Spectral expansion for finite temperature two-point functions and clustering

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Abstract

Recently, the spectral expansion of finite temperature two-point functions in integrable quantum field theories was constructed using a finite volume regularization technique and the application of multidimensional residues. In the present work, the original calculation is revisited. By clarifying some details in the residue evaluations, we find and correct some inaccuracies of the previous result. The final result for contributions involving no more than two particles in the intermediate states is presented. The result is verified by proving a symmetry property which follows from the general structure of the spectral expansion, and also by numerical comparison to the discrete finite volume spectral sum. A further consistency check is performed by showing that the expansion satisfies the cluster property up to the order of the evaluation.

1 Introduction

Correlation functions play a central role in the formulation of many-body quantum systems. Integrable models presents a unique opportunity to study strongly correlated quantum systems in situations where conventional methods break down. Recent experimental advances resulted in renewed interest in integrable models, since it is now possible to realize certain models with the help of optical and magnetic traps [1, 2, 3, 4] or in low-dimensional magnets [5, 6].

In a recent paper [7] finite temperature (i.e. thermal) two-point correlation functions were constructed using the exact form factors in 1+1 dimensional integrable models. In an integrable quantum field theory, the basic object is the factorized S-matrix [8, 9]. The matrix elements of the local operators (form factors) satisfy a certain set of equations (the form factor bootstrap equations) which follow from general field theoretical arguments

supplemented with the special analytic properties of the S-matrix [10, 11, 12, 13]. Solving these equations gives the form factor functions, which can then be used to construct correlation functions by expanding in the basis formed by the infinite volume asymptotic scattering states.

The form factor expansion of zero-temperature correlations in integrable QFT is very well understood. In general, the series has very good convergence properties in massive models and can be evaluated numerically to any desired precision [12, 14]. However, the problem of thermal correlation functions is much more complicated and has been the subject of active research in the last two decades [15, 16, 17, 18, 19, 20, 21, 22, 23]. The form factor construction of the spectral series is plagued with problems due to the presence of disconnected terms in the expansion, which lead to formally divergent expressions. Following Balog it can be shown that the divergent parts cancel with contributions from the partition function [24]. Nevertheless, it is a very non-trivial task to obtain the correct finite answer. Leclair and Mussardo conjectured an answer for the spectral expansion for one-point and two-point functions in terms of form factors dressed by appropriate occupation number factors containing the pseudo-energy function from the thermodynamical Bethe Ansatz [16]. Their proposal for the two-point function was questioned by Saleur [17]; on the other hand, in the same paper he also gave a proof of the Leclair-Mussardo formula for one-point functions provided the operator considered is the density of some local conserved charge. By comparison to an alternative proposal [19], it was also shown that the results obtained by naive regularization are ambiguous [20].

The idea behind our approach is to use a finite volume setting to regularize the divergences. In [25, 26] this was applied to one-point functions giving a confirmation of the Leclair-Mussardo formula to third order; later a derivation to all orders was also obtained [27]. The crucial point is that finite volume is not an ad hoc, but a physical regulator (since physically realizable systems are always of finite size), therefore one is virtually guaranteed to obtain the correct result when taking the infinite volume limit. The natural small parameter for the finite temperature expansion is the Boltzmann-factor $e^{-m/T}$ where m is the mass gap (which is assumed to be nonzero). The result is an integral series, where the Nth term represents N-particle processes over the Fock-vacuum. The contributions with a low number of particles can be interpreted as disconnected terms of matrix elements calculated in a thermal state with a large number of particles [23, 27]. In this sense the approach is similar to the one used in algebraic Bethe Ansatz [28, 29].

Besides correlation functions, the finite volume regularization can also be applied to numerous other problems. The finite volume form factor approach was extended to boundary operators as well [30], which was used to compute finite temperature one-point functions of boundary operators [31]. Another application of the bulk finite volume form factors is the construction of one-point functions of bulk operators on a finite interval [32]. It was also used [33] to construct the form factor perturbation expansion in non-integrable field theories (originally proposed by Delfino et al. [34]) beyond the leading order. It also turned out that this approach can be applied to quenches in field theory [32, 35, 36].

Regarding thermal two-point functions, the finite volume regularization method was first applied by Essler and Konik [37, 38]; however, their methods do not have any obvious extension to higher order. Despite this shortcoming, their results are very useful as shown by their relevance to inelastic neutron scattering experiments [6]. An independent early calculation of the one-particle–one-particle contribution can also be found in [39].

In [7] we developed a systematic method to compute the finite temperature form factor expansion to arbitrary orders. It turns out that the machinery of multidimensional residues provides an appropriate formalism to evaluate higher order corrections systematically, and this was demonstrated to all orders which involve only intermediate states with at most two

particles. To verify the result, we applied two consistency checks. The first of them was that the correlator should have a finite limit when the volume is taken to infinity, therefore all terms containing positive powers of the volume had to cancel, which was indeed true. The second one took into account that for some contributions there are two independent ways to arrive at the answer, and agreement between them also provides validating evidence. However, for the term D_{22}, which contains the contributions when both intermediate states involved in the spectral sum contain two particles, the second one is not available and the first is insufficient to check the structure of the result in detail.

Therefore we decided to provide a numerical evidence, especially since the analytic manipulations themselves are rather tedious and complicated, with many possible sources of mistakes. As it turned out, the result for D_{22} reported in [7] is unfortunately incorrect. By further investigation, it turned out that some fine details of the residue calculation needed to be carried out more carefully.

Here we report the correct version of the computation and its result, and present the final formula for the finite temperature two-point function including all contributions with at most two-particle intermediate states. To be confident in our results, we perform several checks. First we check that the result for D_{22} satisfies a particular symmetry property following from the general form of the spectral expansion. Then we apply a detailed numerical verification of our analytic manipulations, and also verify that the final result satisfies the physically required cluster property.

The layout of the paper is as follows. Section 2 introduces the thermal two-point function and the idea of finite volume regularization. In section 3, the methods to evaluate the resulting spectral expansion is presented. We re-derive the results of [7], including the correct form of the term D_{22} and present the full formula of the two-point function including all two-particle contributions. The numerical verification of D_{22} is performed in Section 4, together with a similar verification for D_{12} in order to establish a benchmark point for numerical accuracy. In Section 5 we prove that the resulting expansion satisfies the cluster property, and in Section 6 the conclusions are presented. There are also three appendices: Appendix A contains the mathematical formulas used for evaluating the residue contributions, while Appendix B contains the end results of the residue evaluations, which are also necessary for the numerical comparison. The proof of the symmetry property of the D_{22} contribution is given in Appendix C.

2 Finite volume regularization

2.1 The thermal two-point function

A field theory with finite temperature T can be defined using a compact Euclidean (Matsubara) time t:

$$t \equiv t + R \quad \text{where} \quad R = 1/T \tag{2.1}$$

We are interested in the two-point function in $1 + 1$ dimensional field theories:

$$\langle \mathcal{O}_1(x, t) \mathcal{O}_2(0) \rangle^R = \frac{\text{Tr}\left(e^{-RH} \mathcal{O}_1(x, t) \mathcal{O}_2(0)\right)}{\text{Tr}\left(e^{-RH}\right)} \tag{2.2}$$

A naive spectral sum leads to an ill-defined expression due to the presence of disconnected contributions (cf. e.g. the discussion in [7]). However, one can put the system in a finite spatial volume L with periodic boundary conditions

$$x \equiv x + L \tag{2.3}$$

so that

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0)\rangle_L^R = \frac{\mathrm{Tr}_L\left(\mathrm{e}^{-RH_L}\mathcal{O}_1(x,t)\mathcal{O}_2(0)\right)}{\mathrm{Tr}_L\left(\mathrm{e}^{-RH_L}\right)} \tag{2.4}$$

where Tr_L denotes the trace over the finite-volume states, H_L is the Hamiltonian in volume L. This expression can be expanded inserting two complete sets of states

$$\mathrm{Tr}_L\left(\mathrm{e}^{-RH_L}\mathcal{O}_1(x,t)\mathcal{O}_2(0)\right) = \sum_{m,n}\mathrm{e}^{-RE_n(L)}\langle n|\mathcal{O}_1(x,t)|m\rangle_L\langle m|\mathcal{O}_2(0)|n\rangle_L \tag{2.5}$$

where the matrix elements of local operators are also taken in the finite volume system. To evaluate it, we need an expression for form factors in finite volume.

2.2 The form factor bootstrap

In a $1+1$ dimensional field theory, the energy and the momentum of an on-shell particle is parametrized by the rapidity variable as $E = m\cosh\theta$ and $p = m\sinh\theta$. For the sake of simplicity let us suppose that the spectrum of the model consists of a single particle mass m. Incoming and outgoing asymptotic states are defined as:

$$|\theta_1,\ldots,\theta_n\rangle = \begin{cases} |\theta_1,\ldots,\theta_n\rangle^{in} & : \theta_1 > \theta_2 > \cdots > \theta_n \\ |\theta_1,\ldots,\theta_n\rangle^{out} & : \theta_1 < \theta_2 < \cdots < \theta_n \end{cases}$$

Integrability leads to factorized scattering, which can be summarized by the relation

$$|\theta_1,\ldots,\theta_k,\theta_{k+1},\ldots,\theta_n\rangle = S(\theta_k - \theta_{k+1})|\theta_1,\ldots,\theta_{k+1},\theta_k,\ldots,\theta_n\rangle$$

where S denotes the two-particle amplitude; from this any multi-particle scattering process can be obtained by reordering the particles. States are normalized as:

$$\langle\theta'|\theta\rangle = 2\pi\delta(\theta' - \theta) \tag{2.6}$$

The form factors of a local operator $\mathcal{O}(t,x)$ are defined as

$$F_{mn}^{\mathcal{O}}(\theta'_1,\ldots,\theta'_m|\theta_1,\ldots,\theta_n) = \langle\theta'_1,\ldots,\theta'_m|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_n\rangle \tag{2.7}$$

With the help of the crossing relations

$$F_{mn}^{\mathcal{O}}(\theta'_1,\ldots,\theta'_m|\theta_1,\ldots,\theta_n) = F_{m-1n+1}^{\mathcal{O}}(\theta'_1,\ldots,\theta'_{m-1}|\theta'_m + i\pi,\theta_1,\ldots,\theta_n) \tag{2.8}$$

$$+ \sum_{k=1}^{n} 2\pi\delta(\theta'_m - \theta_k)\prod_{l=1}^{k-1}S(\theta_l - \theta_k)F_{m-1n-1}^{\mathcal{O}}(\theta'_1,\ldots,\theta'_{m-1}|\theta_1,\ldots,\theta_{k-1},\theta_{k+1}\ldots,\theta_n)$$

all form factors can be expressed in terms of the elementary form factors

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n) = \langle 0|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_n\rangle$$

which satisfy the form factor bootstrap equations [10, 40, 11]

Lorentz symmetry: $\quad F_n^{\mathcal{O}}(\theta_1 + \Lambda, \theta_2 + \Lambda, \ldots, \theta_n + \Lambda) =$ (2.9)
$$\exp\left(s_{\mathcal{O}}\Lambda\right)F_n^{\mathcal{O}}(\theta_1,\theta_2,\ldots,\theta_n)$$

Exchange: $\quad F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_k,\theta_{k+1},\ldots,\theta_n) =$ (2.10)
$$S(\theta_k - \theta_{k+1})F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{k+1},\theta_k,\ldots,\theta_n)$$

Cyclic property: $\quad F_n^{\mathcal{O}}(\theta_1 + 2i\pi, \theta_2, \ldots, \theta_n) = F_n^{\mathcal{O}}(\theta_2,\ldots,\theta_n,\theta_1)$ (2.11)

Kinematical poles: $\quad -i\operatorname*{Res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta + i\pi, \theta', \theta_1, \ldots, \theta_n) =$ (2.12)
$$\left(1 - \prod_{k=1}^{n}S(\theta' - \theta_k)\right)F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$

where $s_{\mathcal{O}}$ denotes the Lorentz spin of the operator $\mathcal{O}$. There is also a further equation related to bound states which we do not need in the sequel.

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2.3 Form factors in finite volume

A formalism that gives the exact quantum form factors to all orders in L^{-1} was introduced in [25, 26]. The finite volume multi-particle states can be denoted

$$|\{I_1, \ldots, I_n\}\rangle_L$$

where the I_k are momentum quantum numbers, ordered as $I_1 \geq \cdots \geq I_n$ by convention. The corresponding energy levels are determined by the Bethe-Yang equations

$$e^{imL \sinh \tilde{\theta}_k} \prod_{l \neq k} S(\tilde{\theta}_k - \tilde{\theta}_l) = 1$$

Defining the two-particle phase shift $\delta(\theta)$ by the relation

$$S(\theta) = -e^{i\delta(\theta)} \tag{2.13}$$

The derivative of δ will be denoted by

$$\varphi(\theta) = \frac{d\delta(\theta)}{d\theta} \tag{2.14}$$

due to unitarity, δ is an odd and φ is an even function. We can write

$$Q_k(\tilde{\theta}_1, \ldots, \tilde{\theta}_n) = mL \sinh \tilde{\theta}_k + \sum_{l \neq k} \delta(\tilde{\theta}_k - \tilde{\theta}_l) = 2\pi I_k \quad , \quad k = 1, \ldots, n \tag{2.15}$$

where the quantum numbers I_k take integer/half-integer values for odd/even numbers of particles respectively. Eqns. (2.15) must be solved with respect to the particle rapidities $\tilde{\theta}_k$, where the energy (relative to the finite volume vacuum state) can be computed as

$$\sum_{k=1}^{n} m \cosh \tilde{\theta}_k \tag{2.16}$$

up to corrections which decay exponentially with L. The density of n-particle states in rapidity space can be calculated as

$$\rho(\theta_1, \ldots, \theta_n) = \det \mathcal{J}^{(n)} \quad , \quad \mathcal{J}_{kl}^{(n)} = \frac{\partial Q_k(\theta_1, \ldots, \theta_n)}{\partial \theta_l} \quad , \quad k, l = 1, \ldots, n \tag{2.17}$$

The finite volume behavior of local matrix elements can be given as [25]

$$\langle \{I_1', \ldots, I_m'\} | \mathcal{O}(0,0) | \{I_1, \ldots, I_n\} \rangle_L = \frac{F_{m+n}^{\mathcal{O}}(\tilde{\theta}_m' + i\pi, \ldots, \tilde{\theta}_1' + i\pi, \tilde{\theta}_1, \ldots, \tilde{\theta}_n)}{\sqrt{\rho(\tilde{\theta}_1, \ldots, \tilde{\theta}_n)\rho(\tilde{\theta}_1', \ldots, \tilde{\theta}_m')}}$$

$$+ \quad O(e^{-\mu L}) \tag{2.18}$$

where $\tilde{\theta}_k$ ($\tilde{\theta}_k'$) are the solutions of the Bethe-Yang equations (2.15) corresponding to the state with the specified quantum numbers $I_1, \ldots, I_n$ ($I_1', \ldots, I_n'$) at the given volume L. The above relation is valid provided there are no disconnected terms i.e. the left and the right states do not contain particles with the same rapidity, i.e. the sets $\left\{\tilde{\theta}_1, \ldots, \tilde{\theta}_n\right\}$ and $\left\{\tilde{\theta}_1', \ldots, \tilde{\theta}_m'\right\}$ are disjoint.

