

ITP Budapest Report No. 617

Finite size effects in quantum field theories with boundary from scattering data

Z. Bajnok^{1*}, L. Palla^{2†}, and G. Takács^{1‡}

¹*Theoretical Physics Research Group, Hungarian Academy of Sciences,
1117 Budapest, Pázmány Péter sétány 1/A, Hungary*

²*Institute for Theoretical Physics, Eötvös University
1117 Budapest, Pázmány Péter sétány 1/A, Hungary*

6th November 2018

Abstract

We derive a relation between leading finite size corrections for a 1+1 dimensional quantum field theory on a strip and scattering data, which is very similar in spirit to the approach pioneered by Lüscher for periodic boundary conditions. The consistency of the results is tested both analytically and numerically using thermodynamic Bethe Ansatz, Destri-de Vega nonlinear integral equation and classical field theory techniques. We present strong evidence that the relation between the boundary state and the reflection factor one-particle couplings, noticed earlier by Dorey et al. in the case of the Lee-Yang model extends to any boundary quantum field theory in 1+1 dimensions.

1 Introduction

Finite size corrections are important tools in QFT and their use is mandatory in interpreting numerical data. Nearly two decades ago M. Lüscher [1] gave a general expression for the leading finite size correction of particle masses and scattering states valid in any QFT with periodic boundary conditions, in terms of the infinite volume scattering data.

*bajnok@afavant.elte.hu

†palla@ludens.elte.hu

‡takacs@ludens.elte.hu

Lüscher's result was specialized to 1+1 dimensional field theory [2], using the exact scattering amplitudes known in integrable field theories. For example, in a theory with a single particle of mass m and elastic scattering amplitude $S(\theta)$ the leading correction to the particle mass in finite volume is the following:

$$\begin{aligned} \frac{\delta m}{m}(L) &= -\frac{\sqrt{3}}{2}i \left(\operatorname{Res}_{\theta=\frac{2\pi i}{3}} S(\theta) \right) e^{-\frac{\sqrt{3}}{2}mL} \\ &\quad - \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta \left(S\left(\theta + \frac{i\pi}{2}\right) - 1 \right) e^{-mL \cosh \theta} + O\left(e^{-\sqrt{3}mL}\right) \end{aligned} \quad (1.1)$$

(similar formulae exist for general particle spectra). The first term (which, if present, dominates over the second) appears whenever the particle occurs as a bound state of itself, and is proportional to the residue of the S -matrix at the pole, while the second term is present in general and is of order $L^{-1/2}e^{-mL}$.

In integrable 1+1 dimensional models restricted to a spacetime strip of finite width L with integrable boundary conditions, some effort has been devoted recently to the determination of the leading corrections to low lying – mainly ground-state – energy levels as functions of L in classical field theory [3, 4]. However, these investigations did not look for a general, model independent form of the asymptotic correction to the infinite volume ground state energy, which is the goal of the present work to consider. In addition, the results of the present work will be derived at the quantum level, and it is shown that they agree to the results in [3, 4] in the classical limit.

In this paper we derive a formula, similar in spirit to Lüscher's result in the case of periodic boundary conditions, which expresses the finite size correction to the energy of a state containing no particles (but possibly excited boundary states) in terms of the reflection amplitudes on the semi-infinite line. Once again, the leading term depends on the presence of certain poles in the reflection factors. When such poles are present we get corrections of the form e^{-mL} (for a particle of mass m) with coefficients that depend on the residue of the respective pole, otherwise the leading correction is given by a more complicated integral containing the reflection amplitudes and is of order $L^{-1/2}e^{-2mL}$. We show that these results are consistent with the boundary version of thermodynamic Bethe Ansatz (TBA). Using the general results we determine the form of leading finite size corrections in boundary sine-Gordon theory, which we check against results from a nonlinear integral equation (NLIE) in the Dirichlet case, and compare them to the classical field theory results in [3, 4].

2 Cluster expansion and finite size corrections

For simplicity let us consider a theory with only one scalar particle in the bulk. Assume further that the theory is defined on a strip of width L , with boundary conditions labeled by α and β such that the elastic reflection amplitudes of the particle on these boundaries on the semi-infinite line, $R_\alpha(\theta)$ and $R_\beta(\theta)$ are known. To derive the asymptotic ($L \rightarrow \infty$)

form of the ground state energy $E_{\alpha\beta}^0(L)$, we consider the partition function of the theory on a cylinder in two alternative forms:

$$Z_{\alpha\beta}(R, L) = \text{Tr} e^{-RH_{\alpha\beta}(L)} = \langle \alpha | e^{-LH(R)} | \beta \rangle$$

where $H(R)$ denotes the Hamiltonian of the theory in the crossed channel (on a circle of circumference R), $|\alpha\rangle$ and $|\beta\rangle$ are the appropriate boundary states á la Ghoshal and Zamolodchikov [5], while $H_{\alpha\beta}(L)$ is the Hamiltonian on the strip. In the limit $R \rightarrow \infty$ we obtain

$$e^{-RE_{\alpha\beta}^0(L)} \sim \langle \alpha | e^{-LH(R)} | \beta \rangle$$

The right hand side can be expanded for large L in a so-called cluster expansion (the analogue of which was recently used in [6] for the TBA in the other channel to settle long standing issues concerning the boundary entropy function). Inserting a complete set of eigenstates of the Hamiltonian $H(R)$ on the circle we obtain

$$\langle \alpha | e^{-LH(R)} | \beta \rangle = \sum_{n \in \mathcal{H}_{\text{bulk}}} \frac{\langle \alpha | n \rangle \langle n | \beta \rangle}{\langle n | n \rangle} e^{-LE_n(R)}$$

For our purposes, only the first few terms are interesting:

$$\begin{aligned} \langle \alpha | e^{-LH(R)} | \beta \rangle &= \frac{\langle \alpha | 0 \rangle \langle 0 | \beta \rangle}{\langle 0 | 0 \rangle} e^{-LE_0(R)} \\ &+ \sum_{\theta_n} \frac{\langle \alpha | \theta_n \rangle \langle \theta_n | \beta \rangle}{\langle \theta_n | \theta_n \rangle} e^{-L(E_0(R) + m \cosh \theta_n)} \\ &+ \sum_{\theta_n, \theta_m} \frac{\langle \alpha | \theta_n, \theta_m \rangle \langle \theta_n, \theta_m | \beta \rangle}{\langle \theta_n, \theta_m | \theta_n, \theta_m \rangle} e^{-L(E_0(R) + m \cosh \theta_n + m \cosh \theta_m)} \\ &+ \text{three particle contributions} + \dots \end{aligned} \quad (2.1)$$

In the limit $R \rightarrow \infty$ the boundary state $|\beta\rangle$ has the general form [5]

$$|\beta\rangle = \left(1 + \int_0^\infty \frac{d\theta}{2\pi} K_\beta(\theta) A^\dagger(-\theta) A^\dagger(\theta) + \dots \right) |0\rangle \quad (2.2)$$

(similarly for $|\alpha\rangle$) where the dots denote terms containing more particles, $A^\dagger(\theta)$ creates an asymptotic particle of rapidity θ and

$$K_\beta(\theta) = R_\beta \left(\frac{i\pi}{2} - \theta \right) \quad .$$

Normalizing the bulk vacuum energy to 0 and collecting the leading (two-particle) terms for large L yields

$$\begin{aligned} Z_{\alpha\beta}(R, L) &= 1 + mR \int_0^\infty \frac{d\theta}{2\pi} \cosh \theta \bar{K}_\alpha(\theta) K_\beta(\theta) e^{-2mL \cosh \theta} + \dots \\ \bar{K}_\alpha(\theta) &= K_\alpha(-\theta) = R_\alpha \left(\frac{i\pi}{2} + \theta \right) \end{aligned}$$

where we replaced the sums by integrals over the density of states

$$\sum_{\theta_n} \rightarrow mR \int \frac{d\theta}{2\pi} \cosh \theta$$

and used real analyticity of the reflection factor

$$R(\theta)^* = R(-\theta^*)$$

For the energy we obtain

$$E_{\alpha\beta}(L) = -m \int_0^\infty \frac{d\theta}{2\pi} \cosh \theta \bar{K}_\alpha(\theta) K_\beta(\theta) e^{-2mL \cosh \theta} \quad (2.3)$$

One can generalize the above argument to the case when there is a multiplet of particle of mass m . In that case the reflection factor is a matrix

$$R_\beta(\theta)_s^r$$

where r and s denote the multiplet labels. The boundary state takes the form [5]

$$|\beta\rangle = \left(1 + \sum_{r,s} \int_0^\infty \frac{d\theta}{2\pi} K_\beta(\theta)^{rs} A_r^\dagger(-\theta) A_s^\dagger(\theta) + \dots \right) |0\rangle$$

where

$$K_\beta(\theta)^{r\bar{s}} = R_\beta\left(i\frac{\pi}{2} - \theta\right)_s^r$$

and \bar{s} denotes the charge conjugate of multiplet member s . The leading finite size correction takes the form

$$E_{\alpha\beta}(L) = -m \int_0^\infty \frac{d\theta}{2\pi} \cosh \theta \text{Tr} \left(\bar{K}_\alpha(\theta) K_\beta(\theta) \right) e^{-2mL \cosh \theta} \quad (2.4)$$

Let us turn now again to the case of diagonal scattering. If the theory is integrable then the boundary state is known exactly

$$|\beta\rangle = \exp \left(\int_0^\infty K_\beta(\theta) A^\dagger(-\theta) A^\dagger(\theta) \frac{d\theta}{2\pi} \right) |0\rangle \quad (2.5)$$

and summing up the cluster expansion one can derive a TBA equation for the ground state energy as done by Leclair et al. in [7], which is analyzed in detail in the next section. The large volume limit of this TBA is exactly given by the formula (2.3).

