

QCD over T_c : hadrons, partons and continuum

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In this paper we provide a physical picture for the QCD phase transition in terms of qualitative changes in the spectral functions. Our approach takes into account the crossover nature of this transition and counts for the observed strong correlation seen in higher order susceptibilities. We demonstrate that the hadron resonance gas, which alone describes the thermodynamics at temperatures $T < T_c$, will appreciably contribute to the total pressure until $T \lesssim 3T_c$. In this intermediate regime the QCD matter consists of strongly correlated excitations, interpretable as either hadrons or partons. As hadronic spectral peaks gradually vanish, the partonic excitations start to form a stand-alone quasiparticle gas. The conventional picture of a quark gluon plasma emerges only at $T \gtrsim 3T_c$.

I. INTRODUCTION

The strongly interacting elementary matter is described by Quantum Chromodynamics (QCD). As it was proven by Monte Carlo (MC) measurements, it changes from hadronic matter to quark-gluon-plasma (QGP) phase by a continuous transition. One distinguishes a (would-be) critical temperature T_c located at the peak of some susceptibilities: e.g. $T_c = 150$ MeV for chiral susceptibility [1], for other observables one finds somewhat different values [2]. It is remarkable, that from strange quark susceptibility one already obtains a significantly (about 15%) higher crossover temperature [3].

The thermodynamics at high temperatures can be reproduced using the fundamental QCD degrees of freedom, the quarks and gluons. For obtaining the pressure we need to perform resummation, recently a three-loop Hard Thermal Loop (HTL) level [4] has been achieved. In this way one obtains values describing the measured Monte Carlo pressure at $T \gtrsim 2T_c$ satisfactorily. At low temperatures, up to $T \lesssim T_c$, the thermodynamical description is based on the non-interacting gas of hadron resonances [5] (Hadron Resonance Gas, HRG). In this description a free gas of hadrons is taken, with masses determined according to the Particle Data Group [6] tables, and the pressure is calculated by the free pressure formula. This simple picture can reproduce different measurements in MC studies [7–11] up to T_c surprisingly well.

The $T_c < T < (2 - 3)T_c$ regime (crossover regime) is much less understood. First of all, since the phase transition is a crossover, we expect that all thermodynamical quantities change continuously. This, combined with the fact that HRG describes the QCD pressure well at T_c , implies that the hadrons do not disappear from the thermal ensemble at $T > T_c$ [12]. In fact, direct MC measurements support this idea [13–17].

The success of HRG demonstrates that, at least at low temperatures, the strongly interacting QCD is dual to a weakly interacting hadron model. Put another way, the bulk effect of interactions is realized by the modification of the spectral functions, and the newly formed quasiparticles are almost interaction-free. Since at T_c the HRG description seems to work, the hadronic interaction must be weak for $T > T_c$. This means that the main properties of the crossover regime can be described by a weakly or even non-interacting mixture of hadrons, quarks and gluons. We must be aware, however, that in the language of the original degrees of freedom the interaction is rather strong, and therefore we shall expect strong deviations from the most naive small width quasiparticle description.

A first approximation for this spectrum modification is to assume that the *masses* of the thermodynamical degrees of freedom are modified [12]. At high temperature the hadronic masses, at low temperature the quark masses are growing rapidly. Since their thermal weight is diminishing, we do not see hadrons at high and quarks at low temperature. Unfortunately, this first approximation fails. The immediate problem is that we do not see this tendency in the MC spectra [18, 19]. Numerical findings are consistent with a thermal mass satisfying $m_{\text{therm}}^2 = m_0^2 + cT^2$, but nothing particular happens at T_c . There are, moreover, indirect arguments, too. One may measure correlator combinations on the lattice that are zero (or constant) in an uncorrelated quasiparticle model [11]. The measurement of such correlators [3, 11] show a clear distinction from an uncorrelated model. Another indirect proof for the insufficiency of considering quasiparticle mass changes only is that in the small-width, weakly interacting quasiparticle model the transport coefficients are large [20], while in the heavy ion collisions one observes rather small viscosity to entropy ratios, η/s [21].

We conclude that the excitation spectrum is modified more than just by a simple mass shift. In fact we expect

that a realistic quasiparticle spectrum consists of a quasiparticle peak, characterized by its mass, width and wave function renormalization, and of the continuum part. Unfortunately we do not have access to the continuum of the different quantum channels in QCD, neither from experiments, nor from lattice studies. Perturbative calculations are of limited use in this regime. There are, on the other hand, some expectations which must be true in any models, and also seen in experimentally measured spectra [22]. Namely while the continuum part must be nonzero for all four-momenta at finite temperatures, we expect it to be enhanced above some threshold value. Apart from occasional resonances we do not anticipate any fast change in the continuum.

Several authors have studied the effect of the spectrum on the thermodynamics in some approximation [23–26] and also on transport coefficients [27–30]. According to general experience quasiparticle peaks yield similar contribution to the thermodynamics as free particles, but the presence of the continuum appreciably reduces the pressure. Thus in order to describe the pressure of QCD we should consider even in a first approximation hadrons as well as partons (i.e. quasiparticles formed by quarks and gluons) with a generic spectral function. In these spectral functions the temperature variation of the masses must be moderate [18, 19], in accordance with MC measurements. The main effect is the change in the width and height of the quasiparticle peaks and the height of the continuum as the temperature varies. These are not fully independent quantities, they are connected by the sum rules.

The main goal we aim at in this paper is finding a complete description of QCD thermodynamics based on the spectral representation of the QCD excitations. Having this we ask then physical questions, like how fast will the hadrons disappear from the $T > T_c$ plasma? For the mathematical details we examine typical spectral functions as inputs and build effective quadratic models which are compatible with the given spectral functions. Using the formulae of [26, 31] we determine the thermodynamics based on these spectral functions. The model spectral functions contain several peaks and one continuum, thermal mass and spatial momentum dependence. We seek for a method to represent the resulting pressure in some common way. This happens by using the notion of the effective number of degrees of freedom, N_{eff} , which describes the ratio of the actual pressure to the ideal pressure. The decrease of this effective number of degrees of freedom represents the melting of the given excitation.

