Mon. Not. R. Astron. Soc. **000**, 1–7 (2008) Printed 30 October 2018 (MN LaTeX style file v2.2)

Properties of analytic transit light curve models

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Accepted .... Received ...; in original form ...

ABSTRACT
In this paper a set of analytic formulae are presented with which the partial derivatives of the flux obscuration function can be evaluated – for planetary transits and eclipsing binaries – under the assumption of quadratic limb darkening. The knowledge of these partial derivatives is crucial for many of the data modeling algorithms and estimates of the light curve variations directly from the changes in the orbital elements. These derivatives can also be utilized to speed up some of the fitting methods. A gain of ~ 8 in computing time can be achieved in the implementation of the Levenberg-Marquardt algorithm, relative to using numerical derivatives.

Key words: Stars: Binaries: Eclipsing – Stars: Planetary Systems – Methods: Analytical

1 INTRODUCTION

In recent years, the discovery and further characterization of transiting extrasolar planets (TEPs) has provided unique information about the nature of planetary systems. The analysis of a planet which periodically eclipses its host star yields the physical radius, the inclination and the mass of the system in addition to the parameters which are gathered from the radial velocity measurements. Since the discovery of the first such system (see Charbonneau et al. 2000; Brown et al. 2001), more than 40 other TEPs were discovered around other stars. Currently operating ground-based surveys are producing numerous new discoveries. The doubling period of the number of known TEPs is below one year. Moreover, existing (CoRoT, see Barge et al. 2008; Alonso et al. 2008) and planned space-borne instruments (e.g. Kepler Mission, see Borucki et al. 2007) are expected to yield hundreds of new discoveries of such systems, even with planetary radii comparable to that of Earth. Also, subsequent observations of a given transiting system can provide some information on the variations in the timing of successive transits and the light curve shape. These detections can be used to constrain other planetary companions (Agol et al. 2005; Steffen & Agol 2007; Holman & Murray 2005) or co-orbital companions (Trojans, see Ford & Holman 2007).

In order to optimize the precision and speed of TEP observations, a careful analysis of the light curves is required. The basis of such light curve analysis is to find an adequate model of planetary obscuration, which causes a small decrease in the stellar flux. Since both the stellar and the planetary body can be well modelled by a spheroidal shape, the decrease in the stellar flux can be estimated from the full or partial overlap of two circles. The star itself has a non-negligible limb darkening which depends on both the stellar properties and the photometric band and quantified by a small set of coefficients (see e.g. Claret 2004, for such tables of limb darkening constants). At present, the most widely used models for this problem have been given by Mandel & Agol (2002) and Giménez (2006). Mandel & Agol (2002) calculate closed form expressions for the flux decrease assuming non-linear or quadratic limb darkening (quantified by 4 or 2 coefficients, respectively) while Giménez (2006) gives an infinite series where the limb darkening can be taken into account up to arbitrary order. In most cases the quadratic case is adequate because the photometric precision of typical data is not good enough for the higher order limb darkening models to make a difference.

Most data modeling algorithms, including the well known nonlinear Levenberg-Marquardt fitting method (see Press et al. 1992) utilize the partial derivatives of the model function with respect to the model parameters. The uncertainties in the model parameters can be well characterized by the Fisher information matrix (see e.g. Finn 1992), also requiring knowledge of the same partial derivatives. Therefore, in the case of planetary transits, the parametric derivatives of the flux decrease function can be extremely valuable. Moreover, partial derivatives can be used to construct a set of uncorrelated parameters of the light curve which is preferred by most of the parameter fitting algorithms (e.g. Levenberg-Marquardt, downhill simplex, Markov Chain Monte Carlo). Also, the analysis of the partial derivatives with respect to the limb darkening coefficients themselves yields a combination of these with which consistent sanity checks can be done verifying the stellar at-

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Table 1. Exclusion cases of different occultation geometries. To figure out the respective case for a certain value of p and z, check the relations step by step: if the current relation is true, that case can be assigned to the given values of p and z, otherwise go to the next step. The final case **G** is when there is no obscuration. Cases with subscripts can only occur if the radius of the planet is not smaller than 1/2.

Step	Relation	Case	M&A case
1	$z = 0$ & $p < 1$	**A**	10
2	$z \leqslant p - 1$	$\mathbf{A_G}$	11
3	$z < p$ & $z < 1 - p$	**B**	9
4	$z < p$ & $z = 1 - p$	$\mathbf{B_T}$	–
5	$z < p$	$\mathbf{B_G}$	8
6	$z = p$ & $z < 1 - p$	**C**	5
7	$z = p = 1/2$	$\mathbf{C_T}$	6
8	$z = p$	$\mathbf{C_G}$	7
9	$z < 1 - p$	**D**	3
10	$z = 1 - p$	**E**	4
11	$z < 1 + p$	**F**	2
12	–	**G**	1

mospheric properties in an independent way. Finally, these derivatives can be used to directly calculate how the variations in the orbital parameters affect transit timing and the shape of the light curve.

In this paper, we present the partial derivatives of the flux decrease function assuming a quadratic limb darkening law. In the next section, the formalism and the derivatives are presented. In Section 3, we apply these derivatives to construct a set of well behaved parameterizations of transiting light curves which yield an always finite and moderate correlation between the adjusted quantities in all important cases and limits. The correlations between the limb darkening coefficients are also discussed. The results are summarized in the last section.

