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The index theorem in QCD with a finite cut-off [1]

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Abstract

The fixed point Dirac operator on the lattice has exact chiral zero modes on topologically non-trivial gauge field configurations independently whether these configurations are smooth, or coarse. The relation $n_L - n_R = Q^{\mathrm{FP}}$, where n_L (n_R) is the number of left (right)-handed zero modes and Q^{FP} is the fixed point topological charge holds not only in the continuum limit, but also at finite cut-off values. The fixed point action, which is determined by classical equations, is local, has no doublers and complies with the no-go theorems by being chirally non-symmetric. The index theorem is reproduced exactly, nevertheless. In addition, the fixed point Dirac operator has no small real eigenvalues except those at zero, i.e. there are no 'exceptional configurations'.

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In the continuum the Dirac operator of massless fermions in a smooth background gauge field with non-zero topological charge has zero eigenvalues. The corresponding eigenfunctions are chiral and the number of left- and right-handed zero modes are related to the topological charge of the gauge configuration: $n_L - n_R = Q$ [1]. It is believed by many that this theorem has important consequences on the low energy properties of QCD. A possible intuitive picture of a typical gauge configuration in QCD is that of a gas or liquid of instantons and anti-instantons with quantum fluctuations [2]. For such configurations, the Dirac operator is expected to have a large number of quasi-zero modes which could be responsible for spontaneous chiral symmetry-breaking [3].

Typical gauge field configurations contributing to the path integral are not smooth, however. In addition, the regularization which is necessary to define the quantum field theory breaks some other conditions of the index theorem as well. Standard lattice formulations, for example, violate the conditions of the index theorem in all possible ways: the topological charge of coarse gauge configurations is not properly defined and the Dirac operator breaks the chiral symmetry in an essential way. In the continuum limit these problems disappear, while close to the continuum some trace of the index theorem can be identified [4]. In general, however, the expected chiral zero modes are washed away and, even worse, unwanted real modes occur as lattice artifacts ('exceptional configurations' [3]) [5].

We are going to show here that a lattice formulation of QCD working with the fixed point (FP) action (classically perfect action) [6–11] and FP topological charge [12, 13] which are defined in the context of Wilson's renormalization group theory [14], solves all these problems: the index theorem remains valid on the lattice even if the cut-off is small (the resolution is poor). We find it amazing that a theorem which is based on differential geometry and topology finds its validity in a context where none of these notions seem to be defined.

The FP action in QCD is determined by classical equations. Many aspects of these equations and of their solutions have been studied earlier. In particular, it can be shown that the symmetries of the continuum theory (infinitesimal translation, rotation, chiral transformation, etc.) are realized on the *solutions* of the FP equations of motion although the FP lattice action itself is not manifestly invariant [11]. Actually, the index theorem on the lattice can be demonstrated by using these FP equations to connect the lattice problem with that of the continuum, where the Atiyah-Singer theorem is valid [10]. Here we shall follow another way which allows to prove the index theorem on the lattice directly and leads to additional results as well.

We shall first give the essential steps of the argument and then present some details in the second part of the paper.

The fermion part of the FP action can be written as [4]

$$S_{\mathrm{f}}^{\mathrm{FP}}(\bar{\psi}, \psi, U) = \sum_{n,n'} \bar{\psi}_n h_{nn'}(U) \psi_{n'} \,, \tag{1}$$

where $h_{nn'}(U)$ is a specific lattice form of the Dirac operator. In eq. (1) the colour and Dirac indices are suppressed. We consider one flavour, $N_f = 1$. The FP Dirac

[3] We call a configuration exceptional if, due to some real eigenvalue close to the origin, the quark propagator becomes singular when the bare mass is still distinctly different from its critical value.

