection of both sides of the equation $\varphi_c(\text{graph}(f)) = \text{graph}(f(c\cdot))$ with $\{z = 0\}$. On the one hand,

$$\begin{aligned} \{z = 0\} \cap \varphi_c(\text{graph}(f)) &= \{z = 0\} \cap \varphi_c^{\vec{v}} \circ \varphi_c^{\text{ort}}(\text{graph}(f)) \\ &= \varphi_c^{\vec{v}}(\{z = 0\} \cap \varphi_c^{\text{ort}}(\text{graph}(f))) \\ &= \varphi_c^{\vec{v}}(\text{graph}(-w_{c,d}g)), \end{aligned}$$

where we used the fact that $\varphi_c^{\vec{v}}$ is horizontal and Lemma 5.2. On the other

hand, it is easy to see that

$$\{z = 0\} \cap \text{graph}(f(c)) = \text{graph}\left(-\frac{1}{cd}g(c)\right).$$

If $\vec{v} = (v_{c,1}, v_{c,2})$ then we obtain for all $c \in C$ and $x \in \mathbb{R}$

$$-w_{c,d}g(x - v_{c,1}) + v_{c,2} = -\frac{1}{cd}g(cx).$$

For every $c \in C$ let $h_c = \frac{1}{cdw_{c,d}} = \sqrt{\frac{d^2+1}{(cd)^2+1}}$, $u_c = cv_{c,1}$ and $v_c = \frac{v_{c,2}}{w_{c,d}}$. Reordering the above functional equations and substituting $x + v_{c,1}$ for x yield for all $c \in C$ and $x \in \mathbb{R}$

$$g(x) = h_c g(cx + u_c) + v_c.$$

Then Theorem 2.1 implies that either g is constant or $h_c = c^{-s}$ for every $c \in C$. But the latter cannot hold, since $\sqrt{\frac{d^2+1}{(xd)^2+1}}$ and x^{-s} are clearly nonidentical analytic functions, so they cannot coincide on the uncountable set C . Hence g is constant, and the proof of Case A is complete.

Case B. $S_f = \mathbb{S}^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$.

S_f is invariant under every ψ_c , and hence so is under every φ_c^{ort} . Then clearly $\varphi_c^{ort}((0, 0, 1)) = (0, 0, 1)$ or $\varphi_c^{ort}((0, 0, 1)) = (0, 0, -1)$ for every $c \in C$. By the same argument as above, we can assume that the former holds for every $c \in C$. Using the argument again we can assume that all φ_c 's are orientation-preserving. But then each of these is a rotation around the z -axis followed by a translation, in other words, an orientation-preserving transformation in the xy -plane followed by a translation in the z -direction. An orientation-preserving transformation in the plane is either a translation or a rotation. If it is a translation for every c then we are done by Theorem 3.1. So let us assume that there exists a c such that φ_c is a proper rotation around $\vec{x} \in \mathbb{R}^2$ followed by a vertical translation. We claim that then f is constant, which will contradict that S_f is nearly the full sphere, finishing the proof of this case. We may assume that $c > 1$, since otherwise we consider $f(c \cdot)$ instead of f and replace c and φ_c by $1/c$ and φ_c^{-1} , respectively. Let $R > 0$ be so large such that $B(\vec{x}/c, R/c) \subseteq B(\vec{x}, R)$. It is enough to show that f is constant on $B(\vec{x}, R)$. Assume on the contrary that this is not the case, and let $\delta > 0$ be the distance between the compact sets where f attains its minimum and maximum on $B(\vec{x}, R)$. Choose one point of the two sets each, \vec{x}_{min} and \vec{x}_{max} , such that $\delta = |\vec{x}_{min} - \vec{x}_{max}|$. On the one hand, the distance between the sets where the minimum and maximum are attained should still be δ for $f(c \cdot)$, since it is not affected by rotations around \vec{x} or vertical translations. But on the other hand, the function $f(c \cdot)$ takes the same values in \vec{x}_{min}/c and \vec{x}_{max}/c as f in \vec{x}_{min} and \vec{x}_{max} , respectively. Therefore $\vec{x}_{min}/c, \vec{x}_{max}/c \in B(\vec{x}/c, R/c) \subseteq B(\vec{x}, R)$. Thus the distance between the sets where $f(c \cdot)$ attains $\min_{B(\vec{x}, R)} f$ and $\max_{B(\vec{x}, R)} f$ is at most $|\vec{x}_{min}/c - \vec{x}_{max}/c| = \delta/c < \delta$, a contradiction.

Case C. There exists an $\vec{x}_0 \in \mathbb{S}^1$ such that $h_3(\vec{x}_0) = 0$ and $h_3(\vec{x}) = 1$ for every $\vec{x} \neq \vec{x}_0$, that is, S_f is ' \mathbb{S}^2 minus two quarters of a great circle'.

Thus S_f is invariant under every ψ_c , and hence so is under every φ_c^{ort} . Hence φ_c^{ort} maps $(0, 0, 1)$ to one of the four endpoints of the two arcs. Therefore we can assume by splitting C into four pieces according to the image of $(0, 0, 1)$ and applying Lemma 5.1 that $(0, 0, 1)$ is a fixed point of every φ_c^{ort} . But then the two arcs are also fixed, and actually φ_c^{ort} is the identity. Hence every φ_c is a translation, and we are done by Theorem 3.1.

Case D. There exists a closed interval I in \mathbb{S}^1 with $0 < \text{length}(I) < \pi$ such that $h_3(\vec{x}) = 0$ if $\vec{x} \in I$ and $h_3(\vec{x}) = 1$ if $\vec{x} \notin I$, that is, S_f is ‘ \mathbb{S}^2 minus two spherical triangles’.

As S_f is invariant under every φ_c^{ort} , vertices of the triangles are mapped to vertices. Hence we may assume (by splitting C into six pieces) that $(0, 0, 1)$ is fixed. But then the triangles are also fixed sets, and every φ_c^{ort} is the identity, so we are done as in the previous case.

This finishes the proof of the Main Theorem. \square

6 Open questions

All three questions below are somewhat analogous to the ones asked in [4].

Question 6.1. *Which notion of largeness of C suffices for the Main Theorem? For example, does the Main Theorem hold if we only assume that C is uncountable or C generates a dense subgroup of $(0, \infty)$ or C has at least two elements different from 1?*

Question 6.2. *Let us call a set $H \subseteq \mathbb{S}^2$ rigid if $\psi_c(H)$ is isometric to H for every $c > 0$. Is there a simple description of rigid sets? Or if we assume some regularity?*

And finally, the most intriguing problem.

Question 6.3. *What can we say if there are more than two variables? Is every continuous horizontally rigid function affine?*

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