2 PARAMETRIC DERIVATIVES OF THE FLUX DECREASE

In this section the partial derivatives of the flux decrease are presented. The surface brightness of a star as a function of the normalized distance $0 \leqslant r \leqslant 1$ assuming quadratic limb darkening is given by the equation

$$I(r) = 1 - \sum_{m=1,2} \gamma_m \left(1 - \sqrt{1-r^2}\right)^m, \qquad (1)$$

where the constants γ_1 and γ_2 quantify the limb darkening. Recalling Mandel & Agol (2002), the relative apparent flux of an eclipsed star with a quadratic limb darkening can be written as $f = 1 - \Delta f$ (assuming a unity flux out of the transit), where flux decrease Δf can be calculated using the equation

$$\begin{aligned} \Delta f &= W_0 F_0 + W_2 F_2 + \\ &\quad W_1 [F_1 + F_K \mathrm{K}(k) + F_E \mathrm{E}(k) + F_\Pi \Pi(n,k)]. \end{aligned} \qquad (2)$$

In this equation the quantities W_i $(i = 0,1,2)$ are only functions of the limb darkening coefficients, namely

$$W_0 = \frac{6 - 6\gamma_1 - 12\gamma_2}{W}, \qquad (3)$$

$$W_1 = \frac{6\gamma_1 + 12\gamma_2}{W}, \qquad (4)$$

$$W_2 = \frac{6\gamma_2}{W}, \qquad (5)$$

where $W = 6 - 2\gamma_1 - \gamma_2$. In equation (2) the terms F_0, F_1, F_K, F_E, F_Π and F_2 are only functions of the occultation geometry, namely the relative planetary radius $p \equiv R_p/R_\star$ and the normalized projected distance z between the center of the star and the center of the planet. In equation (2) the functions $\mathrm{K}(\cdot)$, $\mathrm{E}(\cdot)$ and $\Pi(\cdot,\cdot)$ denote the complete elliptic integrals of the first, second and third kind, respectively. The variation in the occultation geometry yields 12 distinct cases of obscuration, which are summarized in Table 1. This table is an *exclusion* table and should be interpreted as follows. For a given value of (p, z), the first relation (in step 1) is checked. If it is true, the appropriate case can be assigned to the geometry, otherwise the next relation should be checked and so on. The different cases are denoted by bold capitals for planetary radii smaller than 1/2 and capitals with a subscript which can only occur if the radius of the planet is greater than or equal to 1/2 In practice it would barely occur for planetary companions for earlier types of stars but it might happen in the cases when a Jovian planet transits a later main sequence star (M dwarf). For the actually most common planet-like applications ($p < 1/2$), the 7 major cases (**A**,...,**G**) are in the order of growing distance between the geometrical centers of the planet and the star. In equation (2) the expressions for the terms F_i $(i = 0, 1, K, E, \Pi, 2)$, k and n can be found in tables A1 and A2 in Appendix A; after the appropriate geometrical case has been obtained. We should note here that equation (2) is completely equivalent with the equation found in Mandel & Agol (2002), in the first line of the second paragraph in their Section 4 at page L173. However, this expansion of equation (2) clearly separates the terms which depend only on the limb darkening coefficients (W_m) and the terms which depend only on the occultation geometry (F_i).

2.1 Partial derivatives with respect to the limb darkening coefficients

Since in equation (2) the only quantities which depend on the limb darkening coefficients are W_0, W_1 and W_2, the partial derivatives of Δf with respect to γ_m $(m = 1, 2)$ can easily be obtained, namely

$$\begin{aligned} \frac{\partial \Delta f}{\partial \gamma_m} &= \frac{\partial W_0}{\partial \gamma_m} F_0 + \frac{\partial W_2}{\partial \gamma_m} F_2 + \\ &\quad \frac{\partial W_1}{\partial \gamma_m} [F_1 + F_K \mathrm{K}(k) + F_E \mathrm{E}(k) + F_\Pi \Pi(n,k)], \end{aligned} \qquad (6)$$

where the appropriate derivatives $\partial W_i / \partial \gamma_m$ are the following:

$$\frac{\partial W_0}{\partial \gamma_1} = \frac{2W_0 - 6}{W} \qquad (7)$$

$$\frac{\partial W_0}{\partial \gamma_2} = \frac{W_0 - 12}{W} \qquad (8)$$

$$\frac{\partial W_1}{\partial \gamma_1} = \frac{2W_1 + 6}{W} \qquad (9)$$

$$\frac{\partial W_1}{\partial \gamma_2} = \frac{W_1 + 12}{W} \qquad (10)$$

$$\frac{\partial W_2}{\partial \gamma_1} = \frac{2W_2}{W} \quad (11)$$

$$\frac{\partial W_2}{\partial \gamma_2} = \frac{W_2+6}{W}. \quad (12)$$

2.2 Partial derivatives with respect to the geometric parameters

In equation (2), the terms F_i explicitly depend on the relative planetary radius p and the normalized distance z. There is also an implicit dependence via the complete elliptic integrals since their parameters k and n are also functions of p and z. The derivation of these partial derivatives are quite straightforward for the non-degenerate cases, i.e. for the cases $\mathbf{B}$, $\mathbf{B_G}$, $\mathbf{D}$ and $\mathbf{F}$ since all of the appearing functions in these domains are analytic. For the other cases, the partial derivatives can be calculated as the appropriate limits, namely

$$\partial F_i^{\mathbf{A}} \equiv \lim_{z\to 0} \partial F_i^{\mathbf{B}}, \quad (13)$$

$$\partial F_i^{\mathbf{B_T}} \equiv \lim_{z\to(1-p)-0} \partial F_i^{\mathbf{B}}, \quad (14)$$

$$\partial F_i^{\mathbf{C}} \equiv \lim_{z\to p-0} \partial F_i^{\mathbf{B}} = \lim_{z\to p+0} \partial F_i^{\mathbf{D}}, \quad (15)$$

$$\partial F_i^{\mathbf{C_T}} \equiv \lim_{p\to 1/2} \partial F_i^{\mathbf{C}}, \quad (16)$$

$$\partial F_i^{\mathbf{C_G}} \equiv \lim_{z\to p+0} \partial F_i^{\mathbf{F}}, \quad (17)$$

$$\partial F_i^{\mathbf{E}} \equiv \lim_{z\to(1-p)-0} \partial F_i^{\mathbf{B}}. \quad (18)$$

For $\mathbf{A_G}$ and $\mathbf{G}$, all of the derivatives are obviously 0, except for the case ($p=1$, $z=0$) when the partial derivatives do not exist.