However, if there is a pole in the reflection factor at the imaginary rapidity $\theta = i\pi/2$ of the form

$$R_\beta(\theta) \sim \frac{ig_\beta^2}{2\theta - i\pi} \quad (2.6)$$

the boundary state has an extra one-particle contribution

$$|\beta\rangle = \left(1 + \tilde{g}_\beta A^\dagger(0) + \int_0^\infty \frac{d\theta}{2\pi} K_\beta(\theta) A^\dagger(-\theta) A^\dagger(\theta) + \dots\right) |0\rangle \quad (2.7)$$

or in the case of an integrable field theory one has

$$|\beta\rangle = (1 + \tilde{g}_\beta A^\dagger(0)) \exp\left(\int_0^\infty \frac{d\theta}{2\pi} K_\beta(\theta) A^\dagger(-\theta) A^\dagger(\theta)\right) |0\rangle \quad (2.8)$$

(due to the exclusion principle, adding the one-particle term to the exponent makes no difference). Ghoshal and Zamolodchikov [5] identify \tilde{g}_β with g_β , but Dorey et al. in [8] found the relation

$$\tilde{g}_\beta = \frac{g_\beta}{2} \quad (2.9)$$

comparing one-point functions calculated from the boundary state using form factor methods to TCSA results in the case of the Lee-Yang model. As we show below, our results are fully consistent with this latter suggestion. Namely, using the state (2.7) in the expansion (2.1) one gets for the leading term the result

$$Z_{\alpha\beta}(R, L) = 1 + mR\tilde{g}_\alpha\tilde{g}_\beta e^{-mL} + \dots$$

and so

$$E_{\alpha\beta}(L) = -m\frac{g_\alpha g_\beta}{4} e^{-mL} + \dots \quad (2.10)$$

The existence of this contribution requires that both left and right boundary reflection matrices have a pole at $\theta = i\pi/2$. It is also important to note that if one-particle terms are present in the boundary state, the two-particle term of the cluster expansion is divergent and needs to be regularized.

Hereafter we shall refer to (2.3,2.10) as the *boundary Lüscher formulae* because they are a natural generalization of Lüscher's original results [1] to the case of quantum field theory defined on a strip.

3 BTBA in the infrared limit

In order to lend support to relation (2.9) between the one-particle coupling constant appearing in the reflection factor and the coefficient of the one-particle term in the boundary state, we calculate the infrared limit of boundary TBA (BTBA). Although the original derivation given by Leclair et al. [7] only considers the case when there is no one-particle term in the boundary state, it can be easily argued that the presence of one-particle terms makes no difference to the end result, and this is also supported by numerical studies using comparison with truncated conformal space (TCS) by Dorey et al. in [9].

For simplicity we consider a theory with a single particle of mass m on a strip of length L . The BTBA equation takes the form

$$\epsilon(\theta) = 2mL \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi(\theta - \theta') \log\left(1 + \chi(\theta') e^{-\epsilon(\theta')}\right) \quad (3.1)$$

where

$$\Phi(\theta) = -i \frac{d}{d\theta} \log S(\theta)$$

is the derivative of the phase of the two-particle S -matrix $S(\theta)$, and

$$\chi(\theta) = \bar{K}_\alpha(\theta) K_\beta(\theta)$$

using the notations of the previous section. Using crossing-unitarity [5]

$$K_\alpha(\theta) = S(2\theta) \bar{K}_\alpha(\theta) \tag{3.2}$$

and unitarity

$$S(\theta)S(-\theta) = 1$$

it is easy to see that $\chi(\theta)$ is an even function.

The energy of the ground state is given by

$$E_{\alpha\beta}(L) = -m \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta \log(1 + \chi(\theta) e^{-\epsilon(\theta)}) \tag{3.3}$$

In the case where there is no one-particle coupling, $\chi(\theta)$ is regular on the real axis and for large L

$$\epsilon(\theta) = 2mL \cosh \theta + O(e^{-2mL}) \tag{3.4}$$

which gives the following asymptotics for the energy:

$$E_{\alpha\beta}(L) = -m \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh(\theta) \chi \theta e^{-2mL \cosh \theta} + O(e^{-4mL})$$

in perfect agreement with (2.3). The same is true for the case when only one of the boundaries has a nonzero one-particle coupling. The corresponding K factor has a simple pole at $\theta = 0$, but since the other K is regular and χ is an even function, χ has no singularity at all. This can be understood by recalling that generally

$$S(0) = -1$$

so crossing-unitarity (3.2) entails

$$K_\alpha(0) = -K_\alpha(0)$$

and as a result, K factors must have either a zero or a pole at $\theta = 0$. Even if there is a first order pole in one of the K factors corresponding to a one-particle coupling to the boundary, the product of the two K factors is still regular.

However, for a theory with nonzero one-particle coupling on both boundaries χ has a second-order pole at $\theta = 0$. The logarithmic terms in (3.1) and (3.3) remain integrable and

the BTBA makes perfectly good sense, but to obtain the correct asymptotic expansion one needs to be careful. For large L , the energy can be written in the form

$$E_{\alpha\beta}(L) = -m \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta \log (1 + \chi(\theta) e^{-2mL \cosh \theta}) \quad (3.5)$$

Now we cannot use $\log(1+x) \sim x$ to expand the logarithm as $\chi(\theta)$ diverges at $\theta = 0$. In fact, using (2.6) this divergence can be computed as

$$\chi(\theta) \sim \frac{g_\alpha^2 g_\beta^2}{4\theta^2} \sim \frac{g_\alpha^2 g_\beta^2}{4 \sinh^2 \theta}$$

We can use the following integral formula [14]

$$\int_{-\infty}^{\infty} dx \log \frac{a^2 + x^2}{b^2 + x^2} = 2\pi(a - b) \quad , a, b \geq 0 \quad (3.6)$$

to evaluate the energy in the following way

$$\begin{aligned} E_{\alpha\beta}(L) &= -m \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta \left[\log \left(1 + \frac{g_\alpha^2 g_\beta^2}{4 \sinh^2 \theta} e^{-2mL} \right) + \log \left(\frac{1 + \chi(\theta) e^{-2mL \cosh \theta}}{1 + \frac{g_\alpha^2 g_\beta^2}{4 \sinh^2 \theta} e^{-2mL}} \right) \right] \\ &= -m \frac{|g_\alpha g_\beta|}{4} e^{-mL} - m \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta \left[\log \left(\frac{1 + \chi(\theta) e^{-2mL \cosh \theta}}{1 + \frac{g_\alpha^2 g_\beta^2}{4 \sinh^2 \theta} e^{-2mL}} \right) \right] \end{aligned} \quad (3.7)$$

The remaining integral is an expression of order e^{-2mL} , while the leading term agrees with (2.9,2.10) for $g_\alpha g_\beta > 0$.

It was already noted in [9] (using a comparison with truncated conformal space method) that the BTBA equation only gives the correct ground state energy in this case, and for values of boundary parameters such that $g_\alpha g_\beta < 0$ a suitable analytic continuation must be applied.¹ Note also that in the definition of the boundary coupling g_β from (2.6) a branch of the square root function must be chosen. In all known cases (e.g. Lee-Yang in [9] or sine-Gordon in the following section) there exists a branch choice such that the boundary couplings depend analytically on the boundary parameters. In this case a straightforward analytic continuation from the range of parameters for which $g_\alpha g_\beta > 0$ gives the result

$$E_{\alpha\beta}(L) = -m \frac{g_\alpha g_\beta}{4} e^{-mL} + \dots$$

in the parameter range where $g_\alpha g_\beta < 0$, which is consistent with (2.9,2.10). Furthermore, the result (3.7) also yields a formula for the regularized two-particle contribution. The two-particle contribution is regularized by the logarithm in (3.5), which represents a partial

¹Using the notation of paper [9], the regime $g_\alpha g_\beta > 0$ corresponds to the boundary parameter range $-2 < b < 2$ (for boundary conditions $\alpha = \mathbb{I}$ and $\beta = \Phi(h)$, with the boundary parameter h related to b), which is the range in which they found that BTBA compares well with truncated conformal space. For parameters out of this range the authors needed to resort to analytic continuation in order to match the numerical data.

re-summation of the higher multi-particle contributions, and the bulk interaction plays no role at leading order in large volume. We conjecture that the leading asymptotics calculated from (3.7) is valid also for non-integrable theories, since for large L the leading contribution comes from a very small region around $\theta = 0$ and for such small values of the rapidity the inelastic scattering channels should play no role provided m is the smallest particle mass in the spectrum.

