Effective theory for the soft fluctuation modes in the spontaneously broken phase of the N-component scalar field theory

A. Jakovác[a,*], A. Patkós[b,†], P. Petreczky[b,‡] and Zs. Szép[b,§]

(a) Department of Theoretical Physics, Technical University of Budapest, H-1521 Budapest, Hungary

(b) Department of Atomic Physics, Eötvös University, H-1117 Budapest, Hungary

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Abstract

The effective dynamics of the low-frequency modes is derived for the $O(N)$ symmetric scalar field theory in the broken symmetry phase. The effect of the high-frequency fluctuations is taken into account at one-loop level exactly. A new length scale is shown to govern the long-time asymptotics of the linear response function of the Goldstone modes. The large time asymptotic decay of an arbitrary fluctuation is determined in the linear regime. We propose a set of local equations for the numerical solution of the effective non-linear dynamics. The applicability of the usual gradient expansion is carefully assessed.

1 Introduction

In the study of dynamical phenomena with the participation of long-wavelength Higgs and Goldstone-modes one can account for the effect of the high frequency modes most conveniently by deriving a number of effective equations of motion. In a second step one might express the correction terms of the original equations with help of a set of auxiliary fields, allowing local representation of some nonlocal effects. A remarkable feature of this approach is that for pure gauge theories a simple kinetic interpretation can be given to the dynamics of the auxiliary fields [1, 2, 3, 4, 5].

Recently we have shown that a similar interpretation is possible for fluctuations of the one-component self-interacting scalar field in the broken symmetry phase [6]. The evolution of the modes with wave number $|\mathbf{k}|$ was considered under the assumption $M < T$; that is the mass scale set by the spontanous symmetry breakdown is smaller than the temperature. Since $M \sim \sqrt{\lambda}\langle\Phi\rangle$, this condition relates the vacuum expectation value of the field to the temperature and is actually met in systems with second order phase transition not much below the transition

*Zoltán Magyary fellow, e-mail: jakovac@planck.phy.bme.hu
†e-mail: patkos@ludens.elte.hu
‡e-mail: petr@cleopatra.elte.hu
§e-mail: szepzs@cleopatra.elte.hu

point. This condition actually was used when the comparison of the quantum results with the classical dynamics was made.

The high frequency modes have a non-trivial impact on the dynamics of the low frequency modes if the scale characterizing the low-frequency fluctuations is much below the mass scale of the symmetry breaking. In this sense one has to go beyond the usual hard thermal loop approximation, since the effect of loops with momentum $p \simeq M < T$ should be taken into account. The classical statistical mechanical system proposed in [6] reproduces the linear source-amplitude response computed in quantum theory [7].

In this paper we extend our discussion to the broken phase dynamics of the N-component scalar field theory with Φ^4 self-interaction. This model is relevant to the dynamics of the $\pi-\sigma$-system ($N = 4$) [8] and is actively investigated in connection with the phenomenon of the disoriented chiral condensate [9, 10, 11, 12]. The appearance, the evolution and the damping of the low frequency fluctuations and instabilities of the chiral order parameter are carefully studied in these papers. Less attention is paid to the damping of the Goldstone-modes. Though in the realistic case the explicit breaking of the $O(N)$ symmetry leads to massive "Goldstone"-bosons, it is of interest to see what is the intrinsic dynamics of these excitations in the "ideal" spontaneously broken case. It turns out that the damping of the Goldstone-fluctuations with frequency k_0 is essentially different in the respective domains

$$k_0 << \frac{M^2}{T} << M < T \tag{1}$$

and

$$\frac{M^2}{T} << k_0 << M < T. \tag{2}$$

Our conclusion is that both the on-shell dissipation time and the large time asymptotics of the Goldstone fields is qualitatively different in the first region in comparison to the fluctuations of the order parameter.

The physical picture is very appealing. Radial fluctuations along the order parameter relax fairly quickly, what freezes fast the length of the vacuum expectation value of the Φ field. The second stage of the relaxation consists of the slow rotation of the order parameter, which is described by the relaxation of collectively excited low momentum Goldstone fluctuation modes.

The high-temperature one-loop quantum dynamics of scalar models has been investigated very actively recently partly as a kind of theoretical laboratory for developing powerful calculational methods, partly with the aim to provide answers to important questions of inflatory cosmology and the physics of heavy ion collisions. Our work was substantially influenced by References [13, 14, 15, 16, 17].

In this paper we first derive in section 2 the effective equations for the low frequency modes. The coefficients of the terms correcting the classical equations are determined by various n-point functions of the fast modes. For the effective dynamics we work out the modification of the linear part of the equations (the self-energy operator), which is fully determined by the two-point function of the high-frequency fields. A detailed study of this quantity appears in section 3. Stated in more technical terms, we compute one-loop contributions to the two-point functions with full $T = 0$ propagators and restrict the temperature dependent contribution to the $p_0 > \Lambda$ modes. This approach follows the finite temperature renormalization group transformation scheme of D'Attanasio and Pietroni [18]. In section 4 we shall discuss, in particular, whether the linear response of the high frequency modes to long wavelength fluctuations allows a classical kinetic theory interpretation. Using the results obtained for the two-point function of the theory, in section 5 we present the explicit effective field equations, with help of auxiliary fields introduced to handle the nonlocal nature of the effective dynamics. Our results are summarized in section 6. In Appendix A some results of the main text are rederived with help of the conventional perturbation theory in a form explicitly showing its equivalence to the Dyson-Schwinger treatment. Appendix B presents the iterative solution of the dynamics of the classical $O(N)$-model. Some relevant integrals are explicitly evaluated in Appendix C.

2 The effective equations of motion for the slow modes

The Lagrangian of the system is given by

$$L = \frac{1}{2}(\partial_\mu \varphi_a)^2 - \frac{1}{2}m^2(\varphi_a)^2 - \frac{\lambda}{24}((\varphi_a)^2)^2. \tag{3}$$

The aim of this investigation is to integrate out the effect of $p_0 > \Lambda$ high frequency fluctuations. From the point of view of the final result it turns out to be important whether $\Lambda < M$ or $\Lambda > M$, where M is the mass scale spontaneously generated in the broken symmetry phase. In the spirit of the renormalization group the value of Λ will be lowered gradually and the importance of passing the scale M will become evident in this process. Our actual interest will concentrate on the temporal variation of the lowest frequency fluctuations $k_0 << \Lambda$.

We separate in the starting Lagrangian the high-frequency modes ($\phi_a(x)$ with frequency $\omega > \Lambda$) and the low-frequency modes ($\tilde{\Phi}_a(x)$ with frequency $\omega < \Lambda$)

$$\varphi_a(x) \to \tilde{\Phi}_a(x) + \phi_a(x). \tag{4}$$

Averaging over the high frequency fluctuations gives $\langle \phi_a(x) \rangle = 0$ which yields also $\tilde{\Phi}_a(x) = \langle \varphi_a(x) \rangle$. This separation leads to

$$\begin{aligned} L = & \frac{1}{2}(\partial\tilde{\Phi}_a)^2 + \frac{1}{2}(\partial\phi_a)^2 - \frac{m^2}{2}\Big[(\tilde{\Phi}_a)^2 + (\phi_a)^2\Big] - \frac{\lambda}{24}\Big[(\tilde{\Phi}_a)^2(\tilde{\Phi}_b)^2 + (\phi_a)^2(\phi_b)^2\Big] \\ & -\frac{\lambda}{24}\Big[4(\tilde{\Phi}_a\phi_a)(\tilde{\Phi}_b\phi_b) + 4(\tilde{\Phi}_a\phi_a)(\phi_b)^2 + 4(\tilde{\Phi}_a\phi_a)(\tilde{\Phi}_b)^2 + 2(\tilde{\Phi}_a)^2(\phi_b)^2\Big]. \end{aligned} \tag{5}$$

From Eq.(5) the equations for the slow modes can be derived:

$$(\partial^2 + m^2)\tilde{\Phi}_a + \frac{\lambda}{24}[8\phi_a(\tilde{\Phi}_b\phi_b) + 4\tilde{\Phi}_a(\tilde{\Phi}_b)^2 + 4\phi_a(\phi_b)^2 + 4\phi_a(\tilde{\Phi}_b)^2 + 8\tilde{\Phi}_a(\tilde{\Phi}_b\phi_b) + 4\tilde{\Phi}_a(\phi_b)^2] = 0. \tag{6}$$

The effect of the high-frequency fluctuations on the slow ones is obtained by averaging the equations with respect to their statistics. At one-loop level accuracy $\langle \phi_a\phi_b\phi_c \rangle = 0$ and one arrives at

$$(\partial^2 + m^2)\tilde{\Phi}_a + \frac{\lambda}{6}\tilde{\Phi}_a(\tilde{\Phi}_b)^2 + \frac{\lambda}{3}\tilde{\Phi}_b\langle\phi_a\phi_b\rangle + \frac{\lambda}{6}\tilde{\Phi}_a\langle(\phi_b)^2\rangle = 0. \tag{7}$$

We introduce at this point the conventional notation:

$$\Delta_{ab}(x,y) \equiv \langle \varphi(x)\varphi(y) \rangle. \tag{8}$$

Below no summation will be understood when repeated indices appear without the explicit summation symbol. We shall also use specific pieces extracted from the above two-point functions defined as

$$\begin{aligned} \Delta_{ab}^{(0)}(x,y) &\equiv \Delta_{ab}(x,y)|_{\tilde{\Phi}=0}, \\ \Delta_{ab}^{(1)}(x,y) &\equiv \textstyle\int dz \frac{\delta\Delta_{ab}(x,y)}{\delta\tilde{\Phi}_c(z)}|_{\tilde{\Phi}=0} \cdot \tilde{\Phi}_c(z). \end{aligned} \tag{9}$$

In the broken symmetry phase one has to separate the nonzero average value from the slowly varying field, and write the equation only for the fluctuating part:

$$\tilde{\Phi}_a \to \bar{\Phi}\delta_{a1} + \Phi_a, \tag{10}$$

where we have chosen the direction of the average to point along the $a = 1$ direction. We shall analyze the resulting equations for the $a = 1$ and the remaining $a = i \neq 1$ components separately. Since the main effect of the high frequency modes we are interested in is the modification of the mass term (self-energy contribution), therefore we shall restrict our study to the linearized equation of $\Phi_a(x)$. One has to take into account that the two-point functions $\langle\phi_a\phi_b\rangle$ depend on the background $\Phi_a(x)$. For the linearized equations it is sufficient to compute them only up to terms linear in the background.