Utilizing the parametric derivatives of the elliptic integrals (see Appendix B), the final form of the partial derivatives of Δf with respect to z and p is

$$\begin{aligned}\frac{\partial \Delta f}{\partial g} = {} & W_0 F_{0,g} + W_1 F_{1,g} + W_2 F_{2,g} + \\ & W_1 \mathrm{K}(k)\left[F_{K,g} - \frac{(F_K+F_E)k_g}{k} + \frac{F_\Pi n_g}{2n(n-1)}\right] + \\ & W_1 \mathrm{E}(k)\left[F_{E,g} + \frac{F_K k_g}{k(1-k^2)} + \frac{F_E k_g}{k} + \right. \\ & \left. + \frac{F_\Pi k k_g}{(k^2-n)(1-k^2)} + \frac{F_\Pi n_g}{2(k^2-n)(n-1)}\right],\end{aligned} \quad (19)$$

where g denotes either p or z, the appropriate geometric parameter. The expressions for $F_{0,g}$, $F_{1,g}$, $F_{K,g}$, $F_{E,g}$, $F_{2,g}$, k_g and n_g can be figured out for all cases using the tables in Appendix C, namely Table C1 and Table C2.

We note here that the computation of these derivatives are even more simple and faster than the computation of equation (2) since equation (19) lacks the complete elliptic integral of the third kind for which evaluation requires most of the computing time.

3 APPLICATIONS

In this section we present three simple applications which utilize the partial derivatives of the flux decrease function. All of these applications assume a transiting planetary system on a circular orbit with a given semimajor axis (relative to the radius of the star) $a/R_\star$, an impact parameter, $b \equiv (a/R_\star)\cos i$, the planetary companion has a fixed mean motion of $n = 2\pi/P$ and the transit occurs at the instance E. The relative radius of the planet is denoted by $p \equiv R_p/R_\star$. Therefore, the distance between the center of the stellar and planetary disk has a time dependence,

$$z^2(t) = \left(\frac{a}{R_\star}\right)^2 \sin^2[n(t-E)] + b^2\cos^2[n(t-E)]. \quad (20)$$

From now on we assume that the semimajor axis is relatively large, i.e. the distance can be approximated by

$$z^2(t) \cong \left(n\frac{a}{R_\star}\right)^2 (t-E)^2 + b^2. \quad (21)$$

In the following parts of this section, we first calculate the correlations between the limb darkening coefficients, assuming the orbital parameters and the relative planetary radius to be known. In the second part of this section, we construct a set of adjusted parameters which always yields finite (i.e. definitely smaller than unity) correlations between them in the cases of non-grazing eclipses. This is relevant for studies of transiting planets since equation (21) yields a unity correlation between $a/R_\star$ and b in the limit of $p \to 0$ with or without limb darkening for all impact parameters. In the last part of this section we present an analytical calculation how the uncertainties in the light curve parameters depend on the photometric passbands, assuming a Jupiter-sized planet orbiting a solar-type star. In all of these cases we use the Fisher matrix method (Finn 1992) to obtain the uncertainties and correlations of the fitted parameters. This method gives the covariance matrix as

$$\langle \delta a_m \delta a_n \rangle = \left(\Gamma^{-1}\right)_{mn}, \quad (22)$$

where

$$\Gamma_{mn} = \sum_i \frac{\partial_m f(\mathbf{a}, t_i)\partial_n f(\mathbf{a}, t_i)}{\sigma_i^2}, \quad (23)$$

$f(\mathbf{a}, t_i)$ is the observed flux at the instance t_i, $\mathbf{a} = (a_1, a_2) = (\gamma_1, \gamma_2)$ or $\mathbf{a} = (a_1, a_2, a_3, a_4) = (p, a/R_\star, b, E)$ is the vector of the adjusted parameters (depending on the actual application) and $\partial_m \equiv \partial/\partial a_m$. Since we are interested only in the correlations between the parameters and we can expect uniform uncertainties in the measurements and uniform data sampling, therefore Γ can be multiplied by any arbitrary constant and the sum in equation (23) can be replaced by the integral

$$\Gamma_{mn} \propto \int_{t_1}^{t_2} \partial_m f(\mathbf{a}, t)\partial_n f(\mathbf{a}, t)\mathrm{d}t. \quad (24)$$

Here $t_1 < E - (na/R_\star)^{-1}(1+p)$ and $E + (na/R_\star)^{-1}(1+p) < t_2$, assuring that the ingress and egress are completely observed, independently from the impact parameter.

3.1 Correlations between the limb darkening coefficients

Now, we determine the correlations between the limb darkening coefficients when the adjusted parameters are $\mathbf{a} =$

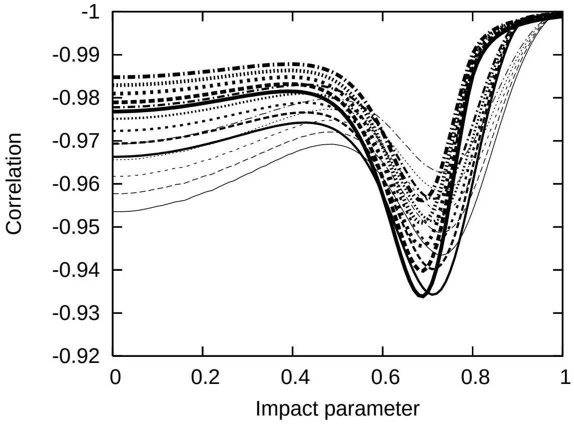

Figure 1. The dependence of the correlation between the limb darkening coefficients, $C(\gamma_1,\gamma_2)$ as a function of the impact parameter. Thin lines are for $p = 0.01$, moderately thick lines are for $p = 0.1$ and thick lines are for $p = 0.2$. The continuous, long dashed, short dashed, dotted and dotted-dashed lines represents the cases when limb darkening coefficients are $\gamma_1 = \gamma_2 = 0$, $\gamma_1 = \gamma_2 = 0.1$, $\gamma_1 = \gamma_2 = 0.2$, $\gamma_1 = \gamma_2 = 0.3$ and $\gamma_1 = \gamma_2 = 0.4$, respectively.