The findings of this study will then be applied to describe QCD pressure. We propose to use a statistical approach for the description of hadrons and a quantum – correlated description for the partons. This latter means that the abundance of the partons depends very strongly on the available hadronic particles: whenever the system is full with hadrons, the partons must have short lifetimes.

The result of our analysis can be summarized in some simple sentences. First of all, due to the melting and correlation effects, the full QCD pressure can be exhausted by our model, moreover the partial pressures of hadrons and partons are obtained separately. We find the hadrons responsible for the full pressure below $\sim T_c$, they dominate the pressure for $T \in [T_c, 2T_c]$ and vanish as thermodynamical degrees of freedom around $\sim 3T_c$. For the partons we obtain the reverse story: they are the only thermodynamical ingredients for $T \gtrsim 3T_c$, dominate the pressure for $T \in [2T_c, 3T_c]$, and vanish from the ensemble at $T \sim T_c$. We realize furthermore that in the intermediate temperature interval *the excitations are not fully particle-like*, their pressure contribution is considerably below the free gas pressure. This conclusion is supported by recent correlation measurements in lattice QCD [3, 11].

This paper is organized as follows: in Section II we establish how a typical spectral function of QCD excitations should be parametrized. Then one-by-one we examine the dependence of the effective number of degrees of freedom on the relevant parameters of the spectrum: on the quasiparticle width and height in subsection II C, on the spatial momentum dependence in subsection II A and on the thermal mass dependence in subsection II B. In Section III we use these findings for building an effective statistical model of QCD, containing melting hadrons and correlated partons as constituents. We compute the pressure and analyze the result in III A. We close this paper with our Conclusions (Section IV).

II. REALISTIC DESCRIPTION OF EXCITATIONS IN QCD

The goal in the present Section is to characterize a generic spectral function by some physically relevant parameters, and to compute the corresponding pressure. This technique is closely related to resummation techniques, where one also parametrically modifies the spectrum of excitations. From this point of view the HRG description itself can be considered as a nonperturbative resummation Ansatz, where we use our knowledge of the low energy QCD spectrum, and single out the most prominent property, the masses. Being an approximation, the success is not guaranteed, but it happens to work for $T < T_c$ quite well. Then we can tell that, by experience, the most relevant degrees of freedom of QCD are the hadrons. At high temperature, $T > (2 - 3)T_c$ the QCD degrees of freedom, after resummation, can also describe the measured QCD pressure.

The temperature interval $T \in [T_c, (2 - 3)T_c]$ seems to be hardly accessible either for the QCD resummation method and for the most naive nonperturbative hadronic excitation “resummation”. QCD resummations are unreliable due to the large gauge coupling, while HRG overshoots the pressure above T_c (in fact it diverges to infinity at the Hagedorn

temperature). Moreover, it is not known how the different resummation methods could join to each other smoothly, as dictated by the observed crossover phase transition. And finally MC measurements suggest considerable correlation between the particle species, suggesting that the free particle description is rather far from the truth.

But before we label this region as strongly interacting and perturbatively not accessible, we should first try to improve the Ansatz for the nonperturbative hadronic “resummation” model beyond the most naive choice of HRG. We maintain the property that it describes non-interacting excitations, but we allow for the most generic hadron, as well as quark and gluon spectra. Unfortunately we do not have access for a real hadronic spectral function, neither from experiments, nor from numerical simulations (although there are some results there). On the other hand we know that near T_c in the HRG model one has to take into account the contribution of $\mathcal{O}(2000)$ hadronic resonances. Therefore what is needed is not the spectrum of each single hadron, which may differ from each other, but only a statistical description. We have to know only the “typical” spectrum. For that we already have a good guess, since in all model calculations a typical spectrum consists of one or more quasiparticle peaks and a continuum. The height of the continuum depends on the number of available scattering channels as well as on the coupling strength to them, while the complete spectral function is subject to generic sum rules. The masses of the quasiparticles at different quantum channels also should be chosen statistically, the Hagedorn-spectrum is a good candidate for that [26, 32, 33].

We still need a method to calculate thermodynamics once the spectrum is given. This procedure has already been worked out in [31]. The key result is that if we have a spectral function ϱ then the pressure of the system is given as

$$P = -\alpha T \int \frac{d^4 p}{(2\pi)^4} \Theta(p_0) \frac{DK_{\varrho}(p)}{p_0} \ln(1 - \alpha e^{-\beta p_0}) \varrho(p), \quad (1)$$

where $\alpha = \pm 1$ for bosonic/fermionic modes. The function $DK_{\varrho}(p)$, besides its momentum dependence, also depends on the spectral function:

$$\mathcal{K}_{\varrho}(p) = \left(\mathcal{P} \int \frac{d\omega}{2\pi} \frac{\varrho(\omega, \mathbf{p})}{p_0 - \omega} \right)^{-1}, \quad DK_{\varrho}(p) = p_0 \frac{\partial \mathcal{K}_{\varrho}}{\partial p^0} - \mathcal{K}_{\varrho}, \quad (2)$$

where \mathcal{P} denotes the principal value integral. Physically \mathcal{K}_{ϱ} represents the kernel of the effective model reproducing the spectral function ϱ .

The consistency of this description is proven in [31]. Here we mention that this theory is a consistent field theory, it is unitary, causal, energy and momentum conserving, and (if needed) Lorentz-invariant. Whenever the spectral function consists of N separate Dirac-delta peaks (with possibly unequal heights), then the pressure formula reproduces the well known bosonic/fermionic free gas expressions with N constituents. One realizes that the pressure is independent of the overall normalization (sum rule).

These formulae use the spectral function as an input. The spectral function is a hardly accessible quantity. In general, at finite temperature we expect a dependence of

$$\varrho(p) = \varrho_Q(p_0, |\mathbf{p}|; T, \mu). \quad (3)$$

The index Q symbolizes the quantum channel: naturally the spectrum depends on the quantum numbers. At finite temperature Lorentz invariance is broken, and so we should have a separate dependence on p_0 and $p = |\mathbf{p}|$. The spectral function also depends on the environmental variables, like T and μ .

Nevertheless, we have some generic expectations. At zero temperature, in a system where all the excitations are stable and massive, the spectrum contains a stand-alone Dirac-delta peak, and, after a gap, the continuum starts beyond a threshold value. In this case we have an asymptotic state. At finite temperature or in the presence of zero mass excitations there is no gap in the spectrum, but we should count with the appearance of one or eventually more broadened quasiparticle peaks and a multiparticle continuum (cf. Fig. 1). The quasiparticle peak and the multiparticle continuum are generally not disjunct.

