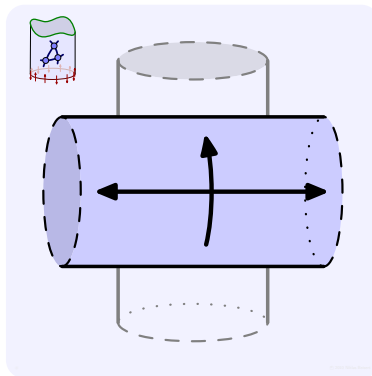# Review of AdS/CFT Integrability, Chapter III.6: Thermodynamic Bethe Ansatz

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Abstract: The aim of the chapter is to introduce in a pedagogical manner the concept of Thermodynamic Bethe Ansatz designed to calculate the energy levels of finite volume integrable systems and to review how it is applied in the planar AdS/CFT setting.

1 Introduction

Thermodynamic Bethe Ansatz (TBA) is a method to calculate exactly the ground-state energy of an integrable quantum field theory in finite volume using its infinite volume scattering data [1][1]. The equations can be extended to excited states as well by analytical continuation [3, 4].

The idea of the TBA is to exploit that the Euclidean partition function is dominated for large imaginary times by the groundstate energy. Calculating the partition function in the doubly Wick rotated (mirror) theory the imaginary time becomes the physical size which is taken to be large. Since the large volume spectrum is under control, the partition function can be evaluated in the saddle point approximation which results in nonlinear integral equation for pseudo energies leading to an exact description of the ground state energy. Excited states on the complex (volume/coupling) plane are connected to the groundstate which enables one to derive nonlinear integral equations for excited states as well.

We start in Section 2 with a toy model containing one single particle with AdS dispersion relation and with scattering matrix which is not a function of the differences of the momenta. Although this is a fictitious system it helps to introduce conceptual notions and steps needed to explain the TBA which is, in analogy, used in Section 3 to present the results for planar AdS/CFT. Finally, we give a guide to the literature in Section 4 and list some open problems.

2 The concept of TBA: a toy model

The application of the TBA method to solve completely the finite volume spectral problem is standard by now and follows the following steps. First the scattering theory has to be solved in infinite volume by determining the scattering matrix from its generic properties such as symmetry, unitarity, crossing relation. The poles of the scattering matrix lying in the physical strip are related to bound-states. These bound-states have to be mapped and their scattering matrices have to be determined from the constituents' scattering matrices. Then in the second step these scattering matrices can be used to describe the spectrum for large volume, which amounts to restrict the allowed particles' momenta via phase shifts and periodicity, and use the dispersion relation to express the energy in terms of the quantized momenta. This method sums up all power like corrections in the inverse of the volume and provides an asymptotical spectrum. The very same asymptotic description of the mirror theory is also needed as it can be used to calculate the exponentially small finite energy corrections from the partition function. Evaluating the Euclidean partition function for large imaginary times (large mirror volumes) in the saddle point approximation

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[1]The method has its origin in the work of Yang and Yang applied for spin chains and for the Bose gas with δ interaction [2].

provides integral equations describing the ground state energy exactly. Finally, these equations can be extended for excited states by analytical continuation. Now let us see how these steps are elaborated in the simplest setting.

Infinite volume characteristics of the model

We consider a toy model with one single particle type only. The dispersion relation is supposed to be the same as in the AdS/CFT correspondence [2]:

$$E(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$$

The sine function indicates lattice behavior and restricts the momentum as $p \in [-\pi, \pi]$. The square root, however, has a relativistic origin. The theory is supposed to be integrable, thus multiparticle scattering matrices factorize into two particle scatterings. As relativistic invariance is not supposed the two particle S-matrix can depend separately on the two momenta $S(p_1, p_2)$ and satisfies unitarity $S(p_1, p_2)S(p_2, p_1) = 1$ and crossing symmetry, which helps to fix it completely. We will not need its explicit form, but will suppose that in the $p_1 = p_2 = p$ particular case $S(p, p) = -1$.

Infinite volume characteristics of the mirror model

The Euclidean version of the model is defined by analytically continuing in the time variable $t = iy$ and considering space x and imaginary time y on an equal footing. The Euclidean theory so obtained can be considered as an analytical continuation of another theory, in which x serves as the analytically continued time $x = i\tau$ and y is the space coordinate. The theory defined in terms of y, τ is called the mirror theory and its dispersion relation can be obtained by the same analytical continuation $E = i\tilde{p}$ and $p = i\tilde{E}$ which results in

$$\tilde{E}(\tilde{p}) = 2\text{arcsinh}\left(\frac{1}{2g}\sqrt{\tilde{p}^2 + 1}\right)$$

Contrary to the original theory the mirror model is not of the lattice type as its momentum can take any real value $\tilde{p} \in \mathbb{R}$. As the scattering matrix is related via the reduction formula to the Euclidean correlator the mirror S-matrix is simply the analytical continuation of the original scattering matrix: $S(\tilde{p}_1, \tilde{p}_2)$.

Very large volume solution: asymptotic Bethe Ansatz for the model

Let us put N particles in a large volume L subject to periodic boundary condition. Integrability ensures that the particle number is conserved and the particles' momenta

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[2]The string tension is related to the 't Hooft coupling as $2\pi g = \sqrt{\lambda}$

are not changed in the consecutive scatterings. The leading effect of the finite volume is the momentum quantization constraint:

$$1 = e^{ip_j L} \prod_{k:k \neq j}^{N} S(p_j, p_k) \qquad (2.1)$$

which is called the Bethe Yang equation or asymptotic Bethe Ansatz (ABA) and follows from the periodicity of the multiparticle wave function. Due to the sine function in the dispersion relation and the periodicity of p consistency of (2.1) requires L to take integer values only.

Bound-states of the theory are manifested in the ABA as complex string-like solutions. Indeed, if the scattering matrix has a pole for $\Im m(p) > 0$, then complex p solutions are also allowed in (2.1). If we take L very large with $p_1 \approx \frac{p}{2} + iq$ then the rhs. of (2.1) would go to ∞ which should be compensated by another complex momentum, $p_2 \approx \frac{p}{2} - iq$ say, such that $S(p_1, p_2)$ exhibits a pole. The two particles with momenta p_1 and p_2 form a bound-state with momentum $p = p_1 + p_2$, energy $E_2(p) = E(p_1) + E(p_2)$ and scattering matrix $S_{21}(p, p_j) = S(\frac{p}{2} + iq, p_j)S(\frac{p}{2} - iq, p_j)$. In general complex solutions built up from more particles are also allowed and they usually form a string-like pattern. Their dispersion relation and scattering matrices can be calculated by extending the method above, which is called the S-matrix bootstrap.

