Casimir effect: running Newton constant or cosmological term

Janos Polonyi[a,b*] and Enikő Regős[b,c†]

[a] *Theoretical Physics Laboratory, CNRS and Louis Pasteur University, Strasbourg, France*[‡]

[b] *Department of Atomic Physics, L. Eötvös University, Budapest, Hungary and*

[c] *Theoretical Physics Group, Hungarian Academy of Sciences, Budapest*

(Dated: October 31, 2018)

We argue that the instability of Euclidean Einstein gravity is an indication that the vacuum is non perturbative and contains a condensate of the metric tensor in a manner reminiscent of Yang-Mills theories. As a simple step toward the characterization of such a vacuum the value of the one-loop effective action is computed for Euclidean de Sitter spaces as a function of the curvature when the unstable conformal modes are held fixed. Two phases are found, one where the curvature is large and gravitons should be confined and another one which appears to be weakly coupled and tends to be flat. The induced cosmological constant is positive or negative in the strongly or weakly curved phase, respectively. The relevance of the Casimir effect in understanding the UV sensitivity of gravity is pointed out.

arXiv:hep-th/0404185v3 14 Nov 2005

I. INTRODUCTION

Quantum Gravity possesses a number of fundamental questions which are unsolved so far. Its apparent conflict with Quantum Mechanics, namely canonical commutation relations imposed on equal time hypersurface require the knowledge of the metric before the operator equations defining it is completed, questions the need of the quantum treatment of gravitation altogether [1]. Let us assume that the metric corresponds either to an elementary or to a composite field and the usual quantum field theoretical path integral quantization applies. The next problem arises from the UV divergences. At this point one either seeks some superstring formalism which might be interpreted as a physical regulator or relies on a formal regulator. We choose the latter being a simpler possibility and do not attempt to remove the cutoff, i.e. we are not interested in the problem of physics at arbitrary high energies. Instead, our goal in this work is to make a small step toward the understanding of the nature of the radiative corrections up to a high but finite energy for a particular background geometry, those of the de Sitter space.

But there are still problems even after such a drastic reduction of the scope of the investigation. Regulators which preserve the local symmetries of the theory and allows analytical computations are available for the Euclidean section of the space-time only. When Euclidean Quantum Gravity is considered then some or all of the conformal modes become unstable [2], rendering the perturbative or saddle point approximation problematical. The usual approach to this complication is to stabilize Euclidean Quantum Gravity on the tree level by an appropriate modification of the Wick rotation [3] or on the quantum level by altering the rules of quantization [4] or the functional measure for the path integral [5].

Similar instability is well known in Yang-Mills models. In fact, Euclidean Yang-Mills theory considered in the presence of a homogeneous magnetic field has unstable modes [6]. Since the Euclidean Yang-Mills action is bounded from below the Euclidean vacuum is stable though the detailed mechanism of this stabilization is still unknown. When the time extent of the Euclidean space can be sufficiently large then the large amplitude modes resulting from the instability can be regarded as an indication of a condensate of the true, physical vacuum of the theory with real time. Can this lesson be relevant for Quantum Gravity and pointing toward the non triviality of its vacuum?

One can imagine three dynamical, consistent stabilization mechanisms for Euclidean Quantum Gravity. (i) It might happen that the true gravitational action contain higher order derivative terms which renders the classical action bounded from below. If these new terms do not involve new dimensional coupling constant then the stable vacuum will be dominated by fluctuations with scale close to the cutoff because the instability of the Euclidean Einstein action is in the UV regime. (ii) Another possibility is that the classical action is unbounded but the theory is stabilized by quantum fluctuations. This happens if the classical instability tries to squeeze the system into a small region of the configuration space and the kinetic energy generated by the uncertainty principle becomes strong. Certain numerical simulations seem to support this scenario for quantum gravity regularized by the Regge-calculus

*Electronic address: polonyi@fresnel.u-strasbg.fr

†Electronic address: eniko@ast.cam.ac.uk

‡URL: http://lpt1.u-strasbg.fr

[7]. (iii) Yet another possibility is that pure gravity is indeed unstable but the stabilization is achieved by the coupling of gravity to matter [8, 9].

But one should bear in mind that either possibility is realized the vacuum is non perturbative and contain a condensate. If the vacuum is stabilized on the n-th order of the loop expansion then it corresponds to the minimum of the effective action computed up to $\mathcal{O}\left(G_B^n\right)$. Here G_B denotes the bare Newton constant which plays the role of $\hbar$ in the absence of matter and counts the order of the loop expansion. The determination of the minimum of this effective action requires the knowledge of all 1PI vertex functions in the given order of the loop expansion. In the case of $n = 0$, c.f. case (i), the solution of the classical equation of motion can be regarded as given by summing over infinitely many tree graphs. When $n > 0$, c.f. cases (ii) and (iii), then an infinite sum of graphs up to n-loops gives the vacuum configuration.

We do not attempt to identify the true stabilization mechanism in this paper. Instead, we shall be contended with a small step along the line of reasoning outlined above and present a study of the dynamical environment in which the stabilization takes place by computing the effective potential for a homogeneous space-time curvature with vanishing unstable modes and by reading off from it the running of the Newton constant with the length scale of such a background geometry.

Before we continue the discussion of the computation of the effective potential let us bring another pertinent element, the Casimir effect, into the discussion. The vacuum energy of quantum field theories in flat space-time depends on a rather complicated manner on the choice of the space-time region $\mathcal{R}$ populated by quantum fields [18]. It is well known that the metric tensor controls all aspects of the geometry, among them the spatial extent of the system. In a space-time of homogeneous, positive curvature the free geodesics are deflected in such a manner that they stay in a region with linear size in the order of

$$L \approx \frac{1}{\sqrt{R}}. \tag{1}$$

Such double role of the curvature, being an external field and an IR cutoff in the same time, provides a new, physical insight into the inherent sensitivity of quantum gravity on the UV sector. The argument is the following. The partition function computed on a homogeneous or at least slowly varying background field in a fixed space-time region $\mathcal{R}$ develops UV finite dependence on the background field after having introduced the usual background field independent counterterms. One would expect the same conclusion when the partition function is considered on a background curvature. But this is not what happens because of the Casimir effect. The reason is that the Casimir effect is always UV divergent. To understand this property let us consider first the number of modes

$$\mathcal{N} \approx (\Lambda L)^d \tag{2}$$

in dimension d where Λ denotes the UV cutoff of dimension energy and L stands for the linear size of the system. The expression (2) diverges when the UV and IR cutoffs are removed because the length scale of a mode can be infinitely large or infinitely small. The result is that the UV and the IR dependences are mixed in $\mathcal{N}$. In a similar manner the Casimir effect mixes the UV and the IR sectors, too. Therefore the dependence of the partition function on the curvature is always UV divergent. It is worthwhile noting that this point is nothing but the throughly studied issue of the coupling of the zero point fluctuations of the vacuum to the gravitation.

Our main goal in this work is to determine the renormalization of the effective potential for the curvature and some effective coupling constants in one-loop order. The effective potential, $\gamma(R)$, is defined by the value of the effective action for homogeneous de Sitter spaces of curvature $R > 0$ and will be computed in $\mathcal{O}\left(G_B\right)$ by restricting the unstable modes to zero, except the homogeneous conformal mode which is kept constant. We do not attempt to remove the UV cutoff and confine our investigation on the finite energy dynamics by keeping Λ fixed and ignoring the formal issue of renormalizability. We find a phase transition when the 'best' de Sitter space is selected by minimizing $\gamma(R)$. For low enough value of the cutoff, $\Lambda << \Lambda_{cr}$ the energetically preferred value of the curvature is $\mathcal{O}\left(\Lambda^4 G\right)$. The theory is weakly coupled at such or larger values of the curvature but the modes with length scale $\ell > 2/\Lambda^2\sqrt{G}$ are non perturbative. The other phase, realized by large cutoff $\Lambda >> \Lambda_{cr}$, prefers small curvature and is perturbative. This phase structure agrees qualitatively with the one found in lattice simulations [7] where a flat and a strongly curved phase were found for large or small values of the bare Newton constant, respectively. One should note in this respect that although the UV cutoff represents an upper limit for the curvature its relation to the Newton constant can be arbitrary, i.e. quantum gravity is a well defined and physically nontrivial theory for either $G\Lambda^2 < 1$ or $G\Lambda^2 > 1$.