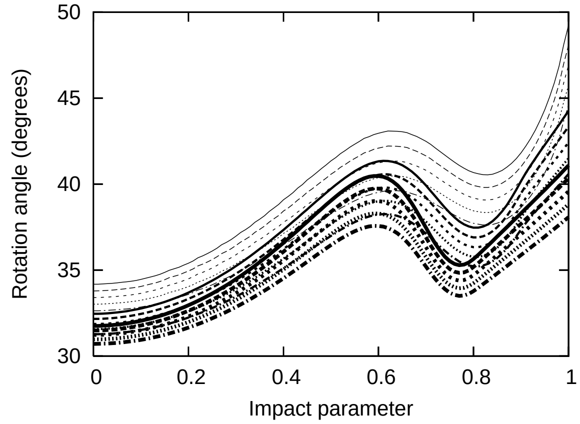

Figure 2. The optimal rotation parameter (in degrees) to avoid correlations between the limb darkening parameters as a function of the impact parameter (see text for definition and further details). Thin lines are for $p = 0.01$, moderately thick lines are for $p = 0.1$ and thick lines are for $p = 0.2$. The continuous, long dashed, short dashed, dotted and dotted-dashed lines represents the cases when limb darkening coefficients are $\gamma_1 = \gamma_2 = 0$, $\gamma_1 = \gamma_2 = 0.1$, $\gamma_1 = \gamma_2 = 0.2$, $\gamma_1 = \gamma_2 = 0.3$ and $\gamma_1 = \gamma_2 = 0.4$, respectively.

(γ_1,γ_2). It is easy to show that this correlation would only depend on the impact parameter, the planetary radius and the two limb darkening parameters themselves while it does not depend on the mean motion, geometrical ratio of $a/R_\star$ and the transit center time E. We have obtained these correlations for very small ($p = 0.01$), average ($p = 0.1$) and large ($p = 0.2$) planetary radii assuming limb darkening coefficients between 0.0 and 0.4, while the impact parameter was varied between 0 and 1. We found that the correlation is always negative, relatively large, i.e. $|C(\gamma_1,\gamma_2)| \gtrsim 0.93$ and it strongly depends on the impact parameter. For these certain values, the correlation is plotted as a function of b on Fig. 1. It can easily be seen that the smallest correlation is around $b \approx 0.7 - 0.8$, almost independent of the limb darkening and the radius of the planet.

Let us now calculate the optimal linear combination of the limb darkening coefficients which can be adjusted to yield no correlation. Define the parameters u_1 and u_2 as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \tag{25}$$

For simplicity, let us denote the above orthogonal matrix by $\mathbf{O} = O_{mn}$. It can be shown that the covariance matrix of (u_1, u_2) and that of (γ_1,γ_2) are related to each other as

$$\langle \delta u_k \delta u_\ell \rangle = O_{km} \langle \delta\gamma_m \delta\gamma_n \rangle \tilde{O}_{n\ell}. \tag{26}$$

To make the matrix $\langle \delta u_k \delta u_\ell \rangle$ diagonal, the rotation parameter φ should be in coincidence with the orientation of the eigenvectors of $\langle \delta\gamma_m \delta\gamma_n \rangle$, namely

$$\varphi = \frac{1}{2} \arctan \frac{\langle \delta\gamma_1\delta\gamma_2 \rangle + \langle \delta\gamma_2\delta\gamma_1 \rangle}{\langle \delta\gamma_1\delta\gamma_1 \rangle - \langle \delta\gamma_2\delta\gamma_2 \rangle}. \tag{27}$$

We have obtained the optimal values of the rotation parameter φ for very small ($p = 0.01$), average ($p = 0.1$) and large ($p = 0.2$) planetary radii assuming limb darkening coefficients between 0.0 and 0.4, while the impact parameter was varied between 0 and 1. We found that like above, this parameter mostly depends on the impact parameter. The results are plotted on Fig. 2. The usefulness of such a plot is somewhat limited if we have no *a priori* knowledge from the limb darkening coefficients themselves since the correlation between them and therefore the optimal rotation parameter depends on the actual values of γ_1 and γ_2. However, in practice if we have a hint for the planetary radius and the impact parameter, this angle can be estimated within a few degrees since the correlation depends more strongly on b and p than γ_1 or γ_2. If we do not know any reasonable value for p or b before the fit, an angle of $\varphi \approx 35^\circ - 40^\circ$ is a plausible selection in general.

