Renormalisability of the 2PI-Hartree approximation of multicomponent scalar models in the broken symmetry phase

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Abstract

Non-perturbative renormalisation of a general class of scalar field theories is performed at the Hartree level truncation of the 2PI effective action in the broken symmetry regime. Renormalised equations are explicitly constructed for the one- and two-point functions. The non-perturbative counterterms are deduced from the conditions for the cancellation of the overall and the subdivergences in the complete Hartree-Dyson-Schwinger equations, with a transparent method. The procedure proposed in the present paper is shown to be equivalent to the iterative renormalisation method of Blaizot *et al.* [1].

1 Motivation

One of the most popular approximation techniques in many-body quantum theory is the Hartree approximation. In quantum field theory it corresponds to the momentum independent two-loop truncation of the two-particle irreducible effective action. It is used extensively both in equilibrium [2, 3, 4] and out-of-equilibrium [5, 6, 7, 8] non-perturbative investigations of phase transition phenomena. Its non-perturbative renormalisability was demonstrated as particular case of the general proof of renormalisability of the physical quantities computed in various 2PI approximations [9, 1, 10]. These proofs are rather involved especially in the broken symmetry phase. For this reason in many practical applications the renormalised equations are not constructed explicitly. For instance, investigations of the finite temperature phase transitions in strongly interacting matter frequently either omit zero temperature quantum corrections in the 2PI approximate equations of the relevant 1- and 2-point functions [3, 4, 11] or take into account vacuum fluctuations by applying some cut-off [12].

The exact generating 2PI-functional $\Gamma[\Phi, G]$ fulfils generalised Ward-Takahashi identities reflecting global internal symmetries of the models. As a consequence the 1PI effective potential $\Gamma[\Phi, G(\Phi)]$

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which arises after substituting the solution of the stationarity condition $\delta\Gamma[\Phi, G]/\delta G = 0$ at fixed Φ generates in the broken symmetry phase an inverse propagator vanishing for $p \to 0$. The same steps lead for a truncated approximate $\Gamma_{tr}[\Phi, G]$ to $\delta^2\Gamma_{tr}/\delta\Phi\delta\Phi$ which has vanishing Fourier transform for $p \to 0$. Since this quantity in general does not coincide with $G^{-1}(p)$ determined self-consistently, Goldstone's theorem gets violated in approximate 2PI computations [13]. This failure might be one of the reasons why 2PI-Hartree approximation does not describe properly the late time dynamics of symmetry breaking, i.e. this approximation does not lead to a thermalised symmetry breaking ground state [14]. Recently a "symmetrized" modification was proposed to replace the original 2PI-functional, which can be uniquely constructed from the requirement of obeying Goldstone's theorem [15].

A further problem concerns the renormalisability of the equations of motion derived from the 2PI-Hartree approximation using a single renormalised quartic coupling [16, 17, 18]. It could be implemented consistently only in large N approximations. Renormalisability in this sense of the above referred "symmetrized" approximation was verified in mass-independent schemes [19]. In parallel studies one succeeded also to implement one-loop renormalisation-group invariance in the Hartree-like equations [18, 20].

Despite all its problems Hartree approximation remains a valuable method in phenomenological studies when looking for the thermodynamical behaviour of complicated multicomponent scalar models. Our aim in the present paper is to clarify fully its renormalisability by providing a simple and transparent explicit construction of the counterterms for a rather general class of scalar field theories along the method proposed in [10, 21].

The paper is organised as follows. In section 2 we present a simplified one-step renormalisation procedure using the familiar example of the $O(N)$ model and show its equivalence with the method of iterative renormalisation [1]. Its correct (exact) large N behaviour will be recovered. We will shortly comment on the counter term structure of the "symmetrized" approximate 2PI-Hartree functional of Ref.[15] in the present scheme. In section 3 we introduce the general class of the models we shall investigate and write the truncated expression of their effective quantum action corresponding to the Hartree-approximation. In section 4 we derive the full set of renormalisation conditions in the broken symmetry phase. The main result is that in the 2PI-Hartree approximation there is an interplay between the symmetry breaking pattern (an infrared feature) and the actual set of counterterms necessary for avoiding ultraviolet divergencies. This circumstance explains the problems signalled earlier concerning the renormalisability of the approximation. We work out details of the procedure for two classes of particular physical interest. In section 5 counterterms are constructed for models which can be embedded into the $O(N) \times O(M)$ symmetry. Section 6 is devoted to the discussion of the $SU(N) \times SU(N)$ symmetric linear sigma model. Throughout we pay special attention to the limit $N \to \infty$. It will be shown for both classes of models that in this limit the "interference" of the symmetry breaking pattern with the counterterms is suppressed. Summary of our results is given in the concluding section 7.

2 One step renormalisation of the 2PI-Hartree approximation of the O(N) model in the broken phase

The essence of our approach can be illustrated on the example of an $O(N)$ symmetric scalar field theory with quartic self-interaction. A compact analysis of the model at the Hartree level of truncation appeared in the literature already as part of a more complete discussion of the 2PI renormalisability in Ref. [10]. The treatment was shown renormalisable by constructing the necessary counterterms iteratively. Now we show an equivalent one-step procedure.

The 2PI-Hartree effective potential for this model is of the form [2]

$$V[v,G]=\frac{1}{2}\mu^2v^2+\frac{1}{24N}F_{abcd}v_av_bv_cv_d-\frac{i}{2}\int_k\ln G^{-1}_{aa}(k)-\frac{i}{2}\int_k\left[D^{-1}_{ab}(k)G_{ba}(k)-N\right]$$
$$+\frac{1}{8N}F_{abcd}\int_k G_{ab}(k)\int_p G_{cd}(p)+V^{\rm ct}[v,G], \tag{1}$$

where $v^2=v_av_a$, with $a=1\dots N$ and $F_{abcd}=\frac{\lambda}{3}\left(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}\right)$ is the renormalised coupling tensor. The three terms in the coupling tensor F_{abcd} are the three rank four invariants of the $O(N)$ group. Because in concrete calculations only the sum of the last two appears, it suffices to introduce only the following invariants:

$$t^1_{abcd}=\delta_{ab}\delta_{cd},\qquad t^2_{abcd}=\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}. \tag{2}$$

These invariant tensors produce two different contractions of the indices in the fifth term of (1): $G_{aa}(k)G_{cc}(p)$ and $G_{ab}(k)G_{ab}(p)$.

In the broken symmetry phase it is useful to introduce the orthogonal projectors [22]

$$P^\sigma_{ab}=\frac{v_av_b}{v^2},\qquad P^\pi_{ab}=\delta_{ab}-P^\sigma_{ab}, \tag{3}$$

projecting on the one dimensional "sigma" and $N-1$ dimensional "Goldstone" subspaces, respectively. (We call the degenerate sector "Goldstone" though in the 2PI-Hartree approximation the Goldstone theorem is not obeyed.) With their help the tree-level propagator in (1) reads as

$$iD^{-1}_{ab}(k)=\left(k^2-\mu^2-\frac{\lambda}{6N}v^2\right)\left(P^\sigma_{ab}+P^\pi_{ab}\right)-\frac{\lambda}{3N}v^2P^\sigma_{ab}. \tag{4}$$

The two terms on the right hand side of the equation correspond to the two invariants given in (2). Patterned after the tree-level propagator one writes the full propagator in the form

$$G_{ab}(k)=G_\sigma(k)P^\sigma_{ab}+G_\pi(k)P^\pi_{ab}, \tag{5}$$

where the coefficient functions are parametrised as $iG^{-1}_{\sigma/\pi}(k)=k^2-M^2_{\sigma/\pi}$.

The truncation of the 2PI approximation of the quantum effective action allows to introduce different definitions of the 4-point function and to each one of them an independent coupling counterterm will be associated. In Ref. [10] two sets of counterterms, indexed with A and B, were introduced corresponding to the two independent $O(N)$ invariant structures, t^1 and t^2, building up the coupling tensor F_{abcd}.