In order to characterize a “typical” spectrum we use trial spectra resembling the above generic picture. We take the following characteristic values: quasiparticle masses, quasiparticle widths which depend on the height of the continuum at the quasiparticle, the threshold value and relative heights of quasiparticle peaks and the continuum. The absolute height is determined by the sum rule, but it drops out from the expression of pressure (1). All of these parameters can be momentum and temperature dependent. Now we take these effects one-by-one, compute the pressure coming from eq. (1) and compare it to the pressure of a free gas. Quantitatively we will introduce the effective number of degrees of freedom as

$$N_{eff}(T) = \frac{P(T)}{P_0(T)}, \quad (4)$$

where $P(T)$ is the actual pressure from (1) and P_0 is the free gas pressure.

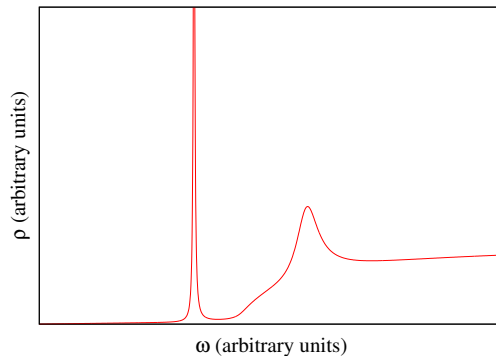


FIG. 1. Schematic plot of a generic spectral function with two quasiparticle peaks for a given momentum.

A. Momentum dependence

First we start from the simplest approach to the realistic excitations of the QCD plasma which is beyond the free particle HRG model. Here we assume that the excitations are free particles, but they feel their environment. Since at finite temperature the plasma singles out a rest frame, a propagating hadron or parton will decay on this thermal background depending on their spatial momenta. This effect results in the spatial momentum dependence of the spectral functions (3). In particular in the crossover temperature regime, the high and low momentum spectra can differ significantly.

Although we do not know exactly the spatial momentum dependence of the spectral functions, we still bring physical arguments how it could behave qualitatively. At large spatial momenta, corresponding to small spatial distance, the thermal effects must be small, and we expect to encounter quark- and gluon-like propagations. On the other hand at small momenta, large scales, we should observe mostly hadronic excitations.

To have a handle on this phenomenological expectation, we study here an oversimplified model. In this model we assume a spectral function which consists of a single Dirac-delta for low spatial momenta, and it is zero¹ for large spatial momenta. This is a model for the hadronic modes. We use a sharp cutoff Λ_c between the two regimes, but this will not be crucial for the results.

We can calculate the pressure from (1), but, since the spectral function is so simple, we can perform the integrals, and obtain the formula similar to the free gas pressure:

$$\frac{P}{T^4} = -\frac{1}{2\pi^2} \int_0^g dx x^2 \ln(1 - e^{-\omega}), \quad g = \beta\Lambda_c, \quad \omega^2 = x^2 + (\beta m)^2. \quad (5)$$

For $m = 0$ and $\Lambda_c \rightarrow \infty$ this provides the usual Stefan-Boltzmann limit $\pi^2/90$. If $\Lambda_c \rightarrow 0$, of course, the pressure is zero.

For the sake of simplicity, we analyze the above formula for constant g values. Since we did not use the temperature independence of g , we are free to choose its T -dependence later. The advantage of the constant g choice is that the resulting $P(T)$ curves are very similar to the ideal gas pressure curve with some effective mass parameter and an overall suppression factor. In Fig. 2 these cutoff model curves are presented, almost covering the fitted

$$P(T) = N_{eff}(T)P_0(m_{eff}(T), T) \quad (6)$$

curves. One inspects an intriguing agreement: in fact even the largest deviation between these two curves (which occurs when the suppression is the largest at $g = 1$), expressed by the relative difference $(P - P_0)/P_{SB}$ does not exceed the few per cent level. We plot the fit parameters in Fig. 3.

We can also try to give fitting functions to these plots, with the sole aim to help to parametrize the numerically observed behavior of the curves. For the effective mass we choose $m_{eff}/m = 1 + ue^{-vg}$ where the best fits come from $u = 1.66$, $v = 0.67$. The function which fits best to the N_{eff} data is given by

$$\text{fit } f(x) = \frac{1}{1 + x^{-2}e^{-(bx)^a}}, \quad x = \frac{g}{g_0}, \quad g_0 = 2.95, \quad a = 1.79, \quad b = 0.58. \quad (7)$$

¹ To be able to fulfill the sum rule, the spectral function should not be zero, but it can be so broad and shallow that it does not contribute to the pressure, cf. Section II C.

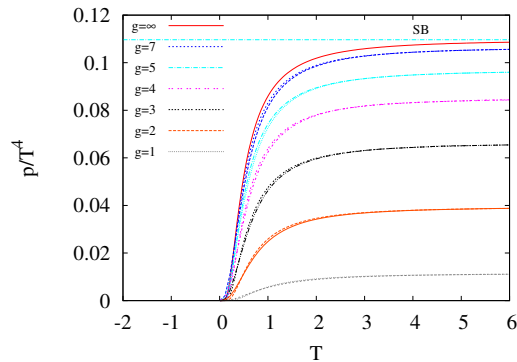


FIG. 2. Pressure curves coming from applying an effective spatial UV cutoff in the pressure integral $\Lambda_c = gT$, and $N_{eff}P_0(m_{eff}, T)$. These two curves cover each other up to a few percent. *SB* indicates the Stefan-Boltzmann limit.