Very large volume solution: ABA for the mirror model

In the mirror model the considerations go along the same line as in the original theory. If we denote the mirror volume by R the ABA reads as

$$1 = e^{i\tilde{p}_j R} \prod_{k:k \neq j} S(\tilde{p}_j, \tilde{p}_k) \qquad (2.2)$$

Since $S(\tilde{p}_1, \tilde{p}_2)$ lives in a different analytical domain than $S(p_1, p_2)$ its pole structure can be also different. If it exhibits poles also at the proper location the mirror theory has also bound-states. Once bound-states exist we can calculate their dispersion relation and scattering matrices from the bootstrap method. Suppose that the bound-states can be labeled with some charge Q, they have energy $\tilde{E}_Q(\tilde{p})$ and their scattering matrix is $S_{Q_j Q_k}(\tilde{p}_j, \tilde{p}_k)$. The generic ABA valid for all the excitations (also for bound-states) will be

$$1 = e^{i\tilde{p}_j R} \prod_{k:k \neq j} S_{Q_j Q_k}(\tilde{p}_j, \tilde{p}_k) \qquad (2.3)$$

Once these equations are solved the energy of the multiparticle state is

$$\tilde{E} = \sum_{j=1}^{N} \tilde{E}_{Q_j}(\tilde{p}_j)$$

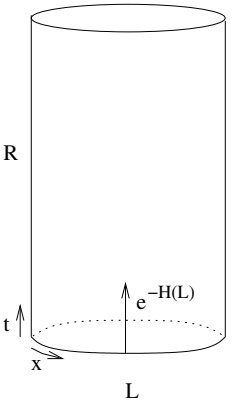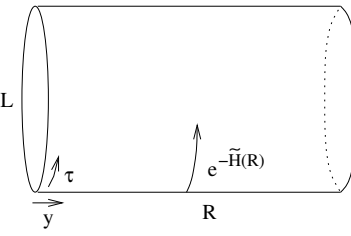which describes the spectrum asymptotically for large volumes R.

original model	mirror model
$(t,x) \equiv (y = it, x)$	$(y, x = i\tau) \equiv (\tau, y)$
$(E,P) \equiv (P_y = iE, P_x = P)$	$(P_y = \tilde{P}, P_x = i\tilde{E}) \equiv (\tilde{E}, \tilde{P})$

Table 1: The relation between the original and the mirror model.

Groundstate TBA equation from the partition function

Let us come back to the original model and see how the exact groundstate energy can be determined in a finite volume L from the Euclidean partition function. We exploit the fact that the imaginary time evolution for large times, R, is dominated by the lowest energy state

$$\lim_{R \to \infty} Z(L,R) = \lim_{R \to \infty} \mathrm{Tr}(e^{-RH(L)}) = \lim_{R \to \infty} e^{-RE_0(L)} + \dots$$

where the ellipsis represents terms exponentially suppressed in R. The same partition function can be determined alternatively, using the time evolution of the mirror theory which is generated by the mirror Hamiltonian $\tilde{H}$:

$$Z(L,R) = \tilde{Z}(R,L) = \mathrm{Tr}(e^{-L\tilde{H}(R)}) = \sum_n e^{-L\tilde{E}_n(R)}$$

The relation between the original model and the mirror model is summarized in Table 1. In switching to the mirror model we ensure that the volume goes to infinity (and not the imaginary time) where the spectrum is controlled by the ABA (2.3).

In the large R limit the sum in the partition function is dominated by finite density particle states. Introducing the density of the particles (and bound-states) in momentum space ($\rho_Q(\tilde{p}) = \frac{\Delta n_Q}{R \Delta \tilde{p}}$) the energy can be expressed as

$$\tilde{E}[\rho] = R \sum_Q \int d\tilde{p} \, \rho_Q(\tilde{p}) \tilde{E}_Q(\tilde{p}) = R \sum_Q \int du \, \rho_Q(u) \tilde{E}_Q(u)$$

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where for later convenience we reparametrized the momentum as $\tilde{p}(u)$, momentum integrations go from $-\infty$ to ∞. The quantization condition comes from taking the logarithm of the mirror ABA

$$\tilde{p}_j(u_j) + \sum_{Q'} \int du'(-i\log S_{Q_jQ'}(u_j,u'))\rho_{Q'}(u') = 2\pi\frac{n_j}{R} \qquad (2.4)$$

where n_j labels the quantized momentum $\tilde{p}_j$ whose charge is Q_j. For a generic multiparticle state there are momenta $\tilde{p}_k$ which satisfy the same equation but which are not excited, not present in the system. They are called holes and their densities in the large volume limit is described by $\bar{\rho}_Q$. Clearly the densities of particles and holes are not independent they are connected by the thermodynamical limit of eq. (2.4) as

$$\partial_u\tilde{p} - 2\pi(\rho_Q + \bar{\rho}_Q) = -\sum_{Q'} \int du' K_{QQ'}(u,u')\rho_{Q'}(u') =: -K_{QQ'} \star \rho_{Q'} \qquad (2.5)$$

where the kernel is defined as

$$K_{QQ'}(u,u') = -i\partial_u\log S_{QQ'}(u,u')$$

The particle density itself does not characterize properly the states we sum over in the partition function. Indeed in a given interval $(u, u+du)$ the occupied $R\rho_Q(u)du$ particles can be distributed $\binom{R(\rho_Q(u)+\bar{\rho}_Q(u))du}{R\rho_Q(u)du}$ different ways leading to an entropy factor in the sum. Since in the large particle number limit the factorials can be approximated with the Stirling formula the partition function will take the form

$$Z(L,R) = \sum_n e^{-L\tilde{E}_n(R)} = \sum_Q \int d[\rho_Q]e^{-L\tilde{E}[\rho_Q]+S[\rho_Q,\bar{\rho}_Q]}$$

where the entropy factor is

$$S[\rho_Q,\bar{\rho}_Q] = R\int du\,[(\rho_Q+\bar{\rho}_Q)\log(\rho_Q+\bar{\rho}_Q) - \rho_Q\log\rho_Q - \bar{\rho}_Q\log\bar{\rho}_Q]$$

One can slightly generalize the partition function by adding a chemical potential term to the energy $-L\tilde{E}_Q[\rho_Q] \to \mu_Q[\rho_Q] - L\tilde{E}_Q[\rho_Q]$ where $\mu_Q[\rho_Q] = R\mu_Q\sum_Q\int du\rho_Q(u)$. For fermions we take $\mu_Q = i\pi$, while for bosons $\mu_Q = 0$. This extended partition function can be evaluated in the saddle point approximation. Taking into account the relation between $\delta\rho_Q$ and $\delta\bar{\rho}_Q$ originating from the variation of (2.5) we obtain the minimizing equation in the so called pseudo energy $\varepsilon_Q = \log\frac{\bar{\rho}_Q}{\rho_Q}$ as

$$\begin{aligned}\varepsilon_Q(u) - L\tilde{E}_Q(u) + \mu_Q &= -\sum_{Q'} \int \frac{du'}{2\pi}K_{Q'Q}(u',u)\log(1+e^{-\varepsilon_{Q'}(u')}) \\ &=: -(\log(1+e^{-\varepsilon_{Q'}}) \star K_{Q'Q})(u)\end{aligned}$$

Once the pseudo energies are determined the ground state energy in volume L can be obtained as

$$E_0(L) = -\sum_Q \int \frac{du}{2\pi} (\partial_u \tilde{p}) \log(1 + e^{-\varepsilon_Q(u)}) \tag{2.6}$$

The nonlinear integral equation which determines the pseudo energies is called the thermodynamic Bethe Ansatz (TBA) equation. Although it is not possible to solve it in general it provides an implicit exact description of the groundstate energy. This implicit solution is a starting point of a systematic large and small volume expansion and can be used to derive either functional relations for the pseudo energies or TBA equations for excited states by analytical continuation.