It is easy to see that in the presence of nontrivial scattering there are only two cases when exact equality of (at least some of) the rapidities can occur [26]:

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1. The two states are identical, i.e. $n = m$ and

$$\{I'_1, \ldots, I'_m\} = \{I_1, \ldots, I_n\}$$

in which case the corresponding diagonal matrix element can be written as a sum over all bipartite divisions of the set of the n particles involved (including the trivial ones when A is the empty set or the complete set $\{1, \ldots, n\}$)

$$\langle\{I_1 \ldots I_n\}|\mathcal{O}|\{I_1 \ldots I_n\}\rangle_L = \frac{\sum_{A \subset \{1,2,\ldots n\}} \mathcal{F}(A)_L \rho(\{1, \ldots, n\} \setminus A)_L}{\rho(\{1, \ldots, n\})_L} + O(e^{-\mu L})$$
(2.19)

where

$$\rho(\{k_1, \ldots, k_r\})_L = \rho(\tilde{\theta}_{k_1}, \ldots, \tilde{\theta}_{k_r})$$

is the r-particle Bethe-Yang Jacobi determinant (2.17) involving only the r-element subset $1 \le k_1 < \cdots < k_r \le n$ of the n particles, and

$$\begin{aligned}
\mathcal{F}(\{k_1, \ldots, k_r\})_L &= F^s_{2r}(\tilde{\theta}_{k_1}, \ldots, \tilde{\theta}_{k_r}) \\
F^s_{2l}(\theta_1, \ldots, \theta_l) &= \lim_{\epsilon \to 0} F^{\mathcal{O}}_{2l}(\theta_l + i\pi + \epsilon, \ldots, \theta_1 + i\pi + \epsilon, \theta_1, \ldots, \theta_l)
\end{aligned}$$

is the so-called symmetric evaluation of diagonal multi-particle matrix elements.

2. Both states are parity symmetric states in the spin zero sector, i.e.

$$\begin{aligned}
\{I_1, \ldots, I_n\} &\equiv \{-I_n, \ldots, -I_1\} \\
\{I'_1, \ldots, I'_m\} &\equiv \{-I'_m, \ldots, -I'_1\}
\end{aligned}$$

Furthermore, both states must contain one (or possibly more, in a theory with more than one species) particle of zero quantum number. Writing $m = 2k+1$ and $n = 2l+1$ and defining

$$\begin{aligned}
\mathcal{F}_{k,l}(\theta'_1, \ldots, \theta'_k|\theta_1, \ldots, \theta_l) &= \\
\lim_{\epsilon \to 0} F_{2k+2l+2}(i\pi + \theta'_1 + \epsilon, \ldots, i\pi + \theta'_k + \epsilon, i\pi - \theta'_k + \epsilon, \ldots, i\pi - \theta'_1 + \epsilon, \\
i\pi + \epsilon, 0, \theta_1, \ldots, \theta_l, -\theta_l, \ldots, -\theta_1)
\end{aligned}$$
(2.20)

the formula for the finite-volume matrix element takes the form

$$\begin{aligned}
&\langle\{I'_1, \ldots, I'_k, 0, -I'_k, \ldots, -I'_1\}|\mathcal{O}|\{I_1, \ldots, I_l, 0, -I_l, \ldots, -I_1\}\rangle_L \qquad\qquad (2.21) \\
&= \left(\rho_{2k+1}(\tilde{\theta}'_1, \ldots, \tilde{\theta}'_k, 0, -\tilde{\theta}'_k, \ldots, -\tilde{\theta}'_1)\rho_{2l+1}(\tilde{\theta}_1, \ldots, \tilde{\theta}_l, 0, -\tilde{\theta}_l, \ldots, -\tilde{\theta}_1)\right)^{-1/2} \\
&\quad \times \Big[\mathcal{F}_{k,l}(\tilde{\theta}'_1, \ldots, \tilde{\theta}'_k|\tilde{\theta}_1, \ldots, \tilde{\theta}_l) \\
&\quad + mL\, F_{2k+2l}(i\pi + \tilde{\theta}'_1, \ldots, i\pi + \tilde{\theta}'_k, i\pi - \tilde{\theta}'_k, \ldots, i\pi - \tilde{\theta}'_1, \tilde{\theta}_1, \ldots, \tilde{\theta}_l, -\tilde{\theta}_l, \ldots, -\tilde{\theta}_1)\Big] \\
&\quad + O(e^{-\mu L})
\end{aligned}$$

2.4 The form factor expansion using finite volume regularization

Using the finite volume description introduced in subsection 2.3 we can write

$$\langle\mathcal{O}_1(x, t)\mathcal{O}_2(0)\rangle^R_L = \frac{1}{Z}\sum_{N,M} C_{NM}$$
(2.22)

where

$$C_{NM} = \sum_{I_1...I_N} \sum_{J_1...J_M} \langle \{I_1 \ldots I_N\} | \mathcal{O}_1(0) | \{J_1 \ldots J_M\} \rangle_L \times$$
$$\langle \{J_1 \ldots J_M\} | \mathcal{O}_2(0) | \{I_1 \ldots I_N\} \rangle_L e^{i(P_1-P_2)x} e^{-E_1(R-t)} e^{-E_2 t} \qquad (2.23)$$

and $E_{1,2}$ and $P_{1,2}$ are the total energies and momenta of the multi-particle states $|\{I_1 \ldots I_N\}\rangle_L$ and $|\{J_1 \ldots J_M\}\rangle_L$. The task is to calculate the sum in finite volume and then take the limit $L \to \infty$.

First we classify the contributions into different multi-particle orders following the procedure in [38, 7]. Introducing two auxiliary variables u and v (at the end both will be set to 1):

$$\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle_L^R = \frac{1}{Z} \sum_{N,M} u^N v^M C_{NM} \qquad (2.24)$$

Similarly for the partition function

$$Z = \sum_N (uv)^N Z_N$$

with Z_N denoting the N-particle contribution. The inverse of the partition function is expanded as

$$Z^{-1} = \sum_N (uv)^N \bar{Z}_N$$

where

$$\bar{Z}_0 = 1 \qquad \bar{Z}_1 = -Z_1 \qquad \bar{Z}_2 = Z_1^2 - Z_2$$

Putting this together we can rewrite the expansion as

$$\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle_L^R = \sum u^N v^N \tilde{D}_{NM} \qquad (2.25)$$

with

$$\tilde{D}_{NM} = \sum_l C_{N-l,M-l} \bar{Z}_l \qquad (2.26)$$

The first few nontrivial terms are given by

$$\tilde{D}_{1M} = C_{1M} - Z_1 C_{0,M-1}$$
$$\tilde{D}_{2M} = C_{2M} - Z_1 C_{1,M-1} + (Z_1^2 - Z_2) C_{0,M-2} \qquad (2.27)$$

In this way we produce a double series expansions in powers of the variables e^{-mt} and $e^{-m(R-t)}$. Since these variables are independent, each quantity $\tilde{D}_{NM}$ must have a well-defined $L \to \infty$ limit which we denote as

$$D_{NM} = \lim_{L \to \infty} \tilde{D}_{NM} \qquad (2.28)$$

and we obtain that

$$\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle^R = \lim_{L \to \infty} \langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle_L^R = \sum_{N,M} D_{NM} \qquad (2.29)$$

A similar reordering was also used for the expansion of the one-point function in powers of e^{-mR} [26], and for the boundary one-point function in [31]. It is evident from (2.23) that the D_{NM} with $N > M$ can be obtained from those with $N < M$ after a trivial exchange of t with $R - t$, x with $-x$ and $\mathcal{O}_1$ with $\mathcal{O}_2$.

3 The spectral expansion for finite temperature correlators

To evaluate the finite temperature two-point function, it is necessary to evaluate the summation over two sets of intermediate states. For a given C_{NM} this involves an N and an M particle state. One can start with any of these; to simplify the calculations, it is best to start with the one containing the smallest number or particles, and do the other later. On the other hand, doing the calculation in the reverse order allows one to cross-check the result [7].

To evaluate the first summation, a systematic method was given in [7] based on a multidimensional residue method. Once this is done, all the singularities from the form factors are tamed, and the second summation can be performed by a simple transition from the discrete sum to an integral using the density of states. Then, after assembling $\tilde{D}_{NM}$ using the lower $C_{N'M'}$ coefficients as in (2.26), and taking the limit $L \to \infty$ the final formula for the contribution D_{NM} can be obtained. Another quick validity check of the calculation is provided by the existence of the infinite volume limit.

3.1 Converting sums to contour integrals

For sums over one-particles states $|\{I\}\rangle_L$ with quantum number $I \in \mathbb{Z}$ we can substitute

$$\sum_I \to \sum_I \oint_{C_I} \frac{d\theta}{2\pi} \frac{\rho_1(\theta)}{e^{iQ_1(\theta)} - 1}$$

where

$$Q_1(\theta) = mL \sinh\theta \qquad \rho_1(\theta) = Q_1'(\theta) = mL \cosh\theta$$

and C_I are small closed curves surrounding the solution of

$$Q_1(\theta) = 2\pi I$$

in the complex θ plane.

For two-particle sums over two-particle states $|\{I_1, I_2\}\rangle_L$ with quantum numbers $I_1, I_2 \in \mathbb{Z} + \frac{1}{2}$ we can use the multidimensional generalization of the residue theorem to write

$$\sum_{I_1 > I_2} \to \sum_{I_1 > I_2} \oint \oint_{C_{I_1 I_2}} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{\rho_2(\theta_1, \theta_2)}{\left(e^{iQ_1(\theta_1, \theta_2)} + 1\right)\left(e^{iQ_2(\theta_1, \theta_2)} + 1\right)}$$

where $C_{I_1 I_2}$ is a multi-contour (a direct product of two curves in the variables θ_1 and θ_2) surrounding the solution of

$$
\begin{aligned}
Q_1(\theta_1, \theta_2) &= mL \sinh\theta_1 + \delta(\theta_1 - \theta_2) = 2\pi I_1 \\
Q_2(\theta_1, \theta_2) &= mL \sinh\theta_2 + \delta(\theta_2 - \theta_1) = 2\pi I_2
\end{aligned}
$$

where due to the definition (2.13) I_1 and I_2 take half-integer values, and

$$\rho_2(\theta_1, \theta_2) = \det \begin{pmatrix} \frac{\partial Q_1}{\partial \theta_1} & \frac{\partial Q_1}{\partial \theta_2} \\ \frac{\partial Q_2}{\partial \theta_1} & \frac{\partial Q_2}{\partial \theta_2} \end{pmatrix} = m^2 L^2 \cosh\theta_1 \cosh\theta_2 + mL(\cosh\theta_1 + \cosh\theta_2)\varphi(\theta_1 - \theta_2)$$

Since form factors vanish when any two of their arguments coincide, we can extend the sum by adding the diagonal:

$$\sum_{I_1 > I_2} \to \frac{1}{2}\left(\sum_{I_1, I_2} - \sum_{I_1 = I_2}\right)$$
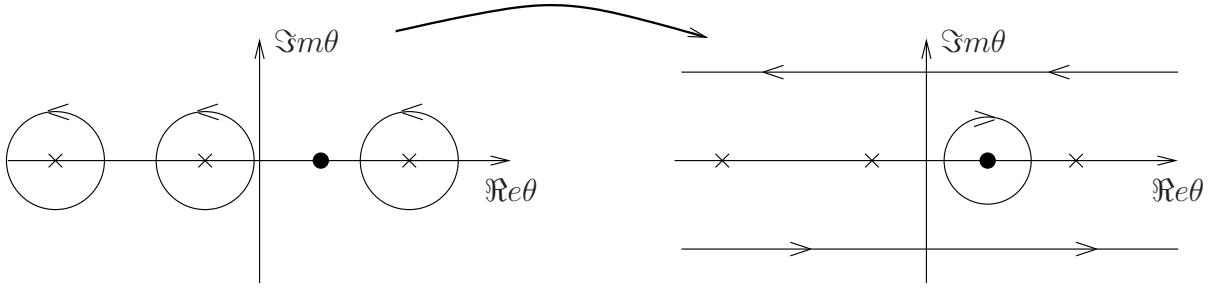
Figure 3.1: Contour deformation procedure. The black dot shows a singularity not enclosed inside the contours following from the spectral sum.

In the next step, the contours are joined together and opened into straight lines, to a product contour whose components in each variable enclose the real axis. However, this can only be done by including other poles (apart from the ones needed for the state summations) in the interior, which come from singularities of the Q-dependent denominators and of the form factors. These must be classified and subtracted. This procedure was discussed in some detail in [7], and for one complex variable it is illustrated in fig. 3.1 (for more complex variable it must be performed in each variables separately). We shall only outline it for the case of the D_{22} contribution, because of the corrections we make to the previous calculation performed in that paper.

3.2 The D_{22} contribution revisited

The D_{22} contribution is given by

$$D_{22} = \lim_{L\to\infty}\left[C_{22} - Z_1 C_{11} + \left(Z_1^2 - Z_2\right)C_{00}\right] = \lim_{L\to\infty}\left[C_{22} - Z_1\tilde{D}_{11} - Z_2 C_{00}\right]$$

where

$$C_{22} = \sum_{I_1>I_2}\sum_{J_1>J_2}\langle\{I_1,I_2\}|\mathcal{O}_1(0)|\{J_1,J_2\}\rangle_L\langle\{J_1,J_2\}|\mathcal{O}_2(0)|\{I_1,I_2\}\rangle_L K_{t,x}^{(R)}(\tilde{\vartheta}_1,\tilde{\vartheta}_2|\tilde{\vartheta}_1',\tilde{\vartheta}_2')$$

with the notation

$$K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2') = e^{imx\left(\sinh\vartheta_1+\sinh\vartheta_2-\sinh\vartheta_1'-\sinh\vartheta_2'\right)}e^{-m(R-t)(\cosh\vartheta_1+\cosh\vartheta_2)}e^{-mt\left(\cosh\vartheta_1'+\cosh\vartheta_2'\right)}$$

(3.1)

and where [7]

$$Z_1 = mL\int\frac{d\vartheta_1}{2\pi}\cosh\vartheta_1 e^{-mR\cosh\vartheta_1}$$

$$Z_2 = \frac{1}{2}\iint\frac{d\vartheta_1}{2\pi}\frac{d\vartheta_2}{2\pi}\rho_2(\vartheta_1,\vartheta_2)e^{-mR(\cosh\vartheta_1+\cosh\vartheta_2)} - \frac{1}{2}\int\frac{d\vartheta_1}{2\pi}\rho_1(\vartheta_1)e^{-2mR\cosh\vartheta_1}$$

$$C_{00} = \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle$$

$$D_{11} = \iint\frac{d\vartheta_1}{2\pi}\frac{d\vartheta_2}{2\pi}F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1)e^{imx(\sinh\vartheta_1-\sinh\vartheta_2)}e^{-m(R-t)\cosh\vartheta_1}e^{-mt\cosh\vartheta_2}$$

$$+ \left[\langle\mathcal{O}_1\rangle F_{2s}^{\mathcal{O}_2} + \langle\mathcal{O}_2\rangle F_{2s}^{\mathcal{O}_1}\right]\int\frac{d\vartheta_1}{2\pi}e^{-mR\cosh\vartheta_1}$$

The rapidities are quantized by

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$$Q_1(\vartheta_1, \vartheta_2) = mL \sinh \vartheta_1 + \delta(\vartheta_1 - \vartheta_2) = 2\pi I_1$$
$$Q_2(\vartheta_1, \vartheta_2) = mL \sinh \vartheta_2 + \delta(\vartheta_2 - \vartheta_1) = 2\pi I_2 \tag{3.2}$$

and

$$Q_1'(\vartheta_1', \vartheta_2') = mL \sinh \vartheta_1' + \delta(\vartheta_1' - \vartheta_2') = 2\pi J_1$$
$$Q_2'(\vartheta_1', \vartheta_2') = mL \sinh \vartheta_2' + \delta(\vartheta_2' - \vartheta_1') = 2\pi J_2$$

We perform the J_1, J_2-sum first and separate it into a diagonal and an off-diagonal piece:

$$\sum_{J_1 > J_2} = (\{J_1, J_2\} = \{I_1, I_2\} \text{ term}) + \sum_{J_1 > J_2}{}'$$

because the finite volume form factor expressions are different for the two types of contributions. In the second term, the prime indicates that the diagonal contributions are excluded.