[4] We use dimensionless quantities everywhere. The dimensions are carried by the lattice unit a.

operator is local, has no doublers, satisfies the hermiticity property

$$h^\dagger = \gamma_5 h \gamma_5 \,, \tag{2}$$

and, in complying with the Nielsen-Ninomiya theorem [15], it is not chiral invariant. This is realized, however, in a very special way — the chiral symmetry breaking part of the fermion propagator is given by

$$\frac{1}{2}\{h^{-1}_{nn'}(U), \gamma_5\}\gamma_5 = R_{nn'}(U)\,, \tag{3}$$

where $\{,\}$ denotes the anticommutator. The hermitian matrix $R_{nn'}(U)$ is trivial in Dirac space and it is *local*. This chiral symmetry breaking term appears only due to the non-chiral-invariant blocking transformation of the fermion fields, and not by an ad hoc term added to the action. Due to the locality of R the chiral properties of the propagating quark are not affected by this breaking term. The precise form of R depends on the block transformation whose FP we are considering. There are block transformations which give simply $R_{nn'} = \text{const}\cdot\delta_{nn'}$. When the averaging extends over the hypercube only (as it is the case for all the block transformations considered so far), then $n-n'$ extends only over the hypercube as well, but in general R has an extension of $O(a)$. Eq. (3) with a local R is a highly non-trivial property of the FP action. Indeed, $h^{-1}(U)$ is the propagator in the background gauge field U, hence it is non-local. For a general case, its chiral symmetry breaking part is expected to be non-local as well. For the standard Wilson action and for the currently used other improved actions R is non-local, indeed.

Ginsparg and Wilson observed a long time ago that eq. (3) with a local R is the mildest way a local action can break chiral symmetry [16]. They have also demonstrated that the triangle anomaly for $U = 1$ (free fermions) is correctly reproduced in this case. This work remained, however, largely unnoticed since no solution was known to the remnant chiral symmetry condition in eq. (3) for QCD. The FP action is a solution as we shall demonstrate in the second part of this paper (see also in [10]).

To avoid some singularities at intermediate steps of the calculation we introduce a small quark mass which will be sent to zero at the end:

$$h_{nn'}(U) \to \hat{h}_{nn'}(U) = h_{nn'}(U) + m_{\mathrm{q}}\,\delta_{nn'}\,. \tag{4}$$

Due to eq. (3) the chiral symmetry breaking part of $\hat{h}$ satisfies the relation

$$\frac{1}{2}\{\hat{h}_{nn'}, \gamma_5\}\gamma_5 = (h^\dagger R h)_{nn'} + m_q\,\delta_{nn'}\,. \tag{5}$$

This equation will play a basic role in deriving the following results:

I. The expectation value of the divergence of the singlet axial vector current in a background gauge field has the form

$$\langle \bar{\nabla}_\mu J^5_\mu(n)\rangle = -\langle \bar{\psi}_n \gamma_5 \left(h^\dagger R h \psi\right)_n + \left(\bar{\psi} h^\dagger R h\right)_n \gamma_5 \psi_n\rangle - 2m_{\mathrm{q}}\langle \bar{\psi}_n\gamma_5\psi_n\rangle\,. \tag{6}$$

In eq. (6) the expectation value is defined as

$$\langle \mathcal{O}(\bar{\psi},\psi,U)\rangle = \frac{1}{Z(U)}\int D\bar{\psi}D\psi\,\mathcal{O}(\bar{\psi},\psi,U)\exp\{-\sum_{m,n}\bar{\psi}_m\hat{h}_{mn}(U)\psi_n\}\,, \tag{7}$$

where $Z(U)$ is given by the analogous integral without the $\mathcal{O}(\bar{\psi}, \psi, U)$ factor. In the $m_{\mathrm{q}} \to 0$ limit [5], the last term in eq. (6) is dominated by the zero modes (if there are any)

$$\lim_{m_{\mathrm{q}} \to 0} m_{\mathrm{q}} \langle \bar{\psi}_n \gamma_5 \psi_n \rangle = \sum_{j=1}^{n_{\mathrm{L}}} \psi_{\mathrm{L}}^{(j)*}(n) \psi_{\mathrm{L}}^{(j)}(n) - \sum_{i=1}^{n_{\mathrm{R}}} \psi_{\mathrm{R}}^{(i)*}(n) \psi_{\mathrm{R}}^{(i)}(n) \,, \tag{8}$$