The two equations are

$$\begin{aligned}
&(\partial^2 + \frac{\lambda}{3}\bar{\Phi}^2)\Phi_1(x) + \frac{\lambda\bar{\Phi}}{2}\Phi_1^2(x) + \frac{\lambda\bar{\Phi}}{6}\sum_{i\neq 1}\Phi_i^2(x) + \frac{\lambda}{6}\Phi_1(x)\sum_a \Phi_a^2(x) + J_1(x) = 0,\\
&\partial^2\Phi_i(x) + \frac{\lambda\bar{\Phi}}{3}\Phi_i(x)\Phi_1(x) + \frac{\lambda}{6}\Phi_i(x)\sum_a \Phi_a^2(x) + J_i(x) = 0, \qquad (i \neq 1)
\end{aligned} \tag{11}$$

with the induced currents

$$\begin{aligned}
J_1(x) &= \frac{\lambda}{2}\bar{\Phi}\Delta_{11}^{(1)}(x,x) + \frac{\lambda}{6}(N-1)\bar{\Phi}\Delta_{ii}^{(1)}(x,x),\\
J_i(x) &= \frac{\lambda}{3}(\Delta_{ii}^{(0)}(x,x) - \Delta_{11}^{(0)}(x,x))\Phi_i + \frac{\lambda}{3}\bar{\Phi}\Delta_{i1}^{(1)}(x,x).
\end{aligned} \tag{12}$$

We have simplified the equations using the fact $\Delta_{i,b\neq i}$ has no Φ-independent piece, since with zero background the correlators are diagonal. The equations (12) express the linear response of the fast modes to the presence of a low frequency background producing an effective source term to their classical dynamical equations. Also one has not to forget that $\bar{\Phi}$ is now the solution of the constant part of the equations:

$$m^2\bar{\Phi} + \frac{\lambda}{6}(3\Delta_{11}^{(0)}(x,x) + (N-1)\Delta_{ii}^{(0)}(x,x))\bar{\Phi} + \frac{\lambda}{6}\bar{\Phi}^3 = 0. \tag{13}$$

3 The two-point function of the fast modes

Varying Eq.(5) with respect to $\phi(x)$ one arrives at the equations of motion of the fast modes in the background of $\tilde{\Phi}(x)$:

$$(\partial^2+m^2)\phi_a+\frac{\lambda}{24}(8\tilde{\Phi}_a(\tilde{\Phi}_b\phi_b)+4\phi_a(\phi_b)^2+4\tilde{\Phi}_a(\tilde{\Phi}_b)^2+8(\tilde{\Phi}_b\phi_b)\phi_a+4\tilde{\Phi}_a(\phi_b)^2+4\phi_a(\tilde{\Phi}_b)^2) = 0. \tag{14}$$

The equations of motion linearized in the high frequency fields can be written in the form

$$\left\{(\partial^2 + m^2)\delta_{ab} + \frac{\lambda}{6}[2\tilde{\Phi}_a\tilde{\Phi}_b + (\tilde{\Phi}_c)^2\delta_{ab}]\right\}\phi_b = -\frac{\lambda}{6}\tilde{\Phi}_a(\tilde{\Phi}_b)^2. \tag{15}$$

We introduce the notation

$$m_{ab}^2(x) = m^2\delta_{ab} + \frac{\lambda}{6}[2\tilde{\Phi}_a(x)\tilde{\Phi}_b(x) + (\tilde{\Phi}_c(x))^2\delta_{ab}] \tag{16}$$

and apply Eqs.(15) and (16) to the fields appearing in the definition of the two-point function $\Delta_{ac}(x,y)$:

$$\partial_x^2\Delta_{ac}(x,y)=-m_{ab}^2(x)\Delta_{bc}(x,y),$$
$$\partial_y^2\Delta_{ac}(x,y)=-m_{cb}^2(y)\Delta_{ab}(x,y). \tag{17}$$

In the derivation of these homogeneous equations we have exploited that $\langle\tilde\Phi_a(x)(\tilde\Phi_c(x))^2\phi_b(y)\rangle = 0$. After the Wigner-transformation

$$\Delta(X,p)=\int d^4u e^{ipu}\Delta(x,y),\qquad u=x-y \tag{18}$$

one arrives at the exact linear equations for the Wigner transforms. (The use of the Latin letters k,p,q is reserved for the 4-momenta and pk, kq, etc. denotes their Minkovskian scalar products.) For the two-point functions diagonal in $O(N)$ indices one finds in the broken phase to linear order in the background (after using Eq.(10))

$$[\frac{1}{4}\partial_X^2-ip\cdot\partial_X-p^2+M_a^2]\Delta_{aa}(X,p)+\lambda_a\bar\Phi\int\frac{d^4q}{(2\pi)^4}\Phi_1(q)\Delta_{aa}(X,p-\frac{q}{2})e^{-iqX}=0,$$
$$[\frac{1}{4}\partial_X^2+ip\cdot\partial_X-p^2+M_a^2]\Delta_{aa}(X,p)+\lambda_a\bar\Phi\int\frac{d^4q}{(2\pi)^4}\Phi_1(q)\Delta_{aa}(X,p+\frac{q}{2})e^{-iqX}=0, \tag{19}$$

where for the Goldstone bosons on has

$$M_i^2=0,\qquad \lambda_i=\frac{\lambda}{3}, \tag{20}$$

while for the heavy ("Higgs") mode

$$M_1^2=\frac{\lambda}{3}\bar\Phi^2,\qquad \lambda_1=\lambda. \tag{21}$$

Here the tree level mass relations were used, as they correspond to the actual order of the perturbation theory. This procedure can be improved using Eq.(13) for $\bar\Phi$, what is equivalent to the resummation of the perturbative series.

After performing the Fourier-transformation also with respect to the center-of-mass coordinate and subtracting the resulting two equations one arrives at

$$2pk\Delta_{11}(k,p)=-\lambda\bar\Phi\int\frac{d^4q}{(2\pi)^4}\Phi_1(q)[\Delta_{11}(k-q,p+\frac{q}{2})-\Delta_{11}(k-q,p-\frac{q}{2})],$$
$$2pk\Delta_{ii}(k,p)=-\frac{\lambda}{3}\bar\Phi\int\frac{d^4q}{(2\pi)^4}\Phi_1(q)[\Delta_{ii}(k-q,p+\frac{q}{2})-\Delta_{ii}(k-q,p-\frac{q}{2})]. \tag{22}$$

For the mixed two-point function Δ_{i1} in the same steps a similar equation is derived:

$$(2pk+\frac{\lambda}{3}\bar\Phi^2)\Delta_{i1}(k,p)=-\frac{\lambda}{3}\bar\Phi\int\frac{d^4q}{(2\pi)^4}[\Delta_{ii}(k-q,p+\frac{q}{2})-\Delta_{11}(k-q,p-\frac{q}{2})]\Phi_i(k). \tag{23}$$

In the kinematical region, where the center-of-mass variation is very slow one can assume $q\approx k$. (If there would be no X-dependence one would find $\Delta(k-q,P)\sim\delta(k-q)$). If in addition one restricts the variation of p in the high frequency fields by the relation $|k_\mu| << \Lambda\simeq|p_\mu|$, one can approximate the integrals in these equations by a factorized form. If the functions $\Delta_{aa}(k,p-\frac{k}{2})$ are replaced by the first two terms of their power series with respect to $k/2$, one recovers the drift equations proposed originally by Mrówczynski and Danielewicz [13]. In case of the diagonal two-point functions these equations are identical to the collisionless Boltzmann equation for a gas of scalar particles, whose masses ($M_1^2+\lambda\bar\Phi\Phi_1(x)$ and $M_i^2+\lambda\bar\Phi\Phi_1(x)/3$) are determined by the Φ_1 field:

$$p\cdot\partial_X\Delta_{aa}(X,p)+\frac{\lambda_a}{2}\bar\Phi\partial_X\Phi_1\partial_p\Delta_{aa}(X,p)=0. \tag{24}$$

Below we shall discuss an alternative to this expansion, which does not exploit the assumption of slow X-variation. At the end we will be in position to assess the validity of the assumption which led to the Boltzmannian kinetic equations (24).

At weak coupling one can attempt the recursive solution of Eqs. (22) and (23). The starting point of the iteration is the assumption that *the high frequency modes are close to thermal equilibrium*, therefore one has for the starting 2-point functions [19]

$$\begin{aligned}
\Delta^{(0)}_{ii}(p)&=2\pi\delta(p^2)(\Theta(p_0)+\tilde n(|p_0|)),\\
\Delta^{(0)}_{11}(p)&=2\pi\delta(p^2-M_1^2)(\Theta(p_0)+\tilde n(|p_0|)),\\
\tilde n(x)&=n(x)\Theta(x-\Lambda)=\frac{1}{e^{\beta x}-1}\Theta(x-\Lambda).
\end{aligned} \tag{25}$$

Since the starting distributions are independent of X, the integrals in Eq. (19) factorize exactly and one has for the first corrections the following explicit expressions:

$$\begin{aligned}
\Delta^{(1)}_{aa}(k,p)&=-\frac{1}{2pk}\lambda_a\bar\Phi\Phi_1(k)[\Delta^{(0)}_{aa}(p+\frac{k}{2})-\Delta^{(0)}_{aa}(p-\frac{k}{2})],\\
\Delta^{(1)}_{i1}(k,p)&=-\frac{1}{2pk+M_1^2}\frac{\lambda}{3}\bar\Phi\Phi_i(k)[\Delta^{(0)}_{ii}(p+\frac{k}{2})-\Delta^{(0)}_{11}(p-\frac{k}{2})].
\end{aligned} \tag{26}$$

The retarded nature (ie. forward time evolution) will be taken into account with the Landau prescription $k_0\to k_0+i\epsilon$ (see Appendix A for the equivalent formulas in the conventional perturbation theory).