Excited states by analytical continuation

Here we start with bosonic theories without bound-states and suppose that by analytically continuing in some parameter (say in the volume) we can reach all excited states. The way how excited states appear can be understood by analyzing the energy expression (2.6) integrated by parts

$$E = \int \frac{du}{2\pi} \tilde{p}(u) \partial_u \log(1 + e^{-\varepsilon(u)})$$

Let us suppose that in the analytical continuation singularities of type $1 + e^{-\varepsilon(u_i)} = 0$ appear. When we deform the contour their residue contributions give rise to

$$E = \sum_i E(u_i) - \int \frac{du}{2\pi} \partial_u \tilde{p}(u) \log(1 + e^{-\varepsilon(u)})$$

where we took into account the relation between the energy and the mirror momentum $E(u_j) = i\tilde{p}(u_j)$. Taking the same analytical continuation in the equation for the pseudo energy we obtain

$$\varepsilon(u) = L\tilde{E}(u) + \sum_i \log S(u_i, u) - \int \frac{dw}{2\pi} K(w, u) \log(1 + e^{-\varepsilon(w)})$$

Solving these equations iteratively for large L we can recognize that the $1 + e^{-\varepsilon(u_i)} = 0$ equations, which determine the positions of the singularities, coincide at leading order with the ABA equations (2.1). The subleading order calculation provides a universal formula for the leading finite size correction of multiparticle energy levels [5]. Alternatively for doing the analytical continuation one can think of the final result as choosing a different integration contour which surrounds the $1 + e^{-\varepsilon(u_i)} = 0$ singularities, and when we take the integration contour back to the real axis we pick up the above residue contributions.

Finally we note that if we have more species (labeled by Q) with diagonal scatterings (like in the previous subsection) then a singularity in $1 + e^{-\varepsilon_{Q_i}(u_i)} = 0$ results in the equations

$$\varepsilon_Q(u) = L\tilde{E}_Q(u) + \sum_i \log S_{Q_iQ}(u_i, u) - (\log(1 + e^{-\varepsilon_{Q'}}) \star K_{Q'Q})(u)$$

whose solutions $\varepsilon_Q(u)$ and $\{u_i\}$ have to be plugged into the energy formula

$$E = \sum_i E_{Q_i}(u_i) - \sum_Q \int \frac{du}{2\pi} \partial_u \tilde{p}_Q(u) \log(1 + e^{-\varepsilon_Q(u)})$$

One has to be careful with such an analytical continuation in the presence of bound-states. Bound-states require pole singularities of the scattering matrices which usually cross the integration contour in the analytical continuation and result in extra source terms. See the Lee-Yang model in the relativistic case [3,4] for example.

3 TBA for planar AdS/CFT

In this section we push forward the TBA program for planar AdS/CFT. The main difference compared to the previous discussion lies in the nondiagonal nature of the scattering matrix. There is a way, however, how we can profit from the previous diagonal results: the nondiagonal nature of any theory can be encoded into a diagonal theory but with auxiliary degrees of freedom. These auxiliary excitations do not contribute to the energy merely modifies the allowed momenta. Let us now follow the steps of Section 2.

3.1 Infinite volume characteristics of the model

The symmetry algebra of the theory has a factorized form: $su(2|2) \otimes su(2|2)$. The fundamental particle called magnon transforms in the bifundamental representation whose S-matrix has the structure

$$\mathbb{S}_{11}(p_1, p_2) = S(p_1, p_2)\hat{S}_{11}(p_1, p_2) \otimes \hat{S}_{11}(p_1, p_2) \tag{3.1}$$

where the matrix part $\hat{S}$ is fixed from its covariance under one copy of $su(2|2)$ up to a scalar factor, which is determined from unitarity and crossing symmetry. The scattering matrix has simple poles corresponding to bound-states. There is an infinite tower of bound-states labeled by a positive integer charge Q. They transform under the tensor product of the atypical totally *symmetric* representations of the algebra and have dispersion relation

$$E_Q(p) = \sqrt{Q^2 + 4g^2 \sin^2 \frac{p}{2}}$$

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3.2 Infinite volume characteristics of the mirror model

As the mirror model is derived from the same Euclidean theory the fundamental particles' scattering matrix is the analytical continuation of the scattering matrix (3.1). We are in a different analytical domain, however, and here different poles correspond to bound-states. These bound-states are also labeled by the charge Q but they transform under the atypical totally *antisymmetric* representations and have dispersion relation:

$$\tilde{E}_Q(\tilde{p}) = 2\,\text{arcsinh}\left(\frac{1}{2g}\sqrt{\tilde{p}^2 + Q^2}\right)$$

3.3 Very large volume solution: ABA for the model

If we put N particles in a finite volume L the momenta of the particles will be quantized. The multiparticle wave function has to be periodic in each argument, that is when a particle transported along the cylinder it scatters on all other particles before arriving back to its initial position. In a diagonal theory this results in (2.1). In a non-diagonal theory, however, the multiparticle transfer matrix has to be diagonalized. This can be achieved by introducing new type of (magnonic) particles with vanishing dispersion relations and considering the original problem in terms of them as a diagonal scattering theory.

Here we focus only on the charge $Q = 1$ sector of the theory. We have momentum carrying particles ($\bullet_1$) which scatter on each other as [3]

$$S_{11}^{\bullet\bullet}(p_1, p_2) = S(p_1, p_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}\sigma_{12}^{-2}$$

where $x^{\pm}(p) = \frac{(\cot\frac{p}{2}\pm i)}{2g}\left(1 + \sqrt{1 + 4g^2\sin^2\frac{p}{2}}\right)$ and σ represents the dressing phase.

These particles are extended for each $su(2|2)$ factor with two types of auxiliary particles ($y, \circ_1$), whose parameters are labeled by $y \in \mathbb{R}$ and $w \in \mathbb{R}$. The auxiliary particles have trivial dispersion relations (their energy and momentum are zero) and scatter with the fundamental, momentum carrying ones as

$$S_{1y}^{\bullet y}(p, y) = \frac{x^- - y}{x^+ - y}\sqrt{\frac{x^+}{x^-}} = S_{y1}^{y\bullet}(y, p)^{-1} \quad ; \qquad S_{11}^{\bullet\circ}(p, w) = 1$$

Furthermore, they scatter on each other as

$$S_{11}^{\circ\circ}(w_1, w_2) = S_{-2}(w_1 - w_2) \quad ; \qquad S_{y1}^{y\circ}(y, w) = S_1(v(y) - w) \quad ; \qquad S_{yy}^{yy}(y_1, y_2) = 1$$

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[3]The index 1 in $\bullet_1$ refers to the charge of the particle. This particle is a first member of an infinite series of bound-states labeled by $\bullet_Q$. Similarly we will meet particles of type $\circ_N$ and $\rhd_M$.

where $v(y) = y + y^{-1}$ and we introduced a useful function $S_n(v-w) = \frac{v-w+\frac{in}{g}}{v-w-\frac{in}{g}}$. Any scattering matrix can be extended by unitarity to the opposite order of their particle types/arguments: $S(i,j)S(j,i) = 1$.