The Casimir contribution to the effective action is proportional to the space-time volume which raises the possibility of generating a cosmological constant even in the absence of matter. In fact, an induced cosmological constant is found whose sign distinguishes the two phases mentioned above.

The UV cutoff, Λ, appears as an unavoidable parameter in the main results of this paper, in the renormalization of the effective action and the phase structure. Such a state of affairs is certainly erroneous in the case of a renormalizable theory. But one should remember that whatever is the ultimate theory of quantum gravity the quantum effects are supposed be governed by the Einstein-Hilbert action within the energy and distance range explored up to today. The predictions of this theory, being nonrenormalizable, naturally do depend in an essential manner on the choice of the UV cutoff. But this dependence is restricted to scales comparable to the cutoff. At energy scales far below the cutoff quantum gravity is a useful effective theory because the nonrenormalizable, UV irrelevant terms of the action play no role here [10]. It is worthwhile noting that this energy regime is determined by the cutoff only, the other dimensional parameter of the theory discussed in this work, the Newton constant, may take arbitrary values with respect to the UV cutoff in this effective gravity.

There have already been investigation of effective coupling constants carried out before. The one-loop correction to the Newtonian gravitational potential on flat space-time has been computed in the IR domain in Refs. [11] and a weak, $\mathcal{O}\left(R^{-2}\right)$, UV finite effect was found. The result is expected to be rather different when higher loop contributions are retained because those graphs contain UV divergences. The self interaction of gravitons, considered in our computation, is UV divergent already at the one-loop level and the comparison of the running Newton constant presented in these works with those arising from the computation of the gravitational forces between matter particles is not obvious. In fact, the difference of the dependence on the scale of a space-time variable or of a field amplitude is usually a manifestation of a nontrivial wave function renormalization constant [13]. The non triviality of the wave function renormalization constant can be expected on the simple dimensional ground that the metric tensor is a dimensionless field as opposed to the usual one. The conventional renormalization group has already been applied to determine the running of the coupling constants of Grand Unified Theories on curved geometry and a number of scaling laws were identified [12]. But one should bear in mind that the multiplicative renormalization which serves as the basis for the conventional renormalization group equation is valid for renormalizable models only. The renormalization group flow was computed for the Newton and the cosmological constants in Ref. [14] for de Sitter spaces. Two kinds of fixed points were found, a Gaussian and gauge-dependent non-Gaussian. The flow was extended to arbitrary dimension and value of the gauge fixing parameter in Ref. [15]. But it is difficult to compare thess result with ours, partly because this computation involves a formal one-loop integral over the unstable modes, partly because of the presence of the cosmological constant. The gauge dependence of the renormalization group flow was investigated in Ref. [16]. Though our motivation is different, our work is closest in technics to Ref. [17] where the dynamical suppression of the cosmological constant was studied in de Sitter spaces.

The organization of the paper is the following. The effective potential and the technical details such as the regularization, gauge fixing and the eigenfunctions of the Laplace operator in de Sitter spaces are introduced in Section II. The results of the numerical computation are presented in Section III. The problem of the stability of the conformal modes is discussed in Section IV. Finally, Section V is for the summary and conclusions. A very brief summary of the simplest appearance of the Casimir effect is presented in the Appendix.

II. EFFECTIVE POTENTIAL FOR THE CURVATURE

The computation of the value of the one-loop effective potential for de Sitter spaces is presented in this section.

A. Formal expressions

The generator functional for connected Green functions in the Euclidean section is defined as

$$e^{W[j]} = \int D[g] e^{-S[g]+\int dx j^{\mu\nu} g_{\mu\nu}}, \tag{3}$$

where the Einstein-Hilbert action is given in terms of the metric tensor g,

$$S[g] = -\kappa_B^2 \int dx \sqrt{g}(R - 2\lambda_B), \tag{4}$$

the multiplicative constant in front of the action is $\kappa_B^2 = 1/16\pi G_B$ and λ_B denotes the cosmological constant which will be left vanishing on the tree-level for simplicity. We shall use units $c = \hbar = 1$. The effective potential for the metric, the Legendre transform of $W[j]$,

$$\Gamma[g] = -W[j] + \int dx j^{\mu\nu} g_{\mu\nu} \tag{5}$$

is introduced by the definition

$$g = \frac{\delta W[j]}{\delta j}. \tag{6}$$

Finally, an effective potential for the homogeneous curvature will be identified as the value of the effective action at a homogeneous curvature,

$$\gamma(R) = \Gamma[g^{(R)}] \tag{7}$$

where $g^{(R)}$ denotes the metric tensor of the de Sitter space of curvature R. Notice that $\gamma(R)$ is not the conventional effective potential corresponding to the composite operator R, their difference arises from the fluctuations of the metric when the curvature is held fixed. We shall use the simpler construction, $\gamma(R)$, because our goal is the find the geometry which minimizes the effective action under the assumption of the homogeneity of the space-time.

The conformal modes are instable in the Euclidean theory [2]. In fact, the transformations

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \quad R \to \frac{R}{\Omega^2} - 6\frac{\Box\Omega}{\Omega^3}, \tag{8}$$

induce the change

$$S[g] \to S[g] = -\kappa^2 \int dx\sqrt{g}\left(6D_\mu\Omega D_\nu\Omega g^{\mu\nu} + \Omega^2 R - 2\lambda_B\Omega^4\right) \tag{9}$$

showing that the action is unbounded from below for inhomogeneous Ω. In order to isolate the instability the generating functional will be computed in two consecutive steps, first we integrate over the stable modes and the conformal modes will be dealt with in the second step.

The one-loop approximation yields

$$W[j] = -S[\bar{g}] + \int dx\sqrt{g}j^{\mu\nu}\bar{g}_{\mu\nu} + \frac{1}{2}\mathrm{Tr}\ln\frac{\delta^2 S[\bar{g}]}{\delta g\delta g} \tag{10}$$

where $\bar{g}$ is the saddle point satisfying the classical equation of motion

$$\frac{\delta S[\bar{g}]}{\delta g} = j \tag{11}$$

and $g = \bar{g} + h$ with $h = \mathcal{O}\left(\kappa_B^{-2}\right)$. As a result we find

$$\Gamma[g] = S[g] + \frac{1}{2}\mathrm{Tr}\ln\frac{\delta^2 S[g]}{\delta g\delta g}, \tag{12}$$

in particular for $\bar{g} = g^{(R)}$,

$$\gamma(R) = S[g^{(R)}] + \frac{1}{2}\mathrm{Tr}\ln\frac{\delta^2 S[g^{(R)}]}{\delta g\delta g}, \tag{13}$$

up to terms $\mathcal{O}\left(\kappa_B^{-2}\right)$.

One has to settle two problems in order the manipulation outlined above be well defined. One needs gauge invariant UV regulator and the unstable modes must be kept under control in the functional integration.