4 Non-equilibrium linear dynamics of the slow modes

The quantum improved, linearized equations of motion are (see eq. (11))

$$(\partial^2+M_a^2)\Phi_a(x)+J_a(x)=0. \tag{27}$$

The (retarded) self-energy function is introduced through the relation

$$J_a(x)=\int d^4y\,\Pi_a(x-y)\,\Phi_a(y), \tag{28}$$

where, after Fourier-transformation one obtains, (see eq. (26))

$$\begin{aligned}
\Pi_1(k)=\int\frac{d^4p}{(2\pi)^4}\Bigg[&\left(\frac{\lambda\bar\Phi}{2}\right)^2\frac{1}{pk}\left(\Delta^{(0)}_{11}(p-\frac{k}{2})-\Delta^{(0)}_{11}(p+\frac{k}{2})\right)\\
&+\left(\frac{\lambda\bar\Phi}{6}\right)^2(N-1)\frac{1}{pk}\left(\Delta^{(0)}_{ii}(p-\frac{k}{2})-\Delta^{(0)}_{ii}(p+\frac{k}{2})\right)\Bigg],\\
\Pi_i(k)=\int\frac{d^4p}{(2\pi)^4}\Bigg[&\left(\frac{\lambda\bar\Phi}{3}\right)^2\frac{1}{2pk+M_1^2}\left(\Delta^{(0)}_{11}(p-\frac{k}{2})-\Delta^{(0)}_{ii}(p+\frac{k}{2})\right)\\
&-\frac{\lambda}{3}\left(\Delta^{(0)}_{11}(p)-\Delta^{(0)}_{ii}(p)\right)\Bigg].
\end{aligned} \tag{29}$$

These integrals can be calculated (or at least reduced to 1D integrals). Some details of the calculations can be found in Appendix C, here we just state the results:

$$\Pi_1(k) = \left[\left(\frac{\lambda\bar{\Phi}}{2}\right)^2 R_1(k,M_1) + \left(\frac{\lambda\bar{\Phi}}{6}\right)^2 (N-1)\, R_1(k,0)\right],$$
$$\Pi_i(k) = \left(\frac{\lambda\bar{\Phi}}{3}\right)^2 R_i(k,M_1). \tag{30}$$

The explicit form of their real parts is the following:

$$\mathrm{Re}R_1(k,M) = \int_M^\infty d\omega\,(1+2\tilde{n}(\omega))\,\left[\mathcal{A}(\frac{2|\mathbf{p}||\mathbf{k}|}{2\omega k_0+k^2}) + \mathcal{A}(\frac{2|\mathbf{p}||\mathbf{k}|}{-2\omega k_0+k^2})\right],$$
$$\mathrm{Re}R_i(k,M) = \int_M^\infty d\omega\,\left[(1+\tilde{n}(\omega))\,\mathcal{A}(\frac{2|\mathbf{p}||\mathbf{k}|}{2\omega k_0+k^2+M^2}) + \tilde{n}(\omega)\mathcal{A}(\frac{2|\mathbf{p}||\mathbf{k}|}{-2\omega k_0+k^2+M^2})\right]$$
$$+\int_0^\infty d|\mathbf{p}|\,\left[(1+\tilde{n}(|\mathbf{p}|))\,\mathcal{A}(\frac{2|\mathbf{p}||\mathbf{k}|}{-2|\mathbf{p}|k_0+k^2-M^2}) + \tilde{n}(|\mathbf{p}|)\mathcal{A}(\frac{2|\mathbf{p}||\mathbf{k}|}{2|\mathbf{p}|k_0+k^2-M^2})\right], \tag{31}$$

where $|\mathbf{p}|^2 = \omega^2 - M^2$ and

$$\mathcal{A}(x) = \frac{1}{4\pi^2|\mathbf{k}|}\mathrm{arth}(x) = \frac{1}{8\pi^2|\mathbf{k}|}\ln\left|\frac{1+x}{1-x}\right|. \tag{32}$$

The expressions of the imaginary parts look as follows:

$$\mathrm{Im}R_1(k,M) = \frac{-1}{4\pi|\mathbf{k}|}\left[\Theta(-k^2)\int_{P_c-k_0}^{P_c} dP\,\tilde{n}(P) \;+\; \Theta(k^2-4M^2)\int_{k_0/2}^{P_c} dP\,(1+\tilde{n}(k_0-P)+\tilde{n}(P))\right],$$
$$\mathrm{Im}R_i(k,M) = \frac{-1}{16\pi|\mathbf{k}|}\left[\Theta(M^2-k^2)\left[\int_{Q_+}^{Q_++k_0} dP\,\tilde{n}(P) + \int_{|Q_--k_0|}^{|Q_-|} dP\,\tilde{n}(P)\right]\right.$$
$$\left. +\Theta(k^2-M^2)\int_{Q_+}^{Q_-} dP\,(1+\tilde{n}(k_0-P)+\tilde{n}(P))\right], \tag{33}$$

where

$$P_c = \frac{|\mathbf{k}|}{2}\sqrt{1-\frac{4M^2}{k^2}} + \frac{k_0}{2}, \quad Q_+ = \left|\frac{k^2-M^2}{2(k_0+|\mathbf{k}|)}\right|, \quad Q_- = \frac{k^2-M^2}{2(k_0-|\mathbf{k}|)}. \tag{34}$$

The real part is divergent at zero temperature which cancels against the coupling constant counterterm contribution. Care has to be taken, however, when implementing the regularization, to maintain the Lorentz invariance for the pieces containing $T=0$ parts of the propagators $\Delta^{(0)}$, which is manifest in the original form in Eq. (29).

It is remarkable that the usual domain of Landau damping ($|\mathbf{k}|^2 > |k_0|^2$) is apparently extended up to $M_1^2 + |\mathbf{k}|^2 > |k_0|^2$. An analogous situation has been noted and interpreted recently for the propagation of a light fermion in heavy scalar plasma [20]. Below we find for the damping of soft on-shell Goldstone-modes the same interpretation.

The linearized equations of motion can be analyzed from several points of view. One is the determination of the dispersion relations. This describes the quasiparticles, and physically corresponds to the "dressing" of an (external) particle passing through the thermal medium. The other point of view is the field evolution [14], when one follows the solution of the field equations developing from given initial conditions (history).

4.1 Dispersion relations for on-shell waves

The position of the poles is determined by the equation

$$k^2 - M_a^2 - \Pi_a(k) = 0. \tag{35}$$

We split k_0 into real and imaginary parts: $k_0 = \omega - i\Gamma$, and assume $\Gamma \ll \omega$. Then the perturbative solution can be written as

$$\omega^2 = \omega_0^2 + \mathrm{Re}\Pi(\omega_0, \mathbf{k}), \qquad \Gamma = -\frac{\mathrm{Im}\Pi(\omega_0, \mathbf{k})}{2\omega_0}, \tag{36}$$

where $\omega_0^2 = M_a^2 + \mathbf{k}^2$. The corresponding time dependence for fixed wave vector $\mathbf{k}$ and given initial amplitudes ($\partial_t \Phi(t = 0, \mathbf{k}) = P(\mathbf{k}), \Phi(t = 0, \mathbf{k}) = F(\mathbf{k})$) is found

$$\Phi(t, \mathbf{k}) = \left[P(\mathbf{k})\frac{\sin \omega t}{\omega} + F(\mathbf{k}) \cos \omega t\right] e^{-\Gamma t}. \tag{37}$$

Goldstone modes. The tree level dispersion relation has no mass gap. The second equation of Eq. (29) shows that $\Pi_i(k = 0) = 0$, that is no mass gap is created radiatively neither at finite temperature; this is the manifestation of the Goldstone theorem. The on-shell imaginary part, as Eq. (30) shows, comes from the continuation of the Landau-damping extending up to $k^2 < M_1^2$. Going back to Appendix C (Eq.(111)), one finds that the imaginary part receives contribution from the collision of a hard thermal Goldstone particle with distribution $n(p_0)$ with the soft external Goldstone wave of momentum k producing a hard Higgs-particle minus the inverse reaction, when a hard thermal Higgs with momentum distribution $n(p_0 + k_0)$ decays into a soft and a hard Goldstone. The two contributions can be combined into a single integral leading to the following expression for the damping rate:

$$\Gamma_i(\mathbf{k}) = \left(\frac{\lambda\bar{\Phi}}{3}\right)^2 \frac{1}{32\pi\mathbf{k}^2} \int\limits_{M_1^2/4|\mathbf{k}|}^{M_1^2/4|\mathbf{k}|+|\mathbf{k}|} dp\, \tilde{n}(p). \tag{38}$$

If $|\mathbf{k}| \ll M_1$ we can write with a good approximation

$$\Gamma_i(\mathbf{k}) = \left(\frac{\lambda\bar{\Phi}}{3}\right)^2 \frac{1}{32\pi|\mathbf{k}|}\, \tilde{n}(\frac{M_1^2}{4|\mathbf{k}|}). \tag{39}$$

This contribution survives the IR cut, if $\Lambda < M_1^2/4|\mathbf{k}|$.

The result (39) is interesting from several points of view. First, in HTL approximation, where we neglect all the masses, we would found $\Gamma_i \sim \tilde{n}(0) = 0$. On the other hand the classical approximation corresponds to the substitution $n(x) \to T/x$ in Eq.(39), which results (see Appendix B) in

$$\Gamma_i^{\mathrm{class}} = \frac{\lambda T}{24\pi}. \tag{40}$$

However, for very small momenta ($|\mathbf{k}| << M_1^2/4T$) the correct result is exponentially small, deviating considerably from the classical result. The purely classical simulations therefore cannot reproduce the correct Goldstone dynamics in the long wavelength region, they would overestimate the damping rate.

The most important consequence of the form of the Goldstone damping is the generation of a new dynamical length scale $M_2^{-1} = 4T/M_1^2$. The components of the Goldstone condensate with longer wave-length can not (exponentially) decay, they survive for a longer time.

Higgs modes. From Eq. (29) one can easily read off the on-shell ($k_0^2 = \mathbf{k}^2 + M_1^2$) value of the imaginary part of the self-energy. This is entirely due to Higgs scattering into a soft + hard Goldstone-pair, and leads to

$$\Gamma_1 = \frac{-\mathrm{Im}\Pi_1}{2k_0} = \frac{\lambda M_1^2}{96\pi k_0|\mathbf{k}|}(N-1)\int_0^{|\mathbf{k}|/2} dP\,(1 + \tilde{n}(\frac{k_0}{2} - P) + \tilde{n}(\frac{k_0}{2} + P)). \tag{41}$$

In the $k \ll M_1$ limit it simplifies to

$$\Gamma_1 = \frac{\lambda M_1}{96\pi}(N-1)\left(\frac{1}{2} + \tilde{n}(\frac{M_1}{2})\right). \tag{42}$$

This survives the cut if $2\Lambda < M_1$.