Using (2), (4) and (5) it is an easy exercise to write the terms in the effective potential (1) in terms of the exact σ and π propagators keeping track of the terms which come from contracting with the tensors t_1 and t_2. Then the following counterterm functional can be introduced:

$$
\begin{aligned}
V^{\rm ct}[v,G_\sigma,G_\pi] =& V_4^{\rm ct}[v] + V_2^{\rm ct}[v,G_\sigma,G_\pi] + V_0^{\rm ct}[G_\sigma,G_\pi],\\
V_4^{\rm ct}[v] =& \frac{1}{2}\delta m_0^2 v^2 + \frac{\delta\lambda_4}{24N}v^4,\\
V_2^{\rm ct}[v,G_\sigma,G_\pi] =& \frac{1}{2}\left(\delta m_2^2 + \frac{\delta\lambda_2^A}{6N}v^2\right)\int_p \left(G_\sigma(p) + (N-1)G_\pi(p)\right) + \frac{\delta\lambda_2^B v^2}{6N}\int_p G_\sigma(p),\\
V_0^{\rm ct}[G_\sigma,G_\pi] =& \frac{\delta\lambda_0^A + 2\delta\lambda_0^B}{24N}\left(\int_p G_\sigma(p)\right)^2 + \frac{N-1}{24N}\left((N-1)\delta\lambda_0^A + 2\delta\lambda_0^B\right)\left(\int_p G_\pi(p)\right)^2\\
&+ \frac{\delta\lambda_0^A}{12N}(N-1)\int_p G_\sigma(p)\int_k G_\pi(k).
\end{aligned}
\tag{6}
$$

$V_4^{\rm ct}$ corresponds to the classical potential and the indices of the different counterterm functionals refer to the highest power of the background v occurring in their expression. Each $O(N)$ invariant piece of the 2PI-Hartree effective potential receives an independent counterterm.

The stationarity conditions

$$
\frac{\delta V[v,G_\sigma,G_\pi]}{\delta G_\sigma(p)} = 0, \qquad \frac{\delta V[v,G_\sigma,G_\pi]}{\delta G_\pi(p)} = 0, \qquad \frac{\delta V[v,G_\sigma,G_\pi]}{\delta v} = 0, \tag{7}
$$

give two gap equations for the pole masses and an equation of state, which determines the vacuum condensate v:

$$
M_\sigma^2 = m_\sigma^2 + \delta m_\sigma^2 + \frac{1}{6N}\left(3\lambda + \delta\lambda_0^A + 2\delta\lambda_0^B\right)T(M_\sigma^2) + \frac{N-1}{6N}\left(\lambda + \delta\lambda_0^A\right)T(M_\pi^2), \tag{8}
$$

$$
M_\pi^2 = m_\pi^2 + \delta m_\pi^2 + \frac{1}{6N}\left(\lambda + \delta\lambda_0^A\right)T(M_\sigma^2) + \frac{1}{6N}\left((N+1)\lambda + (N-1)\delta\lambda_0^A + 2\delta\lambda_0^B\right)T(M_\pi^2), \tag{9}
$$

$$
0 = v\left[m_\sigma^2 + \delta m_0^2 + \frac{\delta\lambda_4 - 2\lambda}{6N}v^2 + \left(3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B\right)\frac{T(M_\sigma^2)}{6N} + \frac{N-1}{6N}\left(\lambda + \delta\lambda_2^A\right)T(M_\pi^2)\right]. \tag{10}
$$

For compactness we have introduced the following background dependent masses and counterterms:

$$
\begin{aligned}
m_\sigma^2 &= \mu^2 + \frac{\lambda}{2N}v^2, \quad \delta m_\sigma^2 = \delta m_2^2 + \frac{\delta\lambda_2^A + 2\delta\lambda_2^B}{6N}v^2,\\
m_\pi^2 &= \mu^2 + \frac{\lambda}{6N}v^2, \quad \delta m_\pi^2 = \delta m_2^2 + \frac{\delta\lambda_2^A}{6N}v^2,
\end{aligned}
\tag{11}
$$

and $T(M^2) = \int_p i/(p^2 - M^2)$ is the tadpole integral.

Before proceeding further, we can reduce the number of counterterms. Making use of the gap equation (8) for the sigma field in the equation of state (10), one immediately sees that for the consistent renormalisation of the two equations one has to require

$$
\delta m_0^2 = \delta m_2^2 \equiv \delta m^2 \qquad \delta\lambda_0^A = \delta\lambda_2^A \equiv \delta\lambda^A, \qquad \delta\lambda_0^B = \delta\lambda_2^B \equiv \delta\lambda^B, \qquad \delta\lambda_4 = \delta\lambda_2^A + 2\delta\lambda_2^B. \tag{12}
$$

As a consequence, we are left with a very simple renormalised equation of state:

$$
v\left[M_\sigma^2 - \frac{\lambda}{3N}v^2\right] = 0. \tag{13}
$$

The key point of the renormalisation procedure applied to the two gap equations is to know explicitly the divergence structure of the radiative corrections, which for the tadpole diagram with cut-off regularisation is the following:

$$T(M^2) = \int_k \frac{i}{k^2 - M^2} = \Lambda^2 + T_d M^2 + T_F(M^2). \tag{14}$$

Here we used as a 4d cut-off $\Lambda_{\rm CO} = 4\pi\Lambda$, in terms of which the logarithmic divergence is $T_d = -\ln(e\Lambda_{\rm CO}^2/M_0^2)/(16\pi^2)$. $T_F(M^2)$ is the finite part of the tadpole integral, which depends also on the normalisation scale M_0^2.

When substituting (14) into the two gap equations one readily separates their finite parts:

$$M_\sigma^2 = m_\sigma^2 + \frac{\lambda}{2N} T_F(M_\sigma^2) + (N-1)\frac{\lambda}{6N} T_F(M_\pi^2), \tag{15}$$

$$M_\pi^2 = m_\pi^2 + \frac{\lambda}{6N} T_F(M_\sigma^2) + (N+1)\frac{\lambda}{6N} T_F(M_\pi^2). \tag{16}$$

The infinities should consistently cancel by the appropriate choice of the counterterms:

$$\begin{aligned} 0 = \delta m_\sigma^2 &+ \frac{1}{6N}\left(3\lambda + \delta\lambda^A + 2\delta\lambda^B\right)\left(\Lambda^2 + T_d M_\sigma^2\right) + \frac{N-1}{6N}\left(\lambda + \delta\lambda^A\right)\left(\Lambda^2 + T_d M_\pi^2\right) \\ &+(\delta\lambda^A + 2\delta\lambda^B)\frac{T_F(M_\sigma^2)}{6N} + (N-1)\delta\lambda^A \frac{T_F(M_\pi^2)}{6N}, \end{aligned} \tag{17}$$

$$\begin{aligned} 0 = \delta m_\pi^2 &+ \frac{1}{6N}\left(\lambda + \delta\lambda^A\right)\left(\Lambda^2 + T_d M_\sigma^2\right) + \frac{1}{6N}\left(\lambda(N+1) + (N-1)\delta\lambda^A + 2\delta\lambda^B\right)\left(\Lambda^2 + T_d M_\pi^2\right) \\ &+\delta\lambda^A \frac{T_F(M_\sigma^2)}{6N} + \left((N-1)\delta\lambda^A + 2\delta\lambda^B\right)\frac{T_F(M_\pi^2)}{6N}. \end{aligned} \tag{18}$$

The central step of the proposed procedure consists of making use of the renormalised equations (15) and (16) for M_π^2 and M_σ^2 appearing in the coefficients of T_d. Then one can separate the conditions for the vanishing of the overall divergence and of the subdivergences. The former conditions do not contain any dependence on the finite tadpole T_F and read as

$$\begin{aligned} 0 = \delta m_\sigma^2 &+ \frac{\Lambda^2}{6N}\left((N+2)\lambda + N\delta\lambda^A + 2\delta\lambda^B\right) \\ &+ \frac{T_d}{6N}\left[\left(3\lambda + \delta\lambda^A + 2\delta\lambda^B\right) m_\sigma^2 + (N-1)\left(\lambda + \delta\lambda^A\right) m_\pi^2\right], \\ 0 = \delta m_\pi^2 &+ \frac{\Lambda^2}{6N}\left((N+2)\lambda + N\delta\lambda^A + 2\delta\lambda^B\right) \\ &+ \frac{T_d}{6N}\left[\left(\lambda + \delta\lambda^A\right) m_\sigma^2 + \left(\lambda(N+1) + (N-1)\delta\lambda^A + 2\delta\lambda^B\right) m_\pi^2\right]. \end{aligned} \tag{19}$$

The conditions for the subdivergence cancellation, which are independent of the presence of any background, are given by the separate vanishing of the coefficients of $T_F(M_\sigma^2)$ and $T_F(M_\pi^2)$. It is easy to see that the two conditions as obtained from (17) and (18) coincide, so one remains with

$$\delta\lambda^A = -\frac{\lambda T_d}{6N}[(N+4)\lambda + (N+2)\delta\lambda^A + 2\delta\lambda^B], \qquad \delta\lambda^B = -\frac{\lambda T_d}{3N}(\lambda + \delta\lambda^B). \tag{20}$$

The cancellation of the overall divergency can be split in presence of a background into a background dependent and a background independent piece. The vanishing of the background independent piece gives

$$0 = \delta m^2 + \frac{1}{6N}((N+2)\lambda + N\delta\lambda^A + 2\delta\lambda^B)\left(\Lambda^2 + \mu^2 T_d\right). \tag{21}$$

The conditions for vanishing of the background dependent pieces are the same as those in (20). Had we not used from the beginning the equation of state then, by comparing these conditions to those in (20) we would have obtained at this point $\delta\lambda_2^A = \delta\lambda_0^A$ and $\delta\lambda_2^B = \delta\lambda_0^B$.