3.2.1 The diagonal piece

This calculation is exactly the same as in [7], so we only highlight the main steps. Starting from

$$C_{22}^{diag} = \sum_{I_1 > I_2} \langle I_1, I_2 | \mathcal{O}_1(0) | I_1, I_2 \rangle \langle I_1, I_2 | \mathcal{O}_2(0) | I_1, I_2 \rangle K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2)$$

where

$$\langle I_1, I_2 | \mathcal{O}(0) | I_1, I_2 \rangle = \frac{F_{4s}^{\mathcal{O}}(\vartheta_1, \vartheta_2) + (\rho_1(\vartheta_1) + \rho_1(\vartheta_2)) F_{2s}^{\mathcal{O}} + \rho_2(\vartheta_1, \vartheta_2) \langle \mathcal{O} \rangle}{\rho_2(\vartheta_1, \vartheta_2)}$$

with

$$F_{2s}^{\mathcal{O}}(\vartheta) = F_2^{\mathcal{O}}(i\pi, 0)$$
$$F_{4s}^{\mathcal{O}}(\vartheta_1, \vartheta_2) = \lim_{\varepsilon \to 0} F_4^{\mathcal{O}}(\vartheta_1 + i\pi + \varepsilon, \vartheta_2 + i\pi + \varepsilon, \vartheta_2, \vartheta_1)$$

Writing the sum in terms of contour integrals, after opening the contours and performing the large L limit the diagonal contribution becomes

$$
\begin{aligned}
C_{22}^{diag} = {} & \frac{1}{2} \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \Big[F_{4s}^{\mathcal{O}_1}(\vartheta_1, \vartheta_2) \langle \mathcal{O}_2 \rangle \\
& + F_{4s}^{\mathcal{O}_2}(\vartheta_1, \vartheta_2) \langle \mathcal{O}_1 \rangle \Big] e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} \\
& + \frac{1}{2} \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{[\cosh \vartheta_1 + \cosh \vartheta_2]^2}{\cosh \vartheta_1 \cosh \vartheta_2} F_2^{\mathcal{O}_1}(i\pi, 0) F_2^{\mathcal{O}_2}(i\pi, 0) e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} \\
& - \int \frac{d\vartheta_1}{2\pi} \Big[F_2^{\mathcal{O}_1}(i\pi, 0) \langle \mathcal{O}_2 \rangle + F_2^{\mathcal{O}_2}(i\pi, 0) \langle \mathcal{O}_1 \rangle \Big] e^{-m2R \cosh \vartheta_1} \\
& + \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} mL \cosh \vartheta_1 \Big[F_2^{\mathcal{O}_1}(i\pi, 0) \langle \mathcal{O}_2 \rangle + F_2^{\mathcal{O}_2}(i\pi, 0) \langle \mathcal{O}_1 \rangle \Big] e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} \\
& + Z_2 C_{00} \tag{3.3}
\end{aligned}
$$

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3.2.2 The non-diagonal part

In the non-diagonal part, one can use

$$\langle I_1, I_2 | \mathcal{O}(0) | J_1, J_2 \rangle = \frac{F_4^{\mathcal{O}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2')}{\sqrt{\rho_2(\vartheta_1, \vartheta_2)\rho_2(\vartheta_1', \vartheta_2')}}$$

to write

$$
\begin{aligned}
C_{22}^{nondiag} &= \sum_{I_1 > I_2} \sum_{J_1 > J_2} {}' \left\{ \frac{F_4^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2') F_4^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2', \vartheta_1')}{\rho_2(\vartheta_1, \vartheta_2)\rho_2(\vartheta_1', \vartheta_2')} \right. \\
&\qquad \left. \times K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2') \right\} \\
&= \sum_{I_1 > I_2} \frac{\tilde{C}_{22}}{\rho_2(\vartheta_1, \vartheta_2)}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{C}_{22} &= \sum_{J_1 > J_2} {}' \oint_{\mathcal{C}_{J_1} \times \mathcal{C}_{J_2}} \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} \left\{ \frac{F_4^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2') F_4^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2', \vartheta_1')}{\left[e^{iQ_1'(\vartheta_1', \vartheta_2')} + 1 \right] \left[e^{iQ_2'(\vartheta_1', \vartheta_2')} + 1 \right]} \right. \\
&\qquad \left. \times K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2') \right\}
\end{aligned}
$$

and the prime denotes the omission of the $\{J_1, J_2\} = \{I_1, I_2\}$ term. We can substitute

$$\sum_{J_1 > J_2} {}' \rightarrow \frac{1}{2} \sum_{J_1, J_2} {}'$$

since the form factors vanish when any two of their rapidity arguments are identical.

Now we open the contours to encircle the real axis in ϑ_1' and ϑ_2'. However, that brings more singularities inside the contour whose contribution must then be subtracted. These can be classified as follows:

1. **Spurious QQ-poles.** There are two such terms, which come from including the poles with $J_{1,2} = I_{1,2}$ or $J_{1,2} = I_{2,1}$. Their contribution vanishes for $L \to \infty$ [7], hence the term 'spurious'. However, they must be included in the numerical tests, therefore we provide their form in eqn. (B.9). Note that the form factors are not singular in this case, although their limits in such points are direction dependent.

2. **QF-poles.** In this case one of the integrations has a pole from a form factor, and the other one from a Q-term:

$$
\begin{aligned}
\textbf{QFI}: && \vartheta_1' &= \vartheta_1 & \& && Q_2'(\vartheta_1, \vartheta_2') &= 2\pi J_2 \\
\textbf{QFII}: && \vartheta_1' &= \vartheta_2 & \& && Q_2'(\vartheta_2, \vartheta_2') &= 2\pi J_2 \\
\textbf{QFIII}: && \vartheta_2' &= \vartheta_1 & \& && Q_1'(\vartheta_1', \vartheta_1) &= 2\pi J_1 \\
\textbf{QFIV}: && \vartheta_2' &= \vartheta_2 & \& && Q_1'(\vartheta_1', \vartheta_2) &= 2\pi J_1
\end{aligned}
$$

3. **FF poles.** In this case poles in both integrals come from form factors:

$$
\begin{aligned}
\textbf{FFI}: && \vartheta_1' &= \vartheta_2' = \vartheta_1 \\
\textbf{FFII}: && \vartheta_1' &= \vartheta_2' = \vartheta_2
\end{aligned}
$$

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The poles of the form factors can be separated by introducing the regular connected part F_{4rc}:

$$
\begin{aligned}
F_4^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2') &= \frac{A}{\vartheta_2 - \vartheta_1'} + \frac{B}{\vartheta_2 - \vartheta_2'} + \frac{C}{\vartheta_1 - \vartheta_1'} + \frac{D}{\vartheta_1 - \vartheta_2'} \\
&\quad + F_{4rc}^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta_1', \vartheta_2') \\
F_4^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2', \vartheta_1') &= \frac{E}{\vartheta_1 - \vartheta_2'} + \frac{F}{\vartheta_1 - \vartheta_1'} + \frac{G}{\vartheta_2 - \vartheta_2'} + \frac{H}{\vartheta_2 - \vartheta_1'} \\
&\quad + F_{4rc}^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta_2', \vartheta_1')
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
A &= i\left(S(\vartheta_2 - \vartheta_1) - S(\vartheta_1' - \vartheta_2')\right) F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_2') \\
B &= i\left(S(\vartheta_1' - \vartheta_2')S(\vartheta_2 - \vartheta_1) - 1\right) F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_1') \\
C &= i\left(1 - S(\vartheta_2 - \vartheta_1)S(\vartheta_1' - \vartheta_2')\right) F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_2') \\
D &= i\left(S(\vartheta_1' - \vartheta_2') - S(\vartheta_2 - \vartheta_1)\right) F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1') \\
E &= i\left(S(\vartheta_1 - \vartheta_2) - S(\vartheta_2' - \vartheta_1')\right) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') \\
F &= i\left(S(\vartheta_2' - \vartheta_1')S(\vartheta_1 - \vartheta_2) - 1\right) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_2') \\
G &= i\left(1 - S(\vartheta_1 - \vartheta_2)S(\vartheta_2' - \vartheta_1')\right) F_2^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_1') \\
H &= i\left(S(\vartheta_2' - \vartheta_1') - S(\vartheta_1 - \vartheta_2)\right) F_2^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2')
\end{aligned}
$$

Using the above notation, the pole terms resulting from the form factors can be obtained:

$$
\begin{aligned}
&F_4^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2')F_4^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2', \vartheta_1') = \\
&F_{4rc}^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta_1', \vartheta_2')F_{4rc}^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta_2', \vartheta_1') \\
&+F_{4rc}^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta_1', \vartheta_2')\left[\frac{E}{\vartheta_1 - \vartheta_2'} + \frac{F}{\vartheta_1 - \vartheta_1'} + \frac{G}{\vartheta_2 - \vartheta_2'} + \frac{H}{\vartheta_2 - \vartheta_1'}\right] \\
&+F_{4rc}^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta_2', \vartheta_1')\left[\frac{A}{\vartheta_2 - \vartheta_1'} + \frac{B}{\vartheta_2 - \vartheta_2'} + \frac{C}{\vartheta_1 - \vartheta_1'} + \frac{D}{\vartheta_1 - \vartheta_2'}\right] \\
&+\frac{AH}{(\vartheta_2 - \vartheta_1')^2} + \frac{BG}{(\vartheta_2 - \vartheta_2')^2} + \frac{CF}{(\vartheta_1 - \vartheta_1')^2} + \frac{DE}{(\vartheta_1 - \vartheta_2')^2} \\
&+\frac{AE + DH}{(\vartheta_2 - \vartheta_1')(\vartheta_1 - \vartheta_2')} + \frac{AF + CH}{(\vartheta_2 - \vartheta_1')(\vartheta_1 - \vartheta_1')} + \frac{AG + BH}{(\vartheta_2 - \vartheta_1')(\vartheta_2 - \vartheta_2')} \\
&+\frac{BE + DG}{(\vartheta_2 - \vartheta_2')(\vartheta_1 - \vartheta_2')} + \frac{BF + CG}{(\vartheta_2 - \vartheta_2')(\vartheta_1 - \vartheta_1')} + \frac{CE + DF}{(\vartheta_1 - \vartheta_1')(\vartheta_1 - \vartheta_2')}
\end{aligned}
\tag{3.5}
$$

from which one can identify the terms giving QF and FF type singularities. For the residue calculation, the formulas of Appendix A can be used. This results in certain differences from the result derived in [7], where too simplistic evaluation of residues resulted in some inaccuracies in the end result.

Once the residues are calculated, in the case of the QF terms a further summation remains which must be converted into an integral. It has the general form (here written for the case QFI):

$$
\sum_{J_2 \neq I_2} \frac{G\left(\vartheta_1, \vartheta_2, \vartheta_2'\right)}{\left.\frac{\partial Q_2'(\vartheta_1, \vartheta_2')}{\partial \vartheta_2'}\right|_{Q_2' = 2\pi J_2}}
$$

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where ϑ_2' is a solution to

$$Q_2'(\vartheta_1, \vartheta_2') = 2\pi J_2$$

and the case $J_2 = I_2$ was omitted since it is a spurious QQ singularity. One can convert the J_2 summation into integrals using the residue formula

$$-\sum_{J_2 \neq I_2} \oint_{\mathcal{C}_{J_2}} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta_2')}{e^{iQ_2'(\vartheta_1, \vartheta_2')} + 1}$$

Opening the contours and taking care to eliminate the contributions resulting from possible poles of the function G lying on the real ϑ_2' axis:

$$-\oint_{\leftrightarrows} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta_2')}{e^{iQ_2'(\vartheta_1, \vartheta_2')} + 1} + \oint_{\mathcal{C}_{\vartheta_2}} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta_2')}{e^{iQ_2'(\vartheta_1, \vartheta_2')} + 1} + \sum_{\text{poles of } G} \oint_{\mathcal{C}_{\vartheta_2^*}} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta_2')}{e^{iQ_2'(\vartheta_1, \vartheta_2')} + 1}$$

$$= -\oint_{\leftrightarrows} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta_2')}{e^{iQ_2'(\vartheta_1, \vartheta_2')} + 1} - \frac{G(\vartheta_1, \vartheta_2, \vartheta_2)}{\left.\frac{\partial Q_2'(\vartheta_1, \vartheta_2')}{\partial \vartheta_2'}\right|_{\vartheta_2' = \vartheta_2}} + \sum_{\text{poles of } G} \oint_{\mathcal{C}_{\vartheta_2^*}} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta_2')}{e^{iQ_2'(\vartheta_1, \vartheta_2')} + 1} \qquad (3.6)$$

where the second term corrects for the subtraction of the $J_2 = I_2$ case and ϑ_2^* denotes the location of the poles of G. The notation $\leftrightarrows$ corresponds to the straight line contours enclosing the real axis as illustrated in fig. 3.1. The full results of the residue calculations are given in Appendix B.

The $J_2 = I_2$ term typically is of order $O(1/L)$, except for second order pole contributions. This results in the following contribution to the QF terms:

$$F_2^{\mathcal{O}_1}(i\pi, 0) F_2^{\mathcal{O}_2}(i\pi, 0) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_2, \vartheta_1)$$
$$\times \left(\frac{[mL \cosh \vartheta_1 - \varphi(\vartheta_1 - \vartheta_2)]}{[mL \cosh \vartheta_2 + \varphi(\vartheta_2 - \vartheta_1)]} + \frac{[mL \cosh \vartheta_2 - \varphi(\vartheta_2 - \vartheta_1)]}{[mL \cosh \vartheta_1 + \varphi(\vartheta_1 - \vartheta_2)]} \right) \qquad (3.7)$$

which is included in $QF6$ in (B.7). This term was omitted by the calculation performed in [7]; its presence is critical for the cluster property.