4 Application to sine-Gordon theory with integrable boundary conditions

Next we apply these results to sine-Gordon theory on a strip with integrable boundary conditions of Ghoshal-Zamolodchikov type [5]. The model has a complicated spectrum with non-diagonal bulk and boundary scattering. However, to derive the leading term (2.3,2.10) we used only (2.2,2.7) which is valid in general field theories (even non-integrable ones) in two dimensions, as follows from the derivation of [5]. Integrability is only needed to derive the exact exponentiated form of the boundary state (2.5,2.8). Besides that, the derivation of Section 2 carries over to the case of non-diagonal boundary scattering in a straightforward manner, while the effects of bulk scattering appear only in higher correction terms.

The action of the model is

$$\mathcal{A} = \int_{-\infty}^{\infty} dt \left[\int_0^L dx \mathcal{L}_{SG}(x, t) - V_B^{(0)}(\Phi(0, t)) - V_B^{(L)}(\Phi(L, t)) - \alpha \frac{\partial \Phi(L, t)}{\partial t} \right] \quad (4.1)$$

where the bulk Lagrangian density is given by

$$\mathcal{L}_{SG}(x, t) = \frac{1}{2} (\partial_\mu \Phi(x, t))^2 - \frac{m^2}{\beta^2} (1 - \cos \beta \Phi)$$

and the boundary interaction is described by the potentials

$$V_B^{(z)}(\Phi(z, t)) = M_z \left[1 - \cos \frac{\beta}{2} (\Phi(z, t) - \Phi_z) \right] \quad , \quad z = 0, L \quad (4.2)$$

(the $M_z \rightarrow \infty$ limit corresponds to Dirichlet boundary conditions). The last term (proportional to α) is allowed by integrability, and on the strip – in contrast to the theory on the half line – it cannot be eliminated from the action by an appropriate “gauge” transformation [10]. The spectrum is constituted by the solitons and breathers, and the scattering theory on the half line is known. The reflection amplitudes on the two boundaries can be parameterized with the Ghoshal-Zamolodchikov parameters η_z and ϑ_z ($z = 0, L$).

Using a perturbed conformal field theory description one can write the action of the quantum theory as

$$\mathcal{A}_{\text{pCFT}} = \mathcal{A}_{c=1} + \mu \int_{-\infty}^{\infty} dt \int_{-\infty}^0 dx : \cos \beta \Phi : + \tilde{\mu}_0 \int_{-\infty}^{\infty} dt : \cos \frac{\beta}{2} (\Phi(0, t) - \Phi_0) :$$

$$\begin{aligned}
& + \tilde{\mu}_L \int_{-\infty}^{\infty} dt : \cos \frac{\beta}{2} (\Phi(L, t) - \Phi_L) : \\
\mathcal{A}_{c=1} & = \int_{-\infty}^{\infty} dt \int_{-\infty}^0 dx \frac{1}{2} (\partial_\mu \Phi(x, t))^2 - \alpha \int_{-\infty}^{\infty} dt \frac{\partial \Phi(L, t)}{\partial t}
\end{aligned}$$

The parameters of this action are related to the boundary parameters η, ϑ by the UV-IR relation [11]

$$\begin{aligned}
\frac{\tilde{\mu}_z}{\tilde{\mu}_{\text{crit}}} \sin \frac{\beta \Phi_z}{2} & = -(-1)^{z/L} \sin \frac{\eta_z}{\lambda + 1} \sinh \frac{\vartheta_z}{\lambda + 1} \\
\frac{\tilde{\mu}_z}{\tilde{\mu}_{\text{crit}}} \cos \frac{\beta \Phi_z}{2} & = \cos \frac{\eta_z}{\lambda + 1} \cosh \frac{\vartheta_z}{\lambda + 1}
\end{aligned} \tag{4.3}$$

where

$$\tilde{\mu}_{\text{crit}} = \sqrt{\frac{2\mu}{\sin \frac{\beta^2}{8}}} \quad , \quad \lambda = \frac{8\pi}{\beta^2} - 1 \tag{4.4}$$

and the sign factor $-(-1)^{z/L}$ arises from the fact that on the left end ($z = 0$) of the strip the role of the soliton and antisoliton are interchanged with respect to the right end ($z = L$). In the classical limit $\tilde{\mu}_z$ can be replaced by M_z and μ by m^2/β^2 from (4.1), and the η, ϑ parameters must be scaled as [13]

$$\eta = \eta_{\text{cl}} (1 + \lambda) \quad , \quad \vartheta = \vartheta_{\text{cl}} (1 + \lambda) \tag{4.5}$$

keeping $\eta_{\text{cl}}, \vartheta_{\text{cl}}$ finite. This leaves us with the classical UV-IR relations

$$\begin{aligned}
\frac{M_z}{M_{\text{crit}}} \sin \frac{\beta \Phi_z}{2} & = -(-1)^{z/L} \sin \eta_{\text{cl}} \sinh \vartheta_{\text{cl}} \\
\frac{M_z}{M_{\text{crit}}} \cos \frac{\beta \Phi_z}{2} & = \cos \eta_{\text{cl}} \cosh \vartheta_{\text{cl}} \\
M_{\text{crit}} & = \frac{4m}{\beta^2}
\end{aligned} \tag{4.6}$$

In the regime $\beta^2 < 4\pi$ the spectrum contains the first breather, which has the reflection matrix [12]

$$R_{\lfloor}^{(1)}(\theta)_z = \frac{\left(\frac{1}{2}\right) \left(\frac{1}{2\lambda} + 1\right) \left(\frac{\eta_z}{\pi\lambda} - \frac{1}{2}\right) \left(\frac{i\vartheta_z}{\pi\lambda} - \frac{1}{2}\right)}{\left(\frac{1}{2\lambda} + \frac{3}{2}\right) \left(\frac{\eta_z}{\pi\lambda} + \frac{1}{2}\right) \left(\frac{i\vartheta_z}{\pi\lambda} + \frac{1}{2}\right)} \quad , \quad (x) = \frac{\sinh\left(\frac{\theta}{2} + i\frac{\pi x}{2}\right)}{\sinh\left(\frac{\theta}{2} - i\frac{\pi x}{2}\right)}. \tag{4.7}$$

The breather reflection amplitudes are all independent of the parameter α , which appears only in the soliton reflection matrices. The reflection factor (4.7) does have a pole at $\theta = i\pi/2$ with the singular term taking the form

$$R_{\lfloor}^{(1)}(\theta) \sim 4i \frac{1 + \cos \frac{\pi}{2\lambda} - \sin \frac{\pi}{2\lambda}}{1 - \cos \frac{\pi}{2\lambda} + \sin \frac{\pi}{2\lambda}} \tan^2 \frac{\eta}{2\lambda} \tanh^2 \frac{\vartheta}{2\lambda} \frac{1}{2\theta - i\pi}$$

which gives the following result for the coupling of the first breather to the boundary with parameters η and ϑ

$$g_1(\eta, \vartheta) = 2\sqrt{\frac{1 + \cos \frac{\pi}{2\lambda} - \sin \frac{\pi}{2\lambda}}{1 - \cos \frac{\pi}{2\lambda} + \sin \frac{\pi}{2\lambda}}} \tan \frac{\eta}{2\lambda} \tanh \frac{\vartheta}{2\lambda} \quad (4.8)$$

(The expression under the square root is always positive as long as $\lambda > 1$ which is necessary for the first breather to exist in the spectrum). However, some care must be taken, because the sign of the coupling g_1 must be opposite on the two ends of the strip since these are related by a spatial reflection under which the first breather is odd.

As a result, formula (2.10) together with (2.9) predicts that in the regime $\beta^2 < 4\pi$ ($\lambda > 1$) for generic boundary conditions the leading finite size correction to the ground state on the strip is given by

$$\begin{aligned} E_{\alpha\beta}(L) &= -\frac{1}{4}m_1g_1(\eta_L, \vartheta_L)(-g_1(\eta_0, \vartheta_0))e^{-m_1L} \\ &= m_1\frac{1 + \cos \frac{\pi}{2\lambda} - \sin \frac{\pi}{2\lambda}}{1 - \cos \frac{\pi}{2\lambda} + \sin \frac{\pi}{2\lambda}} \tan \frac{\eta_0}{2\lambda} \tanh \frac{\vartheta_0}{2\lambda} \tan \frac{\eta_L}{2\lambda} \tanh \frac{\vartheta_L}{2\lambda} e^{-m_1L} + \dots \end{aligned} \quad (4.9)$$

where m_1 is the mass of the first breather, η_0, ϑ_0 and η_L, ϑ_L are the boundary parameters characterizing the boundary at the left and right ends of the strip, respectively. Soliton corrections are always subleading, since there is no one-particle coupling of solitons to the boundary and $m_1 < 2M$.