Since $M_1 < T$ we can perform a high temperature expansion. This means at the same time that the classical approximation is applicable. We find

$$\Gamma_1 = \frac{\lambda T}{48\pi}(N-1), \tag{43}$$

which is $(N-1)/2\,\Gamma_i^{\mathrm{class}}$.

4.2 Solution of the initial value problem of the slow fields

Equation(27) describes an integro-differential equation. For its solution we have to know the complete past history of the fields. The previous subsection provides a specific way to look at the problem. In general we can introduce $z_a(x) = \int_{y_0<0} d^4y\,\Pi_a(x-y)\,\Phi_a(y)$, then for $x_0 > 0$

$$\Phi_a(x) = \int \frac{d^4k}{(2\pi)^4}\,e^{-ikx}\,\frac{z_a(k)}{k^2 - M^2 - \Pi_a(k)}. \tag{44}$$

Since all singularities are in the lower k_0 half plane, $\Phi_a(x)$ defined by this relation vanishes for $x_0 \equiv t < 0$. $z(x)$ can be used for setting the initial conditions [14], as it can describe jumps in the field as well as in its time derivative. For example $\Phi_a(x) = 0$ for $x_0 < 0$ and $\Phi_a(t=0) = F_a$, $\partial_t\Phi_a(t=0) = P_a$ corresponds to

$$z_a(x) = -P_a(\mathbf{x})\delta(x_0) - F_a(\mathbf{x})\delta'(x_0), \qquad z_a(k) = ik_0F_a(\mathbf{k}) - P_a(\mathbf{k}). \tag{45}$$

Direct substitution of Eq. (44) into Eq. (27) shows that it satisfies the effective homogenous wave equation. By evaluating the k_0 integral for $t = 0$ (for details, see below) one also can demonstrate that it fulfills the initial conditions set above.

In general, we are faced with the computation of Fourier transforms of functions analytic on the upper complex plane. The procedure of extracting their large time asymptotics was investigated already in Refs. [7, 14]. For $t > 0$ the k_0 integration contour can be closed with an infinite semi-circle in the lower half-plane and we pick up the contribution of the cuts and poles inside the closed integration path:

$$f(t) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} f(k_0)\, e^{-ik_0 t} = \sum_{\omega\in\text{poles}} (-iZ(\omega))e^{-i\omega t} + \sum_{I\in\text{cuts}} \int_I \frac{dk_0}{2\pi i} \rho(k_0)\, e^{-ik_0 t}, \tag{46}$$

where $Z(\omega)$ is the residuum of the physical poles of the f-function [1], $\rho = i\,\text{Disc}f$ is the discontinuity along the cut, $I = [\omega_1, \omega_2]$ is the support of the cut. The second term can be computed again by completing the original integration interval to a closed contour. The discontinuity itself may have poles (but no cuts!) which contribute in the same way as the "normal" poles. After these poles are "encircled", there are two straight contours - parallel to the imaginary axis - left, starting at the two ends of the cut and running in the interval $[\omega, \omega - i\infty]$, where ω is the end (or starting) point (threshold). After this analysis one arrives at the generic form

$$f(t) = \sum_{\omega\in\text{poles}} (-iZ(\omega))e^{-i\omega t} + \sum_{\omega\in\text{thresh.}} (\mp)e^{-i\omega t} \int_0^{\infty} \frac{dy}{2\pi} \rho(\omega - iy)\, e^{-yt}, \tag{47}$$

where the $-$ sign is to be applied for the starting, the $+$ sign for the end point of the cut, and we have to take into account all poles, also the ones on the unphysical Riemann sheet. Expanding ρ around ω into power series of y, the y integration can be performed. The term $\sim y^n$ of the expansion contributes to the time dependence t^{-n-1}. The large time behavior is *dominated by the lowest power term of the expansion.* If ρ cannot be power expanded then the damping for large times is faster than a power-law.

In our case the position of the poles is determined by the dispersion relations (see previous section), and for given initial values we find

$$\Phi_{pole}(t, \mathbf{k}) = Z(\mathbf{k}) \left[P(\mathbf{k}) \frac{\sin \omega t}{\omega} + F(\mathbf{k}) \cos \omega t \right] e^{-\Gamma t}. \tag{48}$$

This expression coincides with eq. (37) apart from the wave function renormalization $Z(\mathbf{k})$. The time dependence of the quasi-particle pole contribution was analyzed before.

For the cuts the factorized form can be used for the spectral function: $\rho_a = z_a \rho_a^G$, where ρ_a^G is the spectral function of the propagator:

$$\rho_a^G = \frac{-2\text{Im}\Pi_a}{(k^2 - M^2 - \text{Re}\Pi_a)^2 + (\text{Im}\Pi_a)^2}. \tag{49}$$

The thresholds and the leading large-time behavior are determined by ImΠ, Eq. (33). The imaginary part of the Goldstone self energy is non-analytic only at $k^2 = 0$, but also there the non-analytic piece vanishes as $\sim \exp(-M^2/(2T|k_0 - |\mathbf{k}||))$. This finally leads to a damping faster than any power which can be seen after a saddle point analysis

$$\Phi_{cut,i} \sim \frac{|\mathbf{k}|}{\lambda M_1^2} \left(\frac{\beta M_1^2}{t^3} \right)^{1/4} e^{-2\sqrt{\beta M_1^2 t}} \left[P(\mathbf{k}) \frac{\sin |\mathbf{k}|t}{|\mathbf{k}|} + F(\mathbf{k}) \cos |\mathbf{k}|t \right]. \tag{50}$$

[1] The poles below a cut do not contribute to this formula, they lie on the unphysical Riemann sheet.

The cut contribution for the Higgs damping, on the other hand, is similar to the zero temperature case

$$\Phi_{cut,1}(t,\mathbf{k}) = -\frac{\lambda T}{24M_1}\frac{1}{(\pi M_1 t)^{3/2}}\left[P(\mathbf{k})\frac{\sin(2M_1 t-\pi/4)}{2M_1}+F(\mathbf{k})\cos(2M_1 t-\pi/4)\right]$$
$$+\frac{\lambda|\mathbf{k}|}{96\pi M_1}\frac{1}{\pi M_1 t}\left[P(\mathbf{k})\frac{\sin(|\mathbf{k}|t+\pi/2)}{|\mathbf{k}|}+F(\mathbf{k})\cos(|\mathbf{k}|t+\pi/2)\right]. \quad (51)$$

The first term comes from the threshold of the Higgs pair production, the second from the threshold of the Goldstone pair production or their Landau damping. The Landau damping of the Higgs particles does not contribute to the power law decay.

The terms decaying as some powers of time will dominate the time evolution after the period of the exponential decay for the Higgs bosons ($|\mathbf{k}|^{-1} \ll M_1^{-1}$). Similar is the case for small Goldstone domains ($|\mathbf{k}|^{-1} \ll M_2^{-1}$), there the exponential decay is followed by a $\sim \exp(-\sqrt{t})$ time evolution. In case of large Goldstone domains ($|\mathbf{k}|^{-1} \gg M_2^{-1}$), however, because of the exponentially small damping rate, the situation is reversed: the $\sim \exp(-\sqrt{t})$ behavior will be dominant for intermediate times, while the amplitude is reduced by a factor of Z^{-1}. Only for very long times will become the exponential damping term, arising from the Goldstone-pole, relevant. Its action will erase completely the large size domains.

5 Nonlinear dynamics

The calculation of the effective non-linear evolution of the low-frequency modes can be based on Eq.(11) using the explicit expressions for the Fourier transforms of the induced currents defined in Eq.(12):

$$J_1(k) = \frac{\lambda^2\bar{\Phi}^2}{4}\int\frac{d^4p}{(2\pi)^4}\frac{1}{pk}\Phi_1(k)[\Delta_{11}^{(0)}(p-k/2)-\Delta_{11}^{(0)}(p+k/2)]$$
$$+(N-1)\frac{\lambda^2\bar{\Phi}^2}{36}\int\frac{d^4p}{(2\pi)^4}\frac{1}{pk}\Phi_1(k)[\Delta_{ii}^{(0)}(p-k/2)-\Delta_{ii}^{(0)}(p+k/2)],$$
$$J_i(k) = -\frac{\lambda}{3}\int\frac{d^4p}{(2\pi)^4}\frac{2pk}{2pk+M_1^2}\Phi_i(k)[\Delta_{11}^{(0)}(p-k/2)-\Delta_{ii}^{(0)}(p+k/2)]. \quad (52)$$

Implicitly in all Wigner-transforms $\Delta^{(0)}$ the Bose-Einstein factor is understood with a low frequency cutoff, well separating the modes treated classically from the almost thermalised high frequency part of the fluctuation spectra.

For the purpose of numerical investigations the above non-local form of the induced currents is difficult to use. In this section we discuss the introduction of auxiliary fields making the numerical solution of the nonlinear dynamics easier to implement. We shall not take into account the nonlinear piece of the source induced at one-loop level, since in weak coupling the leading nonlinear effect comes from the tree-level cubic term. The consistent inclusion of the higher power induced sources will not lead to qualitatively new, leading effects unlike the linear source, which is responsible for damping. We leave this extension for future investigations.

We concentrate first on the induced current J_1. By appropriate shifts of the integration variable p it can be rewritten as

$$J_1(k) = \frac{\lambda^2\bar{\Phi}^2}{2}\int\frac{d^4p}{(2\pi)^4}[\frac{1}{2pk+k^2}-\frac{1}{2pk-k^2}]\Delta_{11}^{(0)}(p)\Phi_1(k)$$
$$+(N-1)\frac{\lambda^2\bar{\Phi}^2}{18}\int\frac{d^4p}{(2\pi)^4}[\frac{1}{2pk+k^2}-\frac{1}{2pk-k^2}]\Delta_{ii}^{(0)}(p)\Phi_1(k). \quad (53)$$

After explicitly performing the p_0 integration one arrives at the following expression which has a clear interpretation as a specific statistical average:

$$J_1(k) = \Phi_1(k)\Big[\frac{\lambda^2\bar\Phi^2}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}(1+2\tilde n(p_0))(\frac{1}{2pk+k^2}-\frac{1}{2pk-k^2})|_{p_0=(\mathbf{p}^2+M_1^2)^{1/2}}$$
$$+(N-1)\frac{\lambda^2\bar\Phi^2}{18}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}(1+2\tilde n(p_0))(\frac{1}{2pk+k^2}-\frac{1}{2pk-k^2})|_{p_0=|\mathbf{p}|}\Big]. \quad (54)$$

The temperature independent piece corresponds to the $T=0$ renormalization of λ, fixed at the scale k^2. It is absorbed into the mass term of Φ_1, therefore we retain in the integrand only the terms proportional to the cutoff Bose-Einstein factors.