3.2.3 Performing the I_1, I_2 sum and the large volume limit

We can write

$$\begin{aligned}
C_{22}^{nondiag} &= \sum_{I_1 > I_2} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{\rho_2(\vartheta_1, \vartheta_2)} = \frac{1}{2} \left(\sum_{I_1, I_2} - \sum_{I_1 = I_2} \right) \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{\rho_2(\vartheta_1, \vartheta_2)} = \\
&= \frac{1}{2} \sum_{I_1, I_2} \oint_{\mathcal{C}_{I_1}} \oint_{\mathcal{C}_{I_2}} \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{\left[e^{iQ_1(\vartheta_1, \vartheta_2)} + 1\right]\left[e^{iQ_2(\vartheta_1, \vartheta_2)} + 1\right]} \\
&\quad + \frac{1}{2} \sum_{I_1 = I_2} \oint_{\mathcal{C}_{I_1}} \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_1)}{\left[e^{iQ_1(\vartheta_1, \vartheta_1)} + 1\right]} \frac{\rho_1(\vartheta_1)}{\rho_2(\vartheta_1, \vartheta_1)}
\end{aligned}$$

Since $\tilde{C}_{22}$ doesn't have any pole we open the contours in the usual way enclosing the real axis as illustrated in fig. 3.1. For the $L \to \infty$ it is necessary to examine the behavior of the Q-functions:

$$\begin{aligned}
iQ_1(\vartheta_1 + i\varepsilon_1, \vartheta_2 + i\varepsilon_2) &= imL \sinh(\vartheta_1 + i\varepsilon_1) + i\delta(\vartheta_1 + i\varepsilon_1 - \vartheta_2 - i\varepsilon_2) = \\
&= imL \sinh \vartheta_1 \cos \varepsilon_1 - mL \cosh \vartheta_1 \sin \varepsilon_1 + i\delta(\vartheta_1 + i\varepsilon_1 - \vartheta_2 - i\varepsilon_2)
\end{aligned}$$

and similarly for Q_2 and $Q'_{1,2}$. This results in the following limits:

$$\lim_{L\to\infty} \frac{1}{e^{iQ_i(\vartheta_1+i\varepsilon_1,\vartheta_2+i\varepsilon_2)}+1} = \begin{cases} 1, & \varepsilon_i \in [0,\pi]+2n\pi \\ 0, & \varepsilon_i \in [\pi,2\pi]+2n\pi \end{cases}$$

$$\lim_{L\to\infty} \frac{1}{e^{iQ_1(\vartheta_1+i\varepsilon_1,\vartheta_1+i\varepsilon_1)}+1} = \begin{cases} 1, & \varepsilon_1 \in [0,\pi]+2n\pi \\ 0, & \varepsilon_1 \in [\pi,2\pi]+2n\pi \end{cases}$$

$$\lim_{L\to\infty} \frac{1}{e^{iQ'_i(\vartheta'_1+i\varepsilon_1,\vartheta'_2+i\varepsilon_2)}+1} = \begin{cases} 1, & \varepsilon_i \in [0,\pi]+2n\pi \\ 0, & \varepsilon_i \in [\pi,2\pi]+2n\pi \end{cases}$$

Therefore only the upper contours need to be kept, since all other terms vanish exponentially for large L:

$$\frac{1}{2}\iint \frac{d\vartheta_1}{2\pi}\frac{d\vartheta_2}{2\pi}\tilde{C}_{22}(\vartheta_1+i\varepsilon,\vartheta_2+i\varepsilon) - \frac{1}{2}\int \frac{d\vartheta_1}{2\pi}\tilde{C}_{22}(\vartheta_1+i\varepsilon,\vartheta_1+i\varepsilon)\frac{\rho_1(\vartheta_1+i\varepsilon)}{\rho_2(\vartheta_1+i\varepsilon,\vartheta_1+i\varepsilon)}$$

In addition, the integrals can be shifted to the real axis. However this leads to singularities in the contribution like QF5 (B.6) due to the term containing

$$\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta'_1,\vartheta_1)\left[S(\vartheta_1-\vartheta_2)-S(\vartheta'_1-\vartheta_1)\right]}{\left[e^{iQ'_1(\vartheta'_1,\vartheta_1)}+1\right](\vartheta_1-\vartheta'_1)} \tag{3.8}$$

which can be treated using the identity

$$\frac{1}{x\pm i\varepsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x)$$

3.3 End result for D_{22}

The terms divergent as $L\to\infty$ drop out when including the contribution $-Z_1\tilde{D}_{11}-Z_2 C_{00}$. We can also combine some terms by introducing the function

$$
\begin{aligned}
F_{4ss}^{\mathcal{O}}(\vartheta_1+i\pi,\vartheta_2+i\pi,\vartheta'_2,\vartheta'_1) &= \\
&= \frac{i\left(S(\vartheta_1-\vartheta_2)-S(\vartheta'_2-\vartheta'_1)\right)}{\vartheta_1-\vartheta'_2}F_2^{\mathcal{O}}(\vartheta_2+i\pi,\vartheta'_1) \\
&+ \frac{i\left(1-S(\vartheta_1-\vartheta_2)S(\vartheta'_2-\vartheta'_1)\right)}{\vartheta_2-\vartheta'_2}F_2^{\mathcal{O}}(\vartheta_1+i\pi,\vartheta'_1) + \frac{i\left(S(\vartheta'_2-\vartheta'_1)-S(\vartheta_1-\vartheta_2)\right)}{\vartheta_2-\vartheta'_1}F_2^{\mathcal{O}}(\vartheta_1+i\pi,\vartheta'_2) \\
&+ F_{4rc}^{\mathcal{O}}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta'_2,\vartheta'_1)
\end{aligned} \tag{3.9}
$$

The end result is

$$
\begin{aligned}
D_{22} &= \frac{1}{4} \iiiint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} F_4^{\mathcal{O}_1} (\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1' + i\varepsilon, \vartheta_2' + i\varepsilon) \qquad (3.10) \\
&\quad \times F_4^{\mathcal{O}_2} (\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2' + i\varepsilon, \vartheta_1' + i\varepsilon) K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta_1' + i\varepsilon, \vartheta_2' + i\varepsilon) \\
&\quad + \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \wp \int \frac{\mathrm{d}\vartheta_1'}{2\pi} \left\{ F_{4ss}^{\mathcal{O}_1} (\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta_1', \vartheta_1) F_2^{\mathcal{O}_2} (\vartheta_2 + i\pi, \vartheta_1') \right. \\
&\quad \left. + F_{4ss}^{\mathcal{O}_2} (\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta_1', \vartheta_1) F_2^{\mathcal{O}_1} (\vartheta_2 + i\pi, \vartheta_1') \right\} K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_1') \\
&\quad + \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \int \frac{\mathrm{d}\vartheta_1'}{2\pi} F_2^{\mathcal{O}_1} (\vartheta_2 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2} (\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_1') \\
&\quad \times [(1 - S(\vartheta_1' - \vartheta_1) S(\vartheta_1 - \vartheta_2)) (mx \cosh \vartheta_1 - imt \sinh \vartheta_1) \\
&\quad \left. - \varphi(\vartheta_1' - \vartheta_1) S(\vartheta_1' - \vartheta_1) S(\vartheta_1 - \vartheta_2) \right] \\
&\quad \underline{- \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} F_2^{\mathcal{O}_1} (\vartheta_2 + i\pi, \vartheta_1) F_2^{\mathcal{O}_2} (\vartheta_2 + i\pi, \vartheta_1) K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_1)} \\
&\quad - \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} F_2^{\mathcal{O}_1} (\vartheta_1 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2} (\vartheta_1 + i\pi, \vartheta_1') K_{t,x}^{(R)} (\vartheta_1, \vartheta_1 | \vartheta_1, \vartheta_1') \\
&\quad + \frac{1}{2} \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \left[F_{4s}^{\mathcal{O}_1} (\vartheta_1, \vartheta_2) \langle \mathcal{O}_2 \rangle + F_{4s}^{\mathcal{O}_2} (\vartheta_1, \vartheta_2) \langle \mathcal{O}_1 \rangle \right] K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2) \\
&\quad \underline{+ \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} F_2^{\mathcal{O}_1} (i\pi, 0) F_2^{\mathcal{O}_2} (i\pi, 0) K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2)} \\
&\quad - \int \frac{\mathrm{d}\vartheta_1}{2\pi} \left[F_2^{\mathcal{O}_1} (i\pi, 0) \langle \mathcal{O}_2 \rangle + F_2^{\mathcal{O}_2} (i\pi, 0) \langle \mathcal{O}_1 \rangle \right] K_{t,x}^{(R)} (\vartheta_1, \vartheta_1 | \vartheta_1, \vartheta_1)
\end{aligned}
$$

where $\mathcal{P}$ denotes a principal value integral, $K_{t,x}^{(R)}$ is defined in (3.1) and F_{4s} is the so-called symmetric evaluation of the form factor used in [26]:

$$
F_{4s}^{\mathcal{O}} (\vartheta_1, \vartheta_2) = \lim_{\epsilon \to 0} F^{\mathcal{O}} (\vartheta_1 + i\pi + \epsilon, \vartheta_2 + i\pi + \epsilon, \vartheta_2, \vartheta_1)
$$

Note that by introducing F_{4ss} we combined the terms (3.8) into the second integral, hence the need for the principal value.

In (3.10), the underlined pieces are the contributions that are different from the earlier calculation performed in [7]. The first underlined term only corrects a typo in [7], where this piece was printed with the wrong sign. The second one comes from the subtraction of poles of the integrand in (3.6) and the careful evaluation of the principal value term (3.8), both of which occur in the manipulation of the QF5 contributions (B.6).

The third underlined term plays a crucial role in the cluster property. It is the left-over from the term

$$
\frac{1}{2} \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{[\cosh \vartheta_1 + \cosh \vartheta_2]^2}{\cosh \vartheta_1 \cosh \vartheta_2} F_2^{\mathcal{O}_1} (i\pi, 0) F_2^{\mathcal{O}_2} (i\pi, 0) e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)}
$$

present in the diagonal contribution (3.3), the dependence on

$$
\frac{[\cosh \vartheta_1 + \cosh \vartheta_2]^2}{\cosh \vartheta_1 \cosh \vartheta_2} = \frac{\cosh \vartheta_1}{\cosh \vartheta_2} + \frac{\cosh \vartheta_2}{\cosh \vartheta_1} + 2
$$

is simplified by the inclusion of the contribution (3.7), coming from the second order pole terms collected in QF6 (B.7). In the large L limit, the terms depending on the cosh ratios cancel, leaving us with the last underlined piece in (3.10). As mentioned before, one of the mistakes made in the evaluation of D_{22} in [7] was the omission of this piece.

3.4   The full two-point function up to D_{22}

For completeness, we also give here the lower contributions to the two-point function. These are exactly the same as in [39, 7], so we do not give the derivations here. The calculations are almost trivial with the exception of D_{12}, where one can use either the derivations presented in [7], or follow the steps outlined above, with slight modifications. The terms D_{NM} with $N \leq M \leq 2$ are

$$
\begin{aligned}
D_{00} &= \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \\
D_{01} &= \int \frac{d\vartheta_1}{2\pi} F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} e^{-imx \sinh \vartheta_1 - mt \cosh \vartheta_1} \\
D_{02} &= \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_1, \vartheta_2) F^{\mathcal{O}_2}(\vartheta_2, \vartheta_1) e^{-imx(\sinh \vartheta_1 + \sinh \vartheta_2) - mt(\cosh \vartheta_1 + \cosh \vartheta_2)} \\
D_{11} &= \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_2) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1) \\
&\quad \times e^{imx(\sinh \vartheta_1 - \sinh \vartheta_2)} e^{-m(R-t)\cosh \vartheta_1} e^{-mt \cosh \vartheta_2} \\
&\quad + \left[\langle \mathcal{O}_1 \rangle F_{2s}^{\mathcal{O}_2} + \langle \mathcal{O}_2 \rangle F_{2s}^{\mathcal{O}_1} \right] \int \frac{d\vartheta_1}{2\pi} e^{-mR \cosh \vartheta_1} \\
D_{12} &= \frac{1}{2} \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_3^{\mathcal{O}_1}(\vartheta_1 + i(\pi + \varepsilon), \vartheta_1', \vartheta_2') F_3^{\mathcal{O}_2}(\vartheta_1 + i(\pi + \varepsilon), \vartheta_2', \vartheta_1') \\
&\quad \times K_{t,x}^{(R)}(\vartheta_1 + i\varepsilon | \vartheta_1', \vartheta_2') \\
&\quad + \iint \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} \left[F_1^{\mathcal{O}_1} F_{3rc}^{\mathcal{O}_2}(\vartheta_1' + i\pi | \vartheta_1', \vartheta_2') + F_1^{\mathcal{O}_2} F_{3rc}^{\mathcal{O}_1}(\vartheta_1' + i\pi | \vartheta_1', \vartheta_2') \right] K_{t,x}^{(R)}(\vartheta_1' | \vartheta_1', \vartheta_2') \\
&\quad + \iint \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} 2i \frac{F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2}}{(\vartheta_1' - \vartheta_2')} (S(\vartheta_1' - \vartheta_2') - 1) K_{t,x}^{(R)}(\vartheta_1' | \vartheta_1', \vartheta_2') \\
&\quad + \iint \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} K_{t,x}^{(R)}(\vartheta_1' | \vartheta_1', \vartheta_2') (S(\vartheta_1' - \vartheta_2') - 1)(mx \cosh \vartheta_1' + im(R-t) \sinh \vartheta_1') \\
&\quad - \int \frac{d\vartheta_1'}{2\pi} F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} K_{t,x}^{(R)}(\vartheta_1' | \vartheta_1', \vartheta_1') \qquad (3.11)
\end{aligned}
$$

and D_{22} is given in (3.10). In D_{12} we defined the regular connected form factor function F_{3rc} via the following separation of the kinematical pole terms

$$
F_3^{\mathcal{O}}(\vartheta_1 + i\pi, \vartheta_1', \vartheta_2') = \frac{i(1 - S(\vartheta_1' - \vartheta_2')) F_1^{\mathcal{O}}}{\vartheta_1 - \vartheta_1'} + \frac{i(S(\vartheta_1' - \vartheta_2') - 1) F_1^{\mathcal{O}}}{\vartheta_1 - \vartheta_2'} + F_{3rc}^{\mathcal{O}}(\vartheta_1 + i\pi | \vartheta_1', \vartheta_2')
$$

and used the abbreviation

$$
K_{t,x}^{(R)}(\vartheta_1 | \vartheta_1', \vartheta_2') = e^{imx(\sinh \vartheta_1 - \sinh \vartheta_1' - \sinh \vartheta_2')} e^{-m(R-t)\cosh \vartheta_1} e^{-mt(\cosh \vartheta_1' + \cosh \vartheta_2')}
$$

The other contributions D_{NM} for $M > N$ can be obtained from D_{MN} by exchanging $\mathcal{O}_1$ with $\mathcal{O}_2$ and replacing $t \to R - t$, $x \to -x$.

3.5   The symmetry of the D_{22} term

The relation between the coefficients D_{NM} and D_{MN} stated above, when applied to D_{22} leads to the property that D_{22} must be symmetric under the following transformation:

$$
\begin{aligned}
t &\to R - t \\
\mathcal{O}^1 &\leftrightarrow \mathcal{O}^2 \\
x &\to -x
\end{aligned}
$$

This is the same as requiring that the result should be independent of which two-particle summation is performed first. However, when implementing such a transformation in (3.10), the signs of the ϵ terms change, and therefore the contours must be pulled back to their original positions. The contour deformation encounters all the singularities on the real axis that were treated previously in this section, so the appropriate residue contributions must be computed. This computation is relegated to Appendix C, where it is demonstrated that the required symmetry property indeed holds, providing the first nontrivial test of the result (3.10).