Returning to the relation between the coupling counterterms given in (20), one can solve the second one for $\delta\lambda^B$:

$$\delta\lambda^B = -\frac{\lambda^2 T_d}{3N}\frac{1}{1+\frac{\lambda T_d}{3N}}. \tag{22}$$

The equation for $\delta\lambda^A$ is a bit complicated, but introducing the bare coupling constants as $\lambda_B^A = \lambda^A + \delta\lambda^A$, $\lambda_B^B = \lambda^B + \delta\lambda^B$ one can rewrite the relations of (20) in the form

$$\begin{aligned} \frac{1}{\lambda} - \frac{1}{\lambda_B^B} &= -\frac{T_d}{3N}, \\ \frac{1}{\lambda} - \frac{1}{\lambda_B^A} &= -\frac{T_d}{6N}\left[N + 4 + (N+2)\frac{\lambda T_d}{3N}\right]. \end{aligned} \tag{23}$$

The method of iterative renormalisation searches for the coupling counterterms in form of infinite series. (see *e.g.* [1, 23] for details). When applied to the coupled gap equations (8) and (9) this method leads to the following recursions for the terms of the three counterterm series $\delta m^2 = \sum_{n=1}^{\infty}\delta m^{2(n)}$, $\delta\lambda^A = \sum_{n=1}^{\infty}\delta\lambda^{A(n)}$ and $\delta\lambda^B = \sum_{n=1}^{\infty}\delta\lambda^{B(n)}$:

$$\begin{aligned} \delta\lambda^{A(n)} &= -\frac{\lambda T_d}{6N}[(N+2)\delta\lambda^{A(n-1)} + 2\delta\lambda^{B(n-1)}], \\ \delta\lambda^{B(n)} &= -\frac{\lambda T_d}{3N}\delta\lambda^{B(n-1)}, \\ \delta m^{2(n)} &= -\frac{1}{6N}(N\delta\lambda^{A(n-1)} + 2\delta\lambda^{B(n-1)})\left(\Lambda^2 + \mu^2 T_d\right), \end{aligned} \tag{24}$$

with the initial values $\delta\lambda^{A(0)} = \delta\lambda^{B(0)} = \lambda$. Summing up the coupling counterterms one obtains the relations (20) and (21) demonstrating the equivalence of our single step approach with the iterative construction of the counterterms.

At leading order of the large N expansion all the nonvanishing coupling counterterms are equal, since then

$$\delta\lambda^A = -\frac{\lambda^2 T_d}{6}\frac{1}{1+\frac{\lambda T_d}{6}}, \qquad \delta\lambda^B \sim \mathcal{O}\left(\frac{1}{N}\right). \tag{25}$$

Note, that only in this limit one finds $\delta\lambda_4 = \delta\lambda^A$. Otherwise $\delta\lambda_4$ differs from $\delta\lambda^A$ which is a peculiar feature of the Hartree approximation.

The result (25) agrees with the scaling obtained in an exact leading order large N treatment of the $O(N)$ model [24]. This is remarkable, since an entire set of diagrams is missing from the 2PI effective action truncated at Hartree level (see *e.g.* [25]), which gives the one-loop bubble series in the self consistent equation of the sigma propagator.

The same procedure can be repeated also for the "symmetrized" 2PI-Hartree functional where an additional piece $\Delta V[G_{ab}]$ is added to (1) enforcing the validity of Goldstone's theorem[15]:

$$\Delta V[G_{ab}] = \frac{\lambda}{24N}\left[3\int_k G_{aa}(k)\int_p G_{bb}(p) - 2(N-1)\int_k G_{ab}(k)\int_p G_{ab}(p)\right]. \tag{26}$$

This means that different renormalised couplings are introduced for the two $O(N)$ invariant 4-rank tensors, $t^{\alpha}_{abcd}G_{ab}G_{cd}, \alpha = 1, 2$.

The corrected gap equations for M_σ^2 and M_π^2 are readily derived (no change is induced in the equation of state of v). Comparing the equation of state and the gap equation of the pions one finds that Goldstone's theorem is fulfilled under the following conditions among the counterterms:

$$\delta m_0^2 = \delta m_2^2, \quad \delta\lambda_4 = \delta\lambda_2^A, \quad \delta\lambda_0^A = \delta\lambda_2^A + 2\delta\lambda_2^B, \quad \delta\lambda_2^A = \delta\lambda_0^A + \frac{2}{N-1}\delta\lambda_0^B. \tag{27}$$

In the present scheme the renormalisation of the two gap equations results in a counterterm structure which apparently does not differ in any essential point from the one determined for the conventional Hartree truncation of the 2PI-functional, e.g. one finds different cut-off dependences for $\delta\lambda_0^a$ and $\delta\lambda_0^B$. In view of the fact that for this modified functional a symmetric and mass-independent renormalisation involving a single quartic counterterm was proven[19], finding the relation of the two renormalisation schemes is an interesting task for future investigation.

The renormalisation procedure presented here is generalised to a wide class of scalar models in the next section.

3 2PI-Hartree approximation for multicomponent scalar models

The method presented in section 2 will be applied with one notable alteration to a general class of multicomponent scalar models possessing various internal symmetries. The difference is that one does not start by writing the propagators in terms of mass eigenstates but one leaves them in matrix form. Substitution of the mass matrix into the mass dependent divergent piece of the gap equations provides in a single step a set of very general relations renormalisation between various coupling and counter coupling tensors. The projection of these equations on the different diagonal eigenblocks of the mass matrix determines the counter couplings appearing in the counterterm tensors as coefficients of the independent invariants. Depending on the number of independent mass eigenvalues in the spectra it might happen that only some combinations of them will get determined.

Let us consider the following general Lagrangian density:

$$\begin{aligned} L = \; & \frac{1}{2}[\partial_\mu\sigma_a\partial^\mu\sigma_a + \partial_\mu\pi_\alpha\partial^\mu\pi_\alpha - \mu_S^2\sigma_a\sigma_a - \mu_P^2\pi_\alpha\pi_\alpha] \\ & -\frac{1}{3}F^S_{abcd}\sigma_a\sigma_b\sigma_c\sigma_d - \frac{1}{3}F^P_{\alpha\beta\gamma\delta}\pi_\alpha\pi_\beta\pi_\gamma\pi_\delta - 2H_{\alpha\beta,cd}\pi_\alpha\pi_\beta\sigma_c\sigma_d. \end{aligned} \tag{28}$$

This model includes variants of $O(N)$ symmetric models and also the $SU(N)_L \times SU(N)_R$ symmetric matrix model when specific expressions are chosen for the coefficient tensors $F^P_{\alpha\beta\gamma\delta}$, F^S_{abcd} and $H_{\alpha\beta,cd}$, which reflect the structure of the group algebra. Assuming that in the broken symmetry phase only the σ_a fields acquire an expectation value v_a one obtains the following tree-level inverse propagators:

$$iD^{-1}_{S,ab}(k) = (k^2 - \mu_S^2)\delta_{ab} - 4F^S_{abcd}v_c v_d, \qquad iD^{-1}_{P,\alpha\beta}(k) = (k^2 - \mu_P^2)\delta_{\alpha\beta} - 4H_{\alpha\beta,cd}v_c v_d. \tag{29}$$

We introduce the economical compact "hypervector" notations:

$$\begin{aligned}
&\delta_{AB} = \begin{pmatrix}\delta_{ab}\\ \delta_{\alpha\beta}\end{pmatrix},\ \mu^2_{AB} = \begin{pmatrix}\mu_S^2\delta_{ab}\\ \mu_P^2\delta_{\alpha\beta}\end{pmatrix},\ D^{-1}_{AB} = \begin{pmatrix}D^{-1}_{S,ab}\\ D^{-1}_{P,\alpha\beta}\end{pmatrix},\ G_{AB} = \begin{pmatrix}G_{S,ab}\\ G_{P,\alpha\beta}\end{pmatrix},\\
&\bar\sigma_A = \begin{pmatrix}v_a\\ 0\end{pmatrix},\ \bar\sigma_A\bar\sigma_B = \begin{pmatrix}v_a v_b\\ 0\end{pmatrix},
\end{aligned} \tag{30}$$

where $G_{P,S}$ are the exact propagator matrices of the 2PI approximation. We also organise the coefficient tensors of (28) in the following hypermatrix:

$$Q_{ABCD} = \begin{pmatrix}F^S_{abcd} & H_{ab,\gamma\delta}\\ H_{\alpha\beta,cd} & F^P_{\alpha\beta\gamma\delta}\end{pmatrix}. \tag{31}$$

This hypermatrix inherits from the component tensors the following symmetries under index permutations:

$$Q_{ABCD} = Q_{CDAB} = Q_{BACD} \tag{32}$$

and all combinations of these transformations.

