The Chiral WZNW Phase Space and its Poisson-Lie Groupoid

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Abstract

The precise relationship between the arbitrary monodromy dependent 2-form appearing in the chiral WZNW symplectic form and the 'exchange r-matrix' that governs the corresponding Poisson brackets is established. Generalizing earlier results related to diagonal monodromy, the exchange r-matrices are shown to satisfy a new dynamical generalization of the classical modified Yang-Baxter equation, which is found to admit an interpretation in terms of (new) Poisson-Lie groupoids. Dynamical exchange r-matrices for which right multiplication yields a classical or a Poisson-Lie symmetry on the chiral WZNW phase space are presented explicitly.

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1 Introduction

Since its invention fifteen years ago [1], the Wess-Zumino-Novikov-Witten (WZNW) model has been the subject of many studies, and still receives considerable attention due to its central role in two-dimensional conformal field theory and to the richness of its structure (see e.g. [2]). One of the fascinating aspects of the model is that in addition to its built-in affine Kac-Moody symmetry it also exhibits certain quantum group properties. The quantum group properties were originally discovered [3] in the quantized model, which raised the question to find their classical analogues. The intense efforts [4, 5, 6, 7] at the beginning of the decade led to the consensus that the origin of the quantum group symmetries lies in the Poisson-Lie (P-L) – gauge – symmetries of the so called chiral WZNW phase space that emerges after splitting the left- and right-moving degrees of freedom.

However, in the definition of the chiral WZNW symplectic structure there appears, unavoidably, a choice to be made [6], [7], and the resulting Poisson brackets (PBs) have been analysed so far only in a small subset of the possible cases. In this letter we wish to present new results on the general case (for details, see [8]).

Following [6], [7], the chiral separation of the space of classical solutions, $\mathcal{M}^{sol}$, of the WZNW model based on a simple Lie group G with Lie algebra $\mathcal{G}$ can be described as follows[1]. $\mathcal{M}^{sol}$ consists of the smooth G-valued functions $g(\sigma, \tau)$ which are 2π-periodic in the space variable σ and satisfy the field equation $\partial_R(\partial_L g \cdot g^{-1}) = 0$. The general solution can be written as $g(\sigma, \tau) = g_L(x_L)g_R^{-1}(x_R)$, where (g_L, g_R) is a pair of G-valued, smooth, quasiperiodic functions on the real line $\mathbf{R}$ with *equal monodromies*: for $C = L, R$ one has $g_C(x_C + 2\pi) = g_C(x_C)M$ with some C-independent $M \in G$. This means that $\mathcal{M}^{sol}$ is the base of a principal G-bundle, whose total space is

$$\widehat{\mathcal{M}} = \{(g_L, g_R)|g_{L,R} \in C^\infty(\mathbf{R}, G), \quad g_{L,R}(x + 2\pi) = g_{L,R}(x)M \quad M \in G\}. \tag{1.1}$$

The free right-action of G on $\widehat{\mathcal{M}}$ is given by $G \ni p : (g_L, g_R) \mapsto (g_L p, g_R p)$, and the bundle projection, $\vartheta : \widehat{\mathcal{M}} \to \mathcal{M}^{sol}$, operates as $\vartheta : (g_L, g_R) \mapsto g = g_L g_R^{-1}$. Being a space of solutions, $\mathcal{M}^{sol}$ has a natural symplectic structure, Ω_{sol}^κ, whose pull-back to $\widehat{\mathcal{M}}$ can be extended to the chirally separated space given by $\widehat{\mathcal{M}}^{ext} = \mathcal{M}_L \times \mathcal{M}_R$ with

$$\mathcal{M}_C = \{g_C|g_C \in C^\infty(\mathbf{R}, G), \quad g_C(x + 2\pi) = g_C(x)M_C \quad M_C \in G\}. \tag{1.2}$$

In the extended space it is natural to require the two chiral factors to be completely decoupled. The most general symplectic structure $\widehat{\Omega}_{ext}^\kappa$ on $\widehat{\mathcal{M}}^{ext}$ with this property which equals $\vartheta^*(\Omega_{sol}^\kappa)$ on $\widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}^{ext}$ has the form $\widehat{\Omega}_{ext}^\kappa(g_L, g_R) = \kappa_L \Omega_{chir}^\rho(g_L) + \kappa_R \Omega_{chir}^\rho(g_R)$. Here $\kappa_L = -\kappa_R = \kappa$ is a parameter (the affine Kac-Moody 'level') and

$$\Omega_{chir}^\rho(g_C) = \Omega_{chir}(g_C) + \rho(M_C) \tag{1.3}$$

with

$$\Omega_{chir}(g_C) = -\frac{1}{2}\int_0^{2\pi} dx_C \operatorname{Tr}\left(g_C^{-1}dg_C\right) \wedge \left(g_C^{-1}dg_C\right)' - \frac{1}{2}\operatorname{Tr}\left((g_C^{-1}dg_C)(0) \wedge dM_C \cdot M_C^{-1}\right) \tag{1.4}$$

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[1]Conventions: We have $x_L = \sigma + \tau$, $x_R = \sigma - \tau$, $\partial_C = \frac{\partial}{\partial x_C}$ for $C = L, R$. For a basis T^α of $\mathcal{G}$, $I^{\alpha\beta} = \operatorname{Tr}\left(T^\alpha T^\beta\right)$ and $[T^\alpha, T^\beta] = f_\gamma^{\alpha\beta}T^\gamma$. The dual basis is denoted by T_α, $\operatorname{Tr}(T_\alpha T^\beta) = \delta_\alpha^\beta$, and the elements $A \in \mathcal{G}$ have components $A_\alpha = \operatorname{Tr}(AT_\alpha)$. The usual summation convention is in force, indices are raised and lowered by $I^{\alpha\beta}$ and its inverse. $\widetilde{\mathcal{G}}$ contains the 2π-periodic functions in $C^\infty(\mathbf{R}, \mathcal{G})$.

and some 2-form ρ depending *only* on the monodromy of g_C. More precisely, since in the extended model the chiral factors $(\mathcal{M}_C, \kappa_C \Omega^\rho_{chir})$ must be *symplectic* manifolds *separately*, $d(\kappa_C \Omega^\rho_{chir}) = 0$, one has the condition [6]

$$d\rho(M_C) = \frac{1}{6} \mathrm{Tr} \left(M_C^{-1} dM_C \wedge M_C^{-1} dM_C \wedge M_C^{-1} dM_C \right), \tag{1.5}$$

which in turn implies that the chiral monodromy matrices M_C must be restricted to a domain $\check{G} \subset G$ on which there exists a smooth 2-form ρ satisfying (1.5).