4   Numerical verification of the analytic results

The goal of this section is to validate the D_{22} formula numerically. For this purpose we evaluated directly the sum for the two-particle states and compare it with the result of the contour integrals. For calculations we used the sinh-Gordon model with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{m^2}{g^2}\cosh g\Phi \tag{4.1}$$

The model contains one massive particle, and its two-particle scattering matrix is simple but nontrivial:

$$S(\theta) = \frac{\sinh\theta - i\sin\frac{\pi B}{2}}{\sinh\theta + i\sin\frac{\pi B}{2}}$$

where

$$B = \frac{2g^2}{8\pi + g^2}$$

The nontrivial S-matrix is important, since our formula contains the scattering matrix and its derivative in an essential way which we would like to verify. For the fields in the correlator, we chose the exponential operators

$$e^{kg\Phi} \tag{4.2}$$

normalized to have vacuum expectation value 1, since their form factors are explicitly known [41, 42]:

$$F_n^{(k)}(\theta_1, \theta_2, \ldots, \theta_n) = H_n \frac{P_n^{(k)}(x_1, x_2 \ldots, x_n)}{\prod_{i<j}(x_i + x_j)} \prod_{i<j} f(\theta_i - \theta_j)$$

$$x_i = e^{\theta_i} \tag{4.3}$$

where the polynomials $P_n^{(k)}$ are given by

$$
\begin{aligned}
P_1^{(k)} &= [k] \\
P_n^{(k)} &= [k]\det M^{(n)}(k) \qquad n > 1 \\
&\quad M_{ij}^{(n)}(k) = [i - j + k]\sigma_{2i-j}^{(n)}(x_1, x_2 \ldots, x_n) \quad i, j = 1, \ldots, n-1
\end{aligned}
$$

with

$$H_n = \left(\frac{4\sin\pi B/2}{f(i\pi)}\right)^{n/2} \qquad \text{and} \qquad [n] = \frac{\sin\frac{n\pi B}{2}}{\sin\frac{\pi B}{2}}$$

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Furthermore, $\sigma_l^{(n)}$ denotes the elementary symmetric polynomials of n variables defined by

$$\prod_{i=1}^{n}(x+x_i) = \sum_{l=1}^{n} x^{n-l}\sigma_l^{(n)}(x_1,\ldots,x_n)$$

$$\sigma_l^{(n)} \equiv 0 \qquad \text{if } l < 0 \text{ or } l > n$$

and the minimal two-particle form factor is given by

$$f(\theta) = \mathcal{N}\exp\left[8\int_0^\infty \frac{dx}{x}\sin^2\left(\frac{x(i\pi-\theta)}{2\pi}\right)\frac{\sinh\frac{xB}{4}\sinh(1-\frac{B}{2})\frac{x}{2}\sinh\frac{x}{2}}{\sinh^2 x}\right] \tag{4.4}$$

where

$$\mathcal{N} = \exp\left[-4\int_0^\infty \frac{dx}{x}\frac{\sinh\frac{xB}{4}\sinh(1-\frac{B}{2})\frac{x}{2}\sinh\frac{x}{2}}{\sinh^2 x}\right] \tag{4.5}$$

4.1 Evaluating the two-particle sum

Numerical evaluation of the sum is only possible at finite volume. The factors $K_{t,x}^{(R)}$ decrease exponentially at large rapidities, so it is possible to choose a rapidity cutoff and restrict the summation up to the corresponding Bethe-Yang quantum number. However for large volume this quantum number cutoff is still too big and it is practically impossible to evaluate the four particle sum. The compromise is to evaluate only the inner two particle sum with fixed outer rapidities at moderate volume. This is enough to check the validity of all the nontrivial contour deformations and residue manipulation in the D_{22} calculation. We can write

$$C_{22}^{nondiag} = \sum_{I_1>I_2}\frac{\tilde{C}_{22}(\vartheta_1,\vartheta_2)}{\rho_2(\vartheta_1,\vartheta_2)}$$

with

$$\tilde{C}_{22}(\vartheta_1,\vartheta_2) = \sum_{J_1>J_2}{}'\frac{F_4^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi,\vartheta_1',\vartheta_2')F_4^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi,\vartheta_2',\vartheta_1')}{\rho_2(\vartheta_1',\vartheta_2')}$$
$$\times K_{t,x}^{(R)}(\vartheta_1,\vartheta_2,\vartheta_1',\vartheta_2')$$

and evaluate $\tilde{C}_{22}$ for some given value of $\vartheta_{1,2}$, corresponding to a solution of the Bethe-Yang equations (3.2) with some quantum numbers $\{I_1,I_2\}$.

The parameters for the evaluation can be chosen to help with the convergence of the summation, while ensuring that the structure of the expression tested remains general. The exponential operators (4.2) in the Sinh-Gordon model can be parametrized by the number k that we chose for our evaluations as $k_1 = 2$ for $\mathcal{O}_1$ and $k_2 = 4$ for $\mathcal{O}_2$. The essential structure of the formula does not depend on this choice. The space-time parameters and the temperature were chosen as $mx = 0.0$, $mt = 0.4$, and $mR = 0.8$. Setting mx to zero does not hide any important structure of the equation, but makes the expression real and that helps in comparing the results with the contour integrals. The sum was evaluated with several values for the volume, the sinh-Gordon coupling constant and the outer rapidities:

$$mL = (10,\ 15,\ 20,\ 25,\ 30)$$
$$B = (0.1,\ 0.2,\ 0.3,\ 0.4\ ,0.55,\ 0.7,\ 0.9)$$
$$\{I_1,I_2\} \in \left\{\left\{\frac{5}{2},\frac{1}{2}\right\},\left\{\frac{11}{2},\frac{-5}{2}\right\},\left\{\frac{-5}{2},\frac{-21}{2}\right\},\left\{\frac{7}{2},\frac{-7}{2}\right\},\left\{\frac{1}{2},\frac{-1}{2}\right\}\right\}$$

The rapidity cutoff for the quantum numbers included in the sum was chosen as $\vartheta = 3.0, 4.0, 5.0, 6.0$, and the numerical results showed that for the value $\vartheta = 6.0$ the discrete sum was evaluated within a relative error of less than 10^{-14}.

4.2 Evaluating the contour integrals

To compare the results of the contour integrals with the direct sum, the calculation must be performed at the same volumes. In this regime the exponential and power corrections in volume L are not negligible, so they must be taken into account. Exponential corrections come from the integration on the contours going under the real axis, while power corrections come from the total derivative contribution in the second order pole calculation and from the $\{J_1, J_2\} = \{I_1, I_2\}$ point in the pole and the double integral contributions. The explicit formulas can be found in the Appendix B.

The integration contours run below and above the real axis, and it is important to find a choice that is optimal for numerical evaluation. The form factors and hence the integrands have poles on the real axis, so it would be better to integrate as far from the real axis as possible. However for imaginary parts of rapidities larger than $\frac{\pi}{2}$ the factor $K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1', \vartheta_2')$ becomes oscillating and exponentially growing in the rapidity parameters instead of decaying. Another issue is that the form factors and scattering matrices are also evaluated at rapidities that lie out of the physical strip. In the sinh-Gordon model the scattering matrix and hence the minimal form factor have poles out of the physical strip, with imaginary positions that are proportional to the coupling parameter B [41]. Therefore the contour must be chosen to lie between these poles on the one hand and the poles on the real axis on the other hand. At the same time it must run as far away from all singularities as possible, and also to be closer to the real axis than $\frac{\pi}{2}$. For small B this leaves little space for the contours so they run relatively close to the poles, resulting in a larger error in the numerical integration. The integration itself was performed using Mathematica[1] and the Cuba library for multidimensional numerical integrations [43].

4.3 Comparing the results

$B \backslash mL$	10	15	20	25	30
0.1	8.56×10^{-8}	1.03×10^{-7}	4.93×10^{-9}	3.29×10^{-7}	1.31×10^{-8}
0.2	7.38×10^{-10}	3.53×10^{-10}	6.29×10^{-11}	9.74×10^{-10}	1.96×10^{-10}
0.3	1.74×10^{-10}	1.13×10^{-10}	1.25×10^{-10}	1.26×10^{-10}	1.26×10^{-10}
0.4	1.42×10^{-10}	1.42×10^{-10}	1.41×10^{-10}	1.4×10^{-10}	1.4×10^{-10}
0.55	1.41×10^{-10}	1.42×10^{-10}	1.42×10^{-10}	1.42×10^{-10}	1.42×10^{-10}
0.7	1.33×10^{-10}	1.34×10^{-10}	1.33×10^{-10}	1.33×10^{-10}	1.33×10^{-10}
0.9	1.3×10^{-10}	1.3×10^{-10}	1.3×10^{-10}	1.3×10^{-10}	1.3×10^{-10}

Table 4.1: Relative error of the difference between the direct sum and the contour integral evaluation of $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ with $\{I_1, I_2\} = \left\{\frac{7}{2}, \frac{-7}{2}\right\}$

Table 4.1 shows the relative deviation between the direct sum and the contour integral evaluation of $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ with $\{I_1, I_2\} = \left\{\frac{7}{2}, \frac{-7}{2}\right\}$. Note that the relative error decreases as B grows which can be understood from the conditions for the choice of the integration contour mentioned above. Based on the above understanding of the deviations in the relative errors for different parameters, and the fact that this pattern of dependence was

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[1]Wolfram Research Inc, *Mathematica*, version 8.0, Champaign Illinois, 2010.

the same for every value of I_1, I_2 we checked, it can be inferred that the difference of the sum and the contour integration is only due to the numerical errors of integration.

To provide a further support for this conclusion, the above numerical test was repeated for C_{12}. The formula of C_{12} is derived in two independent ways in [7] (depending on whether the one-particle or the two-particle summation is performed first), and therefore its validity is quite certain even without a numerical test. As in the case of C_{22}, let us denote by $\tilde{C}_{12}(\vartheta_1)$ the result of performing the two-particle summation first with fixed rapidity of the one-particle state. Table 4.2 shows the relative deviation between the direct sum and the contour integral evaluation of $\tilde{C}_{12}(\vartheta_1)$ with $I_1 = 17$ as the Bethe-Yang quantum number of the one-particle state. The relative deviation has the same pattern as for $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$, and is essentially of the same magnitude. Therefore this evaluation gives an independent support for the assertion that the deviations are caused by errors of numerical integration.

As the derivation of D_{22} from $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ is almost trivial, the above numerical tests also confirm the details of our analytic result for D_{22}.

$B\backslash mL$	10	15	20	25	30
0.1	2.03×10^{-6}	3.56×10^{-6}	3.41×10^{-7}	1.56×10^{-7}	8.81×10^{-8}
0.2	7.5×10^{-9}	5.03×10^{-9}	1.97×10^{-10}	1.52×10^{-9}	7.73×10^{-10}
0.3	1.48×10^{-10}	7.76×10^{-11}	3.1×10^{-11}	6.22×10^{-11}	3.13×10^{-11}
0.4	2.35×10^{-11}	5.29×10^{-11}	5.81×10^{-11}	6.77×10^{-11}	6.59×10^{-11}
0.55	5.87×10^{-11}	6.61×10^{-11}	6.87×10^{-11}	6.9×10^{-11}	6.96×10^{-11}
0.7	2.96×10^{-11}	5.32×10^{-11}	6.02×10^{-11}	6.25×10^{-11}	6.38×10^{-11}
0.9	6.01×10^{-11}	6.48×10^{-11}	6.52×10^{-11}	6.51×10^{-11}	6.52×10^{-11}

Table 4.2: Relative error of the difference between the direct sum and the contour integral evaluation of $\tilde{C}_{12}(\vartheta_1)$ with $I_1 = 17$

5 The cluster property of the two-point function

Another important test of the results is provided by checking that the two-point function has the cluster property

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\rangle^R \sim \langle \mathcal{O}_1(0,0)\rangle^R \langle \mathcal{O}_2(0,0)\rangle^R$$

when the spatial separation x grows large. Using the expansion up to D_{22} one can write

$$\begin{aligned}
\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\rangle^R &= \sum_{N,M} D_{NM} \\
&= D_{00} + D_{01} + D_{10} + D_{11} + D_{02} + D_{20} + D_{12} + D_{21} + D_{22} + \dots
\end{aligned}$$

For $mx \gg 1$ the terms containing

$$e^{imx\left(\sum_k \sinh \vartheta_k - \sum_l \sinh \vartheta_l'\right)}$$

oscillate very fast, and therefore the support of the (multiple) rapidity integrals is restricted to the zero measure set

$$\sum_k \sinh \vartheta_k = \sum_l \sinh \vartheta_l'$$

and the integral vanishes. Although this argument looks simple, there is a possible problem. Namely, the argument only works if the integrands of the x-dependent terms are all regular. A nontrivial example is the term

$$\frac{1}{2} \iint \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} \left\{ 2i \frac{F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2}}{(\vartheta_1' - \vartheta_2')} \left(S(\vartheta_1' - \vartheta_2') - 1\right) K_{t,x}^{(R)}(\vartheta_1' | \vartheta_1', \vartheta_2') + (\vartheta_1' \leftrightarrow \vartheta_2') \right\}$$

in the contribution D_{12} (cf. eqn. (3.11)), which is in fact regular at $\vartheta_1' = \vartheta_2'$ when the two terms inside the braces are added together. In the case of the principal value integral in D_{22} in eqn. (3.10), the regularity of the integrand is ensured by the principal value prescription itself.

As a result, one only needs to examine the terms that are x-independent. We denote these by putting a bar over the respective contribution D_{NM} and they read:

$$
\begin{aligned}
\bar{D}_{00} &= \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle \\
\bar{D}_{01} &= \bar{D}_{10} = \bar{D}_{02} = \bar{D}_{20} = 0 \\
\bar{D}_{11} &= \left[\langle\mathcal{O}_1\rangle F_2^{\mathcal{O}_2}(i\pi, 0) + \langle\mathcal{O}_2\rangle F_2^{\mathcal{O}_1}(i\pi, 0)\right] \int \frac{\mathrm{d}\vartheta_1}{2\pi} e^{-mR\cosh\vartheta_1} \\
\bar{D}_{12} &= \bar{D}_{21} = 0 \\
\bar{D}_{22} &= +\frac{1}{2} \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \left[F_{4s}^{\mathcal{O}_1}(\vartheta_1, \vartheta_2)\langle\mathcal{O}_2\rangle + F_{4s}^{\mathcal{O}_2}(\vartheta_1, \vartheta_2)\langle\mathcal{O}_1\rangle\right. \\
&\quad \times e^{-mR(\cosh\vartheta_1 + \cosh\vartheta_2)} \\
&\quad + \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} F_2^{\mathcal{O}_1}(i\pi, 0) F_2^{\mathcal{O}_2}(i\pi, 0) e^{-mR(\cosh\vartheta_1 + \cosh\vartheta_2)} \\
&\quad - \int \frac{\mathrm{d}\vartheta_1}{2\pi} \left[F_2^{\mathcal{O}_1}(i\pi, 0)\langle\mathcal{O}_2\rangle + F_2^{\mathcal{O}_2}(i\pi, 0)\langle\mathcal{O}_1\rangle\right] e^{-2mR\cosh\vartheta_1}
\end{aligned}
$$

The one-point function up to two-particle order is [26]:

$$
\begin{aligned}
\langle\mathcal{O}\rangle^R &= \langle\mathcal{O}\rangle + \int \frac{\mathrm{d}\vartheta_1}{2\pi} F_2^{\mathcal{O}}(i\pi, 0) e^{-mR\cosh\vartheta_1} - \int \frac{\mathrm{d}\vartheta_1}{2\pi} F_2^{\mathcal{O}}(i\pi, 0) e^{-2mR\cosh\vartheta_1} \\
&\quad + \frac{1}{2} \iint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} F_{4s}^{\mathcal{O}}(\vartheta_1, \vartheta_2) e^{-mR(\cosh\vartheta_1 + \cosh\vartheta_2)} + O\left(e^{-3mR}\right)
\end{aligned}
$$

As a result one obtains that

$$\bar{D}_{00} + \bar{D}_{11} + \bar{D}_{22} = \langle\mathcal{O}_1\rangle^R \langle\mathcal{O}_2\rangle^R + O\left(e^{-3mR}\right)$$

and therefore the cluster property is satisfied to the given order. Note that the same argument shows that the formula for D_{22} derived in [7] violates the cluster property, providing another argument that it needs to be corrected.