In the notation introduced above, the 2PI-Hartree effective potential for the models defined by (28) reads as

$$\begin{aligned}
V[\sigma_A, G_{AB}] = \frac{1}{2}(\mu^2_{AB})^{\mathrm T}\bar\sigma_A\bar\sigma_B + \frac{1}{3}(\bar\sigma_A\bar\sigma_B)^{\mathrm T}Q_{ABCD}(\bar\sigma_C\bar\sigma_D) - \frac{i}{2}\int_k\left[(D^{-1}_{AB}(k))^{\mathrm T}G_{BA}(k) - \delta^{\mathrm T}_{AB}\delta_{BA}\right]\\
-\frac{i}{2}\mathrm{Tr}\int_k(u\otimes\ln G^{-1}_{AB}(k)) + \int_k G^{\mathrm T}_{AB}(k)Q_{ABCD}\int_p G_{CD}(p) + V^{\mathrm{ct}}[\sigma_A, G_{AB}],
\end{aligned} \tag{33}$$

where usual matrix operations were used, $u = \begin{pmatrix}1\\ 1\end{pmatrix}$, $\otimes$ denotes dyadic product and the transpose (T) is to be understood as acting on the blocks of hypermatrices but not within the blocks.

The counterterms are introduced by the method of [10, 21] generalising the case of the $O(N)$ model presented in section 2. One has

$$\begin{aligned}
V^{\mathrm{ct}}[\sigma_A, G_{AB}] &= V^{\mathrm{ct}}_4[\sigma_A] + V^{\mathrm{ct}}_2[\sigma_A, G_{AB}] + V^{\mathrm{ct}}_0[G_{AB}],\\
V^{\mathrm{ct}}_4[\sigma_A] &= \frac{1}{2}(\delta\tilde\mu^2_{AB})^{\mathrm T}\bar\sigma_A\bar\sigma_B + \frac{1}{3}(\bar\sigma_A\bar\sigma_B)^{\mathrm T}\delta\tilde Q_{ABCD}(\bar\sigma_C\bar\sigma_D),
\end{aligned} \tag{34}$$

$$V^{\mathrm{ct}}_2[\sigma_A, G_{AB}] = \frac{1}{2}(\delta\hat\mu^2_{AB})^{\mathrm T}\int_k G_{BA}(k) + 4(\bar\sigma_C\bar\sigma_D)^{\mathrm T}\delta\hat Q^{\mathrm T}_{ABCD}\int_k G_{BA}(k), \tag{35}$$

$$V^{\mathrm{ct}}_0[G_{AB}] = \int_k G^{\mathrm T}_{AB}(k)\delta Q_{ABCD}\int_p G_{CD}(p). \tag{36}$$

Here, we used the freedom to introduce three different coupling counter tensors δQ, $\delta\hat Q$ and $\delta\tilde Q$ for the three different definitions of the 4-point couplings as well as two mass counter terms $\delta\hat\mu^2$ and $\delta\tilde\mu^2$

for the two different definitions of the 2-point couplings [10, 21]. For the symmetry breaking pattern considered, in terms of the components, the counter couplings read as

$$\delta\hat{\mu}^2_{AB} = \begin{pmatrix} \delta\hat{\mu}^2_S\delta_{ab} \\ \delta\hat{\mu}^2_P\delta_{\alpha\beta} \end{pmatrix}, \delta\tilde{\mu}^2_{AB} = \begin{pmatrix} \delta\tilde{\mu}^2_S\delta_{ab} \\ 0 \end{pmatrix}, \delta Q = \begin{pmatrix} \delta F^S & \delta H \\ \delta H & \delta F^P \end{pmatrix}, \delta\hat{Q} = \begin{pmatrix} \delta\hat{F} & 0 \\ \delta\hat{H} & 0 \end{pmatrix}, \delta\tilde{Q} = \begin{pmatrix} \delta\tilde{F} & 0 \\ 0 & 0 \end{pmatrix}, \quad (37)$$

with the same index structure inside each block like in (31). All the components of the hypermatrices are linear combinations of rank-4 invariant tensors of the symmetry group considered.

The equations for the full propagator G_{AB} and the vacuum expectation value v_a follow from the stationarity conditions $\delta V/\delta G_{AB} = 0$ and $\delta V/\delta v_a = 0$. Because of the momentum independence of the self-energy in the Hartree approximation, one can write

$$iG^{-1}_{AB}(k) = k^2\delta^{\mathrm{T}}_{AB} - (M^2_{AB})^{\mathrm{T}}, \qquad (M^2_{AB})^{\mathrm{T}} = (M^2_{ab}, M^2_{\alpha\beta}), \quad (38)$$

where M^2_{ab}, and $M^2_{\alpha\beta}$ are the exact squared mass matrices in the 'S' and 'P' sectors, respectively. With this parametrisation the selfconsistent gap equations and the equation of state are written as

$$M^2_{AB} = m^2_{AB} + 4Q_{ABCD}\int_k G_{AB}(k) + \delta\hat{m}^2_{AB} + 4\delta Q_{ABCD}\int_k G_{CD}(k), \quad (39)$$

$$0 = \sigma^T_B\left[\mu^2_{AB} + \delta\tilde{\mu}^2_{AB} + \frac{4}{3}(Q_{ABCD} + \delta\tilde{Q}_{ABCD})\bar{\sigma}_C\bar{\sigma}_D + 4(Q_{ABCD} + \delta\hat{Q}^{\mathrm{T}}_{ABCD})\int_k G_{CD}(k)\right]. \quad (40)$$

In (39) some convenient short-hand notations were introduced:

$$m^2_{AB} = \mu^2_{AB} + 4Q_{ABCD}\bar{\sigma}_C\bar{\sigma}_D, \qquad \delta\hat{m}^2_{AB} = \delta\hat{\mu}^2_{AB} + 4\delta\hat{Q}_{ABCD}\bar{\sigma}_C\bar{\sigma}_D. \quad (41)$$

4 General renormalisation conditions for the 2PI-Hartree approximation

The real and symmetric exact squared mass matrices in the 'S' and 'P' sectors can be diagonalised with orthogonal transformations:

$$O^S_{ac}M^2_{ab}O^S_{bd} = \tilde{M}^2_c\delta_{cd}, \qquad O^P_{\alpha\gamma}M^2_{\alpha\beta}O^P_{\beta\delta} = \tilde{M}^2_\gamma\delta_{\gamma\delta}, \quad (42)$$

where on the right hand side there is no summation over the repeated indices. With this transformation also the propagator matrices become diagonal and the corresponding tadpole integrals can be evaluated explicitly. Using hypervectors one introduces the following short-hand notations

$$M^2_{CD} = O_{CE}O_{DE}\tilde{M}^2_E, \qquad \int_k G_{CD}(k) = O_{CE}O_{DE}T(\tilde{M}^2_E), \quad (43)$$

with the understanding that in the upper component of the squared mass and propagator hypervectors one diagonalises with O^S and in the lower one with O^P. One makes explicit the divergent piece of the tadpole integral by writing

$$T(\tilde{M}^2_E) = \Lambda^2 u + \tilde{M}^2_E T_d + T_F(\tilde{M}^2_E). \quad (44)$$

With help of (43) and (44) one expresses the integrals of (39) in terms of tadpole integrals of the propagating eigenmodes and obtains for the gap equations

$$\begin{aligned} M^2_{AB} &= \mu^2_{AB} + \delta\hat{\mu}^2_{AB} + 4(Q_{ABCD} + \delta\hat{Q}_{ABCD})\bar{\sigma}_C\bar{\sigma}_D \\ &\quad + 4(Q_{ABCD} + \delta Q_{ABCD})\left[\Lambda^2\delta_{CD} + M^2_{CD}T_d + O_{CE}O_{DE}T_F(\tilde{M}^2_E)\right]. \end{aligned} \tag{45}$$