One of the main points of [7] is that one can construct an appropriate 2-form ρ out of any (suitably normalized) antisymmetric solution $\hat{r} \in \mathcal{G} \wedge \mathcal{G}$ of the modified classical Yang-Baxter equation (YBE) in such a way that the corresponding PBs on $\mathcal{M}_C$ are encoded by the 'classical exchange algebra'

$$\left\{ g_C(x) \overset{\otimes}{,} g_C(y) \right\} = \frac{1}{\kappa_C} \left(g_C(x) \otimes g_C(y) \right) \left(\hat{r} + \frac{1}{2} \hat{I} \, \mathrm{sign} \, (y - x) \right), \quad 0 < x, y < 2\pi, \tag{1.6}$$

where $\hat{I} = I^{\alpha\beta} T_\alpha \otimes T_\beta$. This exchange algebra admits the P-L action $p : g_C(x) \mapsto g_C(x)p$ of the group $G = \{p\}$ endowed with the Sklyanin bracket [9]

$$\kappa_C \{ p \overset{\otimes}{,} p \}_{\hat{R}} = [p \otimes p, \hat{R}] \tag{1.7}$$

with $\hat{R} = \hat{r}$.

By using an arbitrary ρ in (1.3), one expects to obtain a classical exchange algebra governed by some monodromy dependent 'exchange r-matrix' $\hat{r}(M)$. Our first result in fact is that we shall establish the precise relationship between ρ and $\hat{r}(M)$ in the general case. We will then point out that $\hat{r}(M)$ and the $\hat{R}$-matrix of the P-L symmetry group need not always coincide. That is, right multiplication can be a P-L symmetry for certain non-constant exchange r-matrices, too, which we shall give explicitly.

To place our second result into context, we recall that an alternative method of the chiral separation is to construct the general solution of the WZNW field equation out of 'Bloch-waves' that have diagonal monodromy matrix, given by e^ω with $\omega \in \mathcal{H}$, where $\mathcal{H}$ is a splitting Cartan subalgebra with basis H^i. It was pointed out in [5] that the corresponding monodromy dependent r-matrix, $\hat{R}(\omega)$, then satisfies

$$[\hat{R}_{12}(\omega), \hat{R}_{23}(\omega)] + \sum_i H_1^i \frac{\partial}{\partial \omega^i} \hat{R}_{23}(\omega) + \text{cycl. perm.} = -\frac{1}{4} \hat{f}, \quad \hat{f} = f_{\alpha\beta\gamma} T^\alpha \otimes T^\beta \otimes T^\beta. \tag{1.8}$$

This modified classical dynamical YBE appears in other contexts [10, 11, 12], too, and has been much studied recently (see [13], [14] and references therein). In [13] a geometric interpretation of (1.8) has been given in terms of *dynamical P-L groupoids*, generalizing the relationship [15] between the (modified) classical YBE and P-L groups. As our second result, we shall show that an arbitrary exchange r-matrix of the WZNW model is a solution of another dynamical generalization of the modified classical YBE, given by eq. (2.28) below, which also admits an interpretation in terms of (new) P-L groupoids.

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2 Poisson brackets on the chiral WZNW phase space

We here investigate the symplectic structure on the chiral WZNW phase space $\mathcal{M}_C$ introduced above. The analysis is the same for both chiralities, $C = L, R$, and we simplify our notation by putting $\mathcal{M}_{chir}$ for $\mathcal{M}_C$ and g, M, κ for g_C, M_C, κ_C, respectively. Thus $\mathcal{M}_{chir}$ is parametrized by the G-valued, smooth, quasiperiodic field $g(x)$ satisfying the monodromy condition

$$g(x + 2\pi) = g(x)M \qquad M \in G. \tag{2.1}$$

The corresponding chiral current, $J(x) = \kappa g'(x)g^{-1}(x) \in \mathcal{G}$, is a smooth, 2π-periodic function of x. The domain in $\mathcal{M}_{chir}$ that corresponds to $M \in \check{G}$ is denoted by $\check{\mathcal{M}}_{chir}$. Below we show that $\kappa\Omega^\rho_{chir}$ defined by (1.3) is non-degenerate if $\check{G}$ is appropriately chosen and describe the main features of the PBs associated with this symplectic form.

To use $\kappa\Omega^\rho_{chir}$ in practice we need to establish some notation for tangent vectors $X[g]$ at $g \in \mathcal{M}_{chir}$ and vector fields X over the chiral phase space. To this end we consider smooth curves on $\mathcal{M}_{chir}$ described by functions $\gamma(x, t) \in G$ satisfying

$$\gamma(x + 2\pi, t) = \gamma(x, t)M(t) \qquad M(t) \in G; \qquad \gamma(x, 0) = g(x). \tag{2.2}$$

$X[g]$ is obtained as the velocity to the curve at $t = 0$, encoded by the $\mathcal{G}$-valued, smooth function

$$\xi(x) := \frac{d}{dt}g^{-1}(x)\gamma(x, t)\Big|_{t=0}. \tag{2.3}$$

The monodromy properties of $\xi(x)$ can be derived by taking the derivative of the first equation in (2.2): $\xi'(x + 2\pi) = M^{-1}\xi'(x)M$, and this can be solved in terms of a $\mathcal{G}$-valued, smooth, 2π-periodic function, $X_J \in \widetilde{\mathcal{G}}$, and a constant Lie algebra element, ξ_0, as follows:

$$\xi(x) = \xi_0 + \int_0^x dy\, g^{-1}(y)X_J(y)g(y). \tag{2.4}$$

A vector field X on $\mathcal{M}_{chir}$ is an assignment, $g \mapsto X[g]$, of a vector to every point $g \in \mathcal{M}_{chir}$. Thus it can be specified by the assignments $g \mapsto \xi_0[g] \in \mathcal{G}$ and $g \mapsto X_J[g] \in \widetilde{\mathcal{G}}$. Using the curve that defines $X[g]$, X acts on a smooth function, $g \mapsto F[g]$, on $\mathcal{M}_{chir}$ as

$$X(F)[g] = \frac{d}{dt}F[g_t]\Big|_{t=0} \qquad g_t(x) = \gamma(x, t). \tag{2.5}$$