6 Conclusions

In this work we revisited the finite volume regularization of thermal correlators introduced in [7], which is based on the finite volume form factor formalism [25, 26]. We have shown that the original results for the two-particle–two-particle contribution D_{22} need to be slightly corrected, and presented a modified prescription for evaluation of residue contributions. As a result, we now have the expansion up to all terms involving intermediate

states with no more than two particles. In addition, the result for the nontrivial terms D_{12} and D_{22} was cross-checked with a numerical evaluation of the finite volume summation over the intermediate multi-particle states. It was also established that the correlation function given by the final formulas (3.10) and (3.11) satisfies the cluster property. In addition, it was shown to possess a symmetry property which follows from the general structure of the spectral expansion.

The formalism presented here can be extended to compute any higher correction in the series. However, the calculation of D_{22} is already very tedious, and it is expected to become even longer for higher terms. In view of potential applications to condensed matter systems, it is likely that the present evaluation would suffice for most of the cases.

Another reason why the result evaluated up to D_{22} is interesting is that this is the part which generalizes to non-integrable field theories. Breaking integrability in general allows inelastic processes; however, below the inelastic threshold the finite volume levels can still be described using only the elastic phase shift and the same quantization conditions as in (3.2) [44, 45]. Therefore the present computation can be extended self-consistently whenever the states dominating the spectral expansion are below the inelastic threshold.

On the other hand, it remains an open question whether the full expansion for the finite temperature two-point function in integrable models can be recast in a form similar to the expression conjectured by Leclair and Mussardo [16]. Such an expression would represent a partial re-summation of the series, expressing the correlator in terms of Fermi-Dirac distribution for the dressed particles corresponding to the representation of the system as a free gas of quasi-particles under the thermodynamic Bethe Ansatz [46]. It is known that this re-summation is possible for the one-point function [26, 27], even for the case of operators located on a system boundary [31]. In that respect, the explicit dependence of D_{22} on the two-particle S-matrix (cf. the first underlined term in eqn. (3.10)) does not bode well. The TBA equation contains only the derivative of the phase-shift φ, so any dressing term from there is only expected to depend only on φ. Some partial integration tricks can be performed to shift this dependence around, but we have found no way of eliminating it. On the other hand, it was noticed in [7] that the D_{1n} contributions allow some re-summation by dressing the contributions D_{0n-1}. There is a possibility that a redefinition of the form factor terms could help, i.e. if one used some definition for the desingularized form factors different from the F_{rc} or F_{ss}. At present it is not known how to accomplish this; however, there is still hope for recovering some expression similar to the original Leclair-Mussardo conjecture. Evaluation of some higher order corrections could shed light on the structure of the series, and the experience gained in the present work opens the way to performing these calculations, armed with a numerical method to verify the results of the complicated analytic manipulations. We hope to return to this line of thought in the near future.

The present calculation was performed for a theory with a single massive particles. Adding more particles to the spectrum is rather straightforward as long as the scattering remains diagonal; it is only necessary to add particle species labels in appropriate places. For non-diagonal scattering theories, recent progress has made almost all matrix elements available in finite volume [47, 48, 49], except for matrix elements involving disconnected pieces when the states involved in the matrix element are subject to non-diagonal scattering. However, more recently we have solved the issue for two-particle states[2] [50], which means that the series presented here can be evaluated for general integrable field theories, including those with non-diagonal scattering such as the sine-Gordon or O(3) σ models [8].

---

[2]In fact, the paper [50] presents a conjectured solution for any number of particles, but only the two-particle case is backed up by (very strong) numerical evidence.

A Residue evaluations

A.1 First order poles

In this case, the two-dimensional residue formula

$$\oint_{\mathcal{C}_a \times \mathcal{C}_b} \oint \frac{\mathrm{d}z_1}{2\pi i} \frac{\mathrm{d}z_2}{2\pi i} \frac{g(z_1, z_2)}{f_1(z_1, z_2) f_2(z_1, z_2)} = \frac{g(a, b)}{\det \left(\frac{\partial f_i}{\partial z_j} \right) \Big|_{(z_1, z_2) = (a, b)}}$$

can be applied directly as

$$\oint_{\mathcal{C}_a \times \mathcal{C}_J} \oint \frac{\mathrm{d}z_1}{2\pi i} \frac{\mathrm{d}z_2}{2\pi i} \frac{g(z_1, z_2)}{[a - z_1] \left[e^{iQ_2(z_1, z_2)} + 1 \right]} = \frac{g(a, z_2^*)}{i \frac{\partial Q_2(a, z_2)}{\partial z_2} \Big|_{z_2 = z_2^*}}$$

where the $(a - z_1)$ is the pole term coming from the appropriate form factor, and z_2^* is the root of

$$e^{iQ_2(a, z_2^*)} + 1 = 0$$

A.2 Second order poles

These can be evaluated by successive integration, performing first the integral over the second order pole coming from the form factor. The fundamental formula to use is

$$\operatorname*{Res}_{z=a} \frac{h(z)}{g(z)} = \frac{2h'(a)}{g''(a)} - \frac{2g'''(a)}{3 [g''(a)]^2} h(a) \tag{A.1}$$

where for a second order pole $g(a) = g'(a) = 0$ but $g''(a) \neq 0$. The terms we need to evaluate have the form

$$\oint_{\mathcal{C}_a \times \mathcal{C}_J} \oint \frac{\mathrm{d}z_1}{2\pi i} \frac{\mathrm{d}z_2}{2\pi i} \frac{g(z_1, z_2)}{[a - z_1]^2 \left[e^{iQ_2(z_1, z_2)} + 1 \right]}$$

Performing the z_1 integral leads to

$$\oint_{\mathcal{C}_J} \frac{\mathrm{d}z_2}{2\pi i} \left\{ \frac{\frac{\partial g(z_1, z_2)}{\partial z_1} \Big|_{z_1 = a}}{\left[e^{iQ_2(z_1, z_2)} + 1 \right]} - \frac{g(a, z_2) e^{iQ_2(a, z_2)} i \frac{\partial Q(z_1, z_2)}{\partial z_1} \Big|_{z_1 = a}}{\left[e^{iQ_2(z_1, z_2)} + 1 \right]^2} \right\}$$

$$= \frac{i \frac{\partial g(z_1, z_2)}{\partial z_1} \Big|_{z_1 = a}}{\frac{\partial Q(z_1, z_2^*)}{\partial z_1} \Big|_{z_1 = a}} - \oint_{\mathcal{C}_J} \frac{\mathrm{d}z_2}{2\pi i} \frac{g(a, z_2) e^{iQ_2(a, z_2)} i \frac{\partial Q(z_1, z_2)}{\partial z_1} \Big|_{z_1 = a}}{\left[e^{iQ_2(z_1, z_2)} + 1 \right]^2}$$

For the second integral, introducing the notation $l(z_1, z_2) = g(z_1, z_2) i \frac{\partial Q_2(z_1, z_2)}{\partial z_1}$ and using (A.1) with $e^{iQ_2(a, z_2^*)} = -1$:

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$$\oint_{\mathcal{C}_J} \frac{\mathrm{d}z_2}{2\pi i} \frac{l(a,z_2)e^{iQ_2(a,z_2)}}{[e^{iQ_2(z_1,z_2)}+1]^2} = \frac{1}{\frac{\partial Q_2(a,z_2)}{\partial z_2}} \frac{\partial}{\partial z_2}\left[\frac{l(a,z_2)}{\frac{\partial Q_2(a,z_2)}{\partial z_2}}\right]\Bigg|_{z_2=z_2^*}$$

Putting it together

$$\oint_{\mathcal{C}_a\times\mathcal{C}_J}\oint \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{2\pi i\,2\pi i} \frac{g(z_1,z_2)}{[a-z_1]^2\,[e^{iQ_2(z_1,z_2)}+1]}$$

$$= \frac{1}{\frac{\partial Q(z_1,z_2^*)}{\partial z_1}\Big|_{z_1=a}}\left\{i\,\frac{\partial g(z_1,z_2)}{\partial z_1}\bigg|_{z_1=a} - \frac{\partial}{\partial z_2}\left[\frac{g(a,z_2)i\,\frac{\partial Q(z_1,z_2)}{\partial z_1}\big|_{z_1=a}}{\frac{\partial Q(a,z_2)}{\partial z_2}}\right]\right\}\Bigg|_{z_2=z_2^*}$$

where again z_2^* is the root of

$$e^{iQ_2(a,z_2^*)}+1=0$$

A.3 Useful formulas for practical evaluation

Using the form of the Bethe-Yang equations for ϑ_1,ϑ_2

$$\begin{aligned}
-e^{iQ_1(\vartheta_1,\vartheta_2)} &= e^{imL\sinh\vartheta_1}S(\vartheta_1-\vartheta_2)=1\\
-e^{iQ_2(\vartheta_1,\vartheta_2)} &= e^{imL\sinh\vartheta_2}S(\vartheta_2-\vartheta_1)=1
\end{aligned}$$

we can easily substitute their solutions into the Bethe-Yang equations for $\vartheta_1',\vartheta_2'$

$$\begin{aligned}
e^{iQ_1'(\vartheta_1,\vartheta_2')} &= e^{imL\sinh\vartheta_1}\left(-S(\vartheta_1-\vartheta_2')\right)=-S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_2')\\
e^{iQ_1'(\vartheta_2,\vartheta_2')} &= e^{imL\sinh\vartheta_2}\left(-S(\vartheta_2-\vartheta_2')\right)=-S(\vartheta_1-\vartheta_2)S(\vartheta_2-\vartheta_2')\\
e^{iQ_2'(\vartheta_1',\vartheta_1)} &= e^{imL\sinh\vartheta_1}\left(-S(\vartheta_1-\vartheta_1')\right)=-S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_1')\\
e^{iQ_2'(\vartheta_1',\vartheta_2)} &= e^{imL\sinh\vartheta_2}\left(-S(\vartheta_2-\vartheta_1')\right)=-S(\vartheta_1-\vartheta_2)S(\vartheta_2-\vartheta_1')
\end{aligned}$$

Denoting

$$T(\vartheta_1,\vartheta_2,\vartheta_3)=1-S(\vartheta_1-\vartheta_2)S(\vartheta_2-\vartheta_3)$$

we can also write

$$\begin{aligned}
e^{iQ_1'(\vartheta_1,\vartheta_2')}+1 &= T(\vartheta_2,\vartheta_1,\vartheta_2')\\
e^{iQ_1'(\vartheta_2,\vartheta_2')}+1 &= T(\vartheta_1,\vartheta_2,\vartheta_2')\\
e^{iQ_2'(\vartheta_1',\vartheta_1)}+1 &= T(\vartheta_2,\vartheta_1,\vartheta_1')\\
e^{iQ_2'(\vartheta_1',\vartheta_2)}+1 &= T(\vartheta_1,\vartheta_2,\vartheta_1')
\end{aligned}$$

B Pole terms for $\tilde{C}_{22}$

Here we list the complete result for the pole term subtractions that appear in

$$\tilde{C}_{22}(\vartheta_1, \vartheta_2) = \frac{1}{2} \oint_{\leftrightarrows \times \leftrightarrows} \oint \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} \left[K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2') \times \right.$$

$$\left. \frac{F_4^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2') F_4^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2', \vartheta_1')}{\left[e^{iQ_1'(\vartheta_1', \vartheta_2')} + 1 \right] \left[e^{iQ_2'(\vartheta_1', \vartheta_2')} + 1 \right]} \right]$$

$$- QF1(\vartheta_1, \vartheta_2) - QF2(\vartheta_1, \vartheta_2) - QF3(\vartheta_1, \vartheta_2) - QF4(\vartheta_1, \vartheta_2)$$

$$- QF5(\vartheta_1, \vartheta_2) - QF6(\vartheta_1, \vartheta_2) - FF(\vartheta_1, \vartheta_2) - 2SQQ(\vartheta_1, \vartheta_2) \quad \text{(B.1)}$$

where the QF are the contributions from the QF singularities, FF comes from the FF singularities and SQQ is the spurious QQ singularity term. As in the main text, the notation $\leftrightarrows$ corresponds to the straight line contours enclosing the real axis as illustrated in fig. 3.1. The explicit form of the individual contributions to (B.1) are as follows:

$$QF1(\vartheta_1, \vartheta_2) = \oint_{\leftrightarrows} \frac{\mathrm{d}\vartheta_1'}{2\pi} \left(\frac{F_{4rc}^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_1', \vartheta_1) F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{e^{iQ_1'(\vartheta_1', \vartheta_1)} + 1} \right.$$

$$+ \left\{ \vartheta_1 \leftrightarrow \vartheta_2, \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \bigg)$$

$$+ \left(\frac{F_{4rc}^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_2, \vartheta_1) F_2^{\mathcal{O}_1}(i\pi, 0) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_2, \vartheta_1)}{mL \cosh \vartheta_2 + \varphi(\vartheta_2 - \vartheta_1)} \right.$$

$$+ \left\{ \vartheta_1 \leftrightarrow \vartheta_2, \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \bigg) \quad \text{(B.2)}$$

$$QF2(\vartheta_1, \vartheta_2) = \oint_{\leftrightarrows} \frac{\mathrm{d}\vartheta_1'}{2\pi} \left(\frac{F_{4rc}^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_1', \vartheta_1) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{e^{iQ_1'(\vartheta_1', \vartheta_1)} + 1} \right.$$

$$+ \left\{ \vartheta_1 \leftrightarrow \vartheta_2, \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \bigg)$$

$$+ \left(\frac{F_{4rc}^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_2, \vartheta_1) F_2^{\mathcal{O}_2}(i\pi, 0) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_2, \vartheta_1)}{mL \cosh \vartheta_2 + \varphi(\vartheta_2 - \vartheta_1)} \right.$$

$$+ \left\{ \vartheta_1 \leftrightarrow \vartheta_2, \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \bigg) \quad \text{(B.3)}$$

$$QF3(\vartheta_1, \vartheta_2) = -i \oint_{\leftrightarrows} \frac{\mathrm{d}\vartheta_1'}{2\pi} \left\{ \frac{K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1) \left[S(\vartheta_1 - \vartheta_2) - S(\vartheta_1' - \vartheta_1) \right]}{\left[e^{iQ_1'(\vartheta_1', \vartheta_1)} + 1 \right] (\vartheta_2 - \vartheta_1)} \right.$$