The renormalised gap equations are easily extracted from the above sum by separating its finite pieces:

$$M^2_{AB} = \mu^2_{AB} + 4Q_{ABCD}\bar{\sigma}_C\bar{\sigma}_D + 4Q_{ABCD}O_{CE}O_{DE}T_F(\tilde{M}^2_E). \tag{46}$$

One has to choose the counter tensors appropriately ensuring the vanishing of all independent overall divergences and subdivergences in (45). Substituting for the squared mass matrix M^2_{CD} its expression from the renormalised gap equation (46) divergence cancellation imposes the following relation on the counterterms

$$\begin{aligned} 0 = \ & \delta\hat{\mu}^2_{AB} + 4(Q_{ABCD} + \delta Q_{ABCD})\left(\Lambda^2\delta_{CD} + \mu^2_{CD}T_d\right) \\ & + 4\left[\delta\hat{Q}_{ABMN} + 4T_d(Q_{ABCD} + \delta Q_{ABCD})Q_{CDMN}\right]\bar{\sigma}_M\bar{\sigma}_N \\ & + 4\left[\delta Q_{ABMN} + 4T_d(Q_{ABCD} + \delta Q_{ABCD})Q_{CDMN}\right]O_{ME}O_{NE}T_F(\tilde{M}^2_E). \end{aligned} \tag{47}$$

Note, that we split the overall divergence into two sets (the first two lines on the right hand side of (47)), one independent of the background and another depending on it quadratically.

Turning now to the renormalisation of the equation of state we first express μ^2_{AB} from (39) and substitute it into (40). Then one does the same steps as in the case of the gap equations: rewrite the integrals in terms of tadpoles of mass eigenstates using (43), separate the divergent part of the tadpole integral using (44) and make use of the finite gap equations (46). The finite equation of state has a very simple form (compare to (13)):

$$\bar{\sigma}^{\mathrm{T}}_B\left[M^2_{AB} - \frac{8}{3}Q_{ABCD}\bar{\sigma}_C\bar{\sigma}_D\right] = 0. \tag{48}$$

The condition for vanishing of all the overall divergences and subdivergences in the equation of state gives

$$\begin{aligned} 0 = \ & \bar{\sigma}^{\mathrm{T}}_B\Big\{\delta\tilde{\mu}^2_{AB} - \delta\hat{\mu}^2_{AB} + 4(\delta\hat{Q}^{\mathrm{T}}_{ABCD} - \delta Q_{ABCD})\left(\Lambda^2\delta_{CD} + \mu^2_{CD}T_d\right) \\ & + 4\left[\frac{1}{3}\delta\tilde{Q}_{ABMN} - \delta\hat{Q}_{ABMN} + 4T_d(\delta\hat{Q}^{\mathrm{T}}_{ABCD} - \delta Q_{ABCD})Q_{CDMN}\right]\bar{\sigma}_M\bar{\sigma}_N \\ & + 4(\delta\hat{Q}^{\mathrm{T}}_{ABCD} - \delta Q_{ABCD})\left[I_{CM}I_{DN} + 4T_dQ_{CDMN}\right]O_{ME}O_{NE}T_F(\tilde{M}^2_E)\Big\}, \end{aligned} \tag{49}$$

where in the last line we introduced $I_{CM} = \mathrm{diag}(\delta_{cm}, \delta_{\gamma\mu})$.

The counter tensors enter in three different types of combinations in (47) and (49). The first line in these equations is independent on the background. The second line reflects the presence of background dependent overall divergences. The expressions in the third line are due to the presence of subdivergences. These latter expressions are products of a piece independent of the pole masses and another one which through its $\tilde{M}^2_E$ dependence is potentially dependent on the temperature, the chemical potential and other "environmental" parameters. Renormalisability of the approximation is equivalent to ensure the vanishing of all three types of expressions without imposing environment dependent conditions.

A sufficient condition for vanishing of the subdivergences contained in the third line of (49) is to choose the same coupling counter tensor in (35) and (36), that is

$$\delta\hat{Q}^{\mathrm{T}}_{ABCD} = \delta Q_{ABCD}. \tag{50}$$

Then the first line of (49) gives $\delta\tilde{\mu}^2_S = \delta\hat{\mu}^2_S$. The vanishing of background dependent divergences of the second line in (49) requires

$$\bar{\sigma}^{\mathrm{T}}_B\left(\frac{1}{3}\delta\tilde{Q}_{ABMN} - \delta\hat{Q}_{ABMN}\right)\bar{\sigma}_M\bar{\sigma}_N = 0, \tag{51}$$

which in view of (37) and (50) is equivalent to

$$\left(\frac{1}{3}\delta\tilde{F}_{abmn} - \delta F_{abmn}\right)v_b v_m v_n = 0. \tag{52}$$

The vanishing of the overall divergence in the gap equations requires the separate cancellation of the first line in (47):

$$\delta\hat{\mu}^2_{AB} + 4(Q_{ABCD} + \delta Q_{ABCD})\left(\Lambda^2\delta_{CD} + \mu^2_{CD}T_d\right) = 0. \tag{53}$$

The condition for vanishing of the subdivergences demands the vanishing of the coefficient of the finite $\tilde{M}^2_E$-dependent combination of the tadpole integrals appearing in the third line of (47). An obvious sufficient condition is to impose the vanishing of the hypermatrix in the square bracket of this line for all its tensorial components. This would yield the following set of linear equations for δQ_{ABCD}:

$$\delta Q_{ABMN} + 4T_d(Q_{ABCD} + \delta Q_{ABCD})Q_{CDMN} = 0. \tag{54}$$

Due to the choice in (50), in the subspace of the sector 'S' spanned by the nonzero components v_m of the background, the above condition is the same as the one coming from the vanishing of the background dependent overall divergences (second line in (47)). Still, it is unnecessary to impose this condition on elements for which the mass-dependent term, e.g. $O_{ME}O_{NE}T_F(\tilde{M}^2_E)$ vanishes. Taking into account the block-diagonal form of the matrix M^2_{AB}, the matrix condition should be fulfilled within each coupled mass block separately. Degenerate modes of common mass (like the Goldstone modes) also form a sector sharing common $T_F(\tilde{M}^2)$. In this block the product $O_{ME}O_{NE}$ gives a projector P_{MN} onto this subspace. The form of renormalisation condition to be applied on this subspace arises by multiplying (54) by P_{MN}.

Let us analyse first the consequences of (54) in the symmetric phase. There is only one completely degenerate block in both sectors 'S' and 'P'. In this case one has

$$(M^2_{AB})^{\mathrm{T}} = (M^2_S\delta_{ab}, M^2_P\delta_{\alpha\beta}), \qquad (G_{AB}(k))^{\mathrm{T}} = (G_S\delta_{ab}(k), G_P\delta_{\alpha\beta}(k), \tag{55}$$

hence one pair of indices of the coupling tensors will be contracted.

Taking for convenience the trace in the original gap equation (39) separately in sectors 'S' and 'P', one arrives at the following renormalisation condition for the vanishing of the subdivergences

$$\delta Q + 4T_d(Q + \delta Q)Q = 0, \tag{56}$$

where we have introduced $Q = Q_{aa\gamma\gamma}$, and $\delta Q = \delta Q_{aa\gamma\gamma}$. This equation actually determines three scalar counterterms $\delta F^P, \delta F^S$ and δH. Note, that these contracted tensor couplings enter in the renormalisation condition for vanishing of the overall divergences (53). As we will see in concrete examples in the symmetric phase only a combination of the coupling counterterms is determined. This combination is split in the broken symmetry phase.