In formulating the ABA equations for the full theory we have to take into account the two $su(2|2)$ factors and that they commute. The ABA equation for the momentum carrying particles reads as

$$1 = e^{ip_jL} \prod_{k:k\neq j}^{N_1^\bullet} S_{11}^{\bullet\bullet}(p_j,p_k) \prod_{\alpha=1,2}\prod_{l=1}^{N_{y\alpha}} S_{1y}^{\bullet y}(p_j,y_l^\alpha)$$

where $N_1^\bullet$ is the number of fundamental and $N_{y\alpha}$ the number of y type particles, while the $\alpha = 1,2$ index refers to the two $su(2|2)$ factors. Since the two factors commute the ABA equations for the auxiliary particles with rapidities $y^{1,2}$ and $w^{1,2}$ can be written as

$$\prod_{k:k\neq j}^{N_1^\bullet} S_{y1}^{y\bullet}(y_j^\alpha,p_k) \prod_{l=1}^{N_{1,\alpha}^\circ} S_{y1}^{y\circ}(y_j^\alpha,w_l^\alpha) = 1 = \prod_{k:k\neq j}^{N_{y\alpha}} S_{1y}^{\circ y}(w_j^\alpha,y_k^\alpha) \prod_{l:l\neq k}^{N_{1,\alpha}^\circ} S_{11}^{\circ\circ}(w_j^\alpha,w_l^\alpha)$$

Not all solutions of the ABA equations correspond to single trace operators as the level matching/zero momentum condition has to be fulfilled $\sum_j p_j = 0$. The theory contains also bound-states which can be determined from the singularity structure of the scattering matrices. Since from the TBA point of view only the bound-state spectrum of the mirror theory is relevant we will focus only on them.

3.4 Very large volume solution: ABA for the mirror model

In the case of the mirror theory the fundamental scattering matrix is the analytical continuation of the original one $p \to \tilde{p}$. As a result the ABA will be the analytical continuation, too

$$1 = e^{i\tilde{p}_jR} \prod_{k:k\neq j}^{N_1^\bullet} S_{11}^{\bullet\bullet}(\tilde{p}_j,\tilde{p}_k) \prod_{\alpha=1,2}\prod_{l=1}^{N_{y\alpha}} S_{1y}^{\bullet y}(\tilde{p}_j,y_l^\alpha) \tag{3.2}$$

$$-1 = \prod_{k:k\neq j}^{N_1^\bullet} S_{y1}^{y\bullet}(y_j^\alpha,\tilde{p}_k) \prod_{l=1}^{N_{1,\alpha}^\circ} S_{y1}^{y\circ}(y_j^\alpha,w_l^\alpha) \tag{3.3}$$

$$1 = \prod_{k:k\neq j}^{N_y^\alpha} S_{1y}^{\circ y}(w_j^\alpha,y_k^\alpha) \prod_{l:l\neq k}^{N_w^\alpha} S_{11}^{\circ\circ}(w_j^\alpha,w_l^\alpha) \tag{3.4}$$

There are some differences compared to the original ABA. First the domain of $\tilde{p} \in \mathbb{R}$ is different compared to $p \in [-\pi,\pi]$ and the total mirror momentum does not need to vanish. Then, as we are in the mirror theory, the way how $x^\pm$ is expressed in terms

of $\tilde{p}$ is also different: $x^{\pm} = \frac{(\tilde{p}-i)}{2g}\left(\sqrt{1 + \frac{4g^2}{1+\tilde{p}^2}} \mp 1\right)$. Additionally, in the calculations of the ground state energy the sectors with antiperiodic fermions are relevant and this is manifested in a minus sign in the middle equation. The possible bound-states and their ABA equations are the subject of the next section. Let us note that usually in the literature instead of (3.4) its inverse is considered as this will lead to positive particle densities in the thermodynamic limit.

3.5 Exact groundstate energy: TBA

In this section we derive TBA integral equations for the groundstate energy in finite volume R. We treat the theory as if it were diagonal with the scattering matrices specified above. First we analyze whether this "diagonal" theory has bound-states by analyzing the thermodynamic behavior of the equations and calculate the scattering matrices of the bound-states, the so called strings. They are special complex solutions of the ABA equations and they all contribute to the partition function which determines the ground state energy. Then we use the canonical procedure to derive coupled integral equations for the pseudo energies in a raw form, finally, using identities between the scattering matrices originating from the symmetry, we rewrite them in a simplified form and analyze simple excited states.

3.5.1 String hypothesis for the mirror model

The string hypothesis is similar to closing the S-matrix bootstrap program, that is to identify all particles (including bound-states) of the theory and to determine their scattering matrices. Let us premise that we will find bound-states of three infinite types $(\bullet_Q, \triangleright_M, \circ_N)$ for $Q, M, N \in \mathbb{N}$, and also of a finite type y_δ particle with $\delta \in \{\pm\}$. They can be arranged in the two dimensional lattice shown in Figure 1. Let us see how they arise from the ABA equations.

In the following we put R and all particle numbers large (keeping their ratio finite) and analyze the ABA one by one. Let us first note the reality properties of the equations. Unitarity of the mirror scattering matrix implies that the y roots come in complex conjugated pairs $y_i = (y_j^{-1})^*$ or lie on the unit circle $y = (y^{-1})^*$, similarly the roots w come in complex conjugated pairs $w_i = w_j^*$ or are real.

$\bullet_Q$ particles

In looking for momentum bound-states we rewrite the scattering matrix in (3.2) as

$$S_{11}^{\bullet\bullet}(\tilde{p}_1, \tilde{p}_2) = \frac{u_1 - u_2 + \frac{2i}{g}}{u_1 - u_2 - \frac{2i}{g}} \Sigma_{11}^{-2} \qquad ; \qquad \Sigma_{11} = \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \sigma$$

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where the rapidity is introduced as $u \pm \frac{i}{g} = x^{\pm} + \frac{1}{x^{\pm}}$. As R is very large complex values for u_1 with positive imaginary part are allowed. In this case the lhs. of (3.2) for $j = 1$ diverges so there should be another u say u_2 that goes to $u_1 - \frac{2i}{g}$. If u_2 still has a positive imaginary part then by the same argument there should be another u say u_3 which goes to $u_2 - \frac{2i}{g}$. Applying this procedure we arrive at a string of Q roots $u + (Q-1)\frac{i}{g}, u + (Q-3)\frac{i}{g}, \ldots, u - (Q-3)\frac{i}{g}, u - (Q-1)\frac{i}{g}$ or shortly $u_{Q+1-2j} = u + i(Q+1-2j)\frac{i}{g}$ where $j = 1, \ldots, Q$. (Clearly the $Q = 1$ string is the original particle itself.) The scattering of the Q-string with any other particle of type (.), label i and rapidity q is

$$S_{Qi}^{\bullet}(u,q) = \prod_{j=1}^{Q} S_{1i}^{\bullet}(u_{Q+1-2j}, q) = S_{iQ}^{\bullet}(q,u)^{-1}$$

Although naively the scattering matrices seem to depend on the parameters $x^{\pm}$ and such a way the bound-state scattering matrix depends on its constituents, this is not the case when we take into account the contributions of the dressing phase as was shown in [6].