$$\times \left[F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \} \right] + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right\}$$

$$- i \left\{ \frac{K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_2, \vartheta_1) \left[S(\vartheta_1 - \vartheta_2) - S(\vartheta_2 - \vartheta_1) \right]}{\left[mL \cosh \vartheta_2 + \varphi(\vartheta_2 - \vartheta_1) \right] (\vartheta_2 - \vartheta_1)} \right.$$

$$\times \left[F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_2) F_2^{\mathcal{O}_2}(i\pi, 0) + \{ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \} \right] + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right\} \quad \text{(B.4)}$$

$$
\begin{aligned}
QF4\left(\vartheta_1,\vartheta_2\right) \;=\; & -i\oint_{\leftrightarrows}\frac{\mathrm{d}\vartheta_1'}{2\pi}\left\{ \frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\left[S(\vartheta_1'-\vartheta_1)S(\vartheta_1-\vartheta_2)-1\right]}{\left[e^{iQ_1'(\vartheta_1',\vartheta_1)}+1\right](\vartheta_2-\vartheta_1')} \right. \\
& \left. \times\left[F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\right]+\{\vartheta_1\leftrightarrow\vartheta_2\}\right\} \\
& -\left\{ \frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_1)\varphi\left(\vartheta_2-\vartheta_1\right)}{\left[mL\cosh\vartheta_2+\varphi\left(\vartheta_2-\vartheta_1\right)\right]} \right. \\
& \left. \times\left[F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(i\pi,0)+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\right]+\{\vartheta_1\leftrightarrow\vartheta_2\}\right\} \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
QF5\left(\vartheta_1,\vartheta_2\right) \;=\; & -i\oint_{\leftrightarrows}\frac{\mathrm{d}\vartheta_1'}{2\pi}\left\{ \frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\left[S(\vartheta_1'-\vartheta_1)-S(\vartheta_1-\vartheta_2)\right]}{\left[e^{iQ_1'(\vartheta_1',\vartheta_1)}+1\right](\vartheta_1-\vartheta_1')} \right. \\
& \left. \times\left[F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1)+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\right]+\{\vartheta_1\leftrightarrow\vartheta_2\}\right\} \\
& -i\left\{ \frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_1)\left[S(\vartheta_2-\vartheta_1)-S(\vartheta_1-\vartheta_2)\right]}{\left[mL\cosh\vartheta_2+\varphi\left(\vartheta_2-\vartheta_1\right)\right](\vartheta_1-\vartheta_2)} \right. \\
& \left. \times\left[F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1)+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\right]+\{\vartheta_1\leftrightarrow\vartheta_2\}\right\} \\
& -\left\{ K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1,\vartheta_1)S(\vartheta_1-\vartheta_2) \right. \\
& \times\left[F_2^{\mathcal{O}}(\vartheta_2+i\pi,\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1)+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\right] \\
& \left. +\{\vartheta_1\leftrightarrow\vartheta_2\}\right\} \qquad\qquad\qquad\qquad\qquad\qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
QF6\left(\vartheta_1,\vartheta_2\right) \;=\; & -\oint_{\leftrightarrows}\frac{\mathrm{d}\vartheta_1'}{2\pi}\left(\frac{F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{\left[e^{iQ_1'(\vartheta_1',\vartheta_1)}+1\right]} \right. \\
& \times\left\{T\left(\vartheta_1',\vartheta_1,\vartheta_2\right)\left[-mx\cosh\vartheta_1+imt\sinh\vartheta_1\right]-mL\cosh\left(\vartheta_1\right)\right. \\
& \left.\left. +\varphi(\vartheta_1'-\vartheta_1)S(\vartheta_1'-\vartheta_1)S(\vartheta_1-\vartheta_2)\right\}+\{\vartheta_1\leftrightarrow\vartheta_2\}\right) \\
& +\; F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(i\pi,0)K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_1) \\
& \times\left(\frac{\left[mL\cosh\vartheta_1-\varphi\left(\vartheta_1-\vartheta_2\right)\right]}{\left[mL\cosh\vartheta_2+\varphi\left(\vartheta_2-\vartheta_1\right)\right]}+\frac{\left[mL\cosh\vartheta_2-\varphi\left(\vartheta_2-\vartheta_1\right)\right]}{\left[mL\cosh\vartheta_1+\varphi\left(\vartheta_1-\vartheta_2\right)\right]}\right) \\
& +\; \oint_{\leftrightarrows}\frac{\mathrm{d}\vartheta_1'}{2\pi}\left(\frac{i}{\left[e^{iQ_1'(\vartheta_1',\vartheta_1)}+1\right]} \right. \\
& \times\frac{\partial}{\partial\vartheta_1'}\left[\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)T\left(\vartheta_1',\vartheta_1,\vartheta_2\right)\varphi\left(\vartheta_1'-\vartheta_1\right)}{\left[mL\cosh\vartheta_1'+\varphi\left(\vartheta_1'-\vartheta_1\right)\right]} \right. \\
& \left.\left. \times F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')\right]+\{\vartheta_1\leftrightarrow\vartheta_2\}\right) \\
& +\; K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_1)F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(i\pi,0)\varphi\left(\vartheta_2-\vartheta_1\right)^2 \qquad \text{(B.7)} \\
& \times\left(\frac{1}{\left[mL\cosh\vartheta_2+\varphi\left(\vartheta_2-\vartheta_1\right)\right]^2}+\frac{1}{\left[mL\cosh\vartheta_1+\varphi\left(\vartheta_1-\vartheta_2\right)\right]^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
FF \;=\;\; & K_{t,x}^{(R)}\left(\vartheta_1, \vartheta_2, \vartheta_1, \vartheta_1\right) S\left(\vartheta_1 - \vartheta_2\right) F_2^{\mathcal{O}_1}\left(\vartheta_2 + i\pi, \vartheta_1\right) F_2^{\mathcal{O}_2}\left(\vartheta_2 + i\pi, \vartheta_1\right) \\
& + \{\vartheta_1 \leftrightarrow \vartheta_2\}
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
SQQ\left(\vartheta_1, \vartheta_2\right) \;=\;\; & \oint\oint_{\mathcal{C}_{\vartheta_1}\times\mathcal{C}_{\vartheta_2}} \frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}\left\{\frac{F_4^{\mathcal{O}_1}\left(\vartheta_2+i\pi,\vartheta_1+i\pi,\vartheta_1',\vartheta_2'\right)F_4^{\mathcal{O}_2}\left(\vartheta_1+i\pi,\vartheta_2+i\pi,\vartheta_2',\vartheta_1'\right)}{\left[e^{iQ_1'\left(\vartheta_1',\vartheta_2'\right)}+1\right]\left[e^{iQ_2'\left(\vartheta_1',\vartheta_2'\right)}+1\right]}\right. \\
& \left. \times K_{t,x}^{(R)}\left(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2'\right)\right\}
\end{aligned}
\tag{B.9}
$$

We gave these contributions in their exact finite volume form (i.e. including the full volume dependence): albeit they simplify when taking the volume to infinity, and the SQQ term does not even contribute in this limit, all terms must be kept in order for the numerical verification of Section 4 to work properly.

C Symmetry of D_{22}

We want to prove that D_{22} is symmetric under

$$
\begin{aligned}
t & \to R - t \\
\mathcal{O}^1 & \leftrightarrow \mathcal{O}^2 \\
x & \to -x
\end{aligned}
\tag{C.1}
$$

First of all, notice that the diagonal terms contain no t and x factor, and are manifestly symmetric under exchanging $\mathcal{O}_1$ and $\mathcal{O}_2$. So it remains only to treat the non-diagonal part.

C.1 The four-integral term

First we treat the term in (3.10) that contains a fourfold integral. After the transformation we change the variables $\vartheta_{1,2} \leftrightarrow \vartheta'_{1,2}$, and shift every contour with $-2i\varepsilon$. This results in the contour now running under the real axis:

$$
\begin{aligned}
& \frac{1}{4}\iiiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1'-i\varepsilon,\vartheta_2'-i\varepsilon) \\
& \times F_4^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi,\vartheta_1'-i\varepsilon,\vartheta_2'-i\varepsilon)F_4^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi,\vartheta_2'-i\varepsilon,\vartheta_1'-i\varepsilon)
\end{aligned}
$$

By shifting the contour above the real axis we can transform this term back to its form in (3.10), but during this process we pick up some pole contributions from the poles in eqn. (3.5).

C.1.1 First order pole terms containing F_{4rc}

Using eqn. (3.5) we can identify a contribution containing F_{4rc}. In these contribution, all poles are of first order, so one can apply the Cauchy formula directly. One set of such terms is given by

$$\frac{1}{4}\iint_{\mathcal{C}_-}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1',\vartheta_2')\left[\frac{E}{\vartheta_1-\vartheta_2'}+\frac{F}{\vartheta_1-\vartheta_1'}+\frac{G}{\vartheta_2-\vartheta_2'}+\frac{H}{\vartheta_2-\vartheta_1'}\right]$$
$$\times K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')$$
$$=\ -\frac{1}{4}\iint_{\mathcal{C}_+}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1',\vartheta_2')\left[\frac{E}{\vartheta_2'-\vartheta_1}+\frac{F}{\vartheta_1'-\vartheta_1}+\frac{G}{\vartheta_2'-\vartheta_2}+\frac{H}{\vartheta_1'-\vartheta_2}\right]$$
$$\times K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_1'}{2\pi}F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1',\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)$$
$$\times[S(\vartheta_1-\vartheta_2)-S(\vartheta_1-\vartheta_1')]$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1,\vartheta_2')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_2')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1,\vartheta_2')$$
$$\times[S(\vartheta_2'-\vartheta_1)S(\vartheta_1-\vartheta_2)-1]$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_1'}{2\pi}F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1',\vartheta_2)F_2^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2)$$
$$\times[1-S(\vartheta_1-\vartheta_2)S(\vartheta_2-\vartheta_1')]$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_2,\vartheta_2')F_2^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_2')$$
$$\times[S(\vartheta_2'-\vartheta_2)-S(\vartheta_1-\vartheta_2)]$$

and a similar contribution from

$$\frac{1}{4}\iint_{\mathcal{C}_-}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_2',\vartheta_1')\left[\frac{A}{\vartheta_2-\vartheta_1'}+\frac{B}{\vartheta_2-\vartheta_2'}+\frac{C}{\vartheta_1-\vartheta_1'}+\frac{D}{\vartheta_1-\vartheta_2'}\right]$$
$$\times K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')$$
$$=\ -\frac{1}{4}\iint_{\mathcal{C}_+}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_2',\vartheta_1')\left[\frac{A}{\vartheta_1'-\vartheta_2}+\frac{B}{\vartheta_2'-\vartheta_2}+\frac{C}{\vartheta_1'-\vartheta_1}+\frac{D}{\vartheta_2'-\vartheta_1}\right]$$
$$\times K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_2',\vartheta_2)F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_2')$$
$$\times[S(\vartheta_2-\vartheta_1)-S(\vartheta_2-\vartheta_2')]$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_1'}{2\pi}F_{4rc}^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_2,\vartheta_1')F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2)$$
$$\times[S(\vartheta_1'-\vartheta_2)S(\vartheta_2-\vartheta_1)-1]$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_2'}{2\pi}F_{4rc}^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_2',\vartheta_1)F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_2')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1,\vartheta_2')$$
$$\times[1-S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_2')]$$
$$+\ \frac{1}{4}\int\frac{\mathrm{d}\vartheta_1'}{2\pi}F_{4rc}^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_1,\vartheta_1')F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)$$
$$\times[S(\vartheta_1'-\vartheta_1)-S(\vartheta_2-\vartheta_1)]$$

where $\mathcal{C}_\pm$ denote integration running above/below from the real axis, in the direction from left to right.

Since the exchange property (2.10) is valid for F_{4rc} in the first and last two variables, it can be used to simplify the total contribution to

$$\frac{1}{2}\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\left\{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\left\{F_{4rc}^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_1',\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\right\}\right.$$

$$\left.+\{\vartheta_1\leftrightarrow\vartheta_2\}\right\}$$

$$-\frac{1}{2}\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\left\{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\left\{F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1',\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\right\}\right.$$

$$\left.+\{\vartheta_1\leftrightarrow\vartheta_2\}\right\}$$

C.1.2 First order poles without F_{4rc}

The form of these terms is

$$\frac{1}{4}\iiiint_{\mathcal{C}_-}\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')\times$$

$$\times\left[\frac{AE+DH}{(\vartheta_2-\vartheta_1')(\vartheta_1-\vartheta_2')}+\frac{AF+CH}{(\vartheta_2-\vartheta_1')(\vartheta_1-\vartheta_1')}+\frac{AG+BH}{(\vartheta_2-\vartheta_1')(\vartheta_2-\vartheta_2')}\right.$$
$$\left.+\frac{BE+DG}{(\vartheta_2-\vartheta_2')(\vartheta_1-\vartheta_2')}+\frac{BF+CG}{(\vartheta_2-\vartheta_2')(\vartheta_1-\vartheta_1')}+\frac{CE+DF}{(\vartheta_1-\vartheta_1')(\vartheta_1-\vartheta_2')}\right]$$

Their evaluation is tedious, but straightforward. For example

$$\begin{aligned}AE+DH&=-\left(S(\vartheta_2-\vartheta_1)-S(\vartheta_1'-\vartheta_2')\right)\left(S(\vartheta_1-\vartheta_2)-S(\vartheta_2'-\vartheta_1')\right)\\&\quad\times\left\{F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\right\}\end{aligned}$$

and one can evaluate the contributions resulting from the contour shift as

$$\begin{aligned}&+\frac{1}{4}\iiiint_{\mathcal{C}_-}\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')\frac{AE+DH}{(\vartheta_1'-\vartheta_2)(\vartheta_2'-\vartheta_1)}\\=&+\frac{1}{4}\iiiint_{\mathcal{C}_+}\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')\frac{AE+DH}{(\vartheta_1'-\vartheta_2)(\vartheta_2'-\vartheta_1)}\\&+\frac{1}{4}\iiint_{\mathcal{C}_+}\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1'-\vartheta_2)}i\left[AE+DH\right]|_{\vartheta_2'=\vartheta_1}\\&+\frac{1}{4}\iiint_{\mathcal{C}_+}\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_2')}{(\vartheta_2'-\vartheta_1)}i\left[AE+DH\right]|_{\vartheta_1'=\vartheta_2}\\&-\frac{1}{4}\iiint_{\mathcal{C}_+}\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_2,\vartheta_1)\left[AE+DH\right]|_{\vartheta_1'=\vartheta_2,\,\vartheta_2'=\vartheta_1}\\=&+\frac{1}{4}\iiiint_{\mathcal{C}_+}\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')\frac{AE+DH}{(\vartheta_1'-\vartheta_2)(\vartheta_2'-\vartheta_1)}\\&-\frac{i}{2}\iint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1'-\vartheta_2)}\left\{F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\right\}\\&\times\left[S(\vartheta_2-\vartheta_1)-S(\vartheta_1'-\vartheta_1)\right]\left[S(\vartheta_1-\vartheta_2)-S(\vartheta_1-\vartheta_1')\right]\end{aligned}$$

and similarly for the remaining five cases.