For the concrete applications of the next sections we write in a less compact notation the conditions for vanishing of the subdivergences coming from the last line of (47):

$$\begin{aligned}
&\left\{\delta F^S_{abmn} \ + 4T_d[(F^S_{abcd} + \delta F^S_{abcd})F^S_{cdmn} + (H_{ab,\gamma\delta} + \delta H_{ab,\gamma\delta})H_{\gamma\delta,mn}]\right\} O^S_{me}O^S_{ne}T_F(\tilde{M}^2_{S,e}) = 0,\\
&\left\{\delta H_{ab,\mu\nu} \ + 4T_d[(F^S_{abcd} + \delta F^S_{abcd})H_{cd,\mu\nu} + (H_{ab,\gamma\delta} + \delta H_{ab,\gamma\delta})F^P_{\gamma\delta\mu\nu}]\right\} O^P_{\mu\epsilon}O^P_{\nu\epsilon}T_F(\tilde{M}^2_{P,\epsilon}) = 0,\\
&\left\{\delta H_{\alpha\beta,mn} + 4T_d[(F^P_{\alpha\beta\gamma\delta} + \delta F^P_{\alpha\beta\gamma\delta})H_{\gamma\delta,mn} + (H_{\alpha\beta,cd} + \delta H_{\alpha\beta,cd})F^S_{cdmn}]\right\} O^S_{me}O^S_{ne}T_F(\tilde{M}^2_{S,e}) = 0,\\
&\left\{\delta F^P_{\alpha\beta\mu\nu} \ + 4T_d[(F^P_{\alpha\beta\gamma\delta} + \delta F^P_{\alpha\beta\gamma\delta})F^P_{\gamma\delta\mu\nu} + (H_{\alpha\beta,cd} + \delta H_{\alpha\beta,cd})H_{cd,\mu\nu}]\right\} O^P_{\mu\epsilon}O^P_{\nu\epsilon}T_F(\tilde{M}^2_{P,\epsilon}) = 0.
\end{aligned} \tag{57}$$

One sees that products of two rank-4 tensors contracted with two pairs of indices are involved. The coupling and counter coupling tensors are linear combinations of independent rank-4 invariant tensors t^α of a given group:

$$F^{P/S}_{abcd} = \sum_\alpha f^{P/S}_\alpha t^\alpha_{abcd}, \quad H_{ab,cd} = \sum_\alpha h_\alpha t^\alpha_{abcd}, \quad \delta F^{P/S}_{abcd} = \sum_\alpha \delta f^{P/S}_\alpha t^\alpha_{abcd}, \quad \delta H_{ab,cd} = \sum_\alpha \delta h_\alpha t^\alpha_{abcd}. \tag{58}$$

The product of the tensors appearing in (57) can be conveniently worked out in form of a multiplication table for the invariants:

$$t^\alpha_{abcd} t^\beta_{cdef} = \sum_\gamma g_{\alpha\beta\gamma} t^\gamma_{abef}. \tag{59}$$

After projecting the resulting equations onto a given coupled block of the original gap equations one equates the coefficients of the arising independent (tensorial) expressions and determines the coupling counterterms $\delta f^{P/S}_\alpha, \delta h_\alpha$. These steps will be explicitly performed next for some concrete models of physical interest.

5 Analysis of the $O(N) \times O(M)$ symmetric model

The application of the renormalisation conditions to the $O(N) \times O(M)$ symmetry structure proceeds by specifying the invariant tensor structure (recall Eq. (2)) of the coupling and coupling counterterm tensors appearing in (31) and (37) (the 'S' sector will be associated with the $O(N)$ symmetry):

$$\begin{aligned}
F^S_{abcd} &= \lambda^S(t^1_{abcd} + t^2_{abcd}), & F^P_{\alpha\beta\gamma\delta} &= \lambda^P(t^1_{\alpha\beta\gamma\delta} + t^2_{\alpha\beta\gamma\delta}),\\
\delta F^S_{abcd} &= \delta\lambda^S_A t^1_{abcd} + \delta\lambda^S_B t^2_{abcd}, & \delta F^P_{\alpha\beta\gamma\delta} &= \delta\lambda^P_A t^1_{\alpha\beta\gamma\delta} + \delta\lambda^P_B t^2_{\alpha\beta\gamma\delta},\\
H_{ab,\gamma\delta} &= \lambda^H t^1_{ab\gamma\delta}, & \delta H_{ab,\gamma\delta} &= \delta\lambda^H t^1_{ab\gamma\delta}.
\end{aligned} \tag{60}$$

Written in an obvious compact notation, the rank-4 $O(N)$ invariant tensors obey simple multiplication rule (a pair of indices is contracted):

$$t^1 * t^1 = Nt^1, \qquad t^1 * t^2 = 2t^1, \qquad t^2 * t^2 = 2t^2. \tag{61}$$

Similar multiplication table holds for the $O(M)$ invariants of the 'P' sector.

There are three blocks generated by (57) in which the equations for the coupling counter tensors are to be satisfied separately. Since the symmetry breaking occurs in the $O(N)$ sector, it is easy to check that in this sector one has $N-1$ "Goldstone" modes with mass $\tilde{M}_\pi$ and one massive mode with mass $\tilde{M}_\sigma$. The $O(M)$ sector remains fully degenerate, all modes have the common mass $\tilde{M}_P$.

In this latter 'P'-sector one has (see (57))

$$O^P_{\mu\epsilon}O^P_{\nu\epsilon}T_F(\tilde{M}^2_{P,\epsilon}) = T_F(\tilde{M}^2_P)O^P_{\mu\epsilon}O^P_{\nu\epsilon} = T_F(\tilde{M}^2_P)\delta_{\mu\nu}. \tag{62}$$

This means that we have to take the trace with respect two the free Greek indices in the curly bracket of the second and fourth equation of (57). Equating with zero the coefficients of the resulting two tensorial structures δ_{ab} and $\delta_{\alpha\beta}$ one obtains:

$$\begin{aligned}(M+2)\delta\lambda^P =& -4T_d\left[(M+2)^2\lambda^P\left(\lambda^P+\delta\lambda^P\right)+MN\lambda^H(\lambda^H+\delta\lambda^H)\right],\\ \delta\lambda^H =& -4T_d\left[(N+2)\lambda^H\left(\lambda^S+\delta\lambda^S\right)+(M+2)\lambda^P(\lambda^H+\delta\lambda^H)\right],\end{aligned} \tag{63}$$

where we introduced the notations $(M+2)\delta\lambda^P \equiv M\delta\lambda^P_A + 2\delta\lambda^P_B, (N+2)\delta\lambda^S \equiv N\delta\lambda^S_A + 2\delta\lambda^S_B$.

In the 'S'-sector using (60) and (61) in the first and third equation of (57) one obtains for the tensors in the curly brackets of these equations the following expressions

$$\begin{aligned}\{...\} =& \ t^1_{abmn}\left\{\delta\lambda^S_A + 4T_d\left[\lambda^S\left((N+4)\lambda^S+(N+2)\delta\lambda^S_A+2\delta\lambda^S_B\right)+M\lambda^H(\lambda^H+\delta\lambda^H)\right]\right\}\\ &+t^2_{abmn}\left[\delta\lambda^S_B+8T_d\lambda^S(\lambda^S+\delta\lambda^S_B)\right],\\ \{...\} =& \ t^1_{\alpha\beta mn}\left\{\delta\lambda^H+4T_d\left[(M+2)\lambda^H\left(\lambda^P+\delta\lambda^P\right)+(N+2)\lambda^S(\lambda^H+\delta\lambda^H)\right]\right\}.\end{aligned} \tag{64}$$

With help of the projectors given in (3) one writes

$$O^S_{me}O^S_{nf}T_F(\tilde{M}^2_{S,e})\delta_{ef} = P^\sigma_{mn}T_F(\tilde{M}^2_\sigma) + P^\pi_{mn}T_F(\tilde{M}^2_\pi). \tag{65}$$

This means that one has to project both expressions in (64) on the 1-dimensional and the $N-1$ dimensional degenerate parts of the spectra by applying the corresponding projectors and then equate the resulting expressions to zero. Applying this procedure to the first expression of (64) gives the following two relations between the counterterms:

$$\begin{aligned}\delta\lambda^S_A =& -4T_d\Big[\lambda^S\big((N+4)\lambda^S+(N+2)\delta\lambda^S_A+2\delta\lambda^S_B\big)+M\lambda^H(\lambda^H+\delta\lambda^H)\Big],\\ \delta\lambda^S_B =& -8T_d\lambda^S\left(\lambda^S+\delta\lambda^S_B\right).\end{aligned} \tag{66}$$

We have used that the projection of the rank-4 tensors can be expressed with help of P^σ and P^π.

The equations for $\delta\lambda^S_A$ and $\delta\lambda^S_B$ if summed with appropriately chosen coefficients reproduce the "mirror" of the equation found for $(M+2)\delta\lambda^P$ in the $O(M)$-symmetric P-sector: $S \leftrightarrow P$ and $N \leftrightarrow M$. Here, however, we have two separate tensor structures, which determine $\delta\lambda^S_A, \delta\lambda^S_B$, and not only the combination $\delta\lambda^S$. This feature shows that the renormalisability of the 2PI-Hartree approximation requires different counterterm structure in the symmetric case and when one deals with broken symmetry, due to the spectral structure induced by the specific symmetry breaking pattern.