It is also interesting to compute the one-particle term contributed by the second breather, since in the regime where the second breather exists ($\beta^2 < 8\pi/3$ i.e. $\lambda > 2$), its mass $m_2 = 2m_1 \cos \frac{\pi}{2\lambda}$ is lower than $2m_1$ and so it dominates over the two-particle correction from the first breather, giving the next-to-leading finite size correction. Furthermore, if any of the η_z, ϑ_z ($z = 0, L$) parameters is zero, then the leading term (4.9) vanishes, leaving the second breather's one-particle term as the leading one.

The reflection amplitude for the second breather is of the form [12]

$$\begin{aligned} R_{\uparrow}^{(2)}(\theta) &= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2\lambda} + 1\right)\left(\frac{1}{\lambda} + 1\right)\left(\frac{1}{2\lambda}\right)}{\left(\frac{1}{2\lambda} + \frac{3}{2}\right)^2\left(\frac{1}{\lambda} + \frac{3}{2}\right)} \\ &\times \frac{\left(\frac{\eta_z}{\pi\lambda} - \frac{1}{2} - \frac{1}{2\lambda}\right)\left(\frac{i\vartheta_z}{\pi\lambda} - \frac{1}{2} - \frac{1}{2\lambda}\right)\left(\frac{\eta_z}{\pi\lambda} - \frac{1}{2} + \frac{1}{2\lambda}\right)\left(\frac{i\vartheta_z}{\pi\lambda} - \frac{1}{2} + \frac{1}{2\lambda}\right)}{\left(\frac{\eta_z}{\pi\lambda} + \frac{1}{2} - \frac{1}{2\lambda}\right)\left(\frac{i\vartheta_z}{\pi\lambda} + \frac{1}{2} - \frac{1}{2\lambda}\right)\left(\frac{\eta_z}{\pi\lambda} + \frac{1}{2} + \frac{1}{2\lambda}\right)\left(\frac{i\vartheta_z}{\pi\lambda} + \frac{1}{2} + \frac{1}{2\lambda}\right)} \end{aligned} \quad (4.10)$$

from which the boundary coupling can be extracted

$$g_2(\eta, \vartheta) = \frac{2 \tan\left(\frac{\pi}{4\lambda} - \frac{\eta}{2\lambda}\right) \tan\left(\frac{\pi}{4\lambda} + \frac{\eta}{2\lambda}\right) \tan\left(\frac{\pi}{4\lambda} - \frac{i\vartheta}{2\lambda}\right) \tan\left(\frac{\pi}{4\lambda} + \frac{i\vartheta}{2\lambda}\right)}{\tan \frac{\pi}{4\lambda} \sqrt{\tan \frac{\pi}{2\lambda} \tan\left(\frac{\pi}{4} + \frac{\pi}{2\lambda}\right)}} \quad (4.11)$$

and the one-particle contribution to the finite size corrections becomes

$$-m_2 \frac{g_2(\eta_0, \vartheta_0)}{2} \frac{g_2(\eta_L, \vartheta_L)}{2} e^{-m_2L} \quad (4.12)$$

(here the sign of the couplings are the same at the two boundaries as the second breather is even). We remark that in this model there is in fact no sign ambiguity in this coupling once the signs of bulk three-particle couplings are fixed, because the soliton reflection factors contain a pole at $\theta = i\frac{n\pi}{2\lambda}$, $n = 1, 2, \dots, [\lambda]$ whose residue is proportional to

$$f_{+-}^n g_n(\eta, \vartheta)$$

where f_{+-}^n is the bulk soliton-antisoliton-breather coupling, while $g_n(\eta, \vartheta)$ is the one-particle coupling of the n th breather to the boundary. As $g_n(\eta, \vartheta)$ appears linearly, it can be unambiguously calculated and the result matches the formulae (4.8,4.11). We note that in the limit $\vartheta \rightarrow \infty$ (Dirichlet boundary conditions) the couplings simplify to

$$\begin{aligned} g_1(\eta, \infty) &= 2\sqrt{\frac{1 + \cos \frac{\pi}{2\lambda} - \sin \frac{\pi}{2\lambda}}{1 - \cos \frac{\pi}{2\lambda} + \sin \frac{\pi}{2\lambda}}} \tan \frac{\eta}{2\lambda} \\ g_2(\eta, \infty) &= \frac{2 \tan\left(\frac{\pi}{4\lambda} - \frac{\eta}{2\lambda}\right) \tan\left(\frac{\pi}{4\lambda} + \frac{\eta}{2\lambda}\right)}{\tan \frac{\pi}{4\lambda} \sqrt{\tan \frac{\pi}{2\lambda} \tan\left(\frac{\pi}{4} + \frac{\pi}{2\lambda}\right)}} \end{aligned} \quad (4.13)$$

5 Comparison with the Dirichlet NLIE

In this section we check the boundary Lüscher formulae derived previously by comparing them to the exact ground state energies at large but finite L -s in the sine-Gordon model with Dirichlet boundary conditions. We determine the exact ground state energies from the (ground state version of the) NLIE proposed recently in [15], which generalizes [7] by allowing the sine-Gordon field to take two different values at the two boundaries of the strip.

The NLIE is a complex nonlinear integral equation of the form

$$\begin{aligned} Z(\theta) &= 2ML \sinh \theta + P(\theta | H_0, H_L) - i \int_{-\infty}^{\infty} dx G(\theta - x - i\eta) \log(1 - e^{iZ(x+i\eta)}) \\ &\quad + i \int_{-\infty}^{\infty} dx G(\theta - x + i\eta) \log(1 - e^{iZ(x-i\eta)}) \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} P(\theta | H_0, H_L) &= 2\pi \int_0^\theta dx (F(x, H_0) + F(x, H_L) + G(x) + J(x)) \\ G(\theta) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\theta} \frac{\sinh\left(\frac{\pi(1-\lambda)}{2\lambda}k\right)}{\sinh \frac{\pi}{2\lambda}k \cosh \frac{\pi}{2}k} \\ J(\theta) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\theta} \frac{\sinh\left(\frac{\pi(1-\lambda)}{4\lambda}k\right) \cosh\left(\frac{\pi(1+\lambda)}{4\lambda}k\right)}{\sinh \frac{\pi}{2\lambda}k \cosh \frac{\pi}{2}k} \\ F(\theta, H) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\theta} \frac{\sinh\left(\frac{\pi}{2}\left(1 + \frac{1}{\lambda} - H\right)k\right)}{2 \sinh \frac{\pi}{2\lambda}k \cosh \frac{\pi}{2}k} \end{aligned}$$

M is the soliton mass, and λ is the parameter defined in (4.4). The parameters $H_{0,L}$ can be expressed from the boundary values of the sine-Gordon field Φ :

$$H_0 = \frac{1 - \frac{8}{\beta}\Phi_0}{\lambda} \quad , \quad H_L = \frac{1 + \frac{8}{\beta}\Phi_L}{\lambda}$$

and the equation in the present form is valid only for $0 < H_{0,L} < 1 + \lambda^{-1}$.

This equation must be solved for the unknown function $Z(\theta)$. All the functions in the equations are analytic in some strip containing the real θ axis, and η must be chosen to lie inside that strip. Once $Z(\theta)$ is obtained (usually by a straightforward iteration procedure), the ground state energy of the theory can be calculated using

$$E(L) = -M \Im \int_{-\infty}^{\infty} \frac{dx}{2\pi} \sinh(x + i\eta) \log(1 - e^{iZ(x+i\eta)}) \quad (5.2)$$

It satisfies

$$E(L) \rightarrow 0 \quad \text{as} \quad L \rightarrow \infty$$

which means that the vacuum energy extracted from the NLIE is normalized to zero vacuum energy density and zero boundary energy in infinite volume. The true vacuum energy (calculated e.g. in the perturbed conformal field theory framework) has the form

$$E_{\text{PCT}} = \mathcal{E}_{\text{bulk}}L + \epsilon_\alpha + \epsilon_\beta + E_{\text{NLIE}}$$

where the bulk energy constant $\mathcal{E}_{\text{bulk}}$ and the boundary energy terms $\epsilon_{\alpha,\beta}$ are known exactly [11]. The NLIE energy (5.2) is normalized the same way as the boundary Lüscher formulae, so they can be compared directly.

In the repulsive regime $\lambda < 1$ one can calculate the leading large volume asymptotics by substituting

$$Z(\theta) = 2ML \sinh \theta + P(\theta | H_0, H_L)$$

into (5.2) and taking $\eta \rightarrow \frac{\pi}{2}$. The result is

$$E = -M \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta (K_\alpha^{s\bar{s}}(-\theta) K_\beta^{\bar{s}s}(\theta) + K_\alpha^{\bar{s}s}(-\theta) K_\beta^{s\bar{s}}(\theta)) e^{-2ML \cosh \theta} \quad (5.3)$$

where

$$K_\beta^{s\bar{s}}(\theta) = R_\alpha \left(i\frac{\pi}{2} - \theta \right)_s^s \quad , \quad K_\beta^{\bar{s}s}(\theta) = R_\beta \left(i\frac{\pi}{2} - \theta \right)_{\bar{s}}^{\bar{s}}$$

in terms of the soliton and antisoliton reflection factors found by Ghoshal and Zamolodchikov [5]. We made use of the relation [7, 15]

$$P(\theta | H_0, H_L) = i \log S_{++}(2\theta) - i \log R_\alpha(\theta)_s^s - i \log R_\beta(\theta)_{\bar{s}}^{\bar{s}} + \pi$$

(where S_{++} is the soliton-soliton phase shift) and also of bulk crossing symmetry and boundary crossing-unitarity to bring the result to the form (5.3). This is in perfect agreement with the result (2.4) obtained from the cluster expansion.