Note that the evaluation functions $F^x[g] := g(x)$ and $\mathcal{F}^x[g] := J(x)$ are differentiable with respect to any vector field, and their derivatives are given by

$$X(g(x)) = g(x)\xi(x) \qquad \text{and} \qquad X(J(x)) = \kappa X_J(x). \tag{2.6}$$

This clarifies the meaning of X_J as well. It is also obvious from its definition that the monodromy matrix yields a G-valued differentiable function on $\mathcal{M}_{chir}$, $g \mapsto M = g^{-1}(x)g(x + 2\pi)$, whose derivative is characterized by the $\mathcal{G}$-valued function $X(M)M^{-1} = M\xi(x+2\pi)M^{-1}-\xi(x)$. Having defined vector fields, one can also introduce differential forms as usual. We only remark that by (2.6) evaluation 1-forms like $dg(x)$, $dJ(x)$ or $(g^{-1}dg)'(x)$ are perfectly well-defined: e.g. $dg(x)(X) = X(g(x)) = g(x)\xi(x)$.

The inversion of $\kappa \Omega^{\rho}_{chir}$ consists of the solution of the following problem: For a fixed (scalar) function F on the phase space $\check{\mathcal{M}}_{chir}$, find the corresponding vector field, Y^F, satisfying

$$X(F) = \kappa \Omega^{\rho}_{chir}(X, Y^F) \tag{2.7}$$

for all vector fields X. Notice that Y^F does not necessarily exist for a given F. We say that F is an element of the set of *admissible Hamiltonians*, denoted as H, if the corresponding hamiltonian vector field, Y^F, exists.

To compute $\kappa \Omega^{\rho}_{chir}(X, Y^F)$ we assume that X is parametrized by $\xi(x)$ and further by the pair $(\xi_0, X_J(x))$, while the analogous parametrization for Y^F is given by $\eta(x)$ and the pair $(\eta_0, Y_J(x))$. Furthermore we parametrize ρ now as

$$\rho(M) = \frac{1}{2} q^{\alpha\beta}(M) \mathrm{Tr}(T_\alpha M^{-1} dM) \wedge \mathrm{Tr}(T_\beta M^{-1} dM). \tag{2.8}$$

The $q^{\alpha\beta}$, $q^{\alpha\beta} = -q^{\beta\alpha}$, are smooth functions on the domain $\check{G} \subset G$. Studying the right hand side of (2.7) one can establish [8] the following three necessary and sufficient conditions that F must obey to guarantee that Y^F exists:
- There must exist a *smooth* $\mathcal{G}$-valued function on $\mathbf{R}$, $A^F(x)$, and a constant Lie algebra element, a^F, such that for any vector field X

$$X(F) = \kappa \int_0^{2\pi} dx \mathrm{Tr}\left(X_J(x) A^F(x)\right) + \kappa \mathrm{Tr}\left(\xi_0 a^F\right). \tag{2.9}$$

This means that $F \in \mathtt{H}$ must have an exterior derivative parametrized by the assignments $g \mapsto A^F(x)[g]$ and $g \mapsto a^F[g]$. The restriction of $A^F(x)$ to $x \in [0, 2\pi]$ and a^F are uniquely determined by (2.9), and $A^F(x)$ is made a unique function on $\mathbf{R}$ by the next requirement.
- The expression

$$\frac{1}{\kappa}\left[A^F(x), J(x)\right] + A^{F\prime}(x) \tag{2.10}$$

must define a smooth 2π-*periodic* function on $\mathbf{R}$.
- $A^F(x)$ and a^F must be related by

$$a^F = g^{-1}(0)\left[A^F(0) - A^F(2\pi)\right]g(0). \tag{2.11}$$

If these conditions are satisfied, then Y^F is uniquely determined, and is in fact given by

$$g^{-1}(x)Y^F(g(x)) = g^{-1}(x)A^F(x)g(x) - \frac{1}{2}a^F + r(M)(a^F), \tag{2.12}$$

where

$$r(M)(a^F) = T_\alpha r^{\alpha\beta}(M)a^F_\beta \tag{2.13}$$

and the matrix $r^{\alpha\beta}(M)$ is defined as the solution of an interesting linear equation. This equation can be best described in terms of the linear operators $q_\pm(M)$, $r_\pm(M) \in \mathrm{End}(\mathcal{G})$ which are associated with the matrices

$$r^{\alpha\beta}_\pm(M) = r^{\alpha\beta}(M) \pm \frac{1}{2}I^{\alpha\beta} \quad \text{and} \quad q^{\alpha\beta}_\pm(M) = q^{\alpha\beta}(M) \pm \frac{1}{2}I^{\alpha\beta} \tag{2.14}$$

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by the rule
$$Q(A) = T_\alpha Q^{\alpha\beta} A_\beta, \qquad\qquad Q^{\alpha\beta} = r_\pm^{\alpha\beta}(M), \ q_\pm^{\alpha\beta}(M). \qquad (2.15)$$

In fact [8], using these operators the equation determining $r^{\alpha\beta}(M)$ obtains the form:

$$q_+(M) \circ r_-(M) = q_-(M) \circ \mathrm{Ad}\,(M^{-1}) \circ r_+(M). \qquad (2.16)$$

Notice that at the identity $e \in G$ the unique solution is $r^{\alpha\beta}(e) = q^{\alpha\beta}(e)$. It follows by continuity that there is a neighbourhood of $e \in G$ on which a unique solution exists and is a smooth function of M. Moreover, it is easy to see that the solution $r(M)$ is antisymmetric, $r^{\alpha\beta} = -r^{\beta\alpha}$. By choosing $\check{G} \subset G$ appropriately, hence we obtain by (2.16) a *one-to-one* correspondence between the smooth 2-form ρ on $\check{G}$ and the smooth map $\check{G} \ni M \mapsto r(M) \in \mathrm{End}(\mathcal{G})$.

Incidentally, it follows from (2.12) that

$$Y^F(J(x)) = \left[A^F(x), J(x)\right] + \kappa A^{F\prime}(x), \qquad (2.17)$$

which explains why (2.10) must define an element of $\widetilde{\mathcal{G}}$.