3.2 Correlations between the light curve parameters

In this subsection we investigate the correlations between the light curve parameters utilizing the previously obtained partial derivatives and Fisher information matrix method. The classical formalism of adjusting parameters of a transiting system uses the same parameters as in equation (20), namely the ratio $a/R_\star$, the impact parameter b and the instance of the center of the transit E as well as the radius of the planet, $p = R_p/R_\star$. Since the flux decrease depends directly on the radius p and indirectly on $a/R_\star$, b and E (assuming a fixed limb darkening), the partial derivatives of the light curve $f(t) = 1 - \Delta f(t)$ can be obtained by using equation (19) and the chain rule. These derivatives can then be plugged into equation (24) while the adjusted set of parameters will be $\mathbf{a} = (p, b, a/R_\star, E)$. We have obtained the correlations between these variables and we have found that the correlation between b and $a/R_\star$ tends to unity as the radius of the planet is decreased. This correlation is plotted as a function of the impact parameter for planetary radii $p = 0.01$, $p = 0.1$ and $p = 0.2$ and for various limb darkening parameters on Fig. 3. It can clearly be seen that $C(b, a/R_\star)$

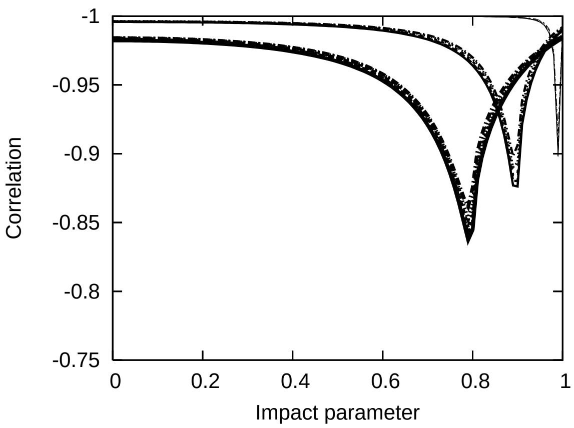

Figure 3. Correlation between the adjusted values of $a/R_\star$ and the impact parameter b for various values of planetary radii and limb darkening coefficients: Thin lines are for $p = 0.01$, moderately thick lines are for $p = 0.1$ and thick lines are for $p = 0.2$. The cases for different limb darkening constants are almost indistinguishable.

is almost independent from the actual limb darkening. From further analysis of the correlation, it turns out that for small impact parameters ($b \lesssim 0.5$), $C(b, a/R_\star)$ can be well approximated by $1 - p^2/2$. Therefore, for very small planets, this correlation can be undesirable and most of the fitting methods would distort the results.

At this point we have checked the correlations between an alternative parametrization proposed by Bakos et al. (2007). In that work the light curve was parametrized by the equation

$$z^2(t) = \left(\frac{\zeta}{R_\star}\right)^2 (1-b^2)(t-E)^2 + b^2, \tag{28}$$

where $\zeta/R_\star = n(a/R_\star)/\sqrt{1-b^2}$. This parameter is related to the duration of the transit, namely $(\zeta/R_\star)^{-1} = T_{\rm duration}/2$, where $T_{\rm duration}$ is the time between the instances when the center of the planet crosses the limb of the star. Since the above parametrization of z^2 is linear in b^2, we have chosen b^2 instead of b as an independent parameter. We have found that utilizing the parameter set $\mathbf{a}' = (p, b^2, \zeta/R_\star, E)$ yields practically no correlation between b^2 and $\zeta/R_\star$ for non-grazing eclipses and increases only up to unity near $b \gtrsim 1 - p$. This correlation is plotted on Fig. 4 for various planetary radii and limb darkening coefficients.

We should mention here that the recent work of Carter et al. (2008) gives an exhaustive analysis of the uncertainties and correlations for various kind of transiting light curve characterizations. Their work focuses on the analytical calculations for light curves with no limb darkening and compares these results with numerical derivations for the limb darkened cases.

3.3 Uncertainties of the light curve parameters

In this subsection we calculate the dependence of uncertainties of the light curve parameters $a/R_\star$, E and b (see equation 20) and the fractional planetary radius p for various photometric passbands from near-ultraviolet to mid-infrared. It is known that the limb-darkening parameters decrease for longer wavelengths, therefore the transits themselves become shallower and flattened. Using the Fisher information matrix method, as described earlier gives a simple and straightforward way to obtain these uncertainties. Assuming a solar-type star – i.e. with metallicity of $[\mathrm{Fe/H}] = 0.00$, surface gravity of $\log g_\star = 4.44\,(\mathrm{CGS})$ and atmospheric temperature of $T_{\rm eff} = 5780\,\mathrm{K}$ – we estimated these uncertainties for photometric passbands u', g', r', i', z', J, H and K, when such a star is transited by a hypotetical planet with the orbital parameters of $a/R_\star = 10$, $P = 3.67$ days and $p = R_{\rm p}/R_\star = 0.1$. The appropriate limb darkening coefficients for each filter have been obtained using the tables provided by Claret (2004). The results are plotted on Fig. 5 for various impact parameters ($b = 0.2$, $b = 0.5$ and $b = 0.8$). During these estimations, the transits are assumed to be observed with a cadence of 10 seconds and with a photometric precision of $\Delta m = 1.4\,\mathrm{mmag}$. Note that this photometric precision is attainable by $\sim 1 - 1.2\,\mathrm{m}$ class telescopes for relatively bright stars ($z' \approx 9.5\,\mathrm{mag}$) and using such cadence (see e.g. Winn et al. 2007).

It can clearly be seen from the plots of Fig. 5 that the uncertainties for all of the parameters decrease for longer wavelengths; moreover, this decrement can reach a factor of ~ 10 (between the near-ultraviolet and mid-infrared bands) for the planetary radius. We note here that these analytical results agrees well with numerical estimations (Joshua N. Winn, personal communication).

4 DISCUSSION AND SUMMARY

In this paper the partial derivatives of the flux decrease function of exoplanetary transits (or stellar binary eclipses) has been calculated assuming a quadratic limb darkening law. These derivatives can then be applied for various analyses from which we have demonstrated the correlation analysis of the limb darkening coefficients and two of the known transit light curve parameterizations. The most time consuming part of the evaluation of the flux decrease (and its derivatives) is the computing of the complete elliptic integrals. Therefore, the calculation of the partial derivatives does not increase significantly the total computing time of both. The presented analytical analysis of light curve parametrization is extremely fast comparing to such an analysis based on Monte-Carlo methods: the integral in equation (24) should be calculated only once instead of the evaluation of the χ^2 function $\mathcal{O}(10^4)$ times[1]. The knowledge of these derivatives also allows straightforward calculations about how the variations in the orbital elements affect the light curves and the timings of the successive transits.