We introduce two complex auxiliary "on-shell" fields $W^a(x,\mathbf{p})$, $a=1,i$. They correspond to the two different mass-shell conditions appearing in the above equation, and fulfill the equations:

$$(2pk-k^2)W^a(k,\mathbf{p}) = \Phi_1(k). \quad (55)$$

Clearly, $J_1(x)$ can be expressed as a well defined combination of the thermal averages of these fields:

$$-J_1(x) = \lambda^2\bar\Phi^2\Big(\langle W^1(x,\mathbf{p})\rangle + \langle W^1(x,\mathbf{p})^*\rangle + \frac{N-1}{9}(\langle W^i(x,\mathbf{p})\rangle + \langle W^i(x,\mathbf{p})^*\rangle)\Big). \quad (56)$$

Thermal averages are defined by the usual formula

$$\langle W^a(x,\mathbf{p})\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}\tilde n(p_0)W^a(x,\mathbf{p})|_{p_0=(\mathbf{p}^2+M_a^2)^{1/2}}, \qquad a=1,i. \quad (57)$$

For the relevant combination one can use in place of Eq. (55) the equations arising after the combination $W^a(x,\mathbf{p})-W^a(x,\mathbf{p})^*$ is eliminated:

$$[(2pk)^2-(k^2)^2](W^a(k,\mathbf{p})+W^a(-k,\mathbf{p})^*) = 2k^2\Phi_1(k). \quad (58)$$

Equations (11), (56) and (58) represent that form of the nonlinear dynamics which is best adapted for numerical solution.

For very small values of k_0 a simplified form can be derived which coincides with the result of the conventional kinetic treatment of the Higgs-modes [6]. One arrives at this approximate expression of $J_1(k)$ if one performs an appropriate three-momentum shift $\mathbf{p}\to\mathbf{p}\mp\mathbf{k}/2$ on the $\mathbf{p}$ variable in Eq. (54). One finds the following expressions for the denominators, accurate to linear order in k_0:

$$2pk\pm k^2 \to 2k_0\sqrt{(\mathbf{p}\mp\mathbf{k}/2)^2+M_a^2}-2\mathbf{pk}\pm k_0^2 \approx 2pk(1\pm k_0/2p_0)+\mathcal{O}(k_0^3,\mathbf{k}^3),$$
$$p_0\to p_0\mp\tfrac{\mathbf{pk}}{2p_0}. \quad (59)$$

Performing the expansion of the denominators and of the arguments of $\tilde n(p_0)/p_0$ to linear order in $\mathbf{k},k_0$ one finds the following approximate expression for J_1:

$$J_1(k) = -\tfrac{\lambda^2\bar\Phi^2}{2}\int\tfrac{d^3p}{(2\pi)^3}\Big[\Big(\tfrac{1}{2p_0^2}(\tfrac{d\tilde n(p_0)}{dp_0}\tfrac{k_0p_0}{pk}+\tfrac{1}{p_0}(\tfrac12+\tilde n(p_0)-p_0\tfrac{d\tilde n(p_0)}{dp_0})\Big)|_{p_0=(\mathbf{p}^2+M_1^2)^{1/2}}$$
$$+\tfrac{N-1}{18}\tfrac{1}{p_0^2}\Big(\tfrac{d\tilde n(p_0)}{dp_0}\tfrac{k_0p_0}{pk}+\tfrac{1}{p_0}(\tfrac12+\tilde n(p_0)-p_0\tfrac{d\tilde n(p_0)}{dp_0})\Big)|_{p_0=|\mathbf{p}|}\Big]\Phi_1(k). \quad (60)$$

The IR-cutoff $\Lambda >> |\mathbf{k}|$ is important to prevent the singularity of the contribution from the Goldstone-modes. [2] The contributions from the second terms in each line are independent of

[2] The proper solution of the effective dynamics of the heavy field surrounded by massless Goldstone-quanta will require the application of methods analogous to the Bloch-Nordsieck resummation at $T=0$ [21]. We thank M. Simionato for an enlightening discussion on this point.

k, therefore they are simply understood as a finite shift in the squared mass of Φ_1. With these steps one arrives at the final expression for the nonlocal part of the induced "Higgs" current:

$$J_1(k) = -\frac{\lambda^2\bar{\Phi}^2}{4}\int\frac{d^3p}{(2\pi)^3}\Big[\frac{1}{p_0^2}\frac{d\tilde{n}(p_0)}{dp_0}|_{p_0=(\mathbf{p}^2+M_1^2)^{1/2}} + \frac{N-1}{9}\frac{1}{p_0^2}\frac{d\tilde{n}(p_0)}{dp_0}|_{p_0=|\mathbf{p}|}\Big]\frac{1}{vk}k_0\Phi_1(k). \quad (61)$$

Here $v^\mu = (1, \mathbf{p}/p_0)$. In this case it is sufficient to introduce only two real auxiliary fields $W^1(x,\mathbf{v})$ and $W^i(x,\mathbf{v})$ in order to make the dynamical equations local. They are defined through the equations:

$$vkW^a(k,\mathbf{v}) = k_0\Phi_1(k), \qquad a = 1, i. \quad (62)$$

The weighting factors in the integrals over the auxiliary field can be interpreted as the deviations from equilibrium of the Higgs and Goldstone particle distributions, $\delta n_1(\mathbf{p})$ and $\delta n_i(\mathbf{p})$, due the scattering of the high frequency particles off the Φ_1 condensate. Then one writes

$$J_1(x) = -\frac{\lambda^2\bar{\Phi}^2}{4}\Big(\delta\langle W^1(x,\mathbf{v})\rangle + \frac{N-1}{9}\delta\langle W^i(x,\mathbf{v})\rangle\Big), \quad (63)$$

where $\delta\langle ...\rangle$ means a phase space integral weighted with the nonequilibrium part of the relevant distributions.

For this case it is easy to construct the energy-momentum vector corresponding to the nonlocal piece of the induced current. One can follow the procedure proposed by Blaizot and Iancu [22] and investigate the divergence of the $(\mu, 0)$ component of the energy-momentum tensor:

$$\partial_\mu T^{\mu 0}_{induced} = -J_1\partial^0\Phi_1, \quad (64)$$

The task is to transform the right hand side into the form of a divergence. Using Eqs.(61) and (62) one finds the following expression:

$$-J_1\partial^0\Phi_1 = \tfrac{\lambda^2\bar{\Phi}^2}{4}\int\tfrac{d^3p}{(2\pi)^3}\Big(W^1(x,\mathbf{v})\tfrac{1}{p_0^2}\tfrac{d\tilde{n}(p_0)}{dp_0}(v\partial_x)W^1(x,\mathbf{v})|_{p_0=(\mathbf{p}^2+M_1^2)^{1/2}}$$
$$+\tfrac{N-1}{9}W^i(x,\mathbf{v})\tfrac{1}{p_0^2}\tfrac{d\tilde{n}(p_0)}{dp_0}(v\partial_x)W^i(x,\mathbf{v})|_{p_0=|\mathbf{p}|}\Big). \quad (65)$$

From here one can read off the induced energy-momentum function of the approximate Higgs dynamics:

$$T^{\mu 0}_{induced} = \frac{\lambda^2\bar{\Phi}^2}{8}\int\frac{d^3p}{(2\pi)^3}v^\mu\Big[\frac{1}{p_0^2}\frac{d\tilde{n}(p_0)}{dp_0}W^1(x,\mathbf{v})^2|_{p_0=(\mathbf{p}^2+M_1^2)^{1/2}}$$
$$+\frac{N-1}{9}\frac{1}{p_0^2}\frac{d\tilde{n}(p_0)}{dp_0}W^i(x,\mathbf{v})^2|_{p_0=|\mathbf{p}|}\Big]. \quad (66)$$

For the exact (essentially non-local) dynamics we did not attempt the construction of an energy functional, but its existence for the above limiting case hints for the Hamiltonian nature also of the full one-loop dynamics as expressed in terms of the auxiliary variables.

The analysis of the induced Goldstone source, appearing in Eq.(52) goes in fully analogous steps. The temperature dependent part to which a statistical interpretation can be linked is rewritten with help of two complex auxiliary fields $V^a(x,\mathbf{p}), a = 1, i$ as

$$J_i(x) = \frac{\lambda}{3}\int\frac{d^3p}{(2\pi)^3}\Big[\frac{1}{p_0}\tilde{n}(p_0 = (\mathbf{p}^2 + M_1^2)^{1/2})\mathrm{Re}\,V^1(x,\mathbf{p}) - \frac{1}{p_0}\tilde{n}(p_0 = |\mathbf{p}|)\mathrm{Re}\,V^i(x,\mathbf{p})\Big], \quad (67)$$

where the auxiliary fields fulfill the equations

$$(2pk + k^2 + M_1^2)V^1(k, \mathbf{p}) = (2pk + k^2)\Phi_i(k),$$
$$(2pk - k^2 + M_1^2)V^i(k, \mathbf{p}) = (2pk - k^2)\Phi_i(k). \tag{68}$$

In this case even for very small values of k_0 we were not able to derive any limiting case in which one could treat the time evolution of the low frequency Goldstone modes as a truly Boltzmannian kinetic evolution.

The solution of the system of equations (11), (56), (58), (67), (68) requires the specification of initial conditions also for the auxiliary fields $W^a + W^{a*}, V^a$. If one uses the "past history" condition for the physical fields $\Phi_1(x) = \text{const}, \Phi_i(x) = \text{const}$ for $t < 0$, then the linear integral equation form of the equations for the auxiliary fields and their retarded nature imply vanishing W^a, V^a for $t = 0$.

6 Conclusions

In this paper we have performed a complete 1-loop analysis of the time evolution of low frequency ($k_0 << M_1$) field configurations in the broken phase of the $O(N)$ symmetric scalar field theory. We have shown explicitly the full formal equivalence of the leading order iterative solution of the Dyson-Schwinger equations and the perturbation theory computation of the two-point function.