B. Gauge fixing and regularization

In order to saturate $W[j]$ by a Gaussian integral one has to fix the gauge. We shall use harmonic gauge by adding

$$S_{gf}[g^{(R)}, h] = \frac{1}{2}\int dx\sqrt{g^{(R)}}g^{(R)\mu\nu}f_\mu[g^{(R)}, h]f_\nu[g^{(R)}, h] \tag{14}$$

to the action where

$$\begin{aligned} f_\mu[g^{(R)}, h] &= \sqrt{2}\kappa_B f_\mu^{\nu\rho}[g^{(R)}]h_{\nu\rho}, \\ f_\mu^{\alpha\beta}[g^{(R)}] &= \frac{1}{2}\delta_\mu^\alpha g^{(R)\beta\gamma}D_\gamma + \frac{1}{2}\delta_\mu^\beta g^{(R)\alpha\gamma}D_\gamma - \frac{1}{2}g^{(R)\alpha\beta}D_\mu, \end{aligned} \tag{15}$$

and D stands for the covariant derivative of the metric $g^{(R)}$. The standard Fadeev-Popov quantization rules lead to the generator functional

$$e^{W[j]} = \int D[h]D[c]D[\bar{c}]e^{-S[g^{(R)}+h]-S_{gf}[g^{(R)},h]+S_{gh}[g^{(R)},h,c,\bar{c}]+\int dx\sqrt{g}j^{\mu\nu}(g^{(R)}_{\mu\nu}+h_{\mu\nu})} \tag{16}$$

where c and $\bar{c}$ denote the anticommuting ghost fields and

$$S_{gh}[g^{(R)},h,c,\bar{c}] = -\sqrt{2}\int dx\sqrt{g^{(R)}}\bar{c}^{\mu}\mathcal{M}_{\mu\nu}[g^{(R)},h]c^{\nu} \tag{17}$$

with

$$\mathcal{M}_{\alpha\beta} = \sqrt{2}\kappa_B f^{\mu\nu}_{\alpha}[g^{(R)}]\left(\frac{\partial g_{\mu\nu}}{\partial x^{\beta}} - \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\beta}}{\partial x^{\nu}}\right). \tag{18}$$

Notice that S_{gh} is independent of the gauge invariant cosmological constant because it is determined by the properties of the gauge orbits in the configuration space. In the same time the dynamics of the metric tensor naturally contains λ_B. This is reminiscent of the mass generation by spontaneous symmetry breaking where the mass term for the gauge field is gauge invariant, too.

The UV regulator will be constructed by a gauge invariant generalization of the momentum cutoff. Since the eigenvalues of the covariant derivative are gauge invariant the theory can be regulated in a non perturbative manner by using the bare action

$$S_B[g] = S[g] + \int dx\sqrt{g}g_{\mu\nu}f_{\Lambda}(-D^2)g^{\mu\nu}, \tag{19}$$

where $f_{\Lambda}(z) \approx 0$ and ∞ for $z < \Lambda^2$ and $z > \Lambda^2$, respectively. Smooth cutoff is realized by either the proper time method [19] or by other, simpler continuous function f_{Λ}. The simplest possibility, followed in this work is the sharp cutoff,

$$f_{\Lambda}(z) = \begin{cases} 0 & z < \Lambda^2, \\ \infty & z > \Lambda^2. \end{cases} \tag{20}$$

The expression for the regulated one-loop effective potential for the curvature [14] can be obtained by following the steps leading from Eq. (3) to Eq. (13) with the regulated and gauge fixed path integral,

$$\begin{aligned} \gamma(R) &= S_B[g^{(R)}] + \gamma^{(1)}(R), \\ \gamma^{(1)}(R) &= \frac{\kappa_B^2}{2}\mathrm{T}r\ln\mathcal{K} - \mathrm{T}r\ln\mathcal{M}, \end{aligned} \tag{21}$$

with

$$\begin{aligned} \mathcal{K}_{(\mu\nu),(\rho\sigma)} &= \kappa_B^2\sqrt{g^{(R)}}\left(-Z_{(\mu\nu),(\rho\sigma)}D^2 + U_{(\mu\nu),(\rho\sigma)}\right), \\ \mathcal{M}_{\mu\nu} &= -\sqrt{2}\kappa_B^2\sqrt{g^{(R)}}\left(g^{(R)}_{\mu\nu}D^2 + R_{\mu\nu}\right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} Z_{(\mu\nu),(\rho\sigma)} &= \frac{1}{4}\left(g^{(R)}_{\mu\rho}g^{(R)}_{\nu\sigma} + g^{(R)}_{\mu\sigma}g^{(R)}_{\nu\rho} - g^{(R)}_{\mu\nu}g^{(R)}_{\rho\sigma}\right), \\ U_{(\mu\nu),(\rho\sigma)} &= Z_{(\mu\nu),(\rho\sigma)}(R - 2\lambda_B) + \frac{1}{2}\left(g^{(R)}_{\mu\nu}R_{\sigma\rho} + R_{\mu\nu}g^{(R)}_{\sigma\rho}\right) - \frac{1}{4}\left(g^{(R)}_{\mu\rho}R_{\nu\sigma} + g^{(R)}_{\mu\sigma}R_{\nu\rho} + R_{\mu\rho}g^{(R)}_{\nu\sigma} + R_{\mu\sigma}g^{(R)}_{\nu\rho}\right) \\ &\quad -\frac{1}{2}\left(R_{\mu\rho}R_{\nu\sigma} + R_{\mu\sigma}R_{\nu\rho}\right), \end{aligned} \tag{23}$$

and $R_{\mu\nu}$ is the Ricci tensor of the de Sitter space.

The evaluation of the functional traces on the right hand side will be carried out in the two steps. First we project the fields into subspaces with well defined spin in order to simplify the computation of the spectrum of $\mathcal{K}$ and $\mathcal{M}$. In the second step we construct the spherical harmonics for each spin sector and obtain the functional traces as regulated, finite sums.

C. Spin projection

The computation of the eigenvalues of the quadratic forms of Eq. (22) is facilitated by separating the different spin components of the fields $h_{\mu\nu}$, c_μ and $\bar{c}_\mu$. In the case of the metric the reduction produces traceless transverse (TT) and longitudinal $(LT),(LL)$ fields and the trace (Tr) [20], $10 = 2\times 1(LL+Tr)+3(LT)+5(TT)$, in particular

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + h^{LT}_{\mu\nu} + h^{LL}_{\mu\nu} + h^{Tr}_{\mu\nu} \tag{24}$$

with $h^{Tr}_{\mu\nu} = g^{(R)}_{\mu\nu}\phi$ where $\phi = g^{(R)}_{\mu\nu}h^{\mu\nu}$ together with $h^{LT}_{\mu\nu} = D_\mu\xi'_\nu + D_\nu\xi'_\mu$ and $h^{LL}_{\mu\nu} = D_\mu D_\nu\sigma' - \frac{1}{4}g^{(R)}_{\mu\nu}D^2\sigma'$. These fields satisfy the auxiliary conditions $D_\mu\xi^\mu = 0$, $g^{(R)\mu\nu}h^{TT}_{\mu\nu} = 0$ and $D^\mu h^{TT}_{\mu\nu} = 0$ and are orthogonal with respect to the scalar product [5, 20]

$$\begin{aligned}
\langle h_1|h_2\rangle &= \int dx\sqrt{g}h_{1;\mu\nu}h_2^{'\mu\nu} \\
&= \int dx\sqrt{g}\Big[h^{TT}_{1;\mu\nu}h_2^{TT'\mu\nu} - 2\xi'_{1;\mu}(g^{(R)\mu\nu}D^2 + \bar{R}^{\mu\nu})\xi'_{2;\nu} - 2\xi'_{1;\mu}\bar{R}^{\mu\nu}D_\nu\sigma'_2 - 2\xi'_{1;\mu}\bar{R}^{\mu\nu}D_\nu\sigma'_2 \\
&\quad +\sigma'_1\left(\frac{3}{4}(D^2)^2 + D_\mu\bar{R}^{\mu\nu}D_\nu\right)\sigma'_2 + \frac{1}{4}\phi_1\phi_2\Big]
\end{aligned} \tag{25}$$

where $g^{(R)\mu\rho}g^{(R)\nu\sigma}h_{\rho\sigma} = h^{\mu\nu}$. In order remove a trivial divergent factor from the path integral the zero modes $D_\mu\xi_\nu + D_\nu\xi_\mu = 0$ and $D_\mu D_\nu\sigma - \frac{1}{4}D^2\sigma = 0$ should be removed from the domain of integration in (3). The comparison of the rescaling (8) and the decomposition (24) reveals

$$\Omega^2 = 1 + \phi. \tag{26}$$

For maximally symmetric spaces $\bar{R}^{\mu\nu} = Cg^{(R)\mu\nu}$ and the $\xi - \sigma$ mixing is absent in the scalar product,

$$\langle h_1|h_2\rangle = \int dx\sqrt{g}\left[h^{TT}_{1;\mu\nu}h_2^{TT'\mu\nu} - 2\xi_1'^{\mu}(D^2 + C)\xi'_{2;\mu} + \sigma'_1 D^2\left(\frac{3}{4}D^2 + C\right)\sigma'_2 + \frac{1}{4}\phi_1\phi_2\right] \tag{27}$$

A further linear transformation [14],

$$\tilde{\xi} = \sqrt{-D^2 - C}\tilde{\xi}', \quad \tilde{\sigma} = \sqrt{(D^2)^2 + \frac{4}{3}CD^2}\tilde{\sigma}' \tag{28}$$

simplifies the scalar product to

$$\langle h_1|h_2\rangle = \int dx\sqrt{g}\left[h^{TT}_{1;\mu\nu}h_2^{TT'\mu\nu} + 2\xi_1^\mu\xi_{2;\mu} + \frac{3}{4}\sigma_1\sigma_2 + \frac{1}{4}\phi_1\phi_2\right]. \tag{29}$$

Similar reduction of the ghost fields gives $4 = 1+3$, $c_\mu = c^T_\mu + D_\mu(-D^2)^{-\frac{1}{2}}\rho$ and $\bar{c}_\mu = \bar{c}^T_\mu + D_\mu(-D^2)^{-\frac{1}{2}}\bar{\rho}$, where $D_\mu c^{T\mu} = 0$.