where $\psi_R^{(i)}(n)$, $\psi_L^{(j)}(n)$ are the normalized $\lambda = 0$ right- and left-handed eigenfunctions of the eigenvalue problem of the Dirac operator

$$\sum_{n'} h_{nn'}(U) \Psi_{n'} = \lambda \Psi_n \,. \tag{9}$$

II. Summing over n in eq. (6) one obtains

$$0 = 2\,\mathrm{Tr}(\gamma_5 h R) + 2(n_R - n_L) \,, \tag{10}$$

where the trace Tr is over colour, Dirac and configuration space. The first term is a pseudoscalar gauge invariant functional of the background gauge field. Further, it can be shown that it is a FP operator. On the other hand, its density behaves for smooth fields U as

$$-\langle \bar{\psi}_n \gamma_5 \left(h^\dagger R h \psi \right)_n + \left(\bar{\psi} h^\dagger R h \right)_n \gamma_5 \psi_n \rangle \to \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu}(n) F^a_{\alpha\beta}(n) \tag{11}$$

up to terms which are of higher order in the vector potential and/or derivatives. It follows then

$$\mathrm{Tr}(\gamma_5 h R) = Q^{\mathrm{FP}} \,, \tag{12}$$

where Q^{FP} is the FP topological charge defined earlier in the Yang-Mills theory [13]. As argued in [12, 13], Q^{FP} (if it is used together with the FP action) defines a correct topological charge: it associates a unique integer number q to any configuration [6], where q satisfies the inequality: action $\geq 8\pi^2 |q|$, with equality for the instanton solutions. Eqs. (10,12) give the index theorem.

III. Eq. (5) constraints the spectrum $\{\lambda\}$ of the Dirac operator. For renormalization group transformations ('blocking out of continuum' [17]) which lead to $R_{nn'} = 1/\kappa_{\mathrm{f}} \cdot \delta_{nn'}$, where κ_{f} is a constant parameter entering the block transformation, the eigenvalues λ lie on a circle whose center is on the real axis in the point $\kappa_{\mathrm{f}}/2$ and has a radius $\kappa_{\mathrm{f}}/2$ (fig. 1). In general, for a local R, the eigenvalues lie between two circles tangent to the imaginary axis – the the circle described previously and a smaller one, as shown on the second picture in fig. 1. This spectrum excludes the existence of exceptional configurations.

We go through the steps now leading to the results above. As usual, the axial vector current is constructed from the chiral symmetric part of the action,

$$h_{\mathrm{SYM}} = \frac{1}{2}[h, \gamma_5]\gamma_5 = \hat{h} - \frac{1}{2}\{\hat{h}, \gamma_5\}\gamma_5 \,. \tag{13}$$

[5] The $m_{\mathrm{q}} \to 0$ limit will always be taken even if it is not indicated explicitly in the following equations.

[6] There exist configurations whose topological charge might change under a small deformation. This is the case when a small (size $O(a)$) topological object falls through the lattice. In the saddle point equation, which defines the fixed point action, there occur two degenerate absolute minima in this situation. [6, 12]. In the very moment when the topological charge changes the number of zero modes will also change as required by the index theorem.

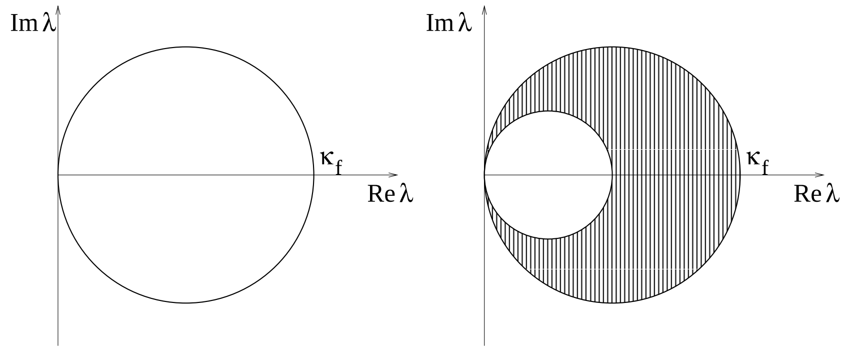

Figure 1: The spectrum of the fixed point Dirac operator: a) for the special case $R_{nn'} = 1/\kappa_{\rm f} \cdot \delta_{nn'}$ the eigenvalues lie on a circle, b) for a more general set of block transformations the eigenvalues lie between the two circles.