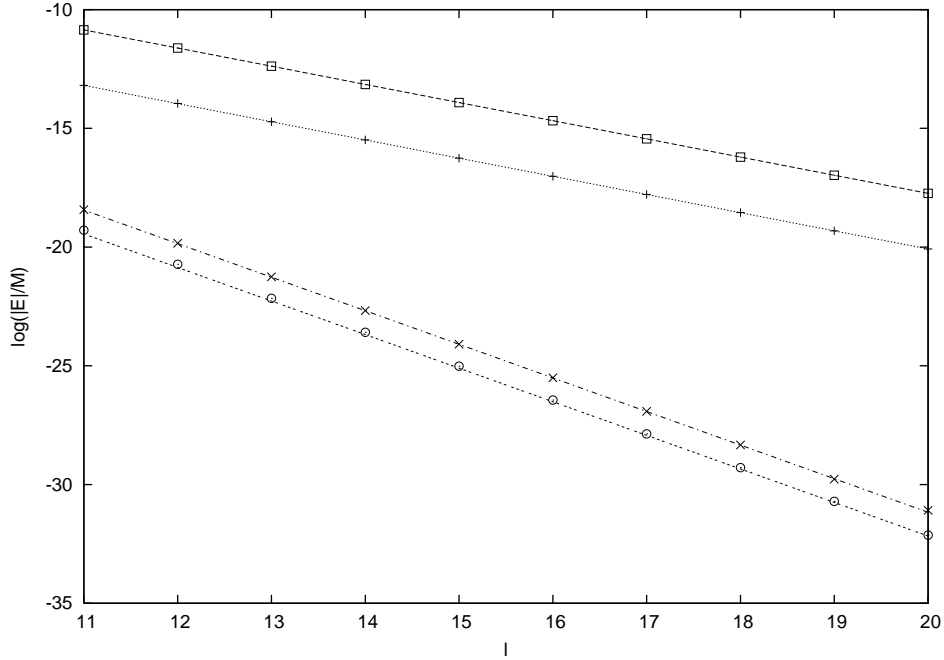


Figure 5.1: Comparison of the Dirichlet NLIE with the boundary one-particle contributions

The above procedure does not work in the attractive regime because the analyticity of Z does not extend far enough to allow taking the limit $\eta \rightarrow \frac{\pi}{2}$. In principle one should be able to recover the cluster expansion analytically from the NLIE expression (5.2) even in the attractive regime; however, this requires more detailed understanding of the analytic structure of the NLIE (5.1) than what we have at present, and is out of the scope of the present paper.

Instead we performed numerical comparison of NLIE results to the boundary Lüscher formulae for several values of the coupling and the boundary parameters. An example is shown in figure 5.1 below, where $\lambda = 4$ and so both the first and the second breathers are present in the bulk spectrum. Varying the boundary values of the sine-Gordon field we investigated the cases when one expects the asymptotics be given by the first or second breathers separately. The results are summarized in figure 5.1, where $\log(|E|/M)$ is plotted against the dimensionless $l = ML$ with M being the soliton mass. (The sign of the exact ground state energy always matched the sign of the appropriate asymptotic correction as calculated from (4.9) and (4.12)).

For simplicity, we parameterize the boundary values Φ_z , $z = 0, L$ at the two ends of the strip in the following way

$$\Phi_z = \frac{2\pi}{\beta} \phi_z$$

and so $\eta_z = -(-1)^{z/L} \pi(1 + \lambda) \phi_z$. The points marked by empty boxes, +s, ×-s and dotted circles are the exact ground state values computed from the NLIE when the boundary values

of the sine-Gordon fields on the upper and lower ends of the strip are $\phi_0 = -0.17$ $\phi_L = 0.1$, $\phi_0 = 0.017$ $\phi_L = 0.1$, $\phi_0 = 0$ $\phi_L = 0$, and $\phi_0 = -0.08$ $\phi_L = 0$, respectively. The two upper continuous lines are the predictions of the first breather's asymptotic contribution (4.9) (with $\vartheta \rightarrow \infty$ to describe the Dirichlet case), while the two lower continuous lines are the predictions coming from the second breather's asymptotic contribution (4.12) when at one end of the strip $\eta_L = 0$:

$$E_{\alpha 0}^{(2)} = -m_2 \frac{g_2(\eta_0, \infty)}{2} \frac{g_2(0, \infty)}{2} e^{-2m_2 L} = -m_2 \frac{\tan\left(\frac{\pi}{4\lambda} + \frac{\eta_0}{2\lambda}\right) \tan\left(\frac{\pi}{4\lambda} - \frac{\eta_0}{2\lambda}\right)}{\tan\left(\frac{\pi}{2\lambda}\right) \tan\left(\frac{\pi}{4} + \frac{\pi}{2\lambda}\right)} e^{-2m_2 L}.$$

The agreement with the NLIE data strongly supports the formulae derived from the cluster expansion. Although for the graphical comparison it is enough to plot only the second breather one-particle term, in order to make the agreement with the NLIE numerically satisfactory the two-particle contribution from the first breather also has to be included. The detailed numerical comparison is performed in the appendix and an excellent agreement is found. There's no such problem with the first breather term, however, because it is possible to go to large enough volume so that it is still numerically measurable and at the same time dominates over the subleading corrections by several orders of magnitude.

6 Classical limit of scaling functions

Motivated by the success of Mussardo et al. [16] who showed that in the case of periodic boundary conditions the (semi)classical limit of Lüscher's finite size correction to the sine-Gordon soliton mass coincides with the infrared limit ($l = mL \rightarrow \infty$) of the soliton's classical energy in finite volume, we show next that the classical limit(s) of the previous Lüscher type corrections to the ground state energy in the boundary sine-Gordon theory on the strip coincide with the asymptotic corrections to the classical energies of the various static (ground state) solutions of the model.

In refs. [3, 4] collectively the following static solutions are described in details in terms of Jacobi elliptic functions:

- The ground state (+) and the solitonic excited solution (-) with the $\Phi(0) = \Phi_0^D$, $\Phi(L) = 0$ Dirichlet-Dirichlet (DD) boundary conditions where in the (+) [(-)] case the sine-Gordon field $\Phi(x)$ decreases [increases] monotonically from Φ_0^D at $x = 0$ to 0 [$2\pi/\beta$] at $x = L$;
- the ground state (A) and the first excited state (B) type solutions in the zero topological charge sector with the

$$0 < \varphi_0 < \frac{\pi}{2}, \quad 0 \leq \varphi_L \leq \pi - \varphi_0, \quad \varphi_{0,L} = \frac{\beta\Phi_{0,L}^D}{2},$$

DD boundary conditions, where, for the case (A) [(B)], $\Phi(x)$ decreases [increases] from Φ_0^D to some turning point ϵ [$\frac{2\pi}{\beta} - \tilde{\epsilon}$] then increases [decreases] from these values to Φ_L^D ;

- the ground state (-1) of the $n = -1$ topological charge sector with the

$$0 < \varphi_0 < \frac{\pi}{2}, \quad \frac{\beta\Phi(L)}{2} = \varphi_L - \pi$$

DD boundary conditions, when $\Phi(x)$ decreases monotonically from positive to negative values,

• and finally the ground state with mixed boundary conditions (DN): Dirichlet at $x = 0$ and a one parametric (M_L) perturbed Neumann type b.c. at $x = L$

$$\frac{\beta}{2}\Phi(0) = \varphi_0, \quad \frac{\beta}{2}\Phi'(x)|_{x=L} = -m\mathcal{A}^{-1}\sin(\varphi_L), \quad \mathcal{A}^{-1} = \frac{M_L\beta^2}{4m}.$$

For all these static solutions the dimensionless width of the strip $l = mL$ and the energies of the solutions can be expressed in terms of the integrals

$$l(a, b, C) = \int_a^b \frac{dv}{\sqrt{\sin^2 v + C}}, \quad I(a, b, C) = \int_a^b dv \sqrt{\sin^2 v + C}.$$