We have analyzed explicitly the case when $T > M_1$, which occurs, for instance, in the vicinity of the second order phase transition of the model. Here we could compare our results with the results arising from the dynamical equations of the classical $O(N)$ field theory with appropriately chosen parameters. For the "Higgs" particle we have found complete agreement for the $|\mathbf{k}| << M_1 < T$ modes. Related to this is the fact that we were able to show that the exact one-loop equations derived for the auxiliary fields W^a have a simple Boltzmannian collisionless kinetic form for small $|\mathbf{k}|$.

However, the analysis gave different conclusions for the linear response of the Goldstone-modes. The agreement with the classical theory is restricted to the interval $M_1^2/4T << |\mathbf{k}| << M_1$. Below the new characteristic scale

$$M_2 \equiv \frac{M_1^2}{4T} \tag{69}$$

the damping of the Goldstone modes becomes exponentially small and it vanishes non-analytically for $|\mathbf{k}| \to 0$. This is a very natural manifestation of the Goldstone-theorem in a dynamical situation: no homogeneous ground state will relax to any "rotated" nearby configuration. In the light of this suggestive picture it is not surprising that we could not find a classical kinetic interpretation of the exact one-loop dynamics of the Goldstone modes even for very small values of $|\mathbf{k}|$.

Besides the exponential damping analyzed above, $(1-Z)$ fraction of the initial configuration follows a different time evolution determined by the particle production and Landau-damping cuts. In case of the lowest wave number Higgs fluctuations this leads to the result that the exponential regime will be followed by a power-decay for times $t > M_1^{-1}$. In case of the Goldstone modes the cut contribution is $\sim \exp(-4\sqrt{M_2 t})$, and for modes above the scale M_2 this will dominate for large times. Below the scale M_2 the cut contribution will be observable for intermediate times, while the pole dominated damping, because of the exponentially small damping rate, becomes relevant only for very long times.

It will be interesting to see through numerical investigations, if the onset of the nonlinear regime will influence the damping scenario of the Goldstone fluctuations. The effect of interaction among the high-k modes (in addition to their scattering off the low-k background) on the effective theory merits also further study. Finally, we work on the extension of our analysis to the Gauge+Higgs models in the broken phase, relevant to the physics of the standard model below the electroweak phase transition.

A Appendix

Instead of the generalized Boltzmann-equations described in Section 3 we can use also perturbation theory to evaluate the two-point correlation functions. Here we will demonstrate the equivalence of the one-loop perturbation theory and the iterative solution of the generalized Boltzmann-equations in the case of $\Delta^{(1)}$.

In the perturbation theory we write

$$\langle\phi_a(x)\phi_b(x)\rangle = \frac{1}{Z}\left\langle \mathrm{T}_c\,\phi_a^{(0)}(x)\phi_b^{(0)}(x)\,e^{-i\mathcal{S}_I}\right\rangle,$$

where $\phi^{(0)}$ are the free fields, $\mathcal{S}_I$ is the part of the action which decribes the interaction between the different field components in the presence of the background. T_c stands for the time ordering along a complex time path c specified in [19]. At one loop, to linear order in Φ we need only

$$S_I = \frac{\lambda\bar{\Phi}}{6}\int_c dx_{0c}\int d^3x\left[\Phi_1\,(\phi_b)^2 + 2(\Phi_b\phi_b)\phi_1\right],$$

where the integration variable x_{0c} represents the points on the complex integration contour in the t plane. The field operator contractions are performed with help of the matrix propagators

$$iG_{ab}(x) = \begin{pmatrix} iG^C_{ab}(x) & iG^<_{ab}(x) \\ iG^>_{ab}(x) & iG^A_{ab}(x)\end{pmatrix} = \begin{pmatrix} \langle\mathrm{T}\phi_a(x)\phi_b(0)\rangle & \langle\phi_b(0)\phi_a(x)\rangle \\ \langle\phi_a(x)\phi_b(0)\rangle & \langle\mathrm{T}^*\phi_a(x)\phi_b(0)\rangle\end{pmatrix}, \tag{70}$$

where T^* denotes anti time ordering. The tree level propagators are diagonal $G_{ab}(x) = \delta_{ab}G_a(x)$. Since the background depends on the real (not the contour) time we find

$$\langle\phi_a(x)\phi_b(x)\rangle^{(1)} \equiv -i\left\langle\phi_a^{(0)}(x)\phi_b^{(0)}(x)\,\mathcal{S}_I\right\rangle = \frac{\lambda\bar{\Phi}}{3}\int d^4y\,[\Phi_1(y)\delta_{ab}+\Phi_a(y)\delta_{b1}+\Phi_b(y)\delta_{a1}]S_{ab}(x-y), \tag{71}$$

where

$$iS_{ab}(z) = G^C_a(z)G^C_b(z) - G^<_a(z)G^<_b(z). \tag{72}$$

The propagators can be expressed with help of the spectral function ρ as

$$G^<(p) = n(p_0)\rho(p), \qquad G^C(t,\mathbf{p}) = \Theta(t)\rho(t,\mathbf{p}) + G^<(t,\mathbf{p}). \tag{73}$$

Using the $(t,\mathbf{p})$ representation and finally performing time Fourier transfromation leads to

$$S_{ab}(k) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{dp_0}{2\pi}\frac{dp_0'}{2\pi}\,\frac{\rho_a(p_0,\mathbf{p})\rho_b(p_0',\mathbf{p}+\mathbf{k})}{k_0-p_0-p_0'+i\epsilon}\,(1+n(p_0)+n(p_0')). \tag{74}$$

Using free spectral functions $\rho_a(p) = (2\pi)\epsilon(p_0)\delta(p^2-m_a^2)$ we arrive at the known expression (see for example [14])

$$S_{ab}(k) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{4\omega_a\omega_b}\left[\frac{1+n_a+n_b}{k_0-\omega_a-\omega_b+i\epsilon} - \frac{1+n_a+n_b}{k_0+\omega_a+\omega_b+i\epsilon} + \frac{n_a-n_b}{k_0+\omega_a-\omega_b+i\epsilon} + \frac{n_b-n_a}{k_0-\omega_a+\omega_b+i\epsilon}\right]. \tag{75}$$

On the other hand introducing in (74) $\Delta^{(0)}(p) = G^{>}(p) = (1+n(p_0))\rho(p) = G^{<}(-p)$, next performing one of the p_0 integrals, and shifting properly the $\mathbf{p}$ integral we obtain

$$S_{ab}(k) = \int \frac{d^4p}{(2\pi)^4}\left[\frac{\Delta^{(0)}_{aa}(p)}{k^2+p^2-2kp-m_b^2} + \frac{\Delta^{(0)}_{bb}(-p)}{k^2+p^2-2kp-m_a^2}\right]. \tag{76}$$

The $i\epsilon$ is assigned to k_0 by the Landau-prescription. Exploiting that $\rho(p) \sim \delta(p^2-m^2)$ one can write

$$S_{ab}(k) = \int \frac{d^4p}{(2\pi)^4}\left[\frac{\Delta^{(0)}_{aa}(p)}{k^2-2kp-M^2} + \frac{\Delta^{(0)}_{bb}(p)}{k^2+2kp+M^2}\right], \tag{77}$$

where $M^2 = m_b^2 - m_a^2$. Finally performing a $\pm k/2$ shift the result is

$$S_{ab}(k) = \int \frac{d^4p}{(2\pi)^4}\,\frac{\Delta^{(0)}_{bb}(p-k/2) - \Delta^{(0)}_{aa}(p+k/2)}{2kp+M^2}. \tag{78}$$

Finally, introducing for the Fourier-transform of the left hand side of Eq. (71) the representation

$$\langle\varphi_a\varphi_b\rangle = \int \frac{d^4p}{(2\pi)^4}\Delta^{(1)}(k,p), \tag{79}$$

we obtain from Eqs. (71) and (78)

$$\begin{aligned} 2kp\,\Delta^{(1)}_{aa}(k,p) &= -\lambda_a\bar\Phi\Phi_1(k)\left[\Delta^{(0)}_{aa}(p+k/2) - \Delta^{(0)}_{aa}(p-k/2)\right], \\ (2kp+M_1^2)\,\Delta^{(1)}_{1i}(k,p) &= -\frac{\lambda}{3}\bar\Phi\Phi_i(k)\left[\Delta^{(0)}_{ii}(p+k/2) - \Delta^{(0)}_{11}(p-k/2)\right], \end{aligned} \tag{80}$$

which exactly coincides with Eq. (26).

B Appendix

An alternative approach for calculating characteristic quantities of real-time correlation functions is based on real-time dimensional reduction [23, 24] and solving the resulting classical effective theory relevant for the modes with high occupation numbers. On-shell as well as the off-shell damping rates were successfully reproduced [25, 26] in the symmetric phase of ϕ^4 theory using this approach. In this appendix we will discuss the application of the classical approach to the $O(N)$ model and compare its results with the exact one-loop dynamics. The Lagrangian of the classical $O(N)$ theory has the following form:

$$L_{cl} = \frac{1}{2}(\partial_\mu\tilde\Phi_a)^2 - \frac{1}{2}m_{cl}^2(\Lambda)\tilde\Phi_a^2 - \frac{\lambda_{cl}}{24}(\tilde\Phi_a^2)^2 - j_a\tilde\Phi_a. \tag{81}$$

The classical equation of motion corresponding to this Lagrangian is:

$$(\partial^2 + m_{cl}^2(\Lambda))\tilde\Phi_a + \frac{\lambda_{cl}}{6}\tilde\Phi_a(\tilde\Phi_b^2) + j_a = 0. \tag{82}$$

The external currents $j_a(x)$ were introduced in order to prepare the derivation of the classical response theory. They should not be confused with the induced currents J_a appearing in the effectice quantum equations of motion. The classical mass $m_{cl}(\Lambda)$ is different from the mass parameter m of the quantum theory (see Eq. (3)). The same is true for the coupling λ_{cl}. The classical mass parameter depends on the ultraviolet cutoff Λ. The results of the classical and quantum calculation could be matched by a suitable choice of $m_{cl}(\Lambda)$ and λ_{cl}. In particular we shall see below (eq.(91)), that the ultraviolet divergencies of the classical theory can be eliminated if the divergent part of $m_{cl}(\Lambda)$ is suitably chosen [23, 24, 25, 26]. Therefore we will separate out the divergent part from the classical mass and write $m_{cl}^2(\Lambda) = m_T^2 + \delta m^2(\Lambda)$ [3].