The auxiliary particles exist for both $su(2|2)$ factors. Here we focus only on one of them and omit to write out its index.

y_δ particles

Let us analyze (3.3). If we suppose that the number of momentum carrying particles $N_1^{\bullet}$ goes to infinity then

$$\prod_{k:k \neq j}^{N_1^{\bullet}} S_{y1}^{y\bullet}(y_j, \tilde{p}_k) \rightarrow \begin{cases} 0 & \text{if} \quad |y_j| < 1 \\ \pm 1 & \text{if} \quad |y_j| = 1 \\ \infty & \text{if} \quad |y_j| > 1 \end{cases} \tag{3.5}$$

In the middle case y roots lying on the unit circle are allowed. As the scattering matrix $S_{y1}^{y\circ}(y,w)$ has a difference form in the variable $v(y) = y + y^{-1}$ we might use the parameter v instead of y. The inverse of the relation, however, is not unique. Defining $y_-(v) = \frac{1}{2}(v - i\sqrt{4-v^2})$ with the branch cuts running from $\pm\infty$ to ± 2 we can describe any y with $\Im m(y) < 0$ for $v \in [-2,2]$. Clearly $y_+(v) = y_-(v)^{-1}$ describes the other $\Im m(y) > 0$ case and in the scattering matrices $S_{y1}^{y\bullet}$ which depends on y, and not on v, we have to specify which root is taken. As a consequence we have two types of y particles y_δ with $\delta = \pm$ and the scattering matrices split as $S_{y1}^{y\bullet}(y,q) \rightarrow S_{\delta 1}^{y\bullet}(y_\delta(v),q) =: S_{\delta 1}^{y\bullet}(v,q)$.

$\triangleright_M$ particles

If $|y_1| < 1$ in (3.5) then the rhs. of (3.3) goes to zero which has to be compensated by a w_1 root which goes to $v_1 - \frac{i}{g} = y_1 + y_1^{-1} - \frac{i}{g}$. But then taking the ABA for w_1

means that the rhs. of (3.4) will diverge which has to be compensated by a root $v_2 = w_1 - \frac{i}{g}$. If the corresponding y_2 satisfies $|y_2| > 1$ then (3.3) is consistent with (3.5) and reality requires $y_1 = (y_2^{-1})^*$, $w_1 = w_1^*$. The three roots $y_1 \leftrightarrow v_1 = v + \frac{i}{g}$ and $w_1 = v$ and $v - \frac{i}{g} = v_2 \leftrightarrow y_2$ form an $M = 1$ string which we denote by $\triangleright_1$. In the case when $|y_2| < 1$ then we have to repeat the same arguments for y_2 leading to w_2 and y_3 and so on. Finally we arrive at the notion of a $\triangleright_M$ string. It consists of $2M$ y particles with $y_j = (y_{-j}^{-1})^*$ and M $\circ$ particles with synchronized parameters $w_{M+1-2j} = v + (M + 1 - 2j)\frac{i}{g}$ and $y_j \to v_{\text{sign}(j)(M+2-2j)} = v + \text{sign}(j)(M + 2 - 2j)\frac{i}{g}$ for $j = 1, \ldots, M$. The composite scattering matrix of the $\triangleright_M$ particle with all other particles is simply the product of the scatterings of its each individual constituents

$$S_{Mi}^{\triangleright\cdot}(v,q) = \prod_{j=1}^{M+1} S_{-i}^{y\cdot}(v_{M+2-2j}, q) \prod_{j}^{M} S_{1i}^{\circ\cdot}(w_{M+1-2j}, q) \prod_{j=1}^{M-1} S_{+i}^{y\cdot}(v_{M-2j}, q) = S_{iM}^{\cdot\triangleright}(i,v)^{-1}$$

$\circ_N$ particles

Suppose we have a large number of y particles and that w_1 has a positive imaginary part. Then the first factor of the rhs. of (3.4) will go to zero which has to be compensated by a root $w_2 = w_1 - \frac{2i}{g}$. If $\Im m(w_2) < 0$ then we obtain a $\circ_2$ string. In the opposite case we repeat to previous argumentation leading to an N string $w_{N+1-2j} = w + (N + 1 - 2j)\frac{i}{g}$. Clearly a single w is just a $\circ_1$ string. The scattering of the N string with any other particle is

$$S_{Ni}^{\circ\cdot}(w,i) = \prod_{j=1}^{N} S_{wi}^{\circ\cdot}(w_{N+1-2j}, i)$$

Scattering matrices

Summarizing, the mirror AdS theory in the thermodynamic limit could be replaced by a diagonal theory having constituents of infinite type $(\bullet, \triangleright, \circ)$ and index Q, M, N for $Q, M, N \in \mathbb{N}$, and also of finite type y particles with $\delta \in \{\pm\}$. See also Figure 1.

	$\bullet_{Q'}$	$\triangleright_{M'}$	$\circ_{N'}$	$y_{\delta'}$
$\bullet_Q$	$S_{QQ'}^{\bullet\bullet}$	$S_{QM'}^{\bullet\triangleright}$	1	$S_{Q\delta'}^{\bullet y}$
$\triangleright_M$	$S_{MQ'}^{\triangleright\bullet}$	$S_{MM'}^{\triangleright\triangleright}$	1	$S_{M\delta'}^{\triangleright y}$
$\circ_N$	1	1	$S_{NN'}^{\circ\circ}$	$S_{N\delta'}^{\circ y}$
y_δ	$S_{\delta Q'}^{y\bullet}$	$S_{\delta M'}^{y\triangleright}$	$S_{\delta N'}^{y\circ}$	1

Table 2: Scattering matrices of the various particles

For the readers convenience we summarize the scattering matrices in Table 2.

The scattering matrices are unitary $S_{ij}S_{ji} = 1$ and their explicit forms are

$$S_{QQ'}^{\bullet\bullet}(u,u') = S_{QQ'}(u-u')\Sigma_{QQ'}(u,u')^{-2}$$

$$S_{QQ'}(u-u') = S_{Q+Q'}(u-u')S_{Q'-Q}(u-u')\prod_{j=1}^{Q-1} S_{Q'-Q+2j}(u-u')^2$$

$$\Sigma_{QQ'}(u,u') = \prod_{j=1}^{Q}\prod_{k=1}^{Q'} \sigma(u_{Q+1-2j}, u_{Q'+1-2k}) \frac{1 - \frac{1}{x(u_{Q-2j})x(u_{Q'+2-2k})}}{1 - \frac{1}{x(u_{Q+2-2j})x(u_{Q'-2k})}}$$

where $u_j = u + j\frac{i}{g}$ and we reparametrized the momentum carrying particles in terms of the rapidity via the function $x(u) = \frac{1}{2}(u - i\sqrt{4-u^2})$. Recall also that $S_n(u-w) = \frac{u_n - w}{u_{-n} - w}$. The other matrix elements are

$$S_{QM}^{\bullet\triangleright}(u,v) = \frac{x(u_{-Q}) - x(v_M)}{x(u_Q) - x(v_M)} \frac{x(u_{-Q}) - x(v_{-M})}{x(u_Q) - x(v_{-M})} \frac{x(u_Q)}{x(u_{-Q})} \prod_{j=1}^{M-1} S_{M-Q-2j}(u,v)$$