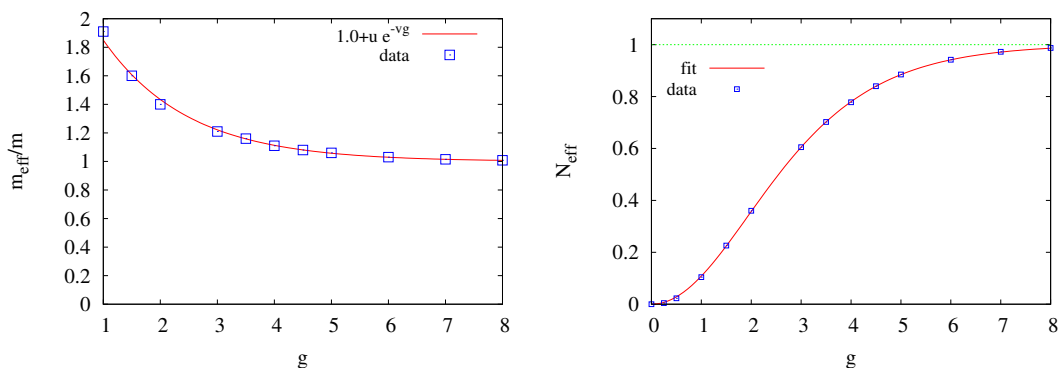


FIG. 3. The effective mass parameter m_{eff} (left figure) and the effective numbers of degrees of freedom N_{eff} (right figure), together with some fit functions (cf. (6) and (7)).

If the spatial momentum cutoff is temperature independent, then we have to choose $g = \Lambda_c/T$. For large temperatures the exponent can be neglected, and effectively we obtain $N_{eff} \sim (1 + T^2)^{-1}$ like dependence.

1. Momentum difference distribution

So far we assumed that the elementary excitations propagate in the matter which singles out a local rest frame, and so the spectral function may depend on the momentum of the particle. But physically the situation is more complicated. Taking a simple model where we have binary collisions between free particles, the cross section in fact is sensitive to the momentum difference between our particle and its colliding partner. Statistically this means a momentum difference distribution, which then leaves a trace in the spectral functions, and, consequently in thermodynamics. To understand better the statistical background physics, here we calculate this effect in the limit of non-interacting massless particles [34].

To begin with, we determine the distribution of *momentum differences* between the elementary excitations. We assume free particles with one-particle distribution function $f(E)$ where E denotes their energy. Since the particles are massless, $E = |\mathbf{p}|$. Using the relativistic kinematics for pairs of massless particles we find

$$Q^2 = -(p_1 - p_2)^2 = 2E_1E_2(1 - \cos\theta). \quad (8)$$

The distribution of Q^2 values is given by

$$P(Q^2) = \frac{\int dE_1 dE_2 d\cos(\theta) E_1^2 E_2^2 f(E_1) f(E_2) \delta(Q^2 - 2E_1E_2(1 - \cos\theta))}{\int dE_1 dE_2 d\cos(\theta) E_1^2 E_2^2 f(E_1) f(E_2)}. \quad (9)$$

Utilizing the Dirac delta functional for the integral over $\cos \theta$ this can be written as

$$P(Q^2) = \frac{\int_0^\infty dE_1 \int_{Q^2/4E_1}^\infty dE_2 \frac{1}{2} E_1 E_2 f(E_1) f(E_2)}{\int_0^\infty dE_1 \int_0^\infty dE_2 2E_1^2 E_2^2 f(E_1) f(E_2)}. \quad (10)$$

This value is always between zero and one, its integral with respect to Q^2 is one, due to the construction by eq.(9). Since the thermal parton distribution, $f(E)$, is a monotonic decreasing function, the numerator is maximal at $Q^2 = 0$. This maximal value is given by

$$P(0) = \frac{1}{4} \frac{\int dE_1 \int dE_2 E_1 E_2 f_1 f_2}{\int dE_1 dE_2 E_1^2 E_2^2 f_1 f_2} = \left\langle \frac{1}{2E} \right\rangle^2 = \frac{c^2}{T^2}, \quad (11)$$

with some constant c depending on the distribution.

For the Boltzmann distribution, $f(E) = e^{-E/T}/Z$, this distribution can be obtained in analytic form:

$$P(Q^2) = \frac{1}{T^2} F(Q^2/T^2), \quad F(x) = \frac{1}{64} \left(x^{3/2} K_1(\sqrt{x}) + 2x K_2(\sqrt{x}) \right). \quad (12)$$

The resulting rescaled momentum difference distribution is shown in the left panel of Fig. 4. Although this result was derived for zero mass, we hope that for massive particles this distribution is similar.

As it was already explained earlier in this section, the hadronic modes have small width for small momentum transfer and large width for large momentum transfer, while the partonic excitations behave in an opposite way. Therefore the pressure contribution of the hadronic modes with small momentum transfer is large, for large momentum transfer is small. So the contribution for the effective number of degrees of freedom N_{eff} must also behave similarly. Just like in the previous model of this section, we apply an extreme approximation to this behavior, and we weight the momentum difference distribution by a cut-off theta-function $\Theta(\Lambda^2 - Q^2)$. Using our earlier notation² $g = \beta\Lambda$, the approximation for the hadronic N_{eff} reads

$$N_{eff}^{hadr} = \int_0^{g^2 T^2} dQ^2 P(Q^2) = \int_0^{g^2} dx F(x) \quad (13)$$

by using the integration variable $x = Q^2/T^2$. This is the integrated distribution function of the thermal Q^2 distribution. For small g the distribution is nearly constant, so we can use (11) to conclude that

$$N_{eff}^{hadr} \approx c^2 g^2. \quad (14)$$

For Boltzmann distribution $c = 1/4$.

The integral can be performed analytically, the formal result is expressed with help of the Meijer G-functions:

$$N_{eff}^{hadr}(g) = \frac{1}{4} \left[G_{1,3}^{2,1} \left(\frac{g}{2}, \frac{1}{2} \left| \begin{matrix} 1 \\ 1, 3, 0 \end{matrix} \right. \right) + G_{1,3}^{2,1} \left(\frac{g}{2}, \frac{1}{2} \left| \begin{matrix} 1 \\ 2, 3, 0 \end{matrix} \right. \right) \right] \quad (15)$$

This result is plotted in Fig. 4. In order to demonstrate the excellent agreement between the functional forms we compare the rescaled $N_{eff}^{hadr}(0.78g)$ against N_{eff} coming from the previous analysis cutting the pressure formula by excluding the large momentum part (cf. (5)). This surprisingly good agreement between the results of two vastly different approaches needs explanation. Although in both models the pressure calculation involved a momentum cut-off, soft part for hadrons, hard part for partons, the fundamental assumptions differ: the single-particle momentum cut relative to the medium, while the Q^2 -cut relative between pairs of particles. Also Lorentz-transformation properties differ essentially. Inspecting further Fig. 4 one realizes that in the transition period, from the 90% hadron dominated phase to the 90% parton dominated one, g increases by a factor of 6. This nicely accords with the ratio 6 between a parton picture based 1 GeV characteristic soft scale and the QCD lattice transition temperature scale of 167 MeV. Based on these two coincidences we believe that the simple cut-off model performs surprisingly well.