Here we note that such derivatives are also helpful in the implementation of the Levenberg-Marquardt algorithm. Since this method requires the partial derivatives of the function to be adjusted, these must be evaluated either analytically or numerically. The numerical evaluation of the partial derivatives requires the computation of the original function twice in all directions of the parameter space. Therefore for such a problem like transit light curve fitting, the numerical

[1] which is necessary to obtain a reliable *a posteriori* distribution of the parameters

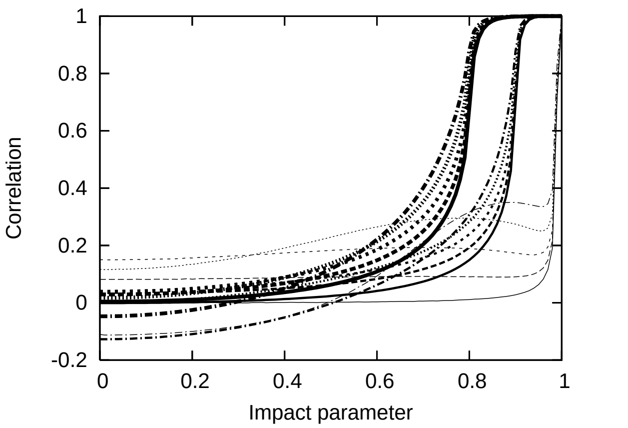

Figure 4. Correlation between the adjusted values of $\zeta/R_\star \equiv n(a/R_\star)/\sqrt{1-b^2}$ and the square of the impact parameter, b^2 for various values of planetary radii and limb darkening coefficients. Thin lines are for $p = 0.01$, moderately thick lines are for $p = 0.1$ and thick lines are for $p = 0.2$. The continuous, long dashed, short dashed, dotted and dotted-dashed lines represents the cases when limb darkening coefficients are $\gamma_1 = \gamma_2 = 0$, $\gamma_1 = \gamma_2 = 0.1$, $\gamma_1 = \gamma_2 = 0.2$, $\gamma_1 = \gamma_2 = 0.3$ and $\gamma_1 = \gamma_2 = 0.4$, respectively.

approximation requires approximately 10 times more computation time (note that in practice the gain will be less, $\sim$ 8 due to other overheads resulted by the computation of increased number of coefficients and the gain will clearly depend on the used programming environment and its features). Moreover – because the derivatives of a transiting light curve function lack the Lipschitz property of continuity – the numerical approach can also be unstable at the points of contacts.

The routines for calculating both the transit decrease function and its derivatives are available[2] in Fortran77 and C languages along with the codes used to calculate the correlations between the limb darkening parameters and the light curve parameters.

The recent review of Southworth (2008) concludes that the determination of some of the light curve parameters, especially the radius of the planet can be sensitive to the applied limb darkening model and its parameters, yielding a possibility of systematic errors. Therefore, analytic description of transit light curves for other different limb darkening laws would also be a point of interest.

ACKNOWLEDGMENTS

The author would like to thank the hospitality and support of the Harvard-Smithsonian Center for Astrophysics where this work was partially carried out. The author acknowledges the support by the HATNet project, and NASA grant NNG04GN74G. The author also thanks Eric Agol, Gáspár Bakos, Bence Kocsis and Joshua Winn for helpful comments and discussions.

[2] http://szofi.elte.hu/~apal/utils/astro/ntiq/

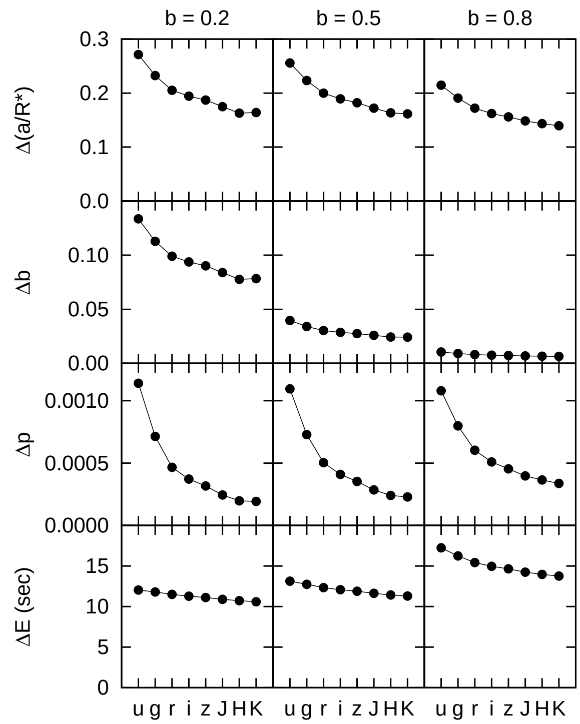

Figure 5. Uncertainties of the geometrical ratio $a/R_\star$, the impact parameter b, the fractional planetary radius $p \equiv R_\mathrm{p}/R_\star$ and the transit center time E for a hypothetical transiting planet with a radius of $p = 0.1$ orbiting its Sun-like host star on an orbit with a semimajor axis of $a/R_\star = 10$. The uncertainties are plotted as the function of the photometric passbands, for various impact parameters. See text for further details.