$$S_{Q\delta}^{\bullet y}(u,v) = \frac{x(u_{-Q}) - x(v)^{\delta}}{x(u_Q) - x(v)^{\delta}} \sqrt{\frac{x(u_Q)}{x(u_{-Q})}}$$

$$S_{MM'}^{\triangleright\triangleright}(u,u) = S_{MM'}(u-u') = S_{MM'}^{\circ\circ}(u,u')^{-1}$$

$$S_{M\delta}^{\triangleright y}(u,v) = S_M(u-v) = S_{M\delta}^{\circ y}(u,v)$$

The ABA equations then have a generic form

$$(-1)^F = e^{i\tilde{p}.(q_j)R} \prod_k S_{jQ_k}^{\cdot\bullet}(q_j, u_{Q_k}) \prod_{\alpha=1,2}\prod_l S_{j\delta_l}^{\cdot y}(q_j, v_l^{\alpha}) \prod_m S_{jM_m}^{\cdot\triangleright}(q_j, v_{M_m}^{\alpha}) \prod_n S_{jN_n}^{\cdot\circ}(q_j, w_{N_n})$$

where . can be any type of $\bullet, \triangleright, \circ, y$ but only the $\bullet$ particles have nonvanishing energy $\tilde{E}_Q$ and momentum $\tilde{p}_Q(u) = g(x(u_{-Q}) - x(u_Q)) + iQ$. The parameter F denotes the fermion number. We also indicated the contributions of the two $su(2|2)$ factors. The energy of such a multiparticle state having $N_{Q_k}^{\bullet}$ of Q_k particles is

$$\tilde{E}(\tilde{p}_1, \ldots, \tilde{p}_k) = \sum_k \tilde{E}_{Q_k}(\tilde{p}_k)$$

Let us note that the ABA equations for the auxiliary particles can be inverted without changing their physical meaning. Taking the inverse of (3.4) is equivalent to redefining simultaneously the scattering matrices $S_{1\delta}^{\circ y} \to (S_{1\delta}^{\circ y})^{-1}$ and $S_{11}^{\circ\circ} \to (S_{11}^{\circ\circ})^{-1}$. Actually these are the equations used in the literature as they give positive particle densities in the thermodynamic limit.

3.5.2 Raw TBA equations

Suppose now that we would like to describe the groundstate energy in the AdS system in volume L. In doing so we follow the steps presented in Section 2 to evaluate the

partition function for large mirror sizes. We introduce densities of particles (strings) $\rho_Q^\bullet(u), \rho_M^\triangleright(u), \rho_N^\circ(u)$ for $u \in \mathbb{R}$ and $\rho_\delta^y(u)$ for $u \in [-2, 2]$ and the analogous densities of holes $\rho \to \bar{\rho}$. They are restricted via the logarithm of the ABA which contains the logarithmic derivatives of the scattering matrices

$$K_{jj'}(u, u') = -i\partial_u \log S_{jj'}(u, u')$$

Clearly $K_{jj'}(u, u') \neq -K_{j'j}(u', u)$ as the scattering matrices are not of the difference type. (Keeping in mind how we obtained the string solutions the densities are naturally ordered $\rho_Q^\bullet \gg \rho^y \gg \rho_N^\circ, \rho_M^\triangleright$.) Then we introduce the entropy factors for the densities, $i\pi$ chemical potential for fermions and calculate the saddle point of the functional integral. This results in integral equations for the pseudo energies $\varepsilon_Q^\bullet, \varepsilon_M^\triangleright, \varepsilon_N^\circ, \varepsilon_\delta^y$ as follows

$$\varepsilon_Q^\bullet = L\tilde{E}_Q - \log(1 + e^{-\varepsilon_{Q'}^\bullet}) \star K_{Q'Q}^{\bullet\bullet} - \log(1 + e^{-\varepsilon_M^\triangleright}) \star K_{MQ}^{\triangleright\bullet} - \log(1 + e^{-\varepsilon_\delta^y}) \star K_{\delta Q}^{y\bullet}$$

where in the contributions of the $\triangleright_M$ and y_δ we have to sum for the contributions of the two $su(2|2)$ factors (which we omitted to write out). The remaining equations are valid separately for the two $su(2|2)$ factors separately:

$$\varepsilon_M^\triangleright = -\log(1 + e^{-\varepsilon_Q^\bullet}) \star K_{QM}^{\bullet\triangleright} - \log(1 + e^{-\varepsilon_{M'}^\triangleright}) \star K_{M'M}^{\triangleright\triangleright} - \log(1 + e^{-\varepsilon_\delta^y}) \star K_{\delta M}^{y\triangleright}$$

$$\varepsilon_N^\circ = \log(1 + e^{-\varepsilon_{N'}^\circ}) \star K_{N'N}^{\circ\circ} + \log(1 + e^{-\varepsilon_\delta^y}) \star K_{\delta N}^{y\circ}$$

$$\varepsilon_\delta^y = -\log(1 + e^{-\varepsilon_Q^\bullet}) \star K_{Q\delta}^{\bullet y} - \log(1 + e^{-\varepsilon_M^\triangleright}) \star K_{M\delta}^{\triangleright y} - \log(1 + e^{-\varepsilon_N^\circ}) \star K_{N\delta}^{\circ y} + i\pi$$

Once these equations are solved the groundstate energy can be obtained as

$$E_0(L) = -\sum_{Q=1}^{\infty} \int \frac{du}{2\pi} \partial_u \tilde{p}_Q \log(1 + e^{-\varepsilon_Q^\bullet})$$

Finally we note that we replaced the magnonic ABA for the particle type $\circ_N$ with its inverse and made the corresponding change in the scattering matrices to ensure the positivity of the magnonic densities ρ_N°. It effectively changed the sign of the related kernels.

3.5.3 Simplified TBA equations and Y-system

In this subsection using identities among the TBA kernels we bring the equations in to a universal local form. This means that pseudo energies can be drawn in a two dimensional lattice, such that only neighboring sites couple to each other with the following universal kernel

$$s I_{MN} = \delta_{MN} - (K + 1)_{MN}^{-1} \quad ; \quad s(u) = \frac{g}{4\cosh\frac{g\pi u}{2}} \tag{3.6}$$

where $I_{MN} = \delta_{M+1,N} + \delta_{M-1,N}$ and $(K+1)^{-1}_{MN} \star (K_{NL} + \delta_{NL}) = \delta_{ML}$. To simplify the notation let us introduce the following Y functions

$$Y_Q^\bullet = e^{-\varepsilon_Q^\bullet} \quad ; \quad Y_M^\triangleright = e^{-\varepsilon_M^\triangleright} \quad ; \quad Y_N^\circ = e^{\varepsilon_N^\circ} \quad ; \quad Y_\delta^y = e^{\delta\varepsilon_\delta^y}$$