Having obtained the conditions of existence and the general expression (2.12) for hamiltonian vector fields, one can analyse which functions belong to H. First note that the evaluation functions $\mathcal{F}^x[g]$ and $F^x[g]$ fail to satisfy the first condition, thus they are not in H. However, their smeared out versions

$$\mathcal{F}_\mu := \int_0^{2\pi} dx \,\mathrm{Tr}\,\Big(\mu(x)J(x)\Big), \qquad F_\phi[g] := \int_0^{2\pi} dx \,\mathrm{Tr}\,\Big(\phi(x)g^\Lambda(x)\Big), \qquad (2.18)$$

(where in defining F_ϕ we use a representation $\Lambda : G \to GL(V)$ of G with $g^\Lambda = \Lambda(g)$ and a smooth test function $\phi : \mathbf{R} \to \mathrm{End}(V)$) can be shown to belong to H, if $\mu(x)$ is a $\mathcal{G}$-valued, smooth, 2π-periodic test function, and ϕ satisfies $\phi^{(k)}(0) = \phi^{(k)}(2\pi) = 0$ for every integer $k \geq 0$. The corresponding hamiltonian vector fields obtained from (2.12) satisfy

$$Y^{\mathcal{F}_\mu}(g(x)) = \mu(x)g(x), \quad Y^{\mathcal{F}_\mu}(J(x)) = [\mu(x), J(x)] + \kappa\mu'(x), \qquad Y^{\mathcal{F}_\mu}(M) = 0, \qquad (2.19)$$

and, for $x \in [0, 2\pi]$,

$$g^{-1}(x)Y^{F_\phi}(g(x)) = \frac{1}{\kappa}T^\alpha \int_x^{2\pi} dy \,\mathrm{Tr}\,\Big(T_\alpha^\Lambda \phi(y)g^\Lambda(y)\Big) - \frac{1}{2}a^{F_\phi} + r(M)(a^{F_\phi}). \qquad (2.20)$$

Eq. (2.19) shows that the $\mathcal{F}_\mu$ generate an infinitesimal action of the loop group on the phase space with respect to which $g(x)$ is an affine Kac-Moody primary field, and the current $J(x)$ transforms according to the co-adjoint action of the centrally extended loop group. The matrix elements M_{kl}^Λ of the monodromy matrix in representation Λ also belong to H. The action of $Y^{M_{kl}^\Lambda}$ on $g_{ij}^\Lambda(x)$ and on M_{ij}^Λ can be written in tensorial form as

$$Y^{M_{kl}^\Lambda}(g_{ij}^\Lambda(x)) = \frac{1}{\kappa}(g(x) \otimes M\,\hat{\Theta}(M))_{ik,jl}^\Lambda, \qquad (2.21)$$

$$Y^{M_{kl}^\Lambda}(M_{ij}^\Lambda) = \frac{1}{\kappa}\Big((M \otimes M)\hat{\Delta}(M)\Big)_{ik,jl}^\Lambda, \qquad (2.22)$$

where our tensor product notation is $(K \otimes L)_{ik,jl} = K_{ij}L_{kl}$, and

$$\hat{\Theta}(M) = \hat{r}_+(M) - M_2^{-1}\hat{r}_-(M)M_2, \qquad \hat{\Delta}(M) = \hat{\Theta}(M) - M_1^{-1}\hat{\Theta}(M)M_1 \qquad (2.23)$$

with $\hat{r}_\pm(M) = r_\pm^{\alpha\beta}(M)T_\alpha \otimes T_\beta$. The matrix $r^{\alpha\beta}(M)$ appearing here is the solution of (2.16), and $M_1 = M \otimes 1$, $M_2 = 1 \otimes M$.

We now wish to interpret the above hamiltonian vector fields in terms of PBs. Recall that the PB of two smooth functions F_1 and F_2 on a *finite dimensional* smooth symplectic manifold is defined by

$$\{F_1, F_2\} = Y^{F_2}(F_1) = -Y^{F_1}(F_2) = \Omega(Y^{F_2}, Y^{F_1}), \qquad (2.24)$$

where Y^{F_i} is the hamiltonian vector field associated with F_i by the symplectic form Ω. One may formally apply the same formula in the infinite dimensional case to the admissible functions that possess a hamiltonian vector field. However, it is then a non-trivial problem to fully specify the set of functions that form a closed Poisson algebra. Setting this question aside, it is clear from (2.19) and (2.22) that the admissible functions of J and those of M will form two closed Poisson subalgebras that centralize each other. Furthermore, we may use the perfectly well-defined expression

$$\{F_\chi, F_\phi\} := Y^{F_\phi}(F_\chi) \qquad (2.25)$$

for the PB of two admissible Hamiltonians of type F in eq. (2.18) to define the ('distribution valued') PB of the evaluation functions $g(x)$ by the equality:

$$\{F_\chi, F_\phi\} := \int_0^{2\pi} \int_0^{2\pi} dxdy \, \mathrm{Tr}_{12}\left(\chi(x) \otimes \phi(y)\{g^\Lambda(x) \,\overset{\otimes}{,}\, g^\Lambda(y)\}\right), \qquad (2.26)$$

where Tr_{12} is the (normalized) trace over $V \otimes V$ and $\{g^\Lambda(x) \,\overset{\otimes}{,}\, g^\Lambda(y)\}_{ik,jl} = \{g_{ij}^\Lambda(x), g_{kl}^\Lambda(y)\}$. With these definitions, our explicit formula of the hamiltonian vector field Y^{F_ϕ} in (2.20) is equivalent to the following quadratic exchange algebra type PB for the chiral field $g(x)$:

$$\left\{g^\Lambda(x) \,\overset{\otimes}{,}\, g^\Lambda(y)\right\} = \frac{1}{\kappa}\left(g^\Lambda(x) \otimes g^\Lambda(y)\right)\left(\hat{r}(M) + \frac{1}{2}\hat{I}\,\mathrm{sign}\,(y-x)\right)^\Lambda, \quad 0 < x, y < 2\pi, \qquad (2.27)$$

where $\hat{r}(M)$ denotes (the appropriate representation of) the element in $\mathcal{G} \otimes \mathcal{G}$ corresponding to the solution of eq. (2.16): $\hat{r}(M) = r^{\alpha\beta}(M)T_\alpha \otimes T_\beta$. Thus eq. (2.16) gives indeed the precise relation between the 2-form ρ in (1.3), (2.8) and the – in general monodromy dependent – 'exchange r-matrix', $\hat{r}(M)$. Proceeding in the same way with the $\{F_\phi, M_{kl}^\Lambda\}$ PB as we did with the $\{F_\chi, F_\phi\}$ one, we conclude that the right hand side of (2.21) should be interpreted as the expression of the $\{g_{ij}^\Lambda(x), M_{kl}^\Lambda\}$ PB, and similarly for (2.22).