Performing a local chiral transformation one obtains for the divergence of the singlet axial vector current

$$\bar{\nabla}_\mu J^5_\mu(n) = \bar{\psi}_n \gamma_5 \left(h_{\rm SYM}\psi\right)_n + \left(\bar{\psi} h_{\rm SYM}\right)_n \gamma_5 \psi_n \,, \tag{14}$$

where $\bar{\nabla}_\mu f(n) = f(n) - f(n-\hat{\mu})$. Eqs. (14,13,5) give

$$\begin{aligned} \langle \bar{\nabla}_\mu J^5_\mu(n) \rangle \quad &= \quad \langle \bar{\psi}_n \gamma_5 (\hat{h}\psi)_n + (\bar{\psi}\hat{h})_n \gamma_5 \psi_n \rangle \\ & \quad -\langle \bar{\psi}_n \gamma_5 \left(h^\dagger R h \psi\right)_n + \left(\bar{\psi} h^\dagger R h\right)_n \gamma_5 \psi_n \rangle \\ & \quad -2m_{\rm q} \langle \bar{\psi}_n \gamma_5 \psi_n \rangle \,. \end{aligned} \tag{15}$$

The first term on the r.h.s. is proportional to the equation of motion $\hat{h}\psi = 0$, the second comes from the remnant chiral symmetry condition, while the last term is due to the explicit symmetry breaking. The contribution of the first term in eq. (15) is zero. Indeed,

$$\langle \bar{\psi}_n \gamma_5 (\hat{h}\psi)_n \rangle = -\frac{1}{Z(U)} \int D\bar{\psi} D\psi \, \bar{\psi}_n \gamma_5 \frac{\delta}{\delta \bar{\psi}_n} \exp\{-\sum_{m',n'} \bar{\psi}_{m'} \hat{h}_{m'n'} \psi_{n'}\} \,, \tag{16}$$

and partial integration by $\bar{\psi}$ gives then ${\rm tr}\gamma_5 = 0$. Integrating out the fermions in the third term in eq. (15) gives $2m_q {\rm tr}(\gamma_5 \hat{h}^{-1}_{nn})$, where the trace is over colour and Dirac space. We insert here the closure relation of eigenvectors of the hermitian matrix $H = h\gamma_5$. It is easy to show using eq. (5) that the zero eigenvalue eigenfunctions of H (if there are any) are chiral and they are zero eigenvalue eigenfunctions of h as well. In the $m_q \to 0$ limit only these eigenfunctions contribute leading to eq. (8).

Summing over n in eq. (6), integrating out the fermions in the first term on the r.h.s. and using eq. (8), one obtains eq. (10) immediately.

Consider now the derivation of eq. (11). We will show first that

$$-2m_{\rm q} \langle \bar{\psi}_n \gamma_5 \psi_n \rangle \;\to\; \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu}(n) F^a_{\alpha\beta}(n) \tag{17}$$

up to terms which are of higher order in the vector potential and/or derivatives. Eq. (17) can equivalently be written as

$$\sum_{m,n} m_\alpha n_\beta \left\{ \frac{\delta}{\delta A^b_\nu(0)} \frac{\delta}{\delta A^a_\mu(m)} \langle 2m_{\rm q} \bar{\psi}_n \gamma_5 \psi_n \rangle \right\} \Bigg|_{A=0} = \frac{1}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} \delta_{ab}. \tag{18}$$

We sketch the derivation only and refer to [18] for the details. Using the notation

$$iJ^a_\mu(m') = -\sum_{m,n} \bar\psi_n \, \frac{\delta}{\delta A^a_\mu(m')} \hat h_{mn}(U)\psi_n \Big|_{A=0} \,, \tag{19}$$

and the definition eq. (7), the l.h.s. of eq. (18) can be written as

$$\sum_{m,n} m_\alpha n_\beta \, \langle \left(-2m_{\rm q}\bar\psi_n\gamma_5\psi_n\right) J^a_\mu(m) J^b_\nu(0)\rangle\big|_{A=0} \,. \tag{20}$$