Since $P^\sigma_{mn}t^1_{\alpha\beta mn} = \delta_{\alpha\beta}$ and $P^\pi_{mn}t^1_{\alpha\beta mn} = (N-1)\delta_{\alpha\beta}$, upon projecting the second equation of (64) on the two blocks of the 'S'-sector one finds

$$\delta\lambda^H = -4T_d\Big[(M+2)\lambda^H\left(\lambda^P+\delta\lambda^P\right)+(N+2)\lambda^S(\lambda^H+\delta\lambda^H)\Big]. \tag{67}$$

This expression is again the "mirror" of that in the second line of (63) under the interchange $S \leftrightarrow P$ and $N \leftrightarrow M$. Its explicit expression can be obtained either from (67) and the first line of (63) or the second line of (63) and the sum of the two equations in (66). The resulting $\delta\lambda^H$ is symmetric, as expected, under the interchange $S \leftrightarrow P$ and $N \leftrightarrow M$:

$$\delta\lambda^H = -\frac{4T_d\lambda^H U}{1+4T_d U}, \qquad U = (M+2)\lambda^P + (N+2)\lambda^S + 4T_d\left[(M+2)(N+2)\lambda^S\lambda^P - MN(\lambda^H)^2\right]. \tag{68}$$

The compatibility condition (52), obtained by confronting the gap equations and the equation of state, imposes condition only on the 'S'-sector. By the symmetry of the classical potential one puts

$$\delta\tilde{F}^S_{abmn} = \delta\tilde{\lambda}(\delta_{ab}\delta_{mn} + \delta_{am}\delta_{bn} + \delta_{an}\delta_{bm}), \tag{69}$$

which leads to the relation

$$\delta\tilde{\lambda} = \delta\lambda^S_A + 2\delta\lambda^S_B. \tag{70}$$

In summary we see that there are four independent coupling counterterms $\delta\lambda^S_A$, $\delta\lambda^S_B$, $\delta\lambda^P$, $\delta\lambda^H$ which renormalise the 2PI-Hartree approximation of the Dyson-Schwinger equations of this model in the broken symmetry phase. Apart from these, there are two mass counterterms which can be easily obtained from (53).

Special cases One recovers the $O(N)$ model from the general Lagrangian (28) if only one series of fields appears, that is in (60) one has

$$F^P_{\alpha\beta\gamma\delta} = \delta F^P_{\alpha\beta\gamma\delta} = H_{ab,\gamma\delta} = \delta H_{ab,\gamma\delta} = 0. \tag{71}$$

The two equations of (66) reduce to the following relations between the two counter-couplings $\delta\lambda_A, \delta\lambda_B$ ($\lambda^S \equiv \lambda$):

$$\begin{aligned}\delta\lambda_A &= -4\lambda T_d\left[(N+4)\lambda + (N+2)\delta\lambda_A + 2\delta\lambda_B\right],\\ \delta\lambda_B &= -8\lambda T_d\left[\lambda + \delta\lambda_B\right].\end{aligned} \tag{72}$$

These equations exactly reproduce those in (20) after rescaling the couplings and the counterterms by $24N$. Equation (70) applies in unchanged form.

Our model can accommodate also the case of two interacting N-plets, if one interprets the fields π_a as a second N-plet with $O(N)$ invariant selfcoupling and assuming that the interaction term between them is unchanged. Its analysis is similar to the general $O(N)\times O(M)$ case just one chooses $N = M$. The coupling counterterms relevant in the large N limit are obtained from the first equations of (63) and (66), and from (67). Using the notation $\lambda^S_A \equiv \lambda^S, \lambda^P_A \equiv \lambda^P$ one has:

$$\begin{aligned}\delta\lambda^P &= -4T_d N\left[\lambda^P\left(\lambda^P + \delta\lambda^P\right) + \lambda^H(\lambda^H + \delta\lambda^H)\right],\\ \delta\lambda^S &= -4T_d N\left[\lambda^S\left(\lambda^S + \delta\lambda^S\right) + \lambda^H(\lambda^H + \delta\lambda^H)\right],\\ \delta\lambda^H &= -4T_d N\left[\lambda^H\left(\lambda^S + \delta\lambda^S\right) + \lambda^P(\lambda^H + \delta\lambda^H)\right].\end{aligned} \tag{73}$$

The solution for $\delta\lambda^H$ is obtained from (68) by taking $M = N$ and making the replacement $N+2 \to N$. In terms of $\delta\lambda^H$ the solution for $\delta\lambda^P$ and $\delta\lambda^S$ can be readily obtained.

6 The $SU(N) \times SU(N)$ meson model in 2PI-Hartree approximation

For $N = 3$ this model is one of the most popular effective meson models. To our knowledge, the proper renormalisation of its 2PI-Hartree approximation appears here for the first time in the literature.

The four-point coupling tensors describing the self-interaction of scalars and pseudoscalar mesons and the interaction between them is usually written in the following form [26, 4]:

$$\begin{aligned} F^S_{abcd} = F^P_{abcd} &= \frac{g_1}{4}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{g_2}{8}(d_{abm}d_{cdm} + d_{acm}d_{bdm} + d_{adm}d_{bcm}), \\ H_{ab,cd} &= \frac{g_1}{4}\delta_{ab}\delta_{cd} + \frac{g_2}{8}(d_{abm}d_{cdm} + f_{acm}f_{bdm} + f_{adm}f_{bcm}). \end{aligned} \tag{74}$$

The sum of the last two terms in the second line of (74) can be rewritten using the following relation between the structure constants of the $SU(N)$ group [27]

$$f_{acm}f_{bdm} = \frac{2}{N}\left(\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}\right) + d_{abm}d_{cdm} - d_{adm}d_{bcm}, \tag{75}$$

resulting in:

$$H_{ab,cd} = \frac{1}{4}\left(g_1 + \frac{2g_2}{N}\right)\delta_{ab}\delta_{cd} - \frac{g_2}{4N}\left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) + \frac{3g_2}{8}d_{abm}d_{cdm} - \frac{g_2}{8}\left(d_{acm}d_{bdm} + d_{adm}d_{bcm}\right). \tag{76}$$

One can express the coupling tensors in terms of the following combinations of only six out of the nine independent invariant rank-4 tensors of the $SU(N)$ group (see eg. [28]):

$$\begin{aligned} t^1_{abcd} &= \delta_{ab}\delta_{cd}, & t^2_{abcd} &= \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}, \\ t^3_{abcd} &= d_{abm}d_{cdm}, & t^4_{abcd} &= d_{acm}d_{bdm} + d_{adm}d_{bcm}. \end{aligned} \tag{77}$$

This set of 4 invariants is closed under multiplication with the following multiplication table:

$$\begin{aligned} &t^1 * t^1 = (N^2-1)t^1, && t^1 * t^2 = 2t^1, \quad t^1 * t^3 = 0, \quad t^1 * t^4 = 2N\left(1 - \frac{4}{N^2}\right)t^1, \\ &t^2 * t^2 = 2t^2, && t^2 * t^3 = 2t^3, \quad t^2 * t^4 = 2t^4, \quad t^3 * t^3 = N\left(1 - \frac{4}{N^2}\right)t^3, \\ &t^3 * t^4 = N\left(1 - \frac{12}{N^2}\right)t^3, && t^4 * t^4 = 2\left(1 - \frac{4}{N^2}\right)(2t^1 + t^2) + N\left(1 - \frac{16}{N^2}\right)t^3 - \frac{8}{N}t^4. \end{aligned} \tag{78}$$

In deriving (78) we have used identities which can be found for example in Appendix A of [29].

The special case of $\boldsymbol{N = 3}$. To specify the analysis we first note that in this case t^4 is not an independent invariant because of the relation $t^4 = (t^1 + t^2)/3 - t^3$ (see [27, 28] for its derivation). Due to this reduction in the set of invariant tensors, the tensorial coupling structure of the $SU(3) \times SU(3)$ model greatly simplifies:

$$\begin{aligned} &F^S = F^P = f(t^1 + t^2), && f = \frac{1}{24}\left(6g_1 + g_2\right), \\ &H = h_1 t^1 + h_2 t^2 + h_3 t^3, && h_1 = \frac{1}{8}\left(2g_1 + g_2\right), \qquad h_2 = -\frac{g_2}{8}, \quad h_3 = \frac{g_2}{2}. \end{aligned} \tag{79}$$

These tensors represent a closed set under multiplication, with a multiplication table which can be read off from (78) by putting $N = 3$. Correspondingly, one can introduce the following counterterm structures for the Dyson-Schwinger equations:

$$\delta F^S = \delta f_1^S t^1 + \delta f_2^S t^2 + \delta f_3^S t^3, \qquad \delta F^P = \delta f_1^P t^1 + \delta f_2^P t^2 + \delta f_3^P t^3,$$
$$\delta H = \delta h_1 t^1 + \delta h_2 t^2 + \delta h_3 t^3. \tag{80}$$