In the broken symmetry phase one separates out the condensate $\bar{\Phi}$,

$$\tilde{\Phi}_a = \bar{\Phi}\delta_{a1} + \Phi_a \tag{83}$$

and the equations of motion read

$$(\partial^2 + m_T^2 + \frac{\lambda_{cl}}{2}\bar{\Phi}^2)\Phi_1 + \frac{\lambda_{cl}}{6}\Phi_1(\Phi_a^2) + \frac{\lambda_{cl}}{2}\bar{\Phi}\Phi_1^2 + \frac{\lambda_{cl}}{3}\bar{\Phi}\Phi_i^2 + \delta m^2\Phi_1 + j_1 = 0, \tag{84}$$

$$(\partial^2 + m_T^2 + \frac{\lambda_{cl}}{6}\bar{\Phi}^2)\Phi_i + \frac{\lambda_{cl}}{6}\Phi_i(\Phi_a^2) + \frac{\lambda_{cl}}{3}\bar{\Phi}\Phi_1\Phi_i + \delta m^2\Phi_i + j_i = 0. \tag{85}$$

In addition the following initial conditions are imposed:

$$\Phi_a(t=0,\mathbf{x}) = F_a(\mathbf{x}), \qquad \partial_t\Phi_a(t,\mathbf{x})|_{t=0} = P_a(\mathbf{x}). \tag{86}$$

The expectation value of some quantity O (e.g. some correlation function) is obtained by averaging over the initial conditions with the Boltzmann factor determined by the classical Hamiltonian $H_{cl}(P_a, F_a, \bar{\Phi})$ corresponding to (81)

$$< O >= \frac{1}{Z}\int DF_a DP_a O \exp(-\beta H_{cl}(P_a, F_a, \bar{\Phi})) \tag{87}$$

$$Z = \int DF_a DP_a \exp(-\beta H_{cl}(P_a, F_a, \bar{\Phi})), \tag{88}$$

The explicit form of $H_{cl}(P_a, F_a, \bar{\Phi})$ is

$$H_{cl}(P_a, F_a, \bar{\Phi}) = \int d^3x\Big[\frac{1}{2}P_a^2 + \frac{1}{2}(\partial_i F_a)^2 + \frac{1}{2}(m_T^2 + \frac{\lambda_{cl}}{2}\bar{\Phi}^2)F_1^2 + \frac{1}{2}(m_T^2 + \frac{\lambda_{cl}}{6}\bar{\Phi}^2)F_i^2 + \frac{\lambda_{cl}}{6}\bar{\Phi}F_1(F_b)^2 + \frac{\lambda_{cl}}{24}(F_a^2)^2 + (m_{cl}^2\bar{\Phi} + \frac{\lambda_{cl}}{6}\bar{\Phi}^3)F_1 + \frac{1}{2}m_{cl}^2\bar{\Phi}^2 + \frac{\lambda_{cl}}{24}\bar{\Phi}^4 + \delta m^2 F_a^2\Big], \tag{89}$$

where we have separated out the classical condansate $\bar{\Phi}$ since no averaging over classical condensate is understood. Since the classical condensate is separated both from the dynamical fields and from the initial conditions the following equation holds:

$$< \Phi_1(t=0,\mathbf{x}) >=< F_1(\mathbf{x}) >= 0. \tag{90}$$

Using the explicit form of H_{cl} (Eq. (89)) this results at one-loop level in the following equation for the classical condensate $\bar{\Phi}$:

$$(m_{cl}^2(\Lambda) + \frac{\lambda_{cl}}{6}\bar{\Phi}^2)\bar{\Phi} + \frac{\lambda_{cl}}{6}\bar{\Phi}\int\frac{d^3p}{(2\pi)^3}\Big[3\frac{1}{\mathbf{p}^2 + m_T^2 + \frac{\lambda_{cl}}{2}\bar{\Phi}^2} + (N-1)\frac{1}{\mathbf{p}^2 + m_T^2 + \frac{\lambda_{cl}}{6}\bar{\Phi}^2}\Big] = 0. \tag{91}$$

Choosing $\delta m^2(\Lambda)$ to cancel the linearly divergent piece on the left hand side this relation reduces in the high temperature limit $(m_T, \lambda_{cl}\bar{\Phi} \ll T)$ to

$$m_T^2 + \frac{\lambda}{6}\bar{\Phi}^2 = 0 \tag{92}$$

[3] The divergent part of the classical mass parameter will be treated as interaction, similarly to quantum field theory

and

$$\delta m^2 = -\frac{\lambda_{cl}}{6}(N+2)\frac{\Lambda T}{2\pi^2}. \tag{93}$$

Our procedure of solving the classical theory perturbatively closely follows Ref. [25]. Equations (84), (85) are rewritten in form of the following integral equations:

$$\begin{aligned}\Phi_1(x,j) &= \Phi_1^0(x) + \int d^4x' D_R^1(x-x')(\frac{\lambda_{cl}}{6}\Phi_1(\Phi_a^2) + \frac{\lambda_{cl}}{2}\bar{\Phi}\Phi_1^2 + \frac{\lambda_{cl}}{3}\bar{\Phi}\Phi_i^2 + \delta m^2\Phi_1 + j_1)\\ \Phi_i(x,j) &= \Phi_i^0(x) + \int d^4x' D_R^i(x-x')(\frac{\lambda_{cl}}{6}\Phi_i(\Phi_a^2) + \frac{\lambda_{cl}}{3}\bar{\Phi}\Phi_1\Phi_i + \delta m^2\Phi_i + j_i),\end{aligned} \tag{94}$$

where

$$D_R^a(x-x') = -\theta(t)\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{qx}}\frac{\sin\omega_q t}{\omega_q} \tag{95}$$

is the classical retarded Green function [25] with $\omega_q = \sqrt{q^2+m_a^2}$, where $m_1^2 = m_T^2 + \frac{\lambda_{cl}}{2}\bar{\Phi}^2 = \frac{\lambda_{cl}}{3}\bar{\Phi}^2$ is the Higgs field mass and $m_i^2 = m_T^2 + \frac{\lambda_{cl}}{6}\bar{\Phi}^2 = 0$ (we have used Eq. (92)). Furthermore, Φ_a^0 are the solutions of the free equations of motion and j in the argument of Φ_a refers to the functional dependence on $j = (j_1, j_i)$. Following Ref. [25] one introduces linear response functions

$$H_R^{ab}(x-x') = \frac{\delta\Phi_a(x,j)}{\delta j_b(x')}. \tag{96}$$

Using the integral equation (94) one can derive a coupled set of integral equations also for the linear response functions H_R^{ab} by functional differentiation of eq. (94) (see Ref. [25] for details). These integral equations can be solved iteratively in the weak coupling limit. When solving the equations, it is important to exploit the fact that only diagonal components of H_R^{ab} have terms $\mathcal{O}(\lambda^0)$. The classical retarded response function is the ensemble average of H_R^{ab} with respect the initial conditions :

$$G_{ab}^{cl}(x-x') =< H_R^{ab}(x-x') > . \tag{97}$$

By eq.(94) one easily finds that it satisfies a Dyson-Schwinger equation of the general form

$$G_{ab}^{cl}(x-x') = D_R^a(x-x')\delta_{ab} + \int d^4y d^4y' D_R^a(x-y)\delta_{ad}\Pi_{dc}^{cl}(y-y')G_{cb}^{cl}(y'-x'), \tag{98}$$

where $\Pi_{ab}^{cl}(y-y')$ is the classical self-energy.

In the perturbative expansion the average is done with the free Hamiltonian and therefore all thermal n-point functions are expressed as products of the two-point function of the free fields (solutions of the free equation of motion). The two point function of the free fields reads [26, 27] as

$$\Delta_a^{cl,0}(x-x') =< \Phi^0(x)_a\Phi^0(x')_a >^0= T\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q(x-x')}}\frac{1}{\omega_q^2}\cos(\omega_q(t-t')) \tag{99}$$

This classical two point function is analogous to the free two point function of the quantum theory $\Delta_{aa}^{(0)}(x,x')$ Using eq. (94) one gets the following self-energies for the Higgs and Goldstone fields at leading order in the coupling constant λ_{cl}.

$$\begin{aligned}&\Pi_{11}^{cl}(\omega,\mathbf{k}) = \delta m^2 + \frac{\lambda_{cl}}{6}\sum_a\Delta_a^{cl,0}(0,0) +\\ &(\lambda_{cl}\bar{\Phi})^2\int dtd^3y e^{i\omega t - i\mathbf{ky}}\Big(\Delta_1^{cl,0}(y)D_R^1(y) + \frac{N-1}{9}\Delta_i^{cl,0}(y)D_R^i(y)\Big),\end{aligned}$$

$$\Pi^{cl}_{ii}(\omega,\mathbf{k}) = \delta m^2 + \frac{\lambda_{cl}}{6}\sum_a \Delta^{cl,0}_a(0,0) +$$

$$(\frac{\lambda_{cl}\bar{\Phi}}{3})^2 \int dt d^3y e^{i\omega t - i\mathbf{ky}}(\Delta^{cl,0}_1(y)D^i_R(y) + \Delta^{cl,0}_i(y)D^1_R(y)), \tag{100}$$

Using the explicit form of $\Delta^{cl,0}_a(0,0)$ and Eq. (93) one can easily verify that all divergencies present in the above expression cancel. Furthermore, in the high temperature limit and at leading order in the coupling constant only the last terms contribute in the expression of $\Pi^{cl}_{11}(\omega,\mathbf{k})$ and $\Pi^{cl}_{ii}(\omega,\mathbf{k})$. One has to evaluate integrals of the following intrinsic form

$$\int dt \int d^3y e^{i\omega t - i\mathbf{ky}} D^a_R(y)\Delta^{cl,0}_b(y) = \int \frac{d^3p}{(2\pi)^3}\int \frac{dp_0}{2\pi}\frac{dp_0'}{2\pi}\frac{\rho_a(p_0,\omega_a)\rho_b(p_0',\omega_b)}{\omega - p_0 - p_0' + i\epsilon}(n^{cl}(p_0)+n^{cl}(p_0')), \tag{101}$$

where $\omega_a = \sqrt{\mathbf{p}^2 + m_a^2}$ and $\omega_b = \sqrt{(\mathbf{p}+\mathbf{k})^2 + m_b^2}$. The above expression coincides with the result of the quantum calculation (74), except the fact the there is no $T=0$ contribution and the Bose-Einstein factors are replaced by the classical distribution: T/p_0. Then it is easy to write down the explicit expression for the classical self-energies using the formal analogy with the result of the one-loop quantum calculations. For example for the classical on-shell damping rate of the Goldstone modes one easily gets the following expression:

$$\Gamma^{cl}_i(\mathbf{k}) = -\frac{\mathrm{Im}\Pi^{cl}_{ii}(\omega = |\mathbf{k}|,\mathbf{k})}{2|\mathbf{k}|} = \frac{\lambda_{cl}^2\bar{\Phi}^2}{288\pi^2|\mathbf{k}|^2}\int_{\frac{m_1^2}{4|\mathbf{k}|}}^{\frac{m_1^2}{4|\mathbf{k}|}+|\mathbf{k}|} dp\, n^{cl}(p) = \frac{\lambda_{cl}^2\bar{\Phi}^2 T}{288\pi^2|\mathbf{k}|^2}\ln\left(1 + \frac{4|\mathbf{k}|^2}{m_1^2}\right). \tag{102}$$