Indeed in the $\Phi_L^D = 0$ Dirichlet-Dirichlet case for the (+) solution one obtains

$$l = l(0, \varphi_0, C), \quad E_+(\varphi_0, l) = \frac{4m}{\beta^2}[I(0, \varphi_0, C) - Cl/2],$$

while for the (-) solution

$$l = l(\varphi_0, \pi, C), \quad E_-(\varphi_0, l) = \frac{4m}{\beta^2}[I(\varphi_0, \pi, C) - Cl/2]$$

results. In the case of general DD boundary conditions, in the $n = 0$ sector, for the type (A) (ground state) solution

$$l = \sum_{i=0, L} l(\epsilon, \varphi_i, -\sin^2 \epsilon), \quad E_A(\varphi_0, \varphi_L, l) = \frac{4m}{\beta^2} \left[\sum_{i=0, L} I(\epsilon, \varphi_i, -\sin^2 \epsilon) + \frac{l}{2} \sin^2 \epsilon \right]$$

is obtained, while for the type (B) (excited state) solution

$$l = \sum_{i=0, L} l(\varphi_i, \pi - \epsilon, -\sin^2 \epsilon), \quad E_B(\varphi_0, \varphi_L, l) = \frac{4m}{\beta^2} \left[\sum_{i=0, L} I(\varphi_i, \pi - \epsilon, -\sin^2 \epsilon) + \frac{l}{2} \sin^2 \epsilon \right]$$

is found. In the $n = -1$ sector of the general DD boundaries for the only ground state we obtain

$$l = l(0, \varphi_0, C) + l(0, \pi - \varphi_L, C), \quad E^{(-1)}(\varphi_0, \varphi_L, l) = \frac{4m}{\beta^2}[I(0, \varphi_0, C) + I(0, \pi - \varphi_L, C) - Cl/2].$$

Finally for the ground state of the DN problem

$$l = l(\varphi_L, \varphi_0, \tilde{C}), \quad E^{(DN)}(\varphi_0, \mathcal{A}, l) = \frac{4m}{\beta^2} \left[-\tilde{C} \frac{l}{2} + I(\varphi_L, \varphi_0, \tilde{C}) - \mathcal{A}^{-1} \cos(\varphi_L) \right]$$

is found with $\tilde{C} = -(1 - \mathcal{A}^{-2}) \sin^2 \varphi_L$. These expressions give the l dependence of the energies in a parametric form: in all cases the integration constant(s) C or ϵ and φ_L are – at least in principle – determined as functions of l (and the boundary parameters) from the first equations and using them in the second equations gives the l dependence of the energy expressions.

The asymptotic form of the energies is obtained in the following way: first in each case the integrals defining l are expressed as an appropriate linear combination of complete and incomplete elliptic integrals. The $l \rightarrow \infty$ limit corresponds to the limit when the parameter of these elliptic integrals goes to unity – sometimes this is also correlated with the amplitude of the incomplete one going to $\pi/2$. Using the relevant asymptotic expansions of the elliptic integrals we determine the asymptotic $C(l)$ ($\epsilon(l)$, φ_L). In the second step the integrals appearing in the expressions for the energy are determined by realizing that their derivatives with respect to C (ϵ , φ_L) can be related to $l(a, b, C)$; viewing this relation as a differential equation we integrate it (the constant is fixed by the known value of $I(a, b, 0)$). In this way the following expressions are obtained at $l \rightarrow \infty$:

$$\begin{aligned}
E_+(\varphi_0, l) &\sim \frac{4m}{\beta^2} \left(1 - \cos \varphi_0 + 4 \tan^2 \frac{\varphi_0}{2} e^{-2l} \right), \\
E_-(\varphi_0, l) &\sim \frac{4m}{\beta^2} \left(1 + \cos \varphi_0 + 4 \cot^2 \frac{\varphi_0}{2} e^{-2l} \right), \\
E_A(\varphi_0, \varphi_L, l) &\sim \frac{4m}{\beta^2} \left(2 - \cos \varphi_0 - \cos \varphi_L - 8 \tan \frac{\varphi_0}{2} \tan \frac{\varphi_L}{2} e^{-l} \right), \\
E_B(\varphi_0, \varphi_L, l) &\sim \frac{4m}{\beta^2} \left(2 + \cos \varphi_0 + \cos \varphi_L - 8 \cot \frac{\varphi_0}{2} \cot \frac{\varphi_L}{2} e^{-l} \right), \\
E^{(-1)}(\varphi_0, \varphi_L, l) &\sim \frac{4m}{\beta^2} \left(2 - \cos \varphi_0 + \cos \varphi_L + 8 \tan \frac{\varphi_0}{2} \tan \frac{\pi - \varphi_L}{2} e^{-l} \right), \\
E^{(DN)}(\varphi_0, \mathcal{A}, l) &\sim \frac{4m}{\beta^2} \left(1 - \cos \varphi_0 - \mathcal{A}^{-1} - 4 \tan^2 \frac{\varphi_0}{2} \frac{1 - \mathcal{A}^{-1}}{1 + \mathcal{A}^{-1}} e^{-2l} \right). \tag{6.1}
\end{aligned}$$

The constant (l -independent) terms of these energy expressions are the classical energies of the static solitons/antisolitons forming the asymptotic ground state (for more details see [3, 4]). Note that some of the asymptotic corrections decrease with e^{-l} , while the others with e^{-2l} . The fact that the asymptotic correction is negative in the $n = 0$ and positive in the $n = -1$ sectors can be understood by realizing that in the former sector the asymptotic solution is a superposition of a soliton and an antisoliton (whose interaction is attractive) while in the latter it is a superposition of two solitons (whose interaction is repulsive). The sign and magnitude of these asymptotic corrections were also verified numerically by comparing to the exact solution of the classical field equations.

We now proceed to show that these asymptotic corrections are indeed the classical limits of the appropriate generalized Lüscher formulae.

It is very important to realize that the classical limit (4.5) is not identical to taking $\beta \rightarrow 0$, in which almost all interesting quantities diverge (this limit cannot even be taken in

the classical formulae for the ground state energies). The general form of the semiclassical expansion for the ground state energy can be obtained from the Euclidean path integral as the limit

$$E(L) = \lim_{T \rightarrow \infty} -\frac{1}{\hbar T} \log \int D\Phi \exp \left\{ -\frac{1}{\hbar} \int_{-T/2}^{T/2} d\tau L_E(\Phi) \right\}$$

where the Euclidean Lagrangian is

$$\begin{aligned} L_E(\Phi) &= \int_0^L dx \left(\frac{1}{2} (\partial_\tau \Phi)^2 + \frac{1}{2} (\partial_x \Phi)^2 + \frac{m^2}{\beta^2} \cos \beta \Phi \right) \\ &+ M_L \left[1 - \cos \frac{\beta}{2} (\Phi(L, \tau) - \Phi_L) \right] + M_0 \left[1 - \cos \frac{\beta}{2} (\Phi(0, \tau) - \Phi_0) \right] \end{aligned}$$

Scaling out the coupling constant and introducing dimensionless variables

$$\Phi = \beta \bar{\Phi} \quad , \quad \bar{x} = mx \quad , \quad \bar{\tau} = m\tau$$

standard semiclassical expansion around the classical solution yields

$$E(L) = \frac{m}{\beta^2} \left(F \left(mL, \frac{M_0 \beta^2}{m}, \frac{M_L \beta^2}{m}, \beta \Phi_0, \beta \Phi_L \right) + O(\hbar \beta^2) \right) \quad (6.2)$$

where F is a dimensionless function of its arguments. The first term is just the classical energy, the large volume asymptotics of which is given in (6.1) and is proportional to β^{-2} . All the quantum corrections are regular as $\beta \rightarrow 0$, and expansion in powers of \hbar corresponds to expansion in β^2 . As a result, the classical limit corresponds to isolating the leading β^{-2} term in the expansion around $\beta = 0$. Thus it is only meaningful to take the large volume asymptotic first and perform the classical limit by considering only the leading terms for small β .

The volume dependence of the finite size correction is determined by the type of the particle giving the leading contribution: for terms of the form e^{-l} it is the first breather, while in the case of e^{-2l} the second (note that $m_2 = 2m_1 = 2m$ in the classical limit).

The UV-IR relations (4.3) imply that the Dirichlet boundary condition is obtained as the $\vartheta \rightarrow \infty$ limit of the general one, while the perturbed Neumann BC as the $\vartheta = 0$ ($\mathcal{A}^{-1} < 1$) or $\eta = 0$ ($\mathcal{A}^{-1} > 1$) case. Using the classical UV-IR relation (4.6) one finds that in the Dirichlet case

$$\eta_{\text{cl},0} = -\varphi_0 = -\frac{\beta \Phi_0}{2} \quad , \quad \eta_{\text{cl},L} = \varphi_L = \frac{\beta \Phi_L}{2}$$

while for the perturbed Neumann BC

$$\cos \eta_{\text{cl}} = \mathcal{A}^{-1} \quad (\mathcal{A}^{-1} < 1) \quad , \quad \cosh \eta_{\text{cl}} = \mathcal{A}^{-1} \quad (\mathcal{A}^{-1} > 1)$$

results.

Using the above identifications, the classical limit of the breather one-particle term (4.9) becomes

$$-\frac{32m}{\beta^2} \tan \frac{\varphi_0}{2} \tan \frac{\varphi_L}{2} e^{-l}$$

which indeed agrees with E_A .

To interpret E_B that belongs to the first excited boundary state let us recall that the reflection amplitudes on it can be obtained by the substitution

$$\eta \rightarrow \bar{\eta} = \pi(\lambda + 1) - \eta \quad (6.3)$$

from the ground state ones [17, 18]. Therefore making this substitution in (4.9) gives the Lüscher type correction to the first excited state energies. Since in the classical limit this substitution simplifies to $\eta_{\text{cl}} \rightarrow \pi - \eta_{\text{cl}}$, it is clear that E_B does coincide with the classical limit of (4.9). Similar reasoning can be used in the case of $E^{(-1)}$.