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APPENDIX A: COEFFICIENTS FOR THE CALCULATION OF THE FLUX DECREASE FUNCTION

In this section, the coefficients required to evaluate the flux decrease function – as it is given by equation (2) – are summarized. The evaluation of the quantities F_0, F_1, F_K, F_E, F_Π, F_2, n and k is done in two steps. First, using Table A1, a set of auxiliary variables should be evaluated, all of them are a function only of p and z, i.e. do not depend on the limb darkening coefficients. Note that for a given geometrical case, not all of these quantities have to be calculated, only those that referred to in the appropriate row of Table A2. (Moreover, it might happen that for some of these equations the value of p or z are out of the allowed domain if they are used in the cases when there is no need for them.) Second, using Table A2, the quantities F_i, k and n should by calculated by substituting the previously obtained values of the auxiliary variables.

APPENDIX B: PARAMETRIC DERIVATIVES OF THE COMPLETE ELLIPTIC INTEGRALS

The derivatives of the complete elliptic integrals of the first and second kind are the following:

$$\frac{\mathrm{d}K(k)}{\mathrm{d}k} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}, \tag{B1}$$

$$\frac{\mathrm{d}E(k)}{\mathrm{d}k} = \frac{E(k)-K(k)}{k}. \tag{B2}$$

If the complete elliptic integral of the third kind is defined with the sign convention as

$$\Pi(n,k) = \int\limits_0^{\pi/2} \frac{\mathrm{d}\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}, \tag{B3}$$

then its partial derivatives are the following:

$$\frac{\partial\Pi(n,k)}{\partial n} = \frac{1}{2(k^2-n)(n-1)}\left[E(k) + \frac{(k^2-n)K(k)}{n} + \right.$$
$$\left. + \frac{(n^2-k^2)\Pi(n,k)}{n}\right], \tag{B4}$$

$$\frac{\partial\Pi(n,k)}{\partial k} = \frac{k}{n-k^2}\left[\frac{E(k)}{k^2-1} + \Pi(n,k)\right]. \tag{B5}$$

Throughout this paper we are using the sign convenction for $\Pi(n,k)$ as it is defined by equation (B3).

APPENDIX C: COEFFICIENTS FOR THE THE CALCULATION OF THE PARTIAL DERIVATIVES OF THE FLUX DECREASE FUNCTION

In this section, the coefficients required to evaluate the partial derivatives of the flux decrease function are summarized, which are needed to evaluate equation (19). The evaluation of these coefficients are also done in two steps. First, using Table C1, a set of auxiliary variables should be evaluated, all of them are a function only of p and z (i.e. do not depend on the limb darkening coefficients). Note that for a given geometrical case, not all of these quantities have to be calculated, only those that are referred to in the appropriate row of Table C2. Second, using Table C2, the quantities $F_{i,g}$, k_g and n_g should by calculated (where g is either p or z), by substituting the previously obtained values of the auxiliary variables.

Table A1. Auxiliary quantities for the calculation of the flux decrease. Note that the quantities a, b, k_0 and k_1 are defined similarly as were defined by Mandel & Agol (2002) and the former two should not be confused with the same notations for the semimajor axis and the impact parameter.

$a=(p-z)^2$	$b=(p+z)^2$	$t^2=p^2+z^2$	$\hat{p}=\sqrt{p(1-p)}$	$p'=\sqrt{1-p^2}$
$C_I=\frac{2}{9\pi\sqrt{1-a}}$	$C_{IK}=1-5z^2+p^2+ab$		$C_{IE}=(z^2+7p^2-4)(1-a)$	$C_{I\Pi}=-3\frac{p+z}{p-z}$
$C_G=\frac{1}{9\pi\sqrt{pz}}$	$C_{GK}=3-6(1-p^2)^2-2pz(z^2+7p^2-4+5pz)$		$C_{GE}=4pz(z^2+7p^2-4)$	$C_{G\Pi}=-3\frac{p+z}{p-z}$
$T_I=\frac{2}{3\pi}\arccos(1-2p)-\frac{4}{9\pi}(3+2p-8p^2)\hat{p}$			$k_0=\arccos\left(\frac{p^2+z^2-1}{2pz}\right)$	$k_1=\arccos\left(\frac{z^2+1-p^2}{2z}\right)$
$G_0=\frac{p^2k_0+k_1-\sqrt{z^2-\frac{1}{4}(1+z^2-p^2)^2}}{\pi}$			$G_2=\frac{k_1+p^2(p^2+2z^2)k_0-\frac{1}{4}(1+5p^2+z^2)\sqrt{(1-a)(b-1)}}{2\pi}$	

Table A2. Coefficients for the flux decrease function.

Step	Case	F_0	F_1	F_K	F_E	F_Π	F_2	k	n
1	**A**	p^2	$\frac{2}{3}(1-(p')^3)$	0	0	0	$\frac{1}{2}p^4$	–	–
2	$\mathbf{A_G}$	1	$\frac{2}{3}$	0	0	0	$\frac{1}{2}$	–	–
3	**B**	p^2	$\frac{2}{3}$	C_IC_{IK}	C_IC_{IE}	$C_IC_{I\Pi}$	$\frac{1}{2}p^2(p^2+2z^2)$	$\sqrt{\frac{4pz}{1-a}}$	$-\frac{4pz}{a}$
4	$\mathbf{B_T}$	p^2	T_I	0	0	0	$\frac{1}{2}p^2(p^2+2z^2)$	–	–
5	$\mathbf{B_G}$	G_0	$\frac{2}{3}$	C_GC_{GK}	C_GC_{GE}	$C_GC_{G\Pi}$	G_2	$\sqrt{\frac{1-a}{4pz}}$	$\frac{a-1}{a}$
6	**C**	p^2	$\frac{1}{3}$	$\frac{2}{9\pi}(1-4p^2)$	$\frac{8}{9\pi}(2p^2-1)$	0	$\frac{3}{2}p^4$	$2p$	–
7	$\mathbf{C_T}$	$\frac{1}{4}$	$\frac{1}{3}-\frac{4}{9\pi}$	0	0	0	$\frac{3}{32}$	–	–
8	$\mathbf{C_G}$	G_0	$\frac{1}{3}$	$-\frac{1}{9\pi p}(1-4p^2)(3-8p^2)$	$\frac{1}{9\pi}16p(2p^2-1)$	0	G_2	$\frac{1}{2p}$	–
9	**D**	p^2	0	C_IC_{IK}	C_IC_{IE}	$C_IC_{I\Pi}$	$\frac{1}{2}p^2(p^2+2z^2)$	$\sqrt{\frac{4pz}{1-a}}$	$-\frac{4pz}{a}$
10	**E**	p^2	T_I	0	0	0	$\frac{1}{2}p^2(p^2+2z^2)$	–	–
11	**F**	G_0	0	C_GC_{GK}	C_GC_{GE}	$C_GC_{G\Pi}$	G_2	$\sqrt{\frac{1-a}{4pz}}$	$\frac{a-1}{a}$
12	**G**	0	0	0	0	0	0	–	–