Now let us discuss the correspondence between classical and quantum calculations. It was shown in Refs. [23, 24, 25, 26] that the result of classical calculations can reproduce the high temperature limit of the corresponding quantum results if the parameters $m^2_{cl}(\Lambda)$ and λ_{cl} are fixed to the values determined by static dimensional reduction. For our case this implies:

$$m^2_{cl}(\Lambda) = m^2 + \frac{\lambda}{6}(N+2)\left(\frac{T^2}{12} - \frac{\Lambda T}{2\pi^2}\right), \quad \lambda_{cl} = \lambda, \tag{103}$$

where m and λ are the mass and the coupling constant of the corresponding quantum theory. From the above equations it is easy to see that the divergent part in m^2_{cl} coincides with δm^2 determined from the ultraviolet finitness of the classical result (cf. Eqs. (91),(93) and (100)) and m_1 is the high temperature limit of M_1 (see Eqs. (92) and (13)). For small values of $|\mathbf{k}|$ ($|\mathbf{k}| \ll m_1$) the logarithm in this expression can be expanded and one obtains the result of Eq. (40), which fails to reproduce the result of the quantum calculation. The same is true for the whole imaginary part of the Goldstone self-energy. The high temperature limit of the imaginary part of the Higgs self-energy (see Eq. (30)), on the other hand, is well reproduced by the classical theory.

This result can be easily understood by looking at the effective quantum equation of motion (11). In the equation for the Goldstone fields the induced current J_i has a non-local contribution from loop momenta $p \sim M_1^2/|\mathbf{k}| >> T$ (cf. (33)). However, no such term is present in the corresponding classical equation of motion on one hand and the classical theory cannot describe fluctuation with wave length much smaller than T^{-1} on the other hand. The induced current J_1 in the effective equation of motion for the Higgs fields receives non-local contribution only from the loop momenta around $p \sim M_1$ which can be described in the framework of the classical theory.

C Appendix

In this Appendix we illustrate the steps of evaluation of the integrals in Eq. (29).

Imaginary parts. As example of the evaluation of a relevant integral we discuss in detail

$$\mathrm{Im}R_1(k,M)=\mathrm{Im}\int\frac{d^4p}{(2\pi)^4}\,\frac{\Delta^{(0)}(p-k/2)-\Delta^{(0)}(p+k/2)}{pk},\tag{104}$$

where the propagators can be either of type 11 or ii. In order to implement the Landau-prescription we transform away any k dependence from the propagators by shifting the p integral by $\pm k/2$. Then we use

$$\lim_{\alpha\to 0}\mathrm{Im}\frac{1}{x+i\alpha}=-\pi\epsilon(\alpha)\delta(x),\tag{105}$$

and finally shift the integrals back. We write the 4D integration measure as

$$\int\frac{d^4p}{(2\pi)^4}=\frac{1}{8\pi^3}\int_0^\infty dp\,p^2\int_{-\infty}^{\infty}dp_0\int_{-1}^{1}dx,\tag{106}$$

where $x=\hat{\mathbf{p}}\hat{\mathbf{k}}$ stands for the cosine of the angle between the spatial momenta. The x integration is trivial because it appears in the Dirac-delta arising from the application of the principal value theorem

$$\int_{-1}^{1}dx\,\delta(p_0k_0-p|\mathbf{k}|x)=\frac{1}{p|\mathbf{k}|}\,\Theta(p|\mathbf{k}|-|p_0k_0|).\tag{107}$$

Using the explicit form of the propagators (see Eq. (25)) and the identity $\Theta(\omega)+n(|\omega|)=\epsilon(\omega)(1+n(\omega))$ we find

$$-\mathrm{Im}R_1(k,M)=\frac{1}{4\pi|\mathbf{k}|}\int_0^\infty dp\,p\int_{-a}^{a}dp_0\,\epsilon(p_0-\frac{k_0}{2})\,\epsilon(p_0+\frac{k_0}{2})\,\delta(p_0^2-S^2)\left[n(p_0-\frac{k_0}{2})-n(p_0+\frac{k_0}{2})\right],\tag{108}$$

where $a=p|\mathbf{k}|/|k_0|$ and $S^2=p^2+M^2-k^2/4$. The p_0 integration over the Dirac-delta gives a constraint for the p integration of the form

$$S<a\qquad\Rightarrow\qquad p^2\frac{k^2}{k_0^2}<\frac{k^2}{4}-M^2.\tag{109}$$

This can be fulfilled only for $k^2>4M^2$ (above the two-particle threshold), or for $k^2<0$ (Landau damping). After elementary algebra one can establish the value of the sign functions and one arrives at the formula appearing in Eq. (33).

A similar analysis can be performed for $\mathrm{Im}R_i$, however, in this case it proved to be more convenient to start from the equivalent form

$$\mathrm{Im}R_i(k,M)=\mathrm{Im}\int\frac{d^4p}{(2\pi)^4}\,\frac{\Delta_{11}^{(0)}(p-k)-\Delta_{ii}^{(0)}(p)}{2pk-k^2+M^2}.\tag{110}$$

After implementing carefully the Landau prescription and performing the x integration as described before, we arrive at

$$-\mathrm{Im}R_i(k,M)=\frac{1}{8\pi|\mathbf{k}|}\int_0^\infty dp\,p\int_{b_-}^{b_+}dp_0\,\epsilon(p_0)\,\epsilon(p_0-k_0)\,\delta(p_0^2-p^2)\,[n(p_0-k_0)-n(p_0)]\,,\tag{111}$$

where

$$b_{\pm} = \frac{k^2 - M^2}{2k_0} \pm p\frac{|\mathbf{k}|}{k_0}. \tag{112}$$

The p_0 integration over the mass-shell delta-function again restricts the domain of integration in the p integral: $b_- < \pm p < b_+$, which, however, does not restrict the possible values for k^2. After the solution of these linear inequalities and the analysis of the sign functions we arrive at the result appearing in Eq. (33).

Real parts. The relevant integrals in $\mathrm{Re}R_1$ are

$$I^{\pm} = \mathrm{Re}\int \frac{d^4p}{(2\pi)^4}\frac{\Delta^{(0)}(p)}{pk \pm k^2/2}, \tag{113}$$

where the propagator can be either of type 11 or ii. With their help we find $\mathrm{Re}R_1 = I^+ - I^-$. The real part comes from the principal value integration. We decompose the integration measure as in the calculation of the imaginary parts. When we use Eq. (25) for the propagator, the value of p_0 is fixed by the delta function. The x integration can be performed as

$$\int_{-1}^{1} dx\, \mathcal{P}\frac{1}{2p_0k_0 - 2p|\mathbf{k}|x \pm k^2} = \frac{1}{p|\mathbf{k}|}\mathrm{arth}(\frac{2p|\mathbf{k}|}{2p_0k_0 \pm k^2}), \tag{114}$$

where $\mathrm{arth}(x) = 1/2\ln|(1+x)/(1-x)|$. Using the properties of the absolute value we find

$$I^+(k_0,\mathbf{k}) = -I^-(-k_0,\mathbf{k}). \tag{115}$$

Then we directly arrive at Eq. (31).

We write $\mathrm{Re}R_i$ in the form

$$\mathrm{Re}R_i = \mathrm{Re}\int \frac{d^4p}{(2\pi)^4}\frac{\Delta_{11}^{(0)}(p)}{2pk + k^2 + M^2} - \mathrm{Re}\int \frac{d^4p}{(2\pi)^4}\frac{\Delta_{ii}^{(0)}(p)}{2pk - k^2 + M^2}. \tag{116}$$

Then the previous scheme of calculation goes through directly.

The only problem still to be discussed is the zero temperature contribution, which diverges logarithmically. The regularization and renormalization prescriptions are better formalized in the language of the propagators and couplings. Using the results of Appendix A to go over to the perturbation theory, we have to evaluate

$$R = \int \frac{d^4p}{(2\pi)^4}\left[G^C(p)G^C(p-k) - G^<(p)G^<(p-k)\right]. \tag{117}$$

The first term at $T = 0$ is the usual time ordered product; with finite four-dimensional cutoff one finds

$$\int \frac{d^4p}{(2\pi)^4} G^C(p)G^C(p-k) = \frac{-1}{16\pi^2}\left[1 + \int_0^1 dx\, \ln\frac{|k^2x(1-x) - m_1^2x - m_2^2(1-x)|}{\Lambda^2}\right]. \tag{118}$$

The divergence is canceled by the coupling constant counterterm. In modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme it reads

$$\frac{-1}{16\pi^2}\int_0^1 dx \ln\frac{|k^2x(1-x) - m_1^2x - m_2^2(1-x)|}{\mu^2}. \tag{119}$$

In the second term of Eq. (117) at $T = 0$ we can use $G^<(p) = \Theta(-p_0)(2\pi)\delta(p^2 - m^2)$. Because of the delta functions this piece yields finite result (as is expected by the arguments of the renormalizability)

$$\int \frac{d^4p}{(2\pi)^4} G^<(p)G^<(p-k) = \frac{1}{8\pi} \begin{cases} \sqrt{1 - \frac{4m^2}{k^2}}\,\Theta(k^2 - 4m^2), \text{ if } m_1 = m_2 = m \\ (1 - \frac{m^2}{k^2})\,\Theta(k^2 - m^2), \quad \text{if } m_1 = 0,\, m_2 = m. \end{cases} \tag{120}$$

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