It is clear from the foregoing discussion that the admissible Hamiltonians of type $\mathcal{F}_\mu$, F_ϕ and M_{kl}^Λ should together generate a closed Poisson algebra. Although at present we cannot fully characterize the set of elements that belong to this algebra, we wish to point out that the Jacobi identity for three functions of type F_ϕ, in any Poisson algebra that contains them, is equivalent to the following equation for $\hat{r}(M)$:

$$[\hat{r}_{12}(M), \hat{r}_{23}(M)] + \Theta_{\alpha\beta}(M)T_1^\alpha \mathcal{R}^\beta \hat{r}_{23}(M) + \text{cycl. perm.} = -\frac{1}{4}\hat{f}. \qquad (2.28)$$

Here $\hat{f}$ is the same as in (1.8) and the cyclic permutation is over the three tensorial factors with $\hat{r}_{23} = r^{\alpha\beta}(1 \otimes T_\alpha \otimes T_\beta)$, $T_1^\alpha = T^\alpha \otimes 1 \otimes 1$ and so on. Furthermore, we use the components

of $\hat{\Theta} = \Theta_{\alpha\beta} T^\alpha \otimes T^\beta$ given by (2.23), and the left-invariant differential operators $\mathcal{R}^\beta$ that act on a function ψ of M by

$$(\mathcal{R}_\alpha \psi)(M) := \frac{d}{dt} \psi(Me^{tT_\alpha})\Big|_{t=0}, \qquad \mathcal{R}^\beta = I^{\beta\alpha} \mathcal{R}_\alpha. \tag{2.29}$$

Eq. (2.28) can be viewed as a dynamical generalization of the classical modified YBE, to which it reduces if the r-matrix is a monodromy independent constant. Of course, (2.28) is satisfied for any $\hat{r}(M)$ that arises as a solution of (2.16) since the Jacobi identity is guaranteed by $d\Omega^\rho_{chir} = 0$.

Next we describe an interesting solution of (2.28) obtained by inverting (2.16) for the r-matrix using a particular 2-form ρ (2.8) as input. For this, we now note that if the monodromy matrix M is near to $e \in G$, then the chiral WZNW field can be uniquely parametrized as

$$g(x) = h(x)e^{x\Gamma}, \tag{2.30}$$

where $h(x)$ is a G-valued, smooth, 2π-periodic function and Γ varies in a neighbourhood of zero in $\mathcal{G}$, $\check{\mathcal{G}} \subset \mathcal{G}$, for which the map $\check{\mathcal{G}} \ni \Gamma \mapsto M = e^{2\pi\Gamma} \in \check{G}$ is a diffeomorhism. An easy computation gives the following formula for Ω_{chir} (1.4) in this parametrization of $\check{\mathcal{M}}_{chir}$:

$$\Omega_{chir}(h, \Gamma) = \Omega^0_{chir}(h, \Gamma) - \rho_0(\Gamma), \tag{2.31}$$

where

$$\Omega^0_{chir}(h, \Gamma) = -\frac{1}{2} \int_0^{2\pi} dx \, \mathrm{Tr} \left(h^{-1}dh \wedge (h^{-1}dh)' \right) + d \int_0^{2\pi} dx \mathrm{Tr} \left(\Gamma h^{-1}dh \right), \tag{2.32}$$

$$\rho_0(\Gamma) = -\frac{1}{2} \int_0^{2\pi} dx \, \mathrm{Tr} \left(d\Gamma \wedge de^{x\Gamma} e^{-x\Gamma} \right). \tag{2.33}$$

Taking into account that $M = e^{2\pi\Gamma}$, it is not difficult to verify that

$$d\Omega^0_{chir} = 0, \qquad d\rho_0(\Gamma) = \frac{1}{6} \mathrm{Tr} \left(M^{-1}dM \wedge M^{-1}dM \wedge M^{-1}dM \right). \tag{2.34}$$

Recalling eq. (1.5), we see that the 2-form ρ in (1.3) in this case can be parametrized by an arbitrary closed 2-form β on $\check{\mathcal{G}}$ as

$$\rho(\Gamma) = \rho_0(\Gamma) + \beta(\Gamma), \qquad d\beta(\Gamma) = 0. \tag{2.35}$$

By (1.3) we thus have $\Omega^\rho_{chir} = \Omega^0_{chir} + \beta$, in particular $\Omega^{\rho_0}_{chir} = \Omega^0_{chir}$. In order to find the exchange r-matrix, $\hat{r}_0$, corresponding to ρ_0, we notice that the integral defining ρ_0 can be computed in closed form and the linear operator, q_0, corresponding by (2.15) to its matrix is given by

$$q_0 = \frac{2\mathcal{Y} + e^{-\mathcal{Y}} - e^{\mathcal{Y}}}{2(e^{\mathcal{Y}} - 1)(1 - e^{-\mathcal{Y}})} \qquad \text{with} \qquad \mathcal{Y} := 2\pi(\mathrm{ad}\,\Gamma). \tag{2.36}$$

Then from eq. (2.16) we find the linear operator[2] version, r_0, of the exchange r-matrix as

$$r_0 = \frac{1}{2} \coth \frac{\mathcal{Y}}{2} - \frac{1}{\mathcal{Y}}. \tag{2.37}$$

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[2]The expressions in eqs. (2.36), (2.37), (2.41) are defined by the power series expansions of the corresponding complex analytic functions around zero. For instance [16], $2r_0 = \sum_{k=1}^\infty \frac{2^{2k} B_{2k}}{(2k)!} \left(\frac{1}{2}\mathcal{Y}\right)^{2k-1}$.