² Here g can be temperature dependent.

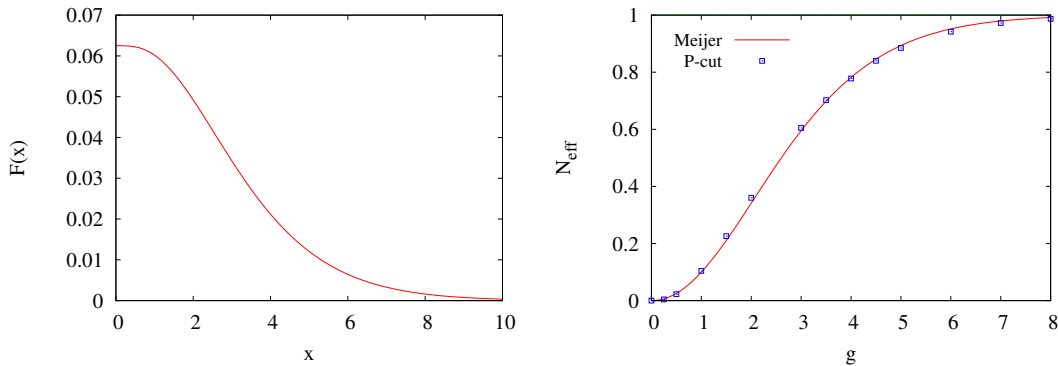


FIG. 4. The left panel shows the momentum difference distribution $F(x)$ defined in (13). On the right panel the rescaled effective numbers of hadronic degrees of freedom (cf. eq. (15)) is plotted as a function of g , together with the similar quantity defined in (5).

B. Thermal mass

To further improve the model of the hadronic/partonic excitations, we may take into account the effect of the medium on the particle masses. Since MC simulations do not show strong temperature variation, there is no signal that near T_c the mass parameters would diverge or behave any extraordinary way: the temperature variation of the masses must not be modified near T_c .

To describe the effect of the variation of the masses with the temperature, we follow a similar path as in the previous subsections. We realize that for a choice $m^2 = m_0^2 + cT^2$ the pressure curve runs similarly to the free gas pressure, only an effective number of excitations and an effective m_0 should be introduced. This is demonstrated in Fig. 5. As

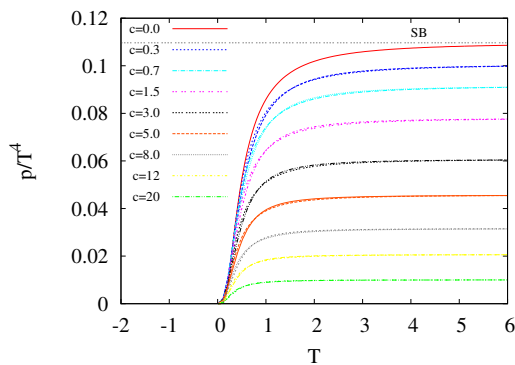


FIG. 5. Pressures coming from using thermal mass $m^2 = m_0^2 + cT^2$, choosing $m_0 = 1$ units, and the $N_{\text{eff}}P_0(m_{\text{eff}}, T)$ assumption, respectively. The two curves almost cover each other. *SB* indicates the Stefan-Boltzmann limit.

before, we plot the pressure with thermal mass and the rescaled free gas pressure (cf. (6)) with some fitted effective mass. The agreement is quite good also here, we find that the scaled deviation, $(P - P_0)/P_{SB}$, is at most on the percent level. We also display the fit parameters in Fig. 6. The fit function for the effective mass reads as

$$\left. \frac{m_{\text{eff}}}{m} \right|_{\text{fit}} = 1 - \frac{wx^v}{1 + ux^v}, \quad u = 0.208, \quad v = 0.666, \quad w = 0.125, \quad (16)$$

while the fit for N_{eff} was performed with

$$N_{\text{eff}}^{(\text{fit})} = \frac{e^{-ax}}{1 + bx^c}, \quad a = 0.0527, \quad b = 0.208, \quad c = 0.871. \quad (17)$$

What we can observe is that for a constant c value the pressure curves do not reach the Stefan-Boltzmann limit. We expect that c decreases at most logarithmically, therefore in the temperature range up to 1 – 2 GeV the constant c approximation must be appropriate.

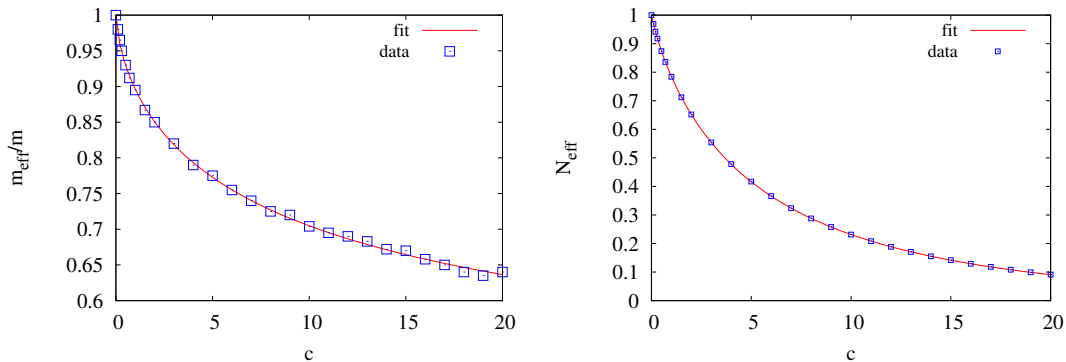


FIG. 6. The fitted effective mass parameter (left figure) and the effective number of degrees of freedom (right figure), together with some fit functions (cf. (16) and (17)).