Clearly we have two copies for $Y_M^{\triangleright,\alpha}, Y_N^{\circ,\alpha}, Y_\delta^{y,\alpha}$. (To conform with the literature we inverted the ABA equations for $\triangleright_M$ and y_-). Acting with the operator (3.6) on these inverted TBA equations and using kernel identities like $(K+1)^{-1}_{MN} \star K_N = s\,\delta_{M,1}$ we arrive at their simplified, universal form

$$\log Y_M^\triangleright = \log(1 + Y_{M+1}^\bullet) \star s - I_{MM'} \log(1 + \frac{1}{Y_{M'}^\triangleright}) \star s + \delta_{M,1} \log \frac{1 + Y_+^y}{1 + \frac{1}{Y_-^y}} \hat{\star} s$$

$$\log Y_N^\circ = I_{NN'} \log(1 + Y_{N'}^\circ) \star s + \delta_{N,1} \log \frac{1 + Y_-^y}{1 + \frac{1}{Y_+^y}} \hat{\star} s$$

where in the convolution $\hat{\star}$ we integrate over the interval $[-2, 2]$ only. The other equations do not behave so nicely.

$$\log Y_Q^\bullet = -I_{QQ'} \log(1 + \frac{1}{Y_{Q'}^\bullet}) \star s + \log(1 + Y_{Q-1}^{\triangleright,1}) \star s + \log(1 + Y_{Q-1}^{\triangleright,2}) \star s \,; \quad Q > 1$$

$$\log Y_1^\bullet = -\log(1 + \frac{1}{Y_2^\bullet}) \star s + (\log(1 + Y_-^{y,1})(1 + Y_-^{y,2})) \star s - \check{\Delta} \star s$$

where $\check{\Delta}$ vanishes on the interval $[-2, 2]$ whose explicit form can be found in [7]. The equation for the y particles are simpler in the original form

$$\delta \log Y_\delta^y = -\log(1 + Y_Q^\bullet) \star K_{Q\delta}^{\bullet y} + \log \frac{1 + Y_M^\triangleright}{1 + \frac{1}{Y_M^\circ}} \star K_M + i\pi$$

These equations for Y_δ^y are not in a local form. However, acting with the inverse of s they can be brought into such form. The operator s^{-1} acts as $(f \star s^{-1})(u) = f(u + \frac{i}{g} - i0) + f(u - \frac{i}{g} + i0)$ and involves the analytical continuation of the functions. It has a large null space, thus when acting on the equation information is lost:

$$\log Y_-^y \star s^{-1} = \log(1 + Y_1^\bullet) + \log(1 + Y_1^\circ) - \log(1 + \frac{1}{Y_1^\triangleright})$$

The advantage of defining s^{-1} in the above manner is that it uses the analytically continued values of the Y functions on the rapidity torus only. If we continue them across the cuts by using $(f \star s^{-1})(u) = f(u + \frac{i}{g} - i0) + f(u - \frac{i}{g} - i0) = f^+(u) + f^-(u)$ then the term $\check{\Delta}$ disappears, but the Y functions have to be extended to an infinite genus Riemann surface. On this surface the Y-system has the universal form

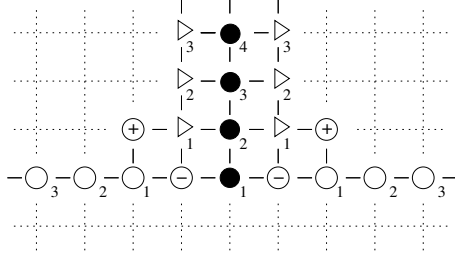$$Y_{N,M}^+ Y_{N,M}^- = \frac{(1 + Y_{N,M+1})(1 + Y_{N,M-1})}{(1 + Y_{N-1,M}^{-1})(1 + Y_{N+1,M}^{-1})} \tag{3.7}$$

Figure 1: Y-system for planar AdS/CFT. Y_- is denoted by $\ominus$ while Y_+ by $\oplus$.

where the N, M indices live on a two dimensional integral lattice. In our situation the identification can be drawn on Figure 1, which explicitly reads as $Y_Q^\bullet = Y_{Q,0}$, $Y_M^{\triangleright,\alpha} = Y_{M+1,v_\alpha}$, $Y_N^{\circ,\alpha} = Y_{1,v_\alpha(N+1)}$, $Y_-^{y,\alpha} = Y_{1,v_\alpha}$ and $Y_+^{y,\alpha} = Y_{2,v_\alpha 2}$ where $v_1 = 1$ and $v_2 = -1$.

3.5.4 Excited states by analytical continuation

Here we focus on the TBA equations for excited states in the sl_2 sector for small coupling. This sector contains particles of type $\bullet_1$ only and have ABA:

$$1 = e^{ip_k L} \prod_{j:j\neq k} S_{11}^{\bullet\bullet}(p_k, p_l)$$

These equations are asymptotic only and the exact system of TBA equations is required to describe the energy of the multiparticle state exactly. As the vacuum is a BPS state it has vanishing energy and its analytical continuation cannot describe excited states. Alternatively we choose an integration contour, such that when it is taken back to the real axis the residue of a singularity of the form $1 + e^{-\varepsilon_1^\bullet(p_k)} = 0$ is picked up resulting in additional source terms in the raw equations as:

$$\varepsilon_Q^\bullet \to \sum_j \log S_{1Q}^{\bullet\bullet}(p_j, u) \quad ; \qquad \varepsilon_M^\triangleright \to \sum_j \log S_{1M}^{\bullet\triangleright}(p_j, u) \quad ; \qquad \varepsilon_\delta^y \to \sum_j \log S_{1\delta}^{\bullet y}(p_j, u)$$

Once the new system of TBA equations are solved the pseudo energies $\varepsilon_Q^\bullet$ have to be plugged into the energy formula:

$$E(L) = \sum_k E_1(p_k) - \sum_{Q=1}^\infty \int \frac{du}{2\pi} \partial_u \tilde{p}_Q \log(1 + e^{-\varepsilon_Q^\bullet})$$

to obtain the energy of the multiparticle system.

We can rewrite the TBA equations in terms of the Y functions into their simplified form. They satisfy the same Y-system relations (3.7) but with a different asymptotical behavior. There is a systematical asymptotical expansion of the Y-system, which reproduces both the ABA and the leading Lüscher correction of these multiparticle

states. This is valid for weak coupling $g \to 0$ (or large sizes) and it is very nontrivial to follow the analytical behavior of the Y functions as one increases the coupling. The ABA solution itself suggests, that additional $1 + Y = 0$ singularities could appear and then the TBA equations have to be modified by additional source terms. These source terms ensure the analytical behavior of the energy around these singular points.

4 Guide to the literature

Here we list the representative papers where the various parts of the TBA program were developed.

The idea that the TBA program can be applied in the planar AdS/CFT setting was presented in [9]. The infinite volume scattering description of theory can be found in chapters [10,11]. The ABA equations for the planar AdS/CFT model was conjectured in [12] (and thoroughly discussed in chapters [13, 14]), while the analogous ABA for the mirror model was described in [15]. As the color structure ($su(2|2)$) of the scattering matrix is the same as that of the Hubbard model, the Hubbard TBA solution can be adopted [16]. This results in the string hypothesis which was formulated explicitly in [17]. The standard procedure leads to raw TBA equations, which were developed in [19, 18, 20]. The simplified form of the TBA equations was presented in [7] and the Y-system relations, presented previously in [21], were derived in [19, 18, 20]. In doing this the analytical properties of the dressing phase [6, 20, 22] had to be investigated. In the AdS/CFT context the volume of the integrable system has to be an integer, which can be seen also on the groundstate TBA [23].