By means of (2.27) this r-matrix defines one of the possible monodromy dependent exchange algebras for the chiral WZWN field, and it also represents a non-trivial special solution of (2.28). To uncover a remarkable property of this solution, we note that the function $F(h, \Gamma) = \Gamma_\alpha = \text{Tr}(T_\alpha \Gamma)$ belongs to H, and its hamiltonian vector field in the $\rho = \rho_0$ case gives rise to the PBs:

$$\{g(x), \bar{\Gamma}_\alpha\} = g(x)T_\alpha \quad \text{and} \quad \{\bar{\Gamma}_\alpha, \bar{\Gamma}_\beta\} = -f_{\alpha\beta}^\gamma \bar{\Gamma}_\gamma \quad \text{for} \quad \bar{\Gamma}_\alpha := 2\pi\kappa\Gamma_\alpha. \tag{2.38}$$

This means that in the case of the symplectic form $\kappa\Omega_{chir}^0$ the $\bar{\Gamma}_\alpha$ (essentially the logarithm of the monodromy matrix) generate a *classical* $\mathcal{G}$-symmetry on $\check{\mathcal{M}}_{chir}$. We remark in passing that a classical $\mathcal{G}$-symmetry is sometimes called 'Abelian' to contrast it with a proper ('non-Abelian') P-L symmetry, for which the symmetry group itself is endowed with a non-zero PB (for P-L symmetry, see e.g. [7], [17] and references therein).

Motivated by the somewhat surprising result obtained above, we now investigate what conditions the monodromy dependent exchange r-matrix should satisfy in general to guarantee that the standard (rigid) right action of G on $\check{\mathcal{M}}_{chir}$ is a proper P-L symmetry. For this purpose we endow the group with the Sklyanin bracket (1.7) (replacing κ_C now by κ), where $\hat{R} = R^{\alpha\beta}T_\alpha \otimes T_\beta \in \mathcal{G} \wedge \mathcal{G}$ is a *constant* r-matrix subject to the requirement

$$[\hat{R}_{12}, \hat{R}_{23}] + \text{cycl. perm.} = -\nu^2 \hat{f} \qquad \nu^2 : \text{some real constant.} \tag{2.39}$$

It is easy to check that the rigid right action[3] of G on $\check{\mathcal{M}}_{chir}$, $p : g(x) \mapsto g(x)p \ \forall p \in G$, is a P-L action if and only if

$$\hat{r}(p^{-1}Mp) - \hat{R} = (p \otimes p)^{-1}(\hat{r}(M) - \hat{R})(p \otimes p). \tag{2.40}$$

This simply means that the $\mathcal{G} \wedge \mathcal{G}$-valued function $(\hat{r}(M) - \hat{R})$ on $\check{G}$ must be *equivariant* with respect to the natural (infinitesimal) actions of G on $\check{G}$ and on $\mathcal{G} \wedge \mathcal{G}$.

Therefore the right multiplication is a P-L symmetry iff the exchange r-matrix $\hat{r}(M)$ is such a solution of (2.28) that the corresponding difference $(\hat{r}(M) - \hat{R})$ is equivariant. Insisting on the parametrization $M = e^{2\pi\Gamma}$, the search for these solutions is made feasible by the observation that any analytic function of $\mathcal{Y} = 2\pi(\text{ad}\Gamma)$ is equivariant. In fact, one of our main results, proved in [8], is that the r-matrix corresponding to the linear operator

$$r = \frac{1}{2}\coth\frac{\mathcal{Y}}{2} - \nu\coth(\nu\mathcal{Y}) + R \tag{2.41}$$

solves (2.28). Some remarks on this formula are in order. First, note that in the $\nu = 0$ case we mean the limit of the corresponding complex analytic function, whereby we recover r_0 in (2.37) if in addition we use $R = 0$ (see also footnote 2). Second, notice that if $\nu = \frac{1}{2}$, then $r = R$, which is the case of the constant exchange r-matrices [7]. Third, it is worth stressing that for a compact Lie algebra $\mathcal{G}$ constant exchange r-matrices do not exist, because of the negative sign on the right hand side of (2.28), but our formula (2.41) gives explicit solutions also in this case using a purely imaginary ν in (2.39). Finally, we remark that in the $\nu = \frac{1}{2}$ case the construction of the 2-form ρ that corresponds to the r-matrix in (2.41) is presented in [7], while in general the existence (and the uniqueness) of a suitable local 2-form is guaranteed by the solvability of (2.16). Further comments and explicit results are contained in [8].

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[3]Since $M \mapsto p^{-1}Mp$, strictly speaking we here have to assume that $\check{G} \subset G$ is invariant under the adjoint action of G, or else the statements should be reformulated in terms of the corresponding $\mathcal{G}$-action.

3 Encoding chiral WZNW by Poisson-Lie groupoids

The classical dynamical YBE (1.8) can be regarded as the guarantee of the Jacobi identity in a P-L groupoid [13]. Below we show that eq. (2.28) admits an analogous interpretation. For this, we introduce a family of P-L groupoids in such a way that a subfamily of these is naturally associated with the possible PBs on the chiral WZNW phase space. Remarkably, these groupoids are finite dimensional Poisson manifolds that encode practically all information about the infinite dimensional chiral WZNW PBs.

Roughly speaking, a groupoid is a set, say P, endowed with a 'partial multiplication' that behaves similarly to a group multiplication in the cases when it can be performed. To understand the following construction one does not need to know details of the notion of a groupoid (see e.g. [18]), since we shall only use the most trivial example of such a structure, for which

$$P = S \times G \times S = \{(M^F, g, M^I)\}, \tag{3.1}$$

where G is a group and S is some set. The partial multiplication is defined for those triples (M^F, g, M^I) and $(\bar{M}^F, \bar{g}, \bar{M}^I)$ for which $M^I = \bar{M}^F$, and the product is

$$(M^F, g, M^I)(\bar{M}^F, \bar{g}, \bar{M}^I) := (M^F, g\bar{g}, \bar{M}^I) \quad \text{for} \quad M^I = \bar{M}^F. \tag{3.2}$$

In other words, the graph of the partial multiplication is the subset of

$$P \times P \times P = \{(M^F, g, M^I)\} \times \{(\bar{M}^F, \bar{g}, \bar{M}^I)\} \times \{(\hat{M}^F, \hat{g}, \hat{M}^I)\} \tag{3.3}$$

defined by the constraints

$$M^I = \bar{M}^F, \quad \hat{M}^F = M^F, \quad \hat{M}^I = \bar{M}^I, \quad \hat{g} = g\bar{g}, \tag{3.4}$$

where the hatted triple encodes the components of the product. A P-L groupoid [19] P is a groupoid and a Poisson manifold in such a way that the graph of the partial multiplication is a *coisotropic* submanifold of $P \times P \times P^-$, where P^- denotes the manifold P endowed with the opposite of the PB on P. In other words, this means that *the constraints that define the graph are first class*. This definition reduces to that of a P-L group in the particular case for which the set S in (3.1) consists of a single point.