The comparison of the first equation in (25) and (29) reveals that the Jacobians corresponding to the change of integration variables $h \to \Phi$, where $\Phi \in \{h^{TT}, \xi, \sigma, \phi, c^T, \bar{c}^T, \rho, \bar{\rho}\}$ is field and curvature independent. The quadratic part of the actions, given by Eq. (22) reads as

$$\begin{aligned}
S^{(2)}[g^{(R)} + h] + S_{gf}[g^{(R)} + h] &= \kappa_B^2\int dx\sqrt{g}\Big[\frac{1}{2}h^{TT}_{\mu\nu}\left(-D^2 + \frac{2R}{3} - 2\lambda_B\right)h^{TT\mu\nu} \\
&\quad +\xi_\mu\left(-D^2 + \frac{R}{4} - 2\lambda_B\right)\xi^\mu + \frac{3}{8}\sigma(-D^2 - 2\lambda_B)\sigma - \frac{1}{6}\phi(-D^2 - 2\lambda_B)\phi\Big]
\end{aligned} \tag{30}$$

and

$$S_{gh}[g^{(R)}, h = 0, c^T, \bar{c}^T, \rho, \bar{\rho}] = -\sqrt{2}\kappa_B^2\int dx\sqrt{g}\left[\bar{c}^T_\mu\left(D^2 + \frac{R}{4}\right)c^{T\mu} + \bar{\rho}\left(D^2 + \frac{R}{4}\right)\rho\right] \tag{31}$$

in terms of the new variables. Notice that all ϕ modes are unstable when $\lambda_B = 0$ and will be dealt with later.

D. Spherical harmonics

The last ingredient needed for the one-loop result is the spectrum of the Laplace operator D^2, the solution of the eigenvalue condition

$$D^2 t^{(s)}_{\ell m} = -R\lambda^{(s)}_\ell t^{(s)}_{\ell m}, \tag{32}$$

for spherical functions of spin s where $\ell = s, s+1, s+2, \cdots$ and the quantum number $m = 1, \cdots, D^{(s)}_\ell$ labels the degenerate eigenvectors. The spherical functions can be constructed either from homogeneous polynomials [21] or in terms of associated Legendre polynomials [22] with the spectrum

$$\begin{aligned} \lambda^{(s)}_\ell &= \frac{\ell(\ell+3)-s}{12}, \\ D^{(s)}_\ell &= \begin{cases} \frac{(2\ell+3)(\ell+1)(\ell+2)}{6} & s=0(LL,Tr), \\ \frac{(2\ell+3)\ell(\ell+3)}{2} & s=1(LT), \\ \frac{5(\ell+4)(\ell-1)(2\ell+3)}{6} & s=2(TT). \end{cases} \end{aligned} \tag{33}$$

The fields are expressed by means of the spherical harmonics as

$$\Phi = \sum_{\ell=\ell^\Phi_{min}}^{\ell^\Phi_{max}} \sum_{m=1}^{D^\Phi_\ell} c^\Phi_{\ell m} t^\Phi_{\ell m}. \tag{34}$$

The spherical harmonics $t^\Phi_{\ell m}$ are normalized by the scalar product

$$\langle t_1|t_2\rangle = \int dx\sqrt{g}t_1^* t_2, \tag{35}$$

the lower limit for the summation over ℓ is $\ell^\Phi_{min} = 2$ except for $\Phi = \phi$ when $\ell^\phi_{min} = 0$. The upper limit is determined by the sharp cutoff, Eq. (20), therefore the integral measure for c_n is finally defined as

$$D[\Phi] \equiv D[c^\Phi] = \prod_{\ell=\ell^\Phi_{min}}^{\ell^\Phi_{max}} \prod_{m=1}^{D^\Phi_\ell} \Lambda^2 d\tilde{c}^\Phi_{\ell m}. \tag{36}$$

The final expression for the effective potential (21) is

$$\begin{aligned} \gamma^{(1)}(R) &= \frac{1}{2}\sum_{\ell=2}^{\infty} D^{(2)}_\ell \ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(\lambda^{(2)}_\ell + \frac{2}{3} - \frac{2\lambda_B}{R}\right)\right] + \frac{1}{2}\sum_{\ell=2}^{\infty} D^{(1)}_\ell \ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(2\lambda^{(1)}_\ell + \frac{1}{2} - \frac{4\lambda_B}{R}\right)\right] \\ &+\frac{1}{2}\sum_{\ell=2}^{\infty} D^{(0)}_\ell \ln\left[\frac{3\kappa_B^2 R}{4\Lambda^4}\left(\lambda^{(0)}_\ell - \frac{2\lambda_B}{R}\right)\right] - \sum_{\ell=1}^{\infty} D^{(1)}_\ell \ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(\lambda^{(1)}_\ell + \frac{1}{4}\right)\right] - \sum_{\ell=0}^{\infty} D^{(0)}_\ell \ln\left[\frac{\kappa_B^2 R}{\Lambda^4}\left(\lambda^{(0)}_\ell + \frac{1}{2}\right)\right]. \end{aligned} \tag{37}$$

These sums are formal because they are divergent. The regularization (19) consists of including finite number of terms only, for which the arguments of the logarithm functions are less then 1.

III. NUMERICAL RESULTS

The main result of the computation above, the effective potential (37), will now be used first to find out the value of the curvature or the homogeneous conformal factor preferred energetically in the vacuum in the absence of bare cosmological constant, $\lambda_B = 0$, and next to obtain the induced, effective Newton and cosmological constants in de Sitter spaces.

One usually introduces first counterterms which render the effective potential UV finite. Instead, we shall use the effective potential (37) without counterterms because our goal is first to find the true vacuum at which the renormalization program, the R-dependent counterterm set in particular, is supposed to be introduced after. Our procedure corresponds to the minimization of the total action, the sum of the finite pieces and the contributions of the counterterms. We use the bare action in finding the saddle point of the path integral which is always given in terms of the bare, cutoff theory. Once the saddle point is found, then the divergences should be removed in the standard fashion, by splitting the bare action into the sum of renormalized and counterterm parts. The minimization of the effective action will be considered in this paper only. The renormalized coupling constant and the counterterms will not be introduced, instead the value of the curvature and the one-loop improved effective coupling constants will be read of at the saddle point.

A. Effective potential

It is easy to read off a *wrong* result from Eq. (37). When a gauge non-invariant cutoff is employed by cutting off the summation over the spherical harmonics at a given in the angular momentum value ℓ_{max} then on finds

$$\gamma^{(1)ninv}(R) = C' \ln \frac{\kappa_B^2 R}{\Lambda^4}, \tag{38}$$

with $C' > 0$. But one should use gauge invariant regularization which requires in our case the definition of the cutoff in terms of the eigenvalues of the Laplace operator. The difference between sharp cutoff (20) and the previous, non invariant regulator is that as the curvature is increased the number of modes contributing to the regulated path integral is decreased or remains constant, respectively. This makes the effective potential changing faster with the curvature when the gauge invariant regulator is used.