C. Quasiparticle width and continuum height

We take an approximation for the spectral function neglecting the momentum dependence and discuss the pressure in terms of the quasiparticle width and wave function renormalization. Because of a fixed sum rule, the quasiparticle wave function renormalization is related to the continuum height.

This situation is partly discussed in Ref. [26]. Now we examine more realistic spectra: these contain several (two or three) quasiparticle excitations above a multiparticle background. The multiparticle contribution is modeled by a two particle cut with imaginary threshold value (for more motivation see [26, 35])

$$\varrho_{cont}(p) = \frac{p}{p^2 + m_{th}^2} \text{Im} \sqrt{m_{th}^2 + iS - p^2}. \quad (18)$$

This function is nowhere zero, it mimics the effect of finite temperature spectral functions or spectral functions with several threshold values. We assume that one quasiparticle has a mass below m_{th} , so it would be a stable excitation if $S = 0$ were true; the remaining ones have masses above m_{th} . The widths of the quasiparticle peaks are determined by the background, and all peaks have the same area. We tested several spectra under such conditions, some sample functions are shown in Fig. 7. The parameters for the samples can be found in Tables I and II. The quasiparticle

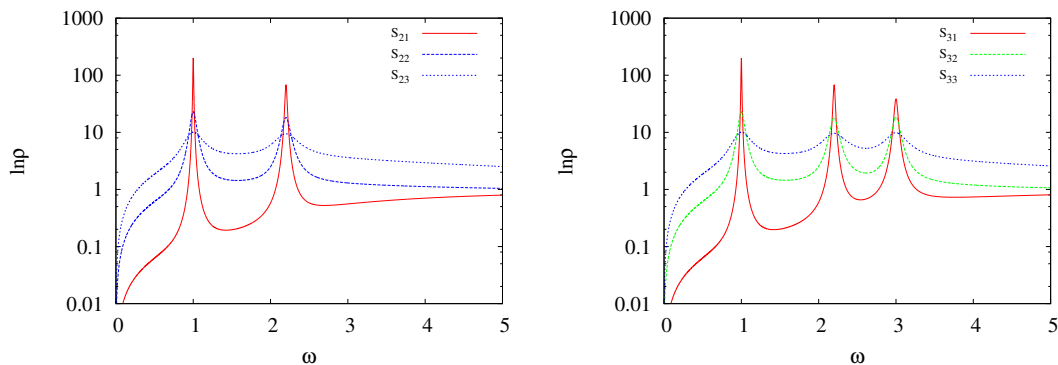


FIG. 7. Spectra with two and three quasiparticle peaks and a continuum. The width parameters of the plots can be found in Table I.

name	S_{21}	S_{22}	S_{23}	S_{24}	S_{25}	S_{26}	S_{27}	S_{28}
γ_1	0.005	0.046	0.138	0.010	0.022	0.032	0.046	0.066
γ_2	0.015	0.059	0.172	0.020	0.032	0.043	0.059	0.083

TABLE I. The parameters of the plots with two peaks.

name	S_{31}	S_{32}	S_{33}	S_{34}	S_{35}	S_{36}	S_{37}	S_{38}
γ_1	0.005	0.046	0.138	0.010	0.022	0.038	0.055	0.079
γ_2	0.015	0.059	0.172	0.020	0.032	0.050	0.070	0.100
γ_2	0.026	0.058	0.161	0.028	0.035	0.050	0.068	0.095

TABLE II. The parameters of the plots with three peaks.

widths are determined by the continuum height at the quasiparticle mass. These spectra are rather typical for hadronic channels.

One inspects the pressure, based on eq. (1), belonging to these spectra on Fig. 8

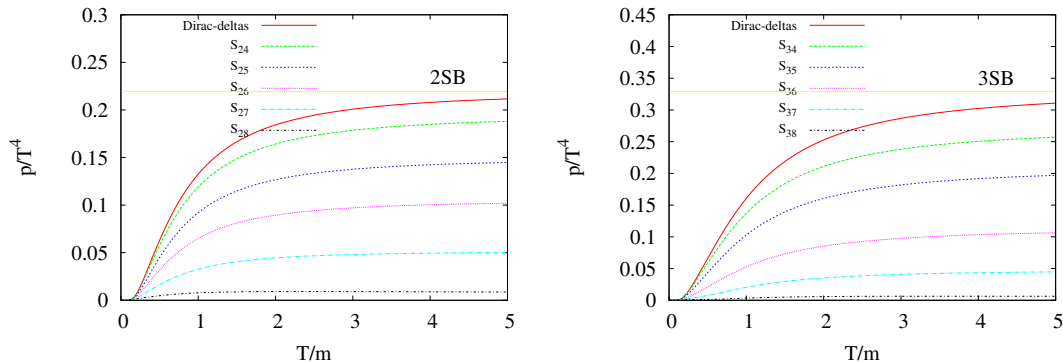


FIG. 8. Pressure coming from the general spectra with two peaks (left figure) or three peaks (right figure). *SB* denotes the Stefan-Boltzmann limit.

Finally we can read out the effective number of dof defined in (4), see Fig. 9. The widths of the data points

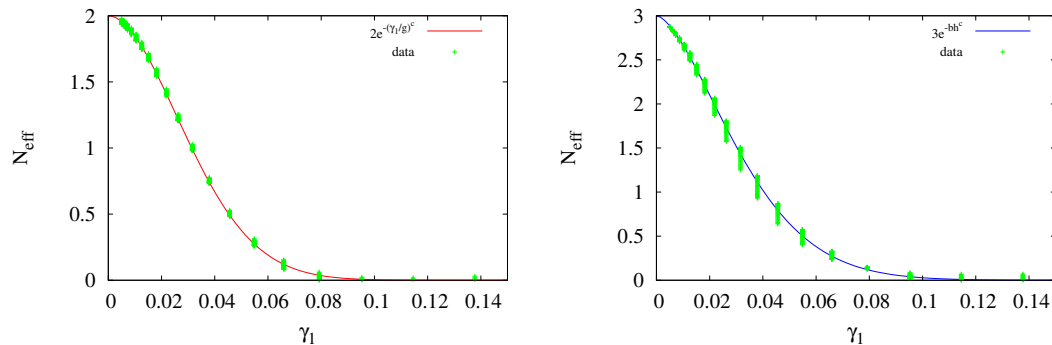


FIG. 9. The effective number of dof for spectra with two peaks (left figure) or three peaks (right figure). In the horizontal axis we see the width of the first peak, which is proportional to the height of the continuum. Data taken from [10].

correspond to the temperature variation of N_{eff} . We see that, in agreement with the experience with the single quasiparticle case in [26], N_{eff} is almost temperature independent. A single curve can be fitted through all points. For the single quasiparticle case a Gaussian was a good choice (cf. [26]), here this Ansatz does not perform nicely. Instead we consider a stretched exponential fit function

$$N_{eff} = \exp(-(\gamma_1/g)^c), \quad (19)$$

where γ_1 is the width of the first peak, proportional to the height of the continuum. This Ansatz produced fits with excellent agreement with the data. The fit parameters can be seen in Table III. As it can be seen, the less peaks the closer is the exponent, c , to the Gaussian ($c = 2$) reached for a single peak.