In the interpretation of (1.8) given in [13] the groupoid P is of the form above with S taken to be a domain in the dual of a Cartan subalgebra of a simple Lie group G. By thinking about a generic monodromy matrix, we now take P to be

$$P = \check{G} \times G \times \check{G}, \tag{3.5}$$

where $\check{G}$ is some open domain in G. On this P, we postulate a PB $\{\ ,\ \}_P$ defined, by using the usual tensorial notation, as follows:

$$\kappa\{g_1, g_2\}_P = g_1 g_2 \hat{r}(M^I) - \hat{r}(M^F) g_1 g_2$$
$$\kappa\{g_1, M_2^I\}_P = g_1 M_2^I \hat{\Theta}(M^I)$$
$$\kappa\{g_1, M_2^F\}_P = M_2^F \hat{\Theta}(M^F) g_1$$
$$\kappa\{M_1^I, M_2^I\}_P = M_1^I M_2^I \hat{\Delta}(M^I)$$
$$\kappa\{M_1^F, M_2^F\}_P = -M_1^F M_2^F \hat{\Delta}(M^F)$$
$$\kappa\{M_1^I, M_2^F\}_P = 0. \tag{3.6}$$

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Here κ is an arbitrary constant included for comparison purposes. The 'structure functions' $\hat{r}$, $\hat{\Theta}$, $\hat{\Delta}$ are $\mathcal{G} \otimes \mathcal{G}$ valued functions on $\check{G}$; in components

$$\hat{r}(M) = r^{\alpha\beta}(M)T_\alpha \otimes T_\beta, \quad \hat{\Theta}(M) = \Theta^{\alpha\beta}(M)T_\alpha \otimes T_\beta, \quad \hat{\Delta}(M) = \Delta^{\alpha\beta}(M)T_\alpha \otimes T_\beta. \qquad (3.7)$$

It is quite easy to verify that a PB given by the ansatz (3.6) always yields a P-L groupoid, since the constraints in (3.4) will be first class for any choice of the structure functions. Of course, the structure functions must satisfy a system of equations in order for the above ansatz to define a PB. The antisymmetry of the PB is ensured by

$$\hat{r} = -\hat{r}_{21} \quad (\hat{r}_{21} := r^{\alpha\beta}T_\beta \otimes T_\alpha) \qquad \text{and} \qquad \hat{\Delta} = -\hat{\Delta}_{21}, \qquad (3.8)$$

while the Jacobi identity is, in fact, equivalent to the following system of equations:

$$[\hat{r}_{12}, \hat{r}_{13}] + \Theta_{\alpha\beta}T_1^\alpha \mathcal{R}^\beta \hat{r}_{23} + \text{cycl. perm.} = \mu \hat{f}, \quad \mu = \text{constant}, \qquad (3.9)$$

$$[\hat{\Delta}_{12}, \hat{\Delta}_{13}] + \Delta_{\alpha\beta}T_1^\alpha \mathcal{R}^\beta \hat{\Delta}_{23} + \text{cycl. perm.} = 0, \qquad (3.10)$$

$$[\hat{r}_{12}, \hat{\Theta}_{13} + \hat{\Theta}_{23}] + [\hat{\Theta}_{13}, \hat{\Theta}_{23}] + \Delta_{\alpha\beta}T_3^\alpha \mathcal{R}^\beta \hat{r}_{12} + \Theta_{\alpha\beta}(T_1^\alpha \mathcal{R}^\beta \hat{\Theta}_{23} - T_2^\alpha \mathcal{R}^\beta \hat{\Theta}_{13}) = 0, \qquad (3.11)$$

$$[\hat{\Theta}_{12} + \hat{\Theta}_{13}, \hat{\Delta}_{23}] + [\hat{\Theta}_{12}, \hat{\Theta}_{13}] + \Theta_{\alpha\beta}T_1^\alpha \mathcal{R}^\beta \hat{\Delta}_{23} + \Delta_{\alpha\beta}(T_3^\alpha \mathcal{R}^\beta \hat{\Theta}_{12} - T_2^\alpha \mathcal{R}^\beta \hat{\Theta}_{13}) = 0. \qquad (3.12)$$

Observe that the left hand side of (3.9) is of the same form as that of (2.28), but in the groupoid context on the right hand side we have an *arbitrary* constant μ. The derivation of the above equations from the various instances of the Jacobi identity is not difficult. What is somewhat miraculous is that one does not obtain more equations than these. This is actually ensured by our choice of the relationship between the PBs that involve M^I and those that involve M^F. As an illustration, let us explain how (3.9) is derived. By evaluating

$$\{\{g_1, g_2\}_P, g_3\}_P + \text{cycl. perm.} = 0, \qquad (3.13)$$

one obtains that this is equivalent to

$$g_1 g_2 g_3 \left([\hat{r}_{12}, \hat{r}_{13}] + \Theta_{\alpha\beta}T_1^\alpha \mathcal{R}^\beta \hat{r}_{23} + \text{cycl. perm.} \right)(M^I) =$$
$$= \left([\hat{r}_{12}, \hat{r}_{13}] + \Theta_{\alpha\beta}T_1^\alpha \mathcal{R}^\beta \hat{r}_{23} + \text{cycl. perm.} \right)(M^F) g_1 g_2 g_3. \qquad (3.14)$$

This holds if and only if the expression in the parenthesis is a constant, Ad-invariant element of $\wedge^3(\mathcal{G})$, and $\mu \hat{f}$ is the only such element for a simple Lie algebra $\mathcal{G}$.

We have seen that the chiral WZNW PBs are encoded by equations (2.27), (2.21) and (2.22), where $\hat{\Theta}$ and $\hat{\Delta}$ are defined by (2.23) in terms of a solution $\hat{r}$ of (2.28). Now our point is the following: *A P-L groupoid can be naturally associated with any Poisson structure on the chiral WZNW phase space by taking the triple $\hat{r}$, $\hat{\Theta}$, $\hat{\Delta}$ that arises in the WZNW model to be the structure functions of a P-L groupoid according to (3.6).*

It can be checked that the Jacobi identities of the P-L groupoid (3.9)–(3.12) are satisfied for any triple $\hat{r}$, $\hat{\Theta}$, $\hat{\Delta}$ that arises in the WZNW model. This actually follows without any computation since, indeed, the Jacobi identities of the chiral WZNW PBs in (2.27), (2.21), (2.22) lead to the same equations, with $\mu = -\frac{1}{4}$, and they are satisfied since they follow from the symplectic form $\kappa \Omega_{chir}^\rho$.