Although we obtained the Y-system from the ground-state TBA equations, in principle, it follows from the hidden $PSU(2,2|4)$ symmetry of the model. An independent alternative approach based on this symmetry is the subject of the next Chapter in this volume [24].

The Y-system plays a crucial role in describing excited states. As it is related to the symmetry of the model [24–26] it is the same for each state. What makes the difference is the asymptotical and analytical behavior of the Y-functions. The analytical properties of the Y-functions was thoroughly analyzed in [19, 28, 23, 8]. Based on the solution of the Y-system of the $O(4)$ model [27] the authors of [21] identified the large volume solution in terms of the transfer matrices of the ABA [14]. This helps to derive excited states TBA equations for the sl_2 sector, which was done in [20, 28]. The excited state TBA equations provide an exact description of the given state and they were used in the Konishi case, [29, 30], to analyze numerically the behavior of the energy for large coupling. The results are summarized in Figure 2[4], see also [31]. It was further shown in [28] how to modify these excited state TBA equations if a $1 + Y = 0$ singularity appears in the analytical continuation in g.

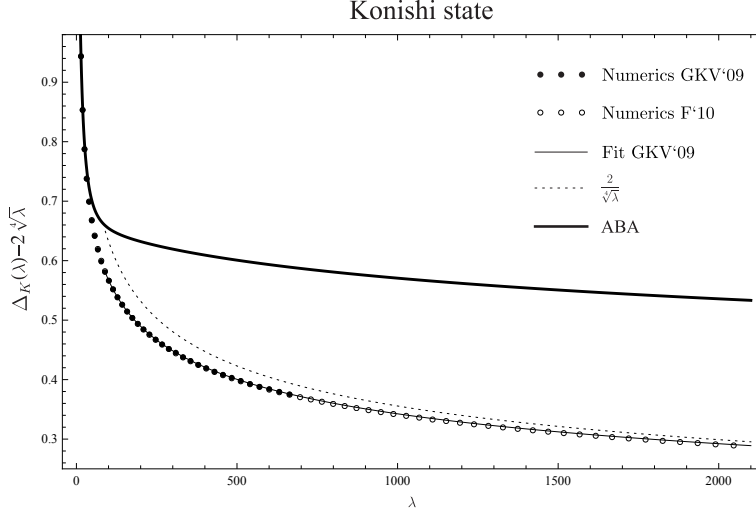[4]We thank the authors of [31] for borrowing their figure.

Figure 2: Numerical solution of the excited TBA equations for the Konishi state [29, 30].

The weak coupling limit of the Y-system equations can be compared to the ABA [14] and Lüscher type correction [32]. The leading order behavior is built in the asymptotic solution [21] of the Y-function, but the next to leading one provides a stringent test of the excited states TBA equations, which was performed numerically for the Konishi operator in [33] and analytically at next to leading order in [34]. Later this analytical calculation was extended to describe the next to leading order Lüscher correction of generic twist two states [35] in [36].

The strong coupling limit of the Y-system for a finite density of string particles was analyzed in [37], where a complete agreement with the one loop string energies including all exponential finite size corrections has been found. The functional Y-system equations were encoded into simpler Q functions in [38, 25, 31].

$Y_{a,s}$	This review	AF	BFT	GKKV
$Y_{Q,0}(u)$	$Y_Q^{\bullet}(u)$	$Y_Q(u)$	$Y_Q(u)$	$Y_{\bullet_Q}(u')$
$Y_{M+1,1}(u)$	$Y_M^{\triangleright}(u)$	$Y_{M\vert vw}^{-1}(u)$	$Y_{v,M}(u)$	$Y_{\triangle_{M+1}}(u')$
$Y_{1,N+1}(u)$	$Y_N^{\circ}(u)$	$Y_{N\vert v}(u)$	$Y_{w,N}(u)$	$Y_{\circ_{N+1}}(u')$
$Y_{2,2}(u)$	$Y_+^y(u)$	$-Y_+(u),$	$Y_y(u)$	$Y_{\oplus}(u')$
$Y_{1,1}(u)$	$Y_-^y(u)$	$-Y_-^{-1}(u),$	$Y_{y^*}(u)$	$Y_{\otimes}(u')$

Table 3: Relating the Y-functions to those in the literature, where $u' = gu$.

Let us mention, how our TBA equations are related to those in the literature. We summarized the relation between the various conventions for half of the Y-system

in Table 2 as the other half is trivially related, see also [8]. Under this replacement our simplified equations are equivalent to AF [7], while the raw equations to BFT [18], except for the chemical potentials of [18]. In comparing to GKKV [20] the indentification is not enough. Comparing our kernel $K_{QQ'}^{\bullet\bullet}$ to the one $K_{\bullet Q \bullet Q'}$ in [20] we observe a slight difference. This is irrelevant, however, for excited states satisfying the level matching/zero momentum condition [5].

The AdS_5/CFT_4 correspondence has a brother theory, the AdS_4/CFT_3 duality [39], where the TBA program has been developed in an analogous way. The ABA together with the string hypothesis of the mirror theory lead to ground state TBA equations and Y-system relations in [40, 41] and extend the previously conjectured Y-system proposal of [21]. This program is further elaborated in [41] by additionally determining excited states TBA equations and comparing them to the asymptotic solution of the Y functions [21] and to the quasi classical string spectrum.

Finally, let us list some open problems.

There are two disagreeing string theory calculations ([42] and [43]) for the anomalous dimension of the Konishi state. Additionally, the numerical solution of the TBA equations for large couplings [29, 30] provides a third result, and calls for improvements both the string theory and the TBA sides. On the string theory side it could be a pure spinor calculation, while on the TBA side one should analyze the analytical behavior of the Y-system and check whether, with increasing g, a singularity of type $1 + Y = 0$ indeed appears, as the asymptotic solution suggests [28]. In principle the effect of such singularities is to make the coupling dependence of the energies analytical, but it has to be established concretely.

The anomalous dimensions of twist operators in the planar limit can be described by integral equations derived directly from the ABA [44]. It would be nice to see, how the exact excited TBA equations reduce to these equations in the large spin limit.

The analytical comparision of the excited state TBA equations to the next to leading order Lüscher corrections [34, 36] tested explicitly only the $\triangleright$ part of the Y-system. A next to next to leading order analysis could test the $\circ$ part as well.

The excited states TBA equations are coupled nonlinear integral equations for infinite unknowns. An ideal system of equations should contain finite unknowns only, and could be developed in analogy to [27, 45] by exploiting the result of [38, 25, 31].

Acknowledgments

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[5] We thank the authors of [20] for pointing out this.

Note added in proof

After this review chapter was finished three string theory calculations based on different methods determined the strong coupling expansion of the anomalous dimension of the Konishi operator [46–48]. All agreed with each other and with the strong coupling expansion of the TBA equation [29, 30]. This gives a strong support not only for the correctness of the TBA equations but also for the integrability approach to planar AdS/CFT.

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