# of peaks	g	c
2	0.038	1.88
3	0.038	1.60

TABLE III. Parameters of the N_{eff} fit $\exp(-(\gamma_1/g)^c)$.

III. QCD THERMODYNAMICS

In the previous section we studied several typical spectral functions, and computed the pressure stemming from them. As we have seen, with an appropriate parametrization of the modification effects, we could factor out an effective number of degrees of freedom N_{eff} . It is, in a fair approximation, temperature independent. For free particles it is $N_{eff} = 1$, in the interacting case it is $N_{eff} \leq 1$. It is a complicated object, partly because the different effects discussed in the previous sections result in complicated fit functions, partly because in reality the fit parameters are temperature dependent themselves in a nontrivial way. On the other hand, to approach QCD thermodynamics, the hadronic degrees of freedom should be treated statistically, so only the robust properties count.

One of the most prominent property of N_{eff} is that it can be a constant for large temperatures. Finite width, presence of the continuum, reduction of the spatial phase space and thermal mass effects all may result in this phenomenon.

The next robust property is that $\ln N_{eff}$ may depend on the parameters as a power law. The dependence of N_{eff} on the quasiparticle width (or continuum height), on the phase space exclusion and on the thermal mass parameter all contain exponential factors. In a realistic case the width usually grows with the temperature. The momentum cutoff parameter is either constant, or – if the physical cutoff Λ_c is constant – decreases with T . The thermal mass coefficient ($m^2 = m_0^2 + cT^2$) is more or less constant. That means that $\ln N_{eff}$ is dominated by the continuum background i.e. by width effects.

Talking about the temperature dependence of the continuum background height we must mention two aspects. One is a generic thermodynamical effect: the radiative corrections in general grow with the temperature, thus the background height, together with the quasiparticle width grows with the temperature. Guided by dimensional reasoning, we shall assume the simple $\gamma^2 = \gamma_0^2 + \kappa T^2$ dependence. The zero temperature parton width γ_0 depends on the hadron mass. We consider a simplified description: we assume that there is a fast rise in $\gamma_0(m)$ meaning an effective cutoff in the number of available hadrons [36]. So in the calculations we just choose $\gamma_0 = 0$ and the number of hadronic excitations goes to some N in the range of [3000 – 5000]. Once γ is given, we use $\ln N_{eff} \sim \gamma^c$.

For the width of the partons (QGP degrees of freedom), we must take into account another effect: the partonic cross section increases with the presence of hadronic excitations. So we assume that the parton width is proportional to some power of N_{eff}^{hadr} , the number of hadronic excitations. It is crucial for the fittability of the MC pressure.

To make life easier, we assume that the width of all hadrons have approximately the same temperature dependence, and partons have another single temperature dependent width, respectively. Moreover, we neglect the variation of the mass parameter of the free gas pressure and the explicit temperature dependence of the partonic width. Then the simplified Ansatz for the QCD pressure is given as the following:

$$\begin{aligned}
P_{hadr}(T) &= N_{eff}^{(hadr)} \sum_{n \in \text{hadrons}}^N P_0(T, m_n), & \ln N_{eff}^{(hadr)} &= (T/T_0)^b, \\
P_{QGP}(T) &= N_{eff}^{(part)} \sum_{n \in \text{partons}} P_0(T, m_n), & \ln N_{eff}^{(part)} &= G_0 + c(N_{eff}^{(hadr)})^d.
\end{aligned} \tag{20}$$

Here $P_0(T, m)$ is the pressure of a free gas with mass m at temperature T . For the masses of the hadronic excitations we assume a Hagedorn-like spectrum [32, 33]

$$m_n = m_\pi + T_H \log n. \tag{21}$$

The lowest mass m_π should be in the order of the pion mass, the parameter T_H is the Hagedorn temperature. For the partons we assume some effective masses, namely $m_{ud} = 300$ MeV, $m_s = 450$ MeV and in the best fits we found $m_{gluons} = 500 - 550$ MeV.

We have quite a few parameters at hand, but the pressure curve found in MC simulations have some well defined regimes. It helps to find the correct values for the parameters. First of all at very low temperatures the pressure comes exclusively from the light hadrons, which helps to find the fit value of m_π . We note here that all hadrons are taken into account with unit multiplicity, therefore m_π is some “smeared” pion mass. Below $T \sim 160$ MeV the pressure is still dominated by the hadrons, this helps to fit T_H . The HRG pressure coming from the hadronic pressure with no

suppression would overshoot the real pressure near $\sim 160 - 200$ MeV. To avoid this, we adjust the fit parameters T_0 and b . We learn that b must be $1.5 - 2$ (cf. Table III), restricting further the allowed fit ranges.

At very large temperatures the pressure is determined by the QGP degrees of freedom. Although there exist perturbative calculations based on the QCD Lagrangian, but to be consistent, here we give an effective, phenomenological description also to this regime. At large temperatures the pressure is more or less constant. In principle there is a slight, logarithmic increase in the pressure, but the lattice data until 1.5 GeV are consistent with a constant (for larger temperatures see [37]). This will determine G_0 . As the temperature starts to decrease hadrons appear, from this the coefficients c and d can be determined. Thus in the QGP sector G_0 is determined by the MC data, there remain only two parameters to fit.