Table C1. Auxiliary quantities for the calculation of the partial derivatives of the flux decrease function.

$C_{IK,p}=+2p(1+2(p^2-z^2))$	$C_{IE,p}=14p(1-a)-2(p-z)(z^2+7p^2-4)$
$C_{IK,z}=-2z(5+2(p^2-z^2))$	$C_{IE,z}=2z(1-a)+2(p-z)(z^2+7p^2-4)$
$C_{GK,p}=-2(p^2(12p+21z)+z(z^2-4)+2p(5z^2-6))$	$C_{GE,p}=4z(-4+21p^2+z^2)$
$C_{GK,z}=2p(4-7p^2-10pz-3z^2)$	$C_{GE,z}=4p(-4+7p^2+3z^2)$
$G_{0,p}=\frac{p}{\pi}(2k_0)$	$G_{2,p}=\frac{p}{\pi}\left(2t^2k_0-4zp\sin k_0\right)$
$G_{0,z}=\frac{p}{\pi}(-2\sin k_0)$	$G_{2,z}=\frac{p}{\pi}\left(2zpk_0-(p^2+z^2+1)\sin k_0\right)$

Table C2. Coefficients for partial derivatives of the flux decrease function.

Step	Case	∂	$F_{0,\partial}$	$F_{1,\partial}$	$F_{K,\partial}$	$F_{E,\partial}$	$F_{2,\partial}$	k_∂	n_∂
1	**A**	p	$2p$	$2pp'$	0	0	$2p^3$	–	–
1	**A**	z	0	0	0	0	0	–	–
2, 12	$\mathbf{A_G}$, **G**	p	0	0	0	0	0	–	–
2, 12	$\mathbf{A_G}$, **G**	z	0	0	0	0	0	–	–
3, 9	**B**, **D**	p	$2p$	0	$C_IC_{IK,p}+C_IC_{IK}\frac{p-z}{1-a}$	$C_IC_{IE,p}+C_IC_{IE}\frac{p-z}{1-a}$	$2pt^2$	$\frac{2z(1+p^2-z^2)}{(1-a)^2k}$	$+\frac{4z(p+z)}{(p-z)a}$
3, 9	**B**, **D**	z	0	0	$C_IC_{IK,z}-C_IC_{IK}\frac{p-z}{1-a}$	$C_IC_{IE,z}-C_IC_{IE}\frac{p-z}{1-a}$	$2p^2z$	$\frac{2p(1-p^2+z^2)}{(1-a)^2k}$	$-\frac{4p(p+z)}{(p-z)a}$
4, 10	$\mathbf{B_T}$, **E**	p	$2p$	$\frac{8p\hat{p}}{\pi}$	0	0	$2pt^2$	–	–
4, 10	$\mathbf{B_T}$, **E**	z	0	$-\frac{8p\hat{p}}{3\pi}$	0	0	$2p^2z$	–	–
5, 11	$\mathbf{B_G}$, **F**	p	$G_{0,p}$	0	$-\frac{F_K}{2p}-C_GC_{GK,p}$	$-\frac{F_E}{2p}+C_GC_{GE,p}$	$G_{2,p}$	$-\frac{1+p^2-z^2}{8kp^2z}$	$+\frac{2}{(p-z)^3}$
5, 11	$\mathbf{B_G}$, **F**	z	$G_{0,z}$	0	$-\frac{F_K}{2z}-C_GC_{GK,z}$	$-\frac{F_E}{2z}+C_GC_{GE,z}$	$G_{2,z}$	$-\frac{1-p^2+z^2}{8kpz^2}$	$-\frac{2}{(p-z)^3}$
6	**C**	p	$2p$	0	$\frac{4p}{9\pi}-\frac{1}{3\pi p}$	$\frac{28p}{9\pi}+\frac{1}{3\pi p}$	$4p^3$	1	–
6	**C**	z	0	0	$-\frac{20p}{9\pi}+\frac{1}{3\pi p}$	$\frac{4p}{9\pi}-\frac{1}{3\pi p}$	$2p^3$	1	–
7	$\mathbf{C_T}$	p	1	$\frac{2}{\pi}$	0	0	$\frac{1}{2}$	–	–
7	$\mathbf{C_T}$	z	0	$-\frac{2}{3\pi}$	0	0	$\frac{1}{4}$	–	–
8	$\mathbf{C_G}$	p	$G_{0,p}$	0	$\frac{3+16p^2(2-9p^2)}{18\pi p^2}$	$\frac{72p^2-2}{9\pi}$	$G_{2,p}$	$-\frac{1}{4p^2}$	–
8	$\mathbf{C_G}$	z	$G_{0,z}$	0	$\frac{3+8p^2(1-6p^2)}{18\pi p^2}$	$\frac{24p^2-14}{9\pi}$	$G_{2,z}$	$-\frac{1}{4p^2}$	–