If the derivatives in eq. (18) act on $Z(U)$, or twice on $\hat h$, the contribution can be shown to be zero. The current in eq. (19) is a correct definition for the colour vector current: it can be shown that $\bar\nabla_\mu J_\mu$ complies with the Noether theorem. The current can also be written as

$$J^a_\mu(m) = \sum_{l,l'} \bar\psi_{m+l} T^a K_\mu(l,l')\psi_{m+l'} \,, \tag{21}$$

where T^a are the colour generators and K_μ is local.

The matrix element in eq. (20) has been considered in [16] when discussing the anomaly (see the considerations in [16] between eqs. (29-41)). It can be shown [18] that K_μ in eq. (21) satisfies the basic sum rules used in the derivation of the anomaly in [16]. The presence of colour in eq. (20) gives an extra factor $1/2\cdot\delta_{ab}$ in eq(41) in [16] leading to eq. (17). In [16], the authors first rewrite $-2m_{\rm q}\bar\psi_n\gamma_5\psi_n$ in eq. (20) using the operator equation for $\bar\nabla_\mu J^5_\mu(n)$ and show that only the term coming from the remnant chiral symmetry condition (proportional to $h^\dagger R h$) contributes. This gives the result claimed in eq. (11).

We want to demonstrate now that

$$Q(U) = {\rm Tr}(\gamma_5 hR) \tag{22}$$

is a FP operator as stated in **II.**. Further, we have to show that the FP Dirac operator satisfies the remnant chiral symmetry condition eq. (3). For both problems we need the classical equations which determine the FP action. The fixed point lies in the $\beta \sim 1/g^2 \to \infty$ hyperplane. In this limit the path integral is reduced to classical saddle point equations. The gauge field part satisfies [7]

$$S^{\rm FP}_{\rm g}(V) = \min_{\{U\}} \left[S^{\rm FP}_{\rm g}(U) + T_{\rm g}(V,U)\right] \,, \tag{23}$$

where $T_g(V,U)$ defines the block transformation relating the block gauge field V to a local average of the fine variables U. Denote the minimizing configuration in eq. (23) by $U_{\rm min}(V)$. The FP Dirac operator h is defined by the recursion equation

$$\begin{aligned} h(V)_{n_B n'_B} &= \kappa_{\rm f}\delta_{n_B n'_B} \\ &-\kappa_{\rm f}^2 b_{\rm f}^2 \sum_{nn'} \omega(U_{\rm min})_{n_B n} \left(h(U_{\rm min}) + \kappa_{\rm f} b_{\rm f}^2 \omega^\dagger(U_{\rm min})\omega(U_{\rm min})\right)^{-1}_{nn'} \omega(U_{\rm min})^\dagger_{n' n'_B} \,. \end{aligned} \tag{24}$$

In case h^{-1} is defined, eq. (24) is equivalent to the somewhat simpler recursion relation [19]

$$h(V)^{-1}_{n_B n'_B} = \frac{1}{\kappa_{\rm f}}\delta_{n_B n'_B} + b_{\rm f}^2 \sum_{n,n'} \omega(U_{\rm min})_{n_B n} h(U_{\rm min})^{-1}_{nn'} \omega(U_{\rm min})^\dagger_{n' n'_B} \,. \tag{25}$$

Here $\omega(U)$, which is trivial in Dirac space, defines the way the block fermion fields are constructed from the fine fermion fields, while $\kappa_{\rm f}$ is a parameter of the block transformation. The coarse fields live on a lattice with a lattice unit $a' = sa$, where a is the lattice unit before the transformation, while s is the scale of the renormalization group transformation. The factor $b_{\rm f}$ is s^{d_ψ}, where d_ψ is the engineering dimension of the field (=3/2), while the averaging function satisfies the normalization condition $\sum_n \omega(U=1)_{n_B n} = 1$.