The best fit for the pressure goes through all the points, as it is shown by the left panel of Fig. 10. We also made

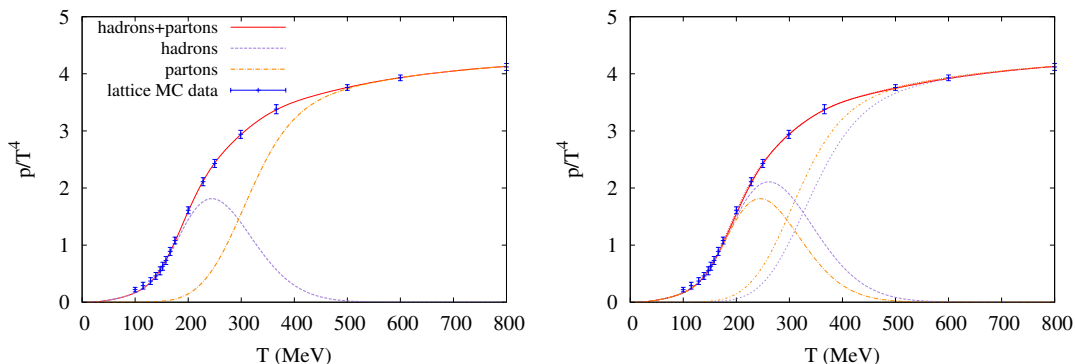


FIG. 10. Pressure of the QCD fitted by our model. The fit curve goes through all points. On the left panel a single selected fit is shown, on the right panel we compare results of two different fits.

further fits, two fits are shown on the right panel of Fig. 10. The fit parameters are summarized in Table IV

nmr.	N_H	m_π	T_0	T_H	d	b	c_g	c_q	G_0
1	5000	140	165	187	0.9	1.95	22	9	0.157
2	3000	120	190	195	0.9	2	9.7	9.7	0.151

TABLE IV. Fit parameters for fitting lattice MC data.

A. Properties of the QCD pressure

As the curves on Fig. 10 show, with the simple hadron+parton model above we could very accurately fit the lattice MC results for the QCD pressure. Despite the fact that we have some fit parameters, this is not self-evident. Most of the parameters of Table IV are not entirely free fit parameters, they are fixed by some physical requirements. Still there remain 5 or 6 parameters (depending on the fitting strategy) which could be played with in order to obtain the best fit.

It was crucial for the fits, as we mentioned before, that the fermionic suppression factor depends on the number of available hadronic modes ($N_{eff}^{(hadr)}$). Without it one can not achieve that partons with mass as low as 300-600 MeV, i.e. lower than most hadronic excitations, would give significant contribution to the pressure only at large temperatures.

There are some robust properties which was shared in all fits we could do. First of all it was known before that up to $T \approx T_c$ the hadron resonance gas alone can describe the full QCD pressure. This we used as an input for the fits. But this also means that the partonic pressure just starts to appear at this temperature. Being a crossover transition the partonic pressure must appear continuously. The question remains, however, that how long does it take before the quarks dominate the thermodynamics.

The conventional view of the hadron-QGP transition considers it as being rather fast, the “width” of the susceptibility curves give a hint for that. Unfortunately this is not a strong argument, since for a single free massive scalar field the p/T^4 curve exhibits very similar properties than the QCD pressure curve, but there any kind of phase transition is missing.

According to the present calculation the rule of thumb for the QCD pressure is that it is dominated by hadrons until $T \approx 2T_c$. The hadronic and QGP contributions are equal approximately at $2T_c$, later the quark degrees of freedom will dominate. Only at $T \approx 3T_c$ do we obtain a system that can be described solely by the QGP excitations.

This finding is more or less in accordance with the perturbative studies [4]. In these studies, according to the authors, the pressure is described down to about approximately $2T_c$, where, in the light of the present calculation, the GQP modes dominate the pressure.

IV. CONCLUSIONS

In the present paper we proposed a model of the strong interactions which is capable to give an account for the numerically determined QCD thermodynamics. The basis of this model is a quadratic field theory with a general spectral function. Once the model is fixed one can calculate the pressure exactly.

For phenomenological applications we have to parametrize the spectral function with some characteristic numbers such as the position (mass) of the peak or peaks, the height and threshold behavior of the continuum, the spatial momentum dependence for finite temperature or density applications, and the sum rule. One can then determine the pressure of the system as a function of these numbers. The most robust finding of these studies is that the pressure of the system decreases and eventually vanishes as the quasiparticle properties become less and less enhanced. We have characterized this tendency by the effective number of degrees of freedom N_{eff} , which is just the ratio of the pressure coming from the actual spectral function and the one with vanishing continuum. By an appropriate choice of the parametrization one can achieve that N_{eff} is approximately temperature independent (cf. Figs. 3, 6, 9): then for each spectral function is modeled by a single number. N_{eff} is a function of the parameter set which determines the spectral function. We have determined this function for several plausible spectral functions and proposed fitting formulas for it.

As a last step we applied our approach for a statistical mixed hadron and parton model of QCD. Ingredients of this model are hadrons, described by a statistical mass distribution and by common, average hadron properties for the spectral function. The other ingredient is a parton gas consisting of u, d and s quarks and gluons. While the hadronic spectral functions are determined on their own, the quark spectral functions, in particular the continuum height, must have depended on the number of available hadrons $N_{eff}^{(hadr)}$. The parameters of the model were adjusted to fit the lattice MC measurement of the QCD pressure.

As a result we do not only fit the lattice MC pressure curve excellently, but we also describe the hadron–parton decomposition of the QCD plasma at a given temperature. We have found that for $T \lesssim T_c \approx 156$ MeV only hadrons are present in the medium. These hadrons do not disappear at the would-be critical temperature: for $T_c \lesssim T \lesssim 2T_c$ they still dominate the pressure, although not in the original hadron resonance gas form, but as excitations with broad spectral functions. In the higher temperature regime the hadrons melt gradually, for $2T_c \lesssim T \lesssim 3T_c$ they still give considerable contribution to the total pressure. Only at $T \approx 3T_c$ do we arrive at the point where the QCD pressure is dominantly given by the fundamental QCD degrees of freedom.

As future prospects we plan to apply our description of QCD EoS to determine further statistical observables of the strongly interacting plasma like the transport coefficients. Moreover we consider to apply this description also at finite chemical potential.

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