Among the 'chiral WZNW P-L groupoids' described above there are those special cases for which $\hat{r}$ satisfies (2.40) with some constant r-matrix $\hat{R}$ in correspondence with a right P-L

action of G on the chiral WZNW phase space. Under this circumstance one can verify that the two maps defined by

$$P \times G \ni ((M^F, g, M^I), p) \mapsto (M^F, gp, p^{-1}M^I p) \in P \qquad (3.15)$$

and respectively by

$$G \times P \ni (p, (M^F, g, M^I)) \mapsto (pM^F p^{-1}, pg, M^I) \in P \qquad (3.16)$$

are *both* Poisson maps. Thus they define (see also footnote 3) respectively *a right and a left* P-L action of the P-L group G, endowed with the PB (1.7) for $\kappa_C = \kappa$, on the P-L groupoid P.

In [13] P-L groupoids are associated with arbitrary subalgebras $\mathcal{K} \subset \mathcal{G}$, although the corresponding dynamical r-matrices are described only if $\mathcal{K}$ is a Cartan subalgebra. The $\mathcal{K} = \mathcal{G}$ special case of their groupoids is in fact equivalent to our P-L groupoid whose structure function is the r-matrix in (2.37). Their P-L groupoids are different from ours in general.

The reader may find a detailed exposition of the subject of this letter and several related issues in [8]. Among the questions for future study, it would be interesting to investigate the quantization of the above introduced P-L groupoids and to find other applications for them in the field of integrable systems. As for the second question, recall that the classical dynamical YBE appears not only in the classical chiral WZNW model, but also in the theory of the Knizhnik-Zamolodchikov-Bernard equation [11], the Calogero-Moser systems [12] etc, and thus perhaps it might be natural to ask if (2.28) and the associated P-L groupoids can have other interesting applications.

References

[1] E. Witten, Commun. Math. Phys. 92 (1984) 455.

[2] P. Di Francesco, P. Mathieu and D. Sénéchal, Conformal Field Theory (Springer, 1996).

[3] A. Tsuchia and Y. Kanie, Lett. Math. Phys. 13 (1987) 303, Adv. Stud. Pure Math. 16 (1988) 297;
L. Alvarez-Gaumé, C. Gómez and G. Sierra, Phys. Lett. B 220 (1989) 142, Nucl. Phys. B 319 (1989) 155, ibid. 330 (1990) 347;
G. Moore and N. Reshetikhin, Nucl. Phys. B 328 (1989) 557;
G. Felder, J. Fröhlich and G. Keller, Commun. Math. Phys. 124 (1989) 647;
A. Ganchev and V. Petkova, Phys. Lett. B 233 (1989) 374;
P. Furlan, A. Ganchev and V. Petkova, Nucl. Phys. B 343 (1990) 205;
C. Ramírez, H. Ruegg and M. Ruiz-Altaba, Phys. Lett. B 247 (1990) 499;
M.R. Gaberdiel, Commun. Math. Phys. 173 (1995) 357;
C. Gómez, M. Ruiz-Altaba and G. Sierra, Quantum Groups in Two-Dimensional Physics (Cambridge University Press, 1996).

[4] O. Babelon, Phys. Lett. B 215 (1988) 523;
B. Blok, Phys. Lett. B 233 (1989) 359;
L. Faddeev, Commun. Math. Phys. 132 (1990) 131;
A. Alekseev and S. Shatashvili, Commun. Math. Phys. 133 (1990) 353;
O. Babelon, F. Toppan and L. Bonora, Commun. Math. Phys. 140 (1991) 93;
M. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. B 266 (1991) 71;
T. Itoh and Y. Yamada, Prog. Theor. Phys. 85 (1991) 751;
A. Alekseev, L. Faddeev, M. Semenov-Tian-Shansky and A. Volkov, The unravelling of the quantum group structure in the WZNW theory, CERN-TH-5981/91;
V.A. Fateev and S.L. Lukyanov, Int. J. Mod. Phys. A 7 (1992) 853, ibid. 7 (1992) 1325;
G. Bimonte, P. Salomonson, A. Simoni and A. Stern, Int. J. Mod. Phys. A 7 (1992) 6159;
A. Alekseev and I.T. Todorov, Nucl. Phys. B 421 (1994) 413;
M. Chu and P. Goddard, Phys. Lett. B 337 (1994) 285, Nucl. Phys. B 445 (1995) 145;
P. Furlan, L.K. Hadjiivanov and I. T. Todorov, Nucl.Phys. B 474 (1996) 497, Int. J. Mod. Phys. A 12 (1997) 23;
L. Caneschi and M. Lysiansky, Nucl. Phys. B 505 (1997) 701.

[5] J. Balog, L. Dąbrowski and L. Fehér, Phys. Lett. B 244 (1990) 227.

[6] K. Gawędzki, Commun. Math. Phys. 139 (1991) 201.

[7] F. Falceto and K. Gawędzki, J. Geom. Phys. 11 (1993) 251.

[8] J. Balog, L. Fehér and L. Palla, `hep-th/9910046`.

[9] E. Sklyanin, J. Sov. Math. 19 (1982) 1546.

[10] J.-L. Gervais and A. Neveu, Nucl. Phys. B 238 (1984) 125;
E. Cremmer and J.-L. Gervais, Commun. Math. Phys. 134 (1990) 619.

[11] G. Felder and C. Wieczerkowski, Commun. Math. Phys. 176 (1996) 133;
G. Felder, pp. 1247-1255 in: Proc. Int. Congr. Math. Zürich, 1994 (`hep-th/9407154`).

[12] J. Avan, O. Babelon and E. Billey, Commun. Math. Phys. 178 (1996) 281.

[13] P. Etingof and A. Varchenko, Commun. Math. Phys. 192 (1998) 77.

[14] Z. J. Liu and P. Xu, Dirac structures and dynamical r-matrices, `math.DG/9903119`.

[15] V.G. Drinfeld, Sov. Math. Dokl. 27 (1983) 68.

[16] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series and Products, edited by A. Jeffrey, Fifth Edition (Academic Press, 1996).

[17] O. Babelon and D. Bernard, Commun. Math. Phys. 149 (1992) 279.

[18] K. Mackanzie, Lie Groupoids and Lie Algebroids in Differential Geometry (Cambridge University Press, 1987).

[19] A. Weinstein, J. Math. Soc. Japan 4 (1988) 705.