Eq. (25) (or eq. (24)) can be solved by iteration. If the scale change s of the transformation is (infinitely) large, the fixed point can be reached in a single renormalization group step. In this case, the fine field U_{min} lives on an infinitely fine lattice, and it is smooth. The Dirac operator h over this field is arbitrarily close to the massless continuum Dirac operator. Since ω is trivial in Dirac space, the second term on the r.h.s. of eq. (25) is chiral invariant, which leads to the remnant chiral symmetry condition in eq. (3) with $R_{nn'} = 1/\kappa_{\rm f} \cdot \delta_{nn'}$. If s is finite, one can start again on an infinitely fine lattice, but the transformation should be iterated. One obtains in every iteration step some contribution to the chiral symmetry breaking part of h^{-1}, and a non-trivial, but local R builds up in this case as claimed in eq. (3) before. If h^{-1} is not defined (due to zero modes), one can use eq. (24) to derive the corresponding remnant symmetry condition for h itself with somewhat more algebra.

By adding to the FP action $S^{\rm FP}(U)$ a small perturbation $\epsilon\mathcal{O}(U)$, the path integral in the saddle point approximation leads to the extra contribution $\epsilon\mathcal{O}'(V) = \epsilon\mathcal{O}(U_{\rm min}(V)$. A FP operator is reproduced by the renormalization group transformation up to a trivial rescaling related to the dimension of the operator:

$$s^{-d_{\mathcal{O}}}\mathcal{O}^{\rm FP}(V) = \mathcal{O}^{\rm FP}\left(U_{\rm min}(V)\right) , \qquad (26)$$

where s is the scale of the RGT and $d_{\mathcal{O}}$ is the dimension of the operator $\mathcal{O}$ [10]. Since the topological charge is a dimensionless quantity, it satisfies the FP equation

$$Q^{\rm FP}(V) = Q^{\rm FP}\left(U_{\rm min}(V)\right) . \qquad (27)$$

We want to show that the operator $\mathrm{Tr}(\gamma_5 hR)$ satisfies eq. (27). Using eqs. (3,24) and the fact that ω is trivial in Dirac space, one obtains the following recursion relation for R

$$R(V)_{n_B n'_B} = \frac{1}{\kappa_{\rm f}}\delta_{n_B n'_B} + \frac{b_{\rm f}^2}{\kappa_{\rm f}}\sum_{n,n'}\omega(U_{\rm min})_{n_B n}R(U_{\rm min})_{nn'}\omega(U_{\rm min})^\dagger_{n' n'_B} . \qquad (28)$$

With the help of eqs. (24,28) one obtains after some algebra that $Q(U)$ in eq. (22) satisfies eq. (27). It is, therefore the FP topological charge studied earlier in spin and gauge models [12, 13].

Finally, let us discuss the spectrum of the FP Dirac operator. Let ψ be a normalized solution of the eigenvalue equation, $h\psi = \lambda\psi$. Eq. (5) (written in the form $\frac{1}{2}(h + h^\dagger) = h^\dagger R h$) implies $\mathrm{Re}\lambda = |\lambda|^2(\psi, R\psi)$. For the special case $R_{nn'} = 1/\kappa_{\rm f} \cdot \delta_{nn'}$ the eigenvalues λ lie on a circle of radius $\kappa_{\rm f}/2$ touching the imaginary axis, as shown in fig. 1. In general, the eigenvalues lie between two circles tangent to the imaginary axis - the circle described previously and a smaller one as shown in fig. 1. The smaller circle has a finite radius which follows from the locality of R. In particular, one can show that for the block transformation used in [8, 20] $1/\kappa_{\rm f} \le (\psi, R\psi) \le 2/\kappa_{\rm f}$ hence the radius of the smaller circle is $\kappa_{\rm f}/4$. This property excludes exceptional configurations, appearing e.g. for the Wilson action and causing serious problems in numerical simulations. As a consequence of eq. (2) the solutions with $\mathrm{Im}\lambda \neq 0$ come in complex conjugate pairs and satisfy $(\psi, \gamma_5 \psi) = 0$. Due to eqs. (3,5), those with $\lambda = 0$ are chiral eigenstates, $\gamma_5 \psi = \pm\psi$.

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