A qualitative estimate of the effective potential can be obtained by exploiting the analogy with the Casimir effect for flat space, summarized from the point of view of our problem in Appendix A. The curvature directly controls the volume of the de Sitter space and indirectly, via the uncertainty principle, the level spacing of the free particle energy spectrum. The former leads to the Casimir effect when sharp cutoff is used and the latter generates the same result for smooth cutoff. The estimate of the one-loop effective potential is based on the equipartition theorem which states that $\gamma(R) \approx \mathcal{N}\bar{f}$ where $\bar{f}$ is the typical contribution of a mode. The factor $\mathcal{N}$ is finite and well defined only if both IR and UV sharp cutoffs are imposed and can easiest be obtained when the dimensions are removed by the UV cutoff as in solid state physics or lattice field theory. In our case $\mathcal{N} \approx \Lambda^4/R^2$. In order to estimate $\bar{f}$ let us recall that the logarithm of the Gaussian integral is the logarithm of the typical, average value of the integral variable, apart of an uninteresting $\ln 2\pi$ term. Since the variables of the path integral are identified by removing every dimension by the UV cutoff the contribution of a mode is $\ln \kappa_B/\Lambda$ which gives $\gamma(R) \approx c\Lambda^4/R^2 \ln \kappa_B^2/\Lambda^2$. Naturally this naive estimate, based on the typical contribution, is reliable for $\mathcal{N} >> 1$ only.

The numerical evaluation of the the effective potential obtained by means of the sharp cutoff is depicted in Fig. 1. The highest value of ℓ contributing to the effective potential (37) is approximately $4 \cdot 10^4$ (small curvature) and 10 (high curvature) along the curves of this Figure. The flat space Casimir effect for a dimensionless scalar field suggests the form

$$\gamma^{(1)}(R) = \begin{cases} M^4(\Lambda)\left(\frac{1}{R^2} - \frac{1}{R^2(\Lambda)}\right) & R \le R(\Lambda) \\ 0 & R > R(\Lambda) \end{cases} \tag{39}$$

with

$$M^4(\Lambda) = c_1 \Lambda^4 \ln \frac{c_2 \kappa_B^2}{\Lambda^2}, \tag{40}$$

and $R(\Lambda) = c_3\Lambda^2$. The second term in the parentheses on the right hand side of Eq. (39) is to cancel the effective potential for high enough curvatures when no eigenvalues are left in the sum by the sharp cutoff. The effective potential is naturally not a smooth function for $R \approx R(\Lambda)$ anymore and one should not consider the theory in this regime where few modes are left only. The fit of the numerical results yields $c_1 \approx 7.201$, $c_2 \approx 2.989$ and $c_3 \approx 0.665$, c.f. Fig. 2.

Let us now consider the complete effective potential

$$\gamma(R) = -\frac{v\kappa_B^2}{R} + c_1\Lambda^4\left(\frac{1}{R^2} - \frac{1}{R^2(\Lambda)}\right)\ln\frac{c_2\kappa_B^2}{\Lambda^2}, \tag{41}$$

where

$$V = \int dx\sqrt{g} = \frac{v}{R^2} \tag{42}$$

is the four volume of the de Sitter space with $v = 3200\pi^2/3$. One finds a quantum phase transition at $\kappa_B^2 = \kappa_{cr}^2 = \Lambda^2/c_2$ when R_{min}, the curvature where $\gamma(R)$ reaches its minimum changes in a discontinuous manner. Deeply inside of the small cutoff phase, $\kappa_{cr}^2 << \kappa_B^2$, the energetically favored curvature is

$$R_{min} = \frac{2c_1\Lambda^4}{v\kappa_B^2}\ln\frac{c_2\kappa_B^2}{\Lambda^2} \tag{43}$$

For large cutoff, $\kappa_B^2 < \kappa_{cr}^2$, the minimum of the effective potential is reached at $R_{min} = 0$.

It is easy to reconstruct the effective potential for the homogeneous conformal mode, $\phi(x) = \Phi$, too. In fact, all needed according to Eq. (26) is the replacement $R \to R/(1+\Phi)$ in the effective potential. Therefore the small or the large cutoff phase prefers $\Phi = R/R_{min} - 1$ or $\Phi = \infty$, respectively. But on should keep in mind that we can not settle the problem of the instability by studying the dynamics of the homogeneous conformal mode only. In fact, the instability of the inhomogeneous and the homogeneous conformal modes comes from the kinetic energy (first) or the potential energy (second) term on the right hand side of Eq. (9). All one can say is that as long as the theory is IR finite the homogeneous mode has a useful diagnostic role, the long wavelength inhomogeneous modes follow a dynamics which is similar to those of the homogeneous mode.

B. Induced cosmological constant

The effective potential (41) suggests the effective action

$$\Gamma[g] = -\kappa_{eff}^2\int dx\sqrt{g(x)}F(R(x)), \tag{44}$$

where $F(R)$ is a polynomial of the curvature. In order to define a unique coupling constant κ_{eff}^2 we impose the condition

$$\frac{dF(R)}{dR}_{|R=0} = 1 \tag{45}$$

on $F(R)$. The inspection of Eq. (41) yields $F(R) = R - 2\lambda - gR^2$ with $\kappa_{eff}^2 = \kappa_B^2$,

$$\lambda = \frac{c_1\Lambda^4}{2v\kappa_B^2}\ln\frac{c_2\kappa_B^2}{\Lambda^2} \tag{46}$$

and

$$g = -\frac{c_1\Lambda^2}{2c_3 v\kappa_B^2}\ln\frac{c_2\kappa_B^2}{\Lambda^2} \tag{47}$$

for $R << \Lambda^2$. Notice that the fourth power of the cutoff in Eq. (46) leads to a constant counterterm by dimensional reason. It indicates that the vacuum fluctuations contribute to the action in a homogeneous manner in space-time which is a manifestation of the Casimir effect, known from the normal ordering prescription in Quantum Field Theory. The one-loop quantum fluctuations leave the Newton constant unchanged but generate a cosmological term and a higher derivative piece.

We see that the difference between the small and large cutoff phase is that the induced cosmological constant is positive and negative, respectively. Furthermore there is a higher order derivative term generated with coupling of the opposite sign as the cosmological constant. The expressions (30) and (37) indicate that the de Sitter background is unstable in the one-loop effective theory for $\lambda > cR$ where

$$c = \min\left(\frac{\lambda_\ell^{(2)}}{2} + \frac{1}{3}, \frac{\lambda_\ell^{(1)}}{2} + \frac{1}{8}, \frac{\lambda_\ell^{(0)}}{2}\right) = \frac{1}{2}. \tag{48}$$

The low cutoff phase is therefore unstable for $R < R_{min}/2$ and the entire large cutoff phase is stable as expected.

C. Running Newton constant

In order to shed more light onto the nature of the two phases we introduce the effective strength of interaction by using the ansatz

$$\Gamma[g] = -\int dx\kappa^2(R(x))\sqrt{g(x)}R(x), \tag{49}$$

where the space-time dependence is explicitly shown. The effective Newton constant, found by comparing Eqs. (41) and (49),

$$G(R) = \frac{1}{16\pi\kappa^2(R)} = \frac{1}{16\pi\kappa_B^2}\frac{1}{1-\frac{c_1\Lambda^4}{v\kappa_B^2}\left(\frac{1}{R}-\frac{R}{R^2(\Lambda)}\right)\ln\frac{c_2\kappa_B^2}{\Lambda^2}} \tag{50}$$

is a measure of the strength of interactions. In fact, $\kappa^2(R)/\kappa_B^2$ measures the suppression of small fluctuations of the metric tensor, Δg, compared to the classical Einstein theory with vanishing cosmological constant. The fluctuations of the one-loop effective theory are suppressed at length scales $\ell >> \sqrt{G}$ where $\sqrt{G}$ "seems small". In fact, simple order of magnitude estimate gives $\Delta g \approx \ell_{Pl}/\ell$ for the typical fluctuations of the metric tensor of length ℓ where $\ell_{Pl} = \sqrt{G}$ denotes the renormalised Planck length. The running Newton constant has a central importance in the phenomenology of quantum effects in gravity. A distance dependent Newton constant may provide a solution to the astrophysical missing mass problem [23, 24].