C.1.3   Second order poles

We get four contributions which contain a second order pole. All four are the same after relabeling the rapidities:

$$
+\frac{1}{4} \iiiint_{\mathcal{C}_-} \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} \left\{ \left[\frac{AH}{(\vartheta_2 - \vartheta_1')^2} + \frac{BG}{(\vartheta_2 - \vartheta_2')^2} + \frac{CF}{(\vartheta_1 - \vartheta_1')^2} + \frac{DE}{(\vartheta_1 - \vartheta_2')^2} \right] \right.
$$
$$
\left. \times K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2') \right\}
$$
$$
= \iiiint_{\mathcal{C}_-} \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta_2')^2} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2')
$$
$$
= \iiiint_{\mathcal{C}_+} \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta_2')^2} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2')
$$
$$
+ i \iiint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} \left. \frac{\partial \left[DE K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2') \right]}{\partial \vartheta_2'} \right|_{\vartheta_2' = \vartheta_1}
$$

After performing the differentiation, the final result is:

$$
\iiiint_{\mathcal{C}_+} \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{\mathrm{d}\vartheta_2'}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta_2')^2} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2')
$$
$$
+ \iiint \frac{\mathrm{d}\vartheta_1}{2\pi} \frac{\mathrm{d}\vartheta_2}{2\pi} \frac{\mathrm{d}\vartheta_1'}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1)
$$
$$
\times \left\{ [mx \cosh \vartheta_1 - imt \sinh \vartheta_1] \, T(\vartheta_2, \vartheta_1, \vartheta_1') \, T(\vartheta_1', \vartheta_1, \vartheta_2) \right.
$$
$$
\left. + [S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') - S(\vartheta_1' - \vartheta_1) S(\vartheta_1 - \vartheta_2)] \, \varphi(\vartheta_1 - \vartheta_1') \right\}
$$

where the notation
$$
T(\vartheta_1, \vartheta_2, \vartheta_3) = 1 - S(\vartheta_1 - \vartheta_2) S(\vartheta_2 - \vartheta_3)
$$
was used.

C.1.4   Putting together the results

Putting together the result for the fourfold integral term one obtains

$$\frac{1}{4}\iiiint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}F_4^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi,\vartheta_1'-i\varepsilon,\vartheta_2'-i\varepsilon)$$

$$\times F_4^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi,\vartheta_2'-i\varepsilon,\vartheta_1'-i\varepsilon)K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1'-i\varepsilon,\vartheta_2'-i\varepsilon)$$

$$=\frac{1}{4}\iiiint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi}F_4^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi,\vartheta_1'+i\varepsilon,\vartheta_2'+i\varepsilon)$$

$$\times F_4^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi,\vartheta_2'+i\varepsilon,\vartheta_1'+i\varepsilon)K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1'+i\varepsilon,\vartheta_2'+i\varepsilon)$$

$$+\iiint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\Big\{F_{4rc}^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_1',\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')$$

$$+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}$$

$$-\iiint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\Big\{F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')$$

$$+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}$$

$$-i\iint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_2-\vartheta_1')}\Big\{F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')$$

$$+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}\left[S(\vartheta_1'-\vartheta_1)-S(\vartheta_2-\vartheta_1)\right]\left[S(\vartheta_1-\vartheta_2)-S(\vartheta_1-\vartheta_1')\right]$$

$$-i\iint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1-\vartheta_1')}\Big\{F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')$$

$$+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}\left[1-S(\vartheta_2-\vartheta_1)S(\vartheta_1'-\vartheta_1)\right]\left[S(\vartheta_1-\vartheta_1')-S(\vartheta_1-\vartheta_2)\right]$$

$$-i\iiint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1,\vartheta_2')}{(\vartheta_2-\vartheta_1)}\Big\{F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')$$

$$+\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}\left[S(\vartheta_2-\vartheta_1)-S(\vartheta_1-\vartheta_1')\right]\left[1-S(\vartheta_1'-\vartheta_1)S(\vartheta_1-\vartheta_2)\right]$$

$$+\iiint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)$$

$$\times\Big\{[mx\cosh\vartheta_1-imt\sinh\vartheta_1]\,T\,(\vartheta_2,\vartheta_1,\vartheta_1')\,T\,(\vartheta_1',\vartheta_1,\vartheta_2)$$

$$+[S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_1')-S(\vartheta_1'-\vartheta_1)S(\vartheta_1-\vartheta_2)]\,\varphi\,(\vartheta_1-\vartheta_1')\Big\}$$

C.2 Other terms

The following two terms in (3.10)

$$-\iint\frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1)K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1,\vartheta_1)$$

$$-\int\frac{\mathrm{d}\vartheta_1}{2\pi}\int\frac{\mathrm{d}\vartheta_1'}{2\pi}F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_1|\vartheta_1,\vartheta_1')$$

transform to each other under the symmetry (C.1).

For the remaining terms in (3.10), after the transformation we redefine $\vartheta_2\leftrightarrow\vartheta_1'$ to have the same K factor as before and get:

$$\iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\int \frac{\mathrm{d}\vartheta_1'}{2\pi} K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)$$

$$\times \left(F_{4rc}^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_1'+i\pi|\vartheta_2,\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') + \{\mathcal{O}^1 \leftrightarrow \mathcal{O}^2\} \right)$$

$$- i\iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1'-\vartheta_1)} [S(\vartheta_1-\vartheta_1')-S(\vartheta_2-\vartheta_1)]$$

$$\times \left\{ F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') + \{\mathcal{O}_1 \leftrightarrow \mathcal{O}_2\} \right\}$$

$$- i\iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1'-\vartheta_2)} [S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_1')-1]$$

$$\times \left\{ F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') + \{\mathcal{O}_1 \leftrightarrow \mathcal{O}_2\} \right\}$$

$$- i\iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int \frac{\mathrm{d}\vartheta_1'}{2\pi} \frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1-\vartheta_2)} [S(\vartheta_2-\vartheta_1)-S(\vartheta_1-\vartheta_1')]$$

$$\times \left\{ F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_1'+i\pi,\vartheta_1) + \{\mathcal{O}_1 \leftrightarrow \mathcal{O}_2\} \right\}$$

$$- \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1) \times$$

$$\times \{T(\vartheta_2,\vartheta_1,\vartheta_1')[-mx\cosh\vartheta_1+im(R-t)\sinh\vartheta_1]+\varphi(\vartheta_2-\vartheta_1)S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_1')\}$$

C.3 Collecting the terms

Using the results so far, the transformed D_{22} is the following:

$$
\begin{aligned}
D_{22}^{trans} &= \frac{1}{4}\iiiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{\mathrm{d}\vartheta_2'}{2\pi} K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1'+i\varepsilon,\vartheta_2'+i\varepsilon) \\
&\quad F_4^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi,\vartheta_1'+i\varepsilon,\vartheta_2'+i\varepsilon)F_4^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_2+i\pi,\vartheta_2'+i\varepsilon,\vartheta_1'+i\varepsilon) \\
&\quad + (F_{4rc}\text{ terms}) + (\text{first order pole terms}) + (\text{second order pole terms}) \\
&\quad - \iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1)K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1,\vartheta_1) \\
&\quad - \int \frac{\mathrm{d}\vartheta_1}{2\pi}\int \frac{\mathrm{d}\vartheta_1'}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_1+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_1|\vartheta_1,\vartheta_1') \\
&\quad + D_{22}^{diag}
\end{aligned}
$$

where the F_{4rc} terms are

$$+ \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi} K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\left\{ F_{4rc}^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2+i\pi|\vartheta_1',\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') \right.$$

$$\left. + \{\mathcal{O}^1 \leftrightarrow \mathcal{O}^2\} \right\}$$

$$+ \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi} K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\left\{ F_{4rc}^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_1'+i\pi|\vartheta_2,\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') \right.$$

$$\left. + \{\mathcal{O}^1 \leftrightarrow \mathcal{O}^2\} \right\}$$

$$- \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi} K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\left\{ F_{4rc}^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') \right.$$

$$\left. + \{\mathcal{O}^1 \leftrightarrow \mathcal{O}^2\} \right\} \tag{C.2}$$

the first order pole terms are

$$
\begin{aligned}
- \ & i \iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_2-\vartheta_1')}\Big\{F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') \\
& +\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}\left[S(\vartheta_1'-\vartheta_1)-S(\vartheta_2-\vartheta_1)\right]\left[S(\vartheta_1-\vartheta_2)-S(\vartheta_1-\vartheta_1')\right] \\
- \ & i \iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1-\vartheta_1')}\Big\{F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') \\
& +\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}\left[1-S(\vartheta_2-\vartheta_1)S(\vartheta_1'-\vartheta_1)\right]\left[S(\vartheta_1-\vartheta_1')-S(\vartheta_1-\vartheta_2)\right] \\
- \ & i \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1,\vartheta_2')}{(\vartheta_2-\vartheta_1)}\Big\{F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1') \\
& +\{\mathcal{O}^1\leftrightarrow\mathcal{O}^2\}\Big\}\left[S(\vartheta_2-\vartheta_1)-S(\vartheta_1-\vartheta_1')\right]\left[1-S(\vartheta_1'-\vartheta_1)S(\vartheta_1-\vartheta_2)\right] \\
- \ & i \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1'-\vartheta_1)}\left[S(\vartheta_1-\vartheta_1')-S(\vartheta_2-\vartheta_1)\right] \\
& \times\Big\{F_2^{\mathcal{O}_1}(\vartheta_1+i\pi,\vartheta_2)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\Big\} \\
- \ & i \iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1'-\vartheta_2)}\left[S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_1')-1\right] \\
& \times\Big\{F_2^{\mathcal{O}_1}(i\pi,0)F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\Big\} \\
- \ & i \iint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\wp\int\frac{\mathrm{d}\vartheta_1'}{2\pi}\frac{K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)}{(\vartheta_1-\vartheta_2)}\left[S(\vartheta_2-\vartheta_1)-S(\vartheta_1-\vartheta_1')\right] \\
& \times\Big\{F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_1'+i\pi,\vartheta_1)+\{\mathcal{O}_1\leftrightarrow\mathcal{O}_2\}\Big\} \quad\text{(C.3)}
\end{aligned}
$$

while the second order pole terms are

$$
\begin{aligned}
+ \ & \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1) \\
& \times\{[mx\cosh\vartheta_1-imt\sinh\vartheta_1]\,T\,(\vartheta_2,\vartheta_1,\vartheta_1')\,T\,(\vartheta_1',\vartheta_1,\vartheta_2) \\
& +\left[S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_1')-S(\vartheta_1'-\vartheta_1)S(\vartheta_1-\vartheta_2)\right]\varphi\,(\vartheta_1-\vartheta_1')\} \\
- \ & \iiint \frac{\mathrm{d}\vartheta_1}{2\pi}\frac{\mathrm{d}\vartheta_2}{2\pi}\frac{\mathrm{d}\vartheta_1'}{2\pi}F_2^{\mathcal{O}_1}(\vartheta_2+i\pi,\vartheta_1')F_2^{\mathcal{O}_2}(\vartheta_2+i\pi,\vartheta_1')K_{t,x}^{(R)}(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_1)\times \quad\text{(C.4)} \\
& \times\{T\,(\vartheta_2,\vartheta_1,\vartheta_1')\,[mx\cosh\vartheta_1+im\,(R-t)\sinh\vartheta_1]-\varphi(\vartheta_2-\vartheta_1)S(\vartheta_2-\vartheta_1)S(\vartheta_1-\vartheta_1')\}
\end{aligned}
$$

C.3.1 F_{4rc} terms

The last two F_{4rc} terms (C.2) cancel due to

$$
F_{4rc}^{\mathcal{O}_{1,2}}(\vartheta_1+i\pi,\vartheta_1'+i\pi|\vartheta_2,\vartheta_1)-F_{4rc}^{\mathcal{O}_{1,2}}(\vartheta_2+i\pi,\vartheta_1+i\pi|\vartheta_1,\vartheta_1')=0
$$

which can be easily proven from the definition (3.4); the remaining one is combined with the first order pole terms below into the F_{4ss} contributions in (3.10).

C.3.2 First order pole terms

The first order pole terms (C.3) can be rearranged into the form

$$
\begin{aligned}
-\quad & i \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \wp \int \frac{d\vartheta_1'}{2\pi} \frac{K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{(\vartheta_2 - \vartheta_1')} \Big\{ F_2^{\mathcal{O}_1}(i\pi, 0) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') \\
& + \{\mathcal{O}^1 \leftrightarrow \mathcal{O}^2\} \Big\} [S(\vartheta_1' - \vartheta_1)S(\vartheta_1 - \vartheta_2) - 1]
\end{aligned}
$$

$$
\begin{aligned}
-\quad & i \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \wp \int \frac{d\vartheta_1'}{2\pi} \frac{K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{(\vartheta_1 - \vartheta_1')} \Big\{ F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') \\
& + \{\mathcal{O}^1 \leftrightarrow \mathcal{O}^2\} \Big\} [S(\vartheta_1' - \vartheta_1) - S(\vartheta_1 - \vartheta_2)]
\end{aligned}
$$

$$
\begin{aligned}
-\quad & i \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1, \vartheta_1')}{(\vartheta_2 - \vartheta_1)} \Big\{ F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') \\
& + \{\mathcal{O}^1 \leftrightarrow \mathcal{O}^2\} \Big\} [S(\vartheta_1 - \vartheta_2) - S(\vartheta_1' - \vartheta_1)]
\end{aligned}
$$

using
$$
S(\vartheta)S(-\vartheta) = 1
$$

These terms are combined with the F_{4rc} terms into the F_{4ss} contributions in (3.10).

C.3.3 Second order pole terms

The second order pole terms (C.4) have a dependence on $R - t$ on the last line. One can make an integration by parts to transform it into a t-dependence as on the second line. Using

$$
\frac{\partial K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{\partial \vartheta_1} = -mR \sinh(\vartheta_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)
$$

we can make the following transformation in the integrand

$$
\begin{aligned}
& -iT(\vartheta_2, \vartheta_1, \vartheta_1') \, mR \sinh\vartheta_1 K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1) = iT(\vartheta_2, \vartheta_1, \vartheta_1') \frac{\partial K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{\partial \vartheta_1} \\
& \text{(partial integration)} \Rightarrow -iK_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1) \frac{T(\vartheta_2, \vartheta_1, \vartheta_1')}{\partial \vartheta_1} \\
& = -K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1) S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') [\varphi(\vartheta_1 - \vartheta_1') - \varphi(\vartheta_2 - \vartheta_1)]
\end{aligned}
$$

and so the contribution is transformed into

$$
\begin{aligned}
+\quad & \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1) \\
& \times \{[mx \cosh\vartheta_1 - imt \sinh\vartheta_1] T(\vartheta_1', \vartheta_1, \vartheta_2) - S(\vartheta_1' - \vartheta_1)S(\vartheta_1 - \vartheta_2)\varphi(\vartheta_1 - \vartheta_1')\}
\end{aligned}
$$

With this the second order pole terms (C.4) reduce to the form of the second order pole term in (3.10).

C.4 End result

After putting together every term, using the definition of F_{4ss} (3.9), and performing some simplifications one obtains
$$
D_{22}^{trans} = D_{22}
$$
which is exactly what was to be proven.

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