For the corrections of order e^{-2l} , however, things are more complicated. Let us take the example of E_+ , the other cases can be treated in the same way. Taking the classical limit of the second breather one-particle term (4.12) gives

$$\frac{32m}{\beta^2} \tan^2 \frac{\varphi_0}{2} e^{-2l} \quad (6.4)$$

which is off by a factor of 2. Fortunately this is not the end of the story. As we already noted above, in the classical limit the masses of the first and second breathers are given by

$$m_1 = m \quad , \quad m_2 = 2m$$

and therefore the first breather two-particle term is exactly of same order as the second breather one-particle term. Let us examine the general formula for the two-particle contribution for a particle of mass m (2.3)

$$-m \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta \bar{K}_\alpha(\theta) K_\beta(\theta) e^{-2mL \cosh \theta} = -m e^{-2mL} \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} f(\theta) e^{-2mL\theta^2/2}$$

where we introduced

$$f(\theta) = \cosh \theta \bar{K}_\alpha(\theta) K_\beta(\theta) e^{-2mL(\cosh \theta - 1 - \theta^2/2)}$$

Since $f(\theta)$ is an even function, it expands in even powers of θ around 0

$$f(\theta) = f_0 + f_1\theta^2 + \dots$$

and so the two-particle contribution can be written as (for $mL \rightarrow 0$)

$$-\frac{m}{4\sqrt{\pi mL}} e^{-2mL} \left(f_0 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2mL)^n} f_n \right) \quad (6.5)$$

At first sight this does not appear to be enough to rescue us, since the two-particle correction is further suppressed by a factor of $L^{-1/2}$. However, it is an interesting and nontrivial issue whether the asymptotic expansion (6.5) can be trusted in the semiclassical domain.

The relevant integral in our case is

$$-m_1 \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta \log (1 + \chi(\theta) e^{-2m_1 L \cosh \theta}) \sim -m_1 \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh \theta \chi(\theta) e^{-2m_1 L \cosh \theta} \quad (6.6)$$

where

$$\chi(\theta) = R_{\downarrow}^{(1)} \left(i\frac{\pi}{2} - \theta \right)_0 R_{\downarrow}^{(1)} \left(i\frac{\pi}{2} + \theta \right)_L$$

and for the case of E_+ the boundary parameters are $\eta_0 = \varphi_0(\lambda + 1)$, $\eta_L = 0$ and $\vartheta_0 = \vartheta_L = \infty$.

Using the known reflection factors, the coefficients f_i can be explicitly calculated to have the following behaviour for small β

$$f_n \sim \lambda^{2(n+1)} \sim \frac{1}{\beta^{4(n+1)}}$$

which means that in the classical limit the series expansion (6.5) is invalid.

The reason why the coefficients f_n diverge and the integral (6.6) must be treated very carefully is that the value of $\chi(\theta)$ and all of its derivatives at $\theta = 0$ go to infinity in the classical limit. The dangerous contribution comes from the particular block

$$\left(\frac{3}{2} + \frac{1}{2\lambda} \right)$$

in the reflection factor (4.7) and can be isolated from χ as follows

$$\chi(\theta) = \frac{\cosh \theta + \cos \frac{\pi}{2\lambda}}{\cosh \theta - \cos \frac{\pi}{2\lambda}} \chi_0(\theta)$$

where χ_0 is a function with a smooth classical limit at the origin

$$\chi_0(0) = \left(\frac{\cos \frac{\pi}{4\lambda} - \sin \frac{\pi}{4\lambda} \tan \frac{\eta_0}{2\lambda}}{\cos \frac{\pi}{4\lambda} + \sin \frac{\pi}{4\lambda} \tan \frac{\eta_0}{2\lambda}} \right)^2 \sim \tan^2 \frac{\varphi_0}{2} + O(\beta^2)$$

while the function

$$g(\theta) = \frac{\cosh \theta + \cos \frac{\pi}{2\lambda}}{\cosh \theta - \cos \frac{\pi}{2\lambda}}$$

has the Taylor coefficients

$$\frac{g^{(2n)}(0)}{(2n)!} = (-1)^n 4 \left(\frac{2\lambda}{\pi} \right)^{2(n+1)} \left(1 + O\left(\frac{1}{\lambda^2} \right) \right)$$

for large λ . As a result, the series (6.5) takes the following form for small β

$$-\frac{256m}{\sqrt{\pi m L \beta^4}} \tan^2 \frac{\varphi_0}{2} e^{-2mL} \left(1 + \sum_{n=1}^{\infty} (2n-1)!! \left(-\frac{128}{mL\beta^4} \right)^n \right)$$

which is a power series with coefficients that grow factorially. It is also obvious that it does not have the right β dependence displayed in eqn. (6.2), and the higher terms show more and more singular behaviour at $\beta = 0$.

In order to get the correct asymptotics, we now evaluate the leading behaviour of the integral (6.6) in the semiclassical regime carefully. First we substitute $x = \sinh \theta$ for the integration variable. For large L , the integrand is concentrated around the origin and therefore we can evaluate the dominant contribution by keeping only the leading terms around $x = 0$:

$$-m_1 \int_{-\infty}^{\infty} \frac{dx}{4\pi} \log \left(1 + \left(\frac{1 + \cos \frac{\pi}{2\lambda} + x^2/2}{1 - \cos \frac{\pi}{2\lambda} + x^2/2} - 1 \right) \chi_0(0) e^{-2m_1 L} \right)$$

where a subtraction of -1 in the coefficient of $\chi_0(0) e^{-2m_1 L}$ was applied to make the integral convergent at the plus and minus infinities (this only affects subleading terms – of order β^0 – as the singular behaviour around $x \sim 0$ is unchanged). Now the integral can be evaluated using (3.6)

$$-m_1 \left[\sqrt{\sin^2 \frac{\pi}{4\lambda} + \cos \frac{\pi}{2\lambda} \chi_0(0) e^{-2m_1 L}} - \sin \frac{\pi}{4\lambda} \right] \sim -\frac{m_1}{2} \chi_0(0) \frac{\cos \frac{\pi}{2\lambda}}{\sin \frac{\pi}{4\lambda}} e^{-2m_1 L}$$

using that L is very large.

Taking the classical limit of small β (i.e. large λ) and keeping only the leading term in β we obtain

$$-\frac{16m}{\beta^2} \tan^2 \frac{\varphi_0}{2} e^{-2mL}$$

which must be added to the second breather one-particle contribution (6.4), thereby giving perfect agreement with the classical formula for E_+ . The same argument works for $E^{(DN)}$, while E_- can be obtained from E_+ by the substitution (6.3).

7 Conclusions

In this paper we succeeded in deriving the boundary analogue of Lüscher's analytic formulae for finite size corrections (which were originally obtained for periodic boundary conditions). The consistency of these boundary Lüscher formulae was checked against boundary TBA, NLIE and classical field theoretic results. The matching of all these different approaches provided ample evidence for the correctness of the formulae, and also a very strong argument for the universality of the relation

$$\tilde{g}_\alpha = \frac{g_\alpha}{2} \tag{7.1}$$

between the one-particle coupling g_α in the reflection factors and the one-particle amplitude \tilde{g}_α in the corresponding boundary state, which was first proposed for the boundary Lee-Yang model in [8] on the basis of matching the form factor expansion of one-point function against truncated conformal space data. Our numerical data based on the Dirichlet NLIE and the matching with the classical finite size corrections show that this relation is valid in boundary sine-Gordon theory for very general values of parameters. The argument based on the boundary TBA shows that it is also valid for any theory with a diagonal bulk and boundary scattering theory. Based on these results we conjecture that this relation is indeed universal.

Using the relation (7.1), the boundary Lüscher formulae proposed in this paper make it possible to compute the leading finite size corrections based only on data from the scattering amplitudes, even in non-integrable theories. It is interesting to note a deep analogy between the boundary Lüscher formulae (2.3, 2.10) and the particle mass correction (1.1) for the periodic case. In both cases the generic correction can be written in terms of an integral over the relevant scattering amplitude, and the two integrals share many other similarities in structure. Besides that, the possible pole terms are also very similar. Therefore we expect that a quantum field theoretic derivation of our results along the lines of [1] must exist. Such a derivation would be very interesting because it would permit generalization to any space-time dimensions (similarly to the case of periodic boundary condition [1]), while the cluster expansion technique is restricted at the moment to 1 + 1 dimensional field theories, albeit not necessarily integrable ones.

While the exponential dependence of the finite size corrections on the volume and the particle masses is obvious from general principles of field theory and TBA/NLIE, there are some highly nontrivial aspects of the results obtained here which we would like to stress. First of all, the relation (7.1) which was originally a (numerical) observation made in a particular model (Lee-Yang) in [8], is shown to be universal by several pieces of evidence (analytic and numeric). Second, we now have a universal analytic form of the exponential finite size corrections expressed in terms of the infinite volume scattering theory, and also a regularized form for the two-particle contribution (eqn. (3.7)) when the naive expression (2.3) is divergent. Third, we have done extensive consistency checks of the results, and in particular we have shown that they agree with classical field theory, but the classical limit of finite size corrections must be treated very carefully.