The tree level theory is weakly coupled when the cutoff is increased as long as we stay in the small cutoff phase. Only the modes beyond the Planck energy which appear in the large cutoff phase are strongly coupled. This feature is changed drastically by the leading order loop corrections because the one-loop theory on the de Sitter space of curvature R is perturbative for length scales

$$\ell^2 > \ell_{Pl}^2(R) = \frac{\ell_B^2}{1+\frac{16\pi c_1}{v}\Lambda^4\ell_B^2\left(\frac{1}{R}-\frac{R}{R^2(\Lambda)}\right)\ln\frac{16\pi\ell_B^2\Lambda^2}{c_2}}, \tag{51}$$

where $\ell_B^2 = 1/16\pi\kappa_B^2$ and the whole theory is perturbative if $\ell_{Pl}\Lambda < 1$,

$$\Lambda^2\ell_B^2 < 1+\frac{16\pi c_1}{v}\Lambda^4\ell_B^2\left(\frac{1}{R}-\frac{R}{R^2(\Lambda)}\right)\ln\frac{16\pi\ell_B^2\Lambda^2}{c_2}. \tag{52}$$

In order to simplify the discussion we shall restrict ourselves into regions far away form the transition region, deeply in the small and large cutoff phases, for $\Lambda^2 << \ell_B^{-2}$ and $\ell_B^{-2} << \Lambda^2$, respectively. There is an IR Landau pole in the small cutoff phase at $R_L = R_{min}/2$ and $G(R)$ is decreasing from infinity to G as R is increased from R_L to $R(\Lambda)$. At the energetically preferred curvature we find $G(R_{min}) = 2G$ and $\ell_{Pl}^2(R_{min}) = 2\ell_B^2$. All modes are non perturbative and the theory strongly coupled unless $R > R_L$. The modes become perturbative rapidly as R is increased beyond the Landau pole and the entire theory is perturbative for $R_{min} \le R$. The running Newton constant is an increasing function of the curvature in the large cutoff phase and one finds

$$G(R) \approx \frac{Rv}{16\pi c_1\Lambda^4\ln\frac{16\pi\ell_B^2\Lambda^2}{c_2}} \tag{53}$$

for $R < R(\Lambda)$ and $G(R(\Lambda)) = G_B$. In a similar manner $\ell_{Pl}(R)$ increases with R, $\ell_{Pl}^2(R) \approx R/\Lambda^4$ as long as $R < R(\Lambda)$ and $\ell_{Pl}^2(R(\Lambda)) = \ell_B^2$. All modes appear perturbative in the effective theory for $R << \Lambda^2$.

The peculiar sensitivity of quantum gravity on the UV sector is reflected in the expression (50), too. When the running coupling constant is computed in flat space-time in the presence of a background field, such as in Yang-Mills theories [6], the cutoff appears in the argument of the logarithmic function only. As mentioned in connection of Eq. (46), that the factor Λ^4 in front of the logarithimc function implies a constant, field independent counterterm, the hallmark of the Casimir effect.

IV. DYNAMICS OF THE CONFORMAL MODES

The unstable conformal modes have been kept frozen at $\Omega = 1$ in the one-loop computation. The complete functional integral can be obtained by factorizing the integral measure for $g_{\mu\nu} = g_{\mu\nu}^{(R)}(1+\phi) + h'_{\mu\nu}$, where h' is obtained by retaining the first three contributions only in the right hand side of Eq. (24), as $D[g] = D[\phi]D[h']$. The regulator is defined as before by means of the eigenvalues of the $\Box$ operator. The integration over h' in Eq. (3) yields

$$e^{W[\phi,j]} = \int D[h']e^{-S[g]+\int dx j^{\mu\nu} g_{\mu\nu}}. \tag{54}$$

The Legendre transformation for a given ϕ configuration is defined by

$$\Gamma[\phi, g] = -W[\phi, j] + \int dx j^{\mu\nu} g_{\mu\nu} \tag{55}$$

with

$$g_{\mu\nu} = g^{(R)}_{\mu\nu}(1 + \phi) + h'_{\mu\nu} = \frac{\delta W[\phi, j]}{\delta j_{\mu\nu}}, \tag{56}$$

cf. Eqs. (5) and (6).

Notice that the dependence on ϕ enters explicitly and through g in the Legendre transform (55). The effective actions appearing in Eqs. (44) or (49) are actually $\Gamma[\phi, g]$, computed at $\phi = 0$. The essence of the anstaz is the upgrading of a simple one-loop result obtained on a fixed de Sitter geometry to a gauge (diffeomorphism) invariant functional. We extend this procedure by requiring the usual transformation rules under conformal transformations, too. This assumtion suppresses the explicit dependence on ϕ and gives rise to

$$e^{W[j]} = \int D[\phi]e^{-S_c[\phi]+\int dx j^{\mu\nu} g_{\mu\nu}} \tag{57}$$

with $S_c[\phi] = \Gamma[0, g]$ for the complete generating functional.

The effective action $S_c[\phi]$ is simpler in terms of $\Omega^2 = 1 + \phi$ and the functional, taken from Eq. (44), leads to

$$S_c[\phi] = \kappa^2 \int dx\sqrt{g}\left[6(1 - 2gR\Omega^{-2})\Omega\Box\Omega + 36g\Omega^{-2}(\Box\Omega)^2 + V(\Omega)\right] \tag{58}$$

according to Eq. (8), where

$$V(\Omega) = -R\Omega^2 + 2\lambda\Omega^4 + gR^2. \tag{59}$$

The ansatz (44) and the assumption about the proper conformal properties amount to a partial resummation of the loop-expansion for the stable modes and the dependence of the effective action on the conformal factor Ω is supposed to become more reliable.

It is easy to identify the stabilization mechanism (ii) mentioned in the Introduction in the low cutoff phase where the potential $V(\Omega)$ is bounded from below ($\lambda > 0$). We write $\Omega = \bar\Omega + \omega$, where the potential fixes $\bar\Omega^2 = R_{min}/4\lambda = 1$. The instability of the inhomogeneous modes is still visible because both the $\mathcal{O}(\Box)$ and the $\mathcal{O}(\Box^2)$ terms are negative definite because $g < 0$. But the increase of the wave number is not a real instability in theories with finite cutoff. In fact, models with unstable dispersion relation are stabilised by the UV regulator and give rise stable, modulated vacuum [25] whose characteristic scale is of the cutoff. The dispersion relations for the excitations above such a vacuum have several branches as in solids. The heavy modes, optical phonons, decouple from the low energy dynamics which is dominated by the light branches, acoustic phonons, and the inhomogeneity of the vacuum is not resolved by measurements with finite resolution for sufficient large values of the cutoff.

But the fate of the high cutoff phase remains unclear. The potential $V(\Omega)$ in unbounded from below and the increase of the amplitude of Ω, the decrease of the curvature and the increase of the volume, remains as instability of the one-loop dynamics. The complication here is that the prefered vacuum of the one-loop dynamics, $R = 0$, can not be reached in the scheme presented here which is valid for $R > 0$. Furthermore, fluctuations around $R = 0$ explore the negative curvature geometries, as well, where the dynamics is not given by the analytical continuation of our formulae.

V. SUMMARY AND CONCLUSION

We argue in this paper that the instability of Euclidean Einstein gravity is an indication of a non trivial vacuum structure whose description requires non perturbative methods. As a simple step in this direction the one-loop renormalization was considered on an Euclidean de Sitter background geometry by suppressing the unstable modes.

The similarity of this problem with the Casimir effect was emphasized together with the role of the curvature as an IR cutoff which makes quantum gravity specially sensitive to the UV regime. The effective action was minimized within the family of homogeneous spaces in order to identify the vacuum.

Two phases were found and the effective strength of interaction behaves in a manner which is contrary to what the tree-level dynamics would suggest. In the small cutoff phase, $\Lambda << 1/\sqrt{\ell_B}$, the conformal modes are stabilised by the one-loop effective action for the other tree-level stable modes and we find a close analogy with the Savvidy vacuum of Yang-Mills theories. The system prefers a non vanishing curvature, $R_{min} \approx \Lambda^4 \ell_B^2$ and $\ell_{Pl} = \sqrt{2}\ell_B$. The tree-level theory is weakly coupled for arbitrary curvature but the fluctuations acquire large amplitude at length scales $\ell > 1/\sqrt{R_{min}}$ and the theory is weakly coupled for $R_{min} \leq R$ only when these modes are excluded due to the smallness of the space-time volume. Deeply in the large cutoff phase when the cutoff is increased far beyond the transition point then the minimum of the effective action is reached at $R_{min} \approx 0$, the fluctuations are suppressed. But the stability of the conformal modes remain an open question.

We proposed two parametrizations for the effective action. The running Newton constant can be defined in the effective theory without cosmological term which amounts to assuming formally massless quantum fluctuations. This coupling constant determines the length scale below which the quantum fluctuations are strong. Another possibility is to allow a radiative correction induced cosmological term explicitly in the effective action. This choice is motivated by the proportionality of the Casimir effect with the space-time volume and is more physical because it produces slower dependence on R in the effective parameters than the previous case. Furthermore, a higher order derivative term is recovered by means of this parametrization. The effective Newton constant agrees with the bare one in this scheme and the two phases are distinguished by the sign of the induced cosmological constant and the coefficient of the higher order derivative term.

The change of the UV cutoff while the other parameters are kept constant modifies the physical content of the theory and even drives the system through a phase transition. The cutoff dependence in general is not surprising in a nonrenormalizable theory. It might turn out that some so far unknown nonperturbative effects render the Einstein-Hilbert theory renormalizable but in lacking these elements one has to accept that the theory which is nonrenormalizable in the framework of the loop-expansion produces cutoff-dependent results at one-loop level. Our conclusions drawn correspond to energy scales well below the cutoff and are therefore independent of the details of the regularization procedure since this latter is always represented by irrelevant operators.

The cutoff dependence of nonrenormalizable models usually remains quantitative. The strong cutoff-effects reported in this work can be traced back to the different sign of the induced cosmological constant in the two phases. This parameter is renormalizable and its sign influences the physics in a more substantial, quantitative level.

We encountered two level of instabilities in this work. In the tree-level theory the conformal modes are unstable but they seem to be stabilized by the quantum fluctuations at least in the long distance region of the high cutoff phase. The one-loop improved effective theory remains unstable in the low cutoff phase. The cosmological constant appears formally as the parameter $-m^2$ in a scalar model. The instability of the low cutoff phase with $\lambda > 0$ should therefore be similar to those of a scalar model with spontaneously broken symmetry. In the latter case the tree-level contributions to the renormalization group equation were found to be crucial [26] in order to establish a systematical approach to the mixed phase of first order phase transitions, in particular the spinodal phase separation [27] where the soft modes arising from the breakdown of space-time symmetries by the inhomogeneous saddle points generate strong nonperturbative effects, among others the Maxwell-cut.

We continue with few comments on the results of the previous Section from the point of view of the case (ii) of the stabilization mechanism, mentioned in the Introduction and identified in the low cutoff phase. It was found that the one-loop effective action may stabilize the vacuum and generate a condensate of the metric tensor even in the absence of matter. It remains to be seen if such a condensate involves the conformal modes only or the spin 2 components, too. Another open question is whether the new vacuum with condensate is weakly or strongly coupled. The dimensionless ratio κ_B/Λ available in this nonrenormalizable theory may generate small or large dimensionless numbers and this question can be settled by detailed computation only. Though the other stabilization mechanisms were not pursued in this work it is clear that the excitations above the condensed vacuum produced by either mechanism listed in the Introduction are different than those above the naive, 'empty' vacuum and the success of general relativity in the classical domain requires more careful justification.

One arrives at a surprising conjecture by following the analogy offered by the scalar model. It is well known that the speed of sound is vanishing in the mixed phase. This is due to the strongly coupled soft modes, the translations of the domain walls. In more physical terms the domain walls absorb the sound. In a similar manner one expects the speed of the propagation of gravitons be vanishing in the low cutoff phase. This situation where the naive plane waves ceases to propagate and drop out from the asymptotical sector is called confinement. If this mechanism is realized then gravitons will be confined in the low cutoff phase due to the quantum liquid nature of the vacuum. The confinement radius which can be approximated by the length scale of the IR Landau pole is in the order of magnitude of the size of the stable de Sitter space, indicating simply that the gravitons should be confined by the horizon.

Finally we mention other natural extensions of this work, the inclusion of matter fields and the cosmological constant in the bare action. Condensate modifies the vacuum structure and it may stabilise the gravitational sector by breaking conformal invariance. The interplay of condensates in the coupled Einstein-Yang-Mills system is a particularly interesting problem. The bare cosmological constant, included from the very beginning in bare action should preserve the topology of the phase structure but make the result more realistic. We plan to report on such results soon.

Acknowledgments

We wish to acknowledge a stimulating discussion with G. Veneziano.

APPENDIX A: CASIMIR EFFECT IN A BOX

It is instructive to consider a dimensionless, real, free, massless scalar field ϕ in a box of linear size L of the Euclidean space-time subject of periodic boundary conditions. The generator functional for connected Green functions is

$$e^{W[j]} = \int D[\phi] e^{\frac{\mu^2}{2}\int dx \phi_x \Box \phi_x + \int dx j_x \phi_x}. \tag{A1}$$

The effective action

$$\Gamma[\phi] = -\frac{\mu^2}{2}\int dx \phi_x \Box \phi_x + \frac{1}{2}\mathrm{T}r \ln \mu^2 \Box \tag{A2}$$

takes the value

$$\Gamma[0] = \frac{1}{2}\mathrm{T}r \ln \mu^2 \Box \tag{A3}$$

in the vacuum.

The evaluation of the formal functional trace requires regularization. We choose lattice regularization and write

$$\phi = \frac{a^2}{L^2}\sum_n c_n e^{i\frac{2\pi}{L}n_\mu x^\mu}, \tag{A4}$$

where a denotes the lattice spacing and the summation is restricted for

$$0 < n^2 \frac{(2\pi)^2}{L^2} < \Lambda^2 = \left(\frac{2\pi}{a}\right)^2 \tag{A5}$$

and the regulated integral measure is given by

$$D[c] = \prod_n dc_n. \tag{A6}$$

One finds finally

$$\Gamma[0] = \frac{1}{2}\sum_{n^\mu \neq 0} \Theta\left(\Lambda^2 - n^2 \frac{(2\pi)^2}{L^2}\right) \ln\left(\mu^2 n^2 \frac{(2\pi)^6}{L^2 \Lambda^4}\right). \tag{A7}$$

One can simplify this expression for $1 << \Lambda L$ as

$$\begin{aligned} \Gamma[0] &= L^4 \int_{|p|\leq\Lambda} \frac{d^4p}{(2\pi)^4} \ln \frac{p^2 \mu^2 (2\pi)^4}{\Lambda^4} \\ &= \frac{L^4 \Lambda^4}{32\pi^2}\left(\ln \frac{\mu^2 (2\pi)^4}{\Lambda^2} - \frac{1}{2}\right). \end{aligned} \tag{A8}$$

[1] S. Adler, Rev. Mod. Phys.**54**,729(1982).