In more practical terms, the boundary Lüscher formulae are very useful for extracting boundary scattering data from finite size effects obtained by any numerical or analytic method like lattice field theory, truncated conformal space or Bethe Ansatz based approaches (TBA, NLIE). Alternatively, when the scattering theory is explicitly known they can be applied to test the validity of numerical methods or conjectured TBA or NLIE equations. Such applications are presently being actively pursued by the authors.

Acknowledgments.

The authors would like to thank G. Watts and C. Rim for illuminating discussions, and R. Nepomechie for pointing out a mistaken (and important) sign. GT would like to thank

	ϕ_0	ϕ_L	α	γ_{measured}	γ_{exact}
$\lambda = 7/2$			exact: 0.867767		
	0.05	0.05	0.867774	-0.024529	-0.024526
	0.03	0.07	0.867812	-0.020640	-0.020625
	0.07	0.07	0.867786	-0.048403	-0.048389
$\lambda = 5/2$			exact: 1.175570		
	0.1	0.2	1.176765	-0.197730	-0.193904
	0.05	0.15	1.175572	-0.069682	-0.069681
	-0.05	0.15	1.175571	0.069681	0.069681

Table A.1: Comparing the first breather one-particle term to NLIE. For each value of λ , we quote the exact value of $\alpha = m_1/M = 2 \sin \frac{\pi}{2\lambda}$, which is compared to the exponent extracted from the NLIE. γ_{measured} is the coefficient extracted from NLIE, while γ_{exact} is the one predicted by (A.1).

the Department of Mathematics, King’s College London, for their hospitality during the course of the work. This research was partially supported by the EC network “EUCLID”, contract number HPRN-CT-2002-00325, and Hungarian research funds OTKA D42209, T037674, T034299 and T043582. GT was also supported by a Széchenyi István Fellowship.

A Numerical comparison to the NLIE

We consider the first breather one-particle contribution first. We normalize the energies and distances in units of the soliton mass M :

$$\frac{E_{\alpha\beta}(L)}{M} = 2 \sin \frac{\pi}{2\lambda} \frac{1 + \cos \frac{\pi}{2\lambda} - \sin \frac{\pi}{2\lambda}}{1 - \cos \frac{\pi}{2\lambda} + \sin \frac{\pi}{2\lambda}} \tan \frac{\eta_0}{2\lambda} \tan \frac{\eta_L}{2\lambda} e^{-2 \sin \frac{\pi}{2\lambda} l} + \dots \quad (\text{A.1})$$

$$l = ML$$

and the boundary values of the field as

$$\Phi_z = \frac{2\pi}{\beta} \phi_z \quad , \quad \eta_z = -(-1)^{z/L} \phi_z \pi (1 + \lambda)$$

Numerical data are obtained by fitting the NLIE results to a curve of the form

$$\gamma e^{-\alpha l}$$

Illustrative examples are given in table A.1; the agreement was tested for many other values of the parameters involved, with results similar to those in the table.

The second breather one-particle term proves more tricky. To eliminate the otherwise dominant first breather one-particle term we choose $\phi_L = 0$. However, the two-particle term for the first breather gives a non-negligible contribution for all values of the parameters that

l		10	12	14	16
$\lambda = 7/2, \phi_0 = 0.1$	$2 B_1$	-4.20321e-10	-1.17697e-11	-3.35392e-13	-9.67996e-15
	B_2	-1.87457e-09	-8.21717e-11	-3.60199e-12	-1.57893e-13
	\sum	-2.29490e-09	-9.39414e-11	-3.93738e-12	-1.67572e-13
	NLIE	-2.29710e-09	-9.39742e-11	-3.93787e-12	-1.67577e-13
$\lambda = 9/2, \phi_0 = 0.013$	$2 B_1$	-6.33781e-09	-3.37019e-10	-1.84416e-11	-1.03073e-12
	B_2	-1.31031e-07	-1.00170e-08	-7.65776e-10	-5.85417e-11
	\sum	-1.37369e-07	-1.03540e-08	-7.84218e-10	-5.95724e-11
	NLIE	-1.37425e-07	-1.03550e-08	-7.84232e-10	-5.95718e-11
$\lambda = 9/2, \phi_0 = 0.007$	$2 B_1$	-1.53201e-08	-8.84074e-10	-5.20261e-11	-3.10504e-12
	B_2	-5.51243e-08	-4.21411e-09	-3.22159e-10	-2.46282e-11
	\sum	-7.04444e-08	-5.09819e-09	-3.74185e-10	-2.77333e-11
	NLIE	-7.04671e-08	-5.09861e-09	-3.74193e-10	-2.77334e-11

Table A.2: NLIE versus second breather one-particle term combined with the first breather two-particle term

are accessible by iterating the NLIE. Therefore we can only compare the NLIE results to the sum of the two terms. The first breather two-particle term can be safely calculated from (2.3) since the integrand is regular. Table A.2 summarizes some of the results obtained this way. The agreement between the NLIE and the boundary Lüscher formula becomes worse with decreasing volume, which is easy to understand since higher order terms of the cluster expansion start playing an increasingly important role. Once again, we have performed a rather extensive numerical check, of which the table contains only a small sample.

References

- [1] M. Lüscher: Volume dependence of the energy spectrum in massive quantum field theories. 1. Stable particle states, *Comm. Math. Phys.* **104** (1986) 177.
M. Lüscher: Volume dependence of the energy spectrum in massive quantum field theories. 2. Scattering states, *Comm. Math. Phys.* **105** (1986) 153-188.
- [2] T. R. Klassen and E. Melzer: On the relation between scattering amplitudes and finite size mass corrections in QFT, *Nucl. Phys.* **B362** (1991) 329-388.
- [3] Z. Bajnok, L. Palla and G. Takács: (Semi)classical analysis of sine-Gordon theory on a strip, *Nucl. Phys.* **B702** (2004) 448-480, hep-th/0406149.
- [4] G. Mussardo, V. Riva and G. Sotkov: Semiclassical energy levels of sine-Gordon model on a strip with Dirichlet boundary conditions, preprint ISAS-44-2004-FM, hep-th/0406246.

- [5] S. Ghoshal and A. B. Zamolodchikov: Boundary S matrix and boundary state in two-dimensional integrable quantum field theory, *Int. J. Mod. Phys.* **A9** (1994) 3841-3886 (Erratum-ibid. **A9** 4353), hep-th/9306002.
- [6] P. Dorey, D. Fioravanti, C. Rim and R. Tateo: Integrable quantum field theory with boundaries: the exact g function, *Nucl. Phys.* **B696** (2004) 445-467, hep-th/0404014.
- [7] A. Leclair, G. Mussardo, H. Saleur and S. Skorik: Boundary energy and boundary states in integrable quantum field theories, *Nucl. Phys.* **B453** (1995) 581-618, hep-th/9503227.
- [8] P. Dorey, M. Pillin, R. Tateo and G. Watts: One point functions in perturbed boundary conformal field theories, *Nucl. Phys.* **B594** (2001) 625-659, hep-th/0007077.
- [9] P. Dorey, A. Pocklington, R. Tateo and G. Watts: TBA and TCSA with boundaries and excited states, *Nucl. Phys.* **B525** (1998) 641-663, hep-th/9712197.
- [10] C. Ahn and R.I. Nepomechie: Finite size effects in the XXZ and sine-Gordon models with two boundaries, *Nucl. Phys.* **B676** (2004) 637-658, hep-th/0309261.
- [11] Al.B. Zamolodchikov, unpublished.
Z. Bajnok, L. Palla and G. Takács: Finite size effects in boundary sine-Gordon theory, *Nucl. Phys.* **B622** (2002) 565-592, hep-th/0108157.
- [12] S. Ghoshal: Bound state boundary S matrix of the sine-Gordon model, *Int. J. Mod. Phys.* **A9** (1994) 4801-4810, hep-th/9310188.
- [13] M. Kormos and L. Palla: Some semi-classical issues in boundary sine-Gordon model, *J. Phys.* **A35** (2002) 5471-5488, hep-th/0201230.
- [14] I.S. Gradshteyn and I.M. Ryzhikh, *Tables of Integrals, Sums and Products*, 5th Edition (New York Academic Press, 1994).
- [15] C. Ahn, M. Bellacosa and F. Ravanini: Excited states NLIE for sine-Gordon model in a strip with Dirichlet boundary conditions, *Phys. Lett.* **B595** (2004) 537-546, hep-th/0312176.
- [16] G. Mussardo, V. Riva and G. Sotkov: Semiclassical scaling functions of sine-Gordon model, *Nucl. Phys.* **B699** (244) 545-574, hep-th/0405139.
- [17] P. Mattsson and P. Dorey: Boundary spectrum in the sine-Gordon model with Dirichlet boundary condition, *J. Phys.* **A33** (2000) 9065-9094, hep-th/0008071.
- [18] Z. Bajnok, L. Palla, G. Takács and G.Zs. Tóth: The spectrum of boundary states in sine-Gordon theory with integrable boundary conditions, *Nucl. Phys.* **B622** (2002) 548-564, hep-th/0106070.