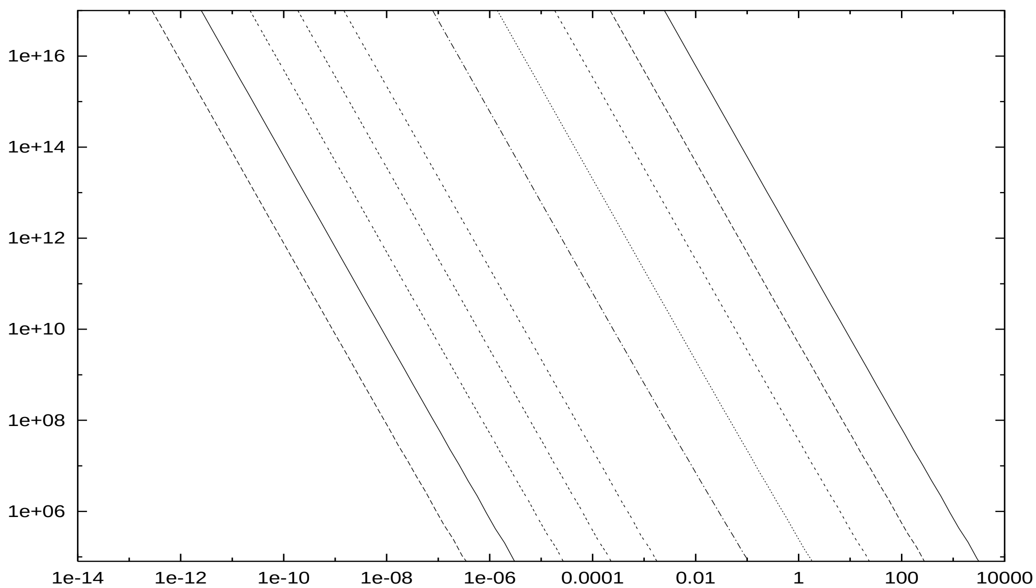

FIG. 1: The effective potential $|\gamma(R)|$ as a function of R/κ_B^2, obtained with sharp cutoff. The lines belong from left to right to $\Lambda^2/\kappa_B^2 = 10^{-5}$, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3, 10^4 and 10^5. Since $\gamma(R) < 0$ in the given range of curvature for $\Lambda^2/\kappa_B^2 \geq 1$ it is $-\gamma(R)$ which is shown for these values of the cutoff. The slightly bigger distance between the curves of $\Lambda^2/\kappa_B^2 = 0.1$ and $\Lambda^2/\kappa_B^2 = 1$ when the sign changes witnesses the logarithmic function in Eq. (41).

[2] V. Moncrief, J. Math. Phys.**16**,493(1975); J. Math. Phys.**16**,1556(1975); J. Math. Phys.**17**,1893(1976).

[3] S. Hawking, in *General Relativiy: An Einstein Centenary Survey*, ed. S. Hawking, W. Israel (Cambridge Univ. Press, Cambridge, 1979).

[4] J. Greensite, Nucl. Phys.**B361**,729(1991); Phys. Let.**B291**,405(1992); Nucl. Phys.**B390**,439(1993).

[5] P. Mazur, E. Mottola, Nucl. Phys.**B341**,187(1990); E. Mottola, J. Math. Phys.**36**,2470(1995).

[6] I. A. Batalin, S. G. Matinyan, G. K. Savvidy, Sov. J. Nucl. Phys.**26**,214(1977); G. K. Savvidy, Phys. Lett.**71B**,133(1977).

[7] B. A. Berg, Phys. Rev. Lett.**55**,904(1985); Phys. Lett.**B176**,39(1986); H. W. Hamber, Phys. Rev.**D45**,507(1992); Nucl. Phys.**B400**,1993(347); E. Bittner, W. Janke, H. Markum, hep-lat/0311031.

[8] S. Deser, M. J. Duff, C. J. Isham, Nucl. Phys.**B111**,45(1976); N. D. Birell, P. C. Davies, *Quantum Fields in Curved Space* (Cambridge University, Cambridge, 1982).

[9] I. Antoniadis, E. Mottola, Phys. Rev.**D45**,2013(1992); S. D. Odintsov, Z. Phys.**C54**,531(1992).

[10] C. P. Burgess, Living Rev. Rel.**7**,5(2004), gr-qc/0311082.

[11] J. Donoghue, Phys. Rev. Lett.**72**,2996(1994); I. J. Muzinich, S. Vokos, Phys. Rev.**D52**,3472(1995).

[12] E.Elizalde, C. O. Lousto, S. D. Odintsov, A. Romeo, Phys. Rev.**D52**,2202(1995).

[13] J. Alexandre, J. Polonyi, Ann. Phys.**288**,37(1999).

[14] M. Reuter, Phys. Rev.**D57**,971(1998); O. Lauscher, M. Reuter, Phys. Rev.**D65**,025013(2002).

[15] D. Litim, Phys. Rev. Lett.**92**,201301(2004).

[16] S. Falkenberg, S. D. Odintsov, *Gauge Dependence of the Effective Average Action in Einstein Gravity*, hep-th/9612019.

[17] T. R. Taylor, G. Veneziano, Nucl. Phys.**B345**,210(1990).

[18] V. M. Mostepanko, N. N. Trunov, *The Casimir Effect and its Application*, Claderon Press, Oxford, 1997; K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero-Point Energy*, World Scientific, River Edge, 2001.

[19] J. Schwinger, Phys. Rev.**82**,664(1951).

[20] J. York, J. Math. Phys.**14**,456(1973).

[21] E. Lifshitz, J. Phys. (Moscow)**10**,116(1946); E. Lifshitz, I. Khalatnikov, Adv. Phys.**12**,185(1963); G. Gibbons, M. Perry, Nucl. Phys.**B146**,90(1978); M. Rubin, C. Ordonez, J. Math. Phys.**25**,2888(1984).

[22] A. Higuchi, J. Math. Phys.**28**,1553(1987).

[23] M. Reuter, H. Weyer, *Do we observe Quantum Gravity Effects at Galactic Scales?*, astro-ph/0509163; in the Proceedings of the 21. IAP meeting on *Mass Profiles and Shapes of Cosmological Structures*, Paris, July, 2005.

[24] B. Qin, U. L. Pen, J. Silk, *Observational Evidence for Extra Dimensions from Dark Matter*, astro-ph/0508572.

[25] G. Gallavotti, V. Rivasseau, Phys. Lett.**B122**,268(1983); J. L. Alonso, J. M. Carmona, J. Clemente Gallardo, L. A. Fernandez, D. Iniguez, A. Tarancon, C. L. Ullod, Phys. Lett.**B376**,148(1996); V. Branchina, H. Mohrbach, J. Polonyi, Phys. Rev.**D60**,45006(1999).

[26] J. Alexandre, V. Branchina, J. Polonyi, Phys. Lett.**445B**,351(1999).

[27] H. Metiu, K. Kitahara, J. Ross in *Fluctuation Phenomena*, Eds. E. W. Montroll, J. L. Lebowitz, North-Holland, Amsterdam, 1976.

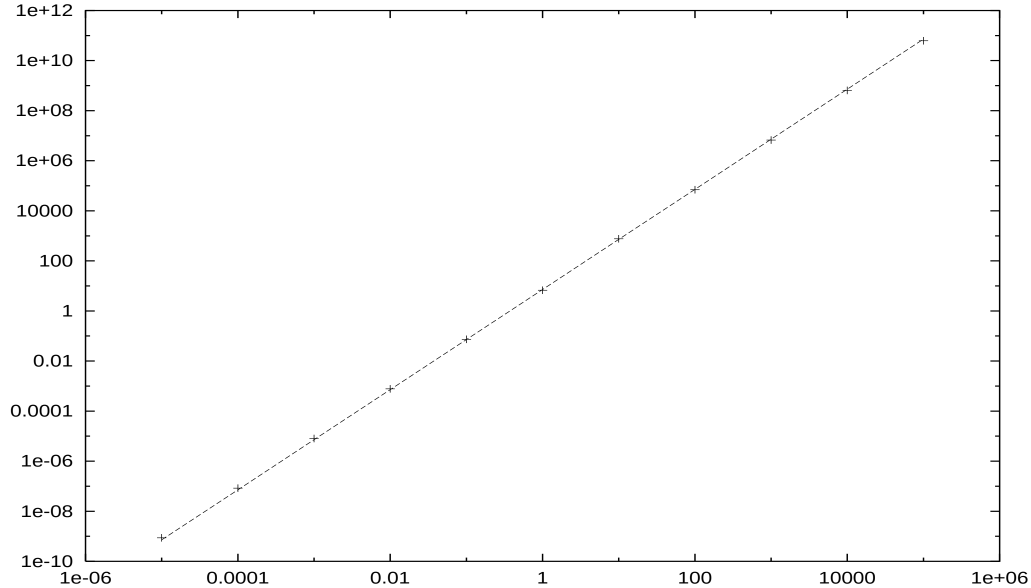

FIG. 2: The numerical and the fitted values for $M^4(\Lambda)/\ln c_2\kappa_B^2/\Lambda^2$ are shown by dots and continuous line, respectively, as functions of Λ^2/κ_B^2.