In order to find the necessary conditions which determine the counterterms in the broken symmetry phase we have to investigate the multiplicity structure in the scalar and pseudoscalar sectors. This is obtained from the renormalised 2PI-Hartree gap-equations (46) which in more detail read as

$$\begin{aligned}(M_S^2)_{ab} &= \mu^2\delta_{ab} + 4F^S_{abcd}\left[v_c v_d + O^S_{ce}O^S_{de}T_F(\tilde M^2_{S,e})\right] + 4H_{ab,cd}O^P_{ce}O^P_{de}T_F(\tilde M^2_{P,e}),\\ (M_P^2)_{ab} &= \mu^2\delta_{ab} + 4H_{ab,cd}\left[v_c v_d + O^S_{ce}O^S_{de}T_F(\tilde M^2_{S,e})\right] + 4F^P_{abcd}O^P_{ce}O^P_{de}T_F(\tilde M^2_{P,e}).\end{aligned} \tag{81}$$

We assume the presence of scalar condensates belonging to the center of the group: $(v_3,\ v_8)$. At tree (classical) level this background results in different multiplet structures in the two sectors, but solving (81) iteratively one can show that the emerging exact mass spectra will have eventually the same structure in both sectors due to the coupling realised by the tensor H. To be specific, when $v_3 \neq 0$, $v_8 \neq 0$ one has 3 degenerate doublets in the "planes" [1,2], [4,5] and [6,7] and one coupled set with unequal eigenvalues in the [3,8] "plane". These "planes" correspond to the pairs of fields $[\pi^+, \pi^-]$, $[K^+, K^-]$, $[K^0, \bar K^0]$, $[\pi^0, \eta]$ in the pseudoscalar sector and to $[a_0^+, a_0^-]$, $[\kappa^+, \kappa^-]$, $[\kappa^0, \bar\kappa^0]$, $[a_0^0, \sigma]$ in the scalar sector. When there is only one condensate then the two middle sectors join in a degenerate quadruplet and the fields of the coupled sector become decoupled mass eigenstates. For $v_3 = 0, v_8 \neq 0$ direction "3" degenerates with the first "plane" resulting in a $3 \oplus 4 \oplus 1$ multiplet structure. For $v_8 \neq 0, v_3 = 0$ directions "3" and "8" remain as separate singlets, so that the multiplet structure is $2 \oplus 1 \oplus 4 \oplus 1$.

In view of the known multiplicity structure one finds sufficient number of independent renormalisation conditions to determine the remaining six coupling counterterms of (80) by requiring the vanishing of the (divergent) coefficient of each independent tadpole appearing in the detailed conditions (57). For the degenerate sectors, since $O^{S/P}_{me} = \delta_{me}$, these conditions result in the vanishing of the trace of the tensor structure in the curly bracket of each equation contained in (57). For the coupled sectors one has to impose the vanishing of those components of the tensor structure in the curly brackets which correspond to the independent elements of these sectors. Important simplification occurs when one realises that $\delta F^P = \delta F^S \equiv \delta F$, since the renormalisation conditions are fully symmetric under the exchange $P \leftrightarrow S$, in view of $F^P = F^S$. Since the number of conditions is sufficiently large all countercouplings $\delta h_i, \delta f_i, i = 1, 2, 3$ can be determined after evaluating products of the tensors and counterterm tensors with help of the multiplication table. Since the number of conditions is large enough, these equations coincide with the equations one obtains from the matrix-equations (54), without investigating the multiplet structure at all:

$$\begin{aligned}\delta f_1 &= -8T_d[5f(f+\delta f_1) + f(f+\delta f_2) + 4h_1(h_1+\delta h_1) + h_1(h_2+\delta h_2) + h_2(h_1+\delta h_1)],\\ \delta f_2 &= -8T_d[f(f+\delta f_2) + h_2(h_2+\delta h_2)],\\ \delta f_3 &= -8T_d[f\delta f_3 + h_3(h_2+\delta h_2) + (h_2+5h_3/6)(h_3+\delta h_3)],\\ \delta h_1 &= -8T_d[(4h_1+h_2)(f+\delta f_1) + h_1(f+\delta f_2) + 5f(h_1+\delta h_1) + f(h_2+\delta h_2)],\\ \delta h_2 &= -8T_d[h_2(f+\delta f_2) + f(h_2+\delta h_2)],\\ \delta h_3 &= -8T_d[h_3(f+\delta f_2) + \delta f_3(h_2+5h_3/6) + f(h_3+\delta h_3)].\end{aligned} \tag{82}$$

d_{m^2-1,m^2-1,n^2-1}	Index relation	$d_{(ij),(ij),n^2-1}$	Index relation
0	$m > n$	0	$n < i < j$
$2-m$	$m = n$	$\frac{1-n}{2}$	$n = i < j$
1	$m < n$	$\frac{1}{2}$	$i < n < j$
		$\frac{2-n}{2}$	$i < n = j$
		1	$i < j < n$

Table 1: $SU(N)$ structure constants $d_{ab(n^2-1)}$ in multiples of $(2/(n^2-n))^{1/2}$. For the index convention $(a, b = m^2-1, (ij))$ see the text. The structure constants are the same for both $\alpha = 1, 2$.

The coupled set for δf_2 and δh_2 is closed in itself. Its solution is substituted into the other four equations which then fall into two coupled two-variable equations for $(\delta f_1, \delta h_1)$ and $(\delta f_3, \delta h_3)$, respectively.

The analysis above shows that lifting the degeneracy of the spectra leads to the determination of the full set of counter couplings in contrast to the renormalisation in the symmetric phase of the model. In the latter case the entire spectra is degenerate and taking the trace results in $t^3_{abcc} = 0$, due to $d_{mcc} = 0$. Renormalisability conditions can be derived from (56) only for two linear combinations $\delta f = 8(4\delta f_1 + \delta f_2)$ and $\delta h = 8(4\delta h_1 + \delta h_2)$. One has

$$\delta f = -T_d\left[f(f+\delta f) + h(h+\delta h)\right], \qquad \delta h = -T_d\left[h(f+\delta f) + f(h+\delta h)\right], \tag{83}$$

where we introduced $f = (6g_1 + g_2)/3$ and $h = 8g_1 + 3g_2$.

Large N analysis. We turn to the analysis of the large N limit. The large N scaling of the parameters and vacuum condensate strengths is established first by the requirement that the potential energy density is proportional to the number of degrees of freedom e.g. $\sim N^2$.

We assume also here that condensates are formed in the center of the group. In order to find the behaviour of the piece $F_{abcd}v_a v_b v_c v_d$ of the potential one has to examine the large N behaviour of the structure constants d_{abc}, where at least one of the indices corresponds to a center element of the Lie algebra. The generalised Gell-Mann matrices building up the center generators are labelled as $\lambda_{n^2-1},\ n = 2, 3, .., N$, while the two sequences of the non-diagonal elements (the analogues of σ^1 and σ^2 for $SU(2)$) hold the obvious double index labelling $\lambda^\alpha(i,j),\ \alpha = 1, 2,\ i = 1, .., N, j = 2, .., N, i < j$ [30]. One finds the expressions of the structure constants necessary for our analysis in Table 1. Note that if one index corresponds to some diagonal λ_{n^2-1} then d_{ab,n^2-1} is nonzero only if $a = b$.

The simplest is to analyse the behaviour in presence of a single component condensate, v_{n^2-1}:

$$V_{tree} = \frac{1}{2}\mu^2 v^2_{n^2-1} + \frac{1}{3}F_{n^2-1,n^2-1,n^2-1,n^2-1}v^4_{n^2-1}. \tag{84}$$

Here

$$F_{n^2-1,n^2-1,n^2-1,n^2-1} = \frac{3}{4}g_1 + \frac{3}{8}g_2\sum_{m=2}^{N} d_{n^2-1,n^2-1,m^2-1}d_{n^2-1,n^2-1,m^2-1}, \tag{85}$$

with

$$\sum_{m=2}^{N} d_{n^2-1,n^2-1,m^2-1} d_{n^2-1,n^2-1,m^2-1} = \frac{2}{n}\left(1 - \frac{n}{N} + \frac{(2-n)^2}{n-1}\right). \tag{86}$$

From the quadratic term of the potential one deduces the scaling of the condensate and using the large N expression of the tensor F from the quartic term one finds the N-scaling of the couplings:

$$v_{n^2-1} \sim N, \quad g_1, g_2 \sim \frac{1}{N^2}. \tag{87}$$

These scaling relations are important when we look for the non-vanishing tadpole contribution to the gap equations in the large N limit. They are different from the scaling one finds in the $U(N)\times U(N)$ symmetric model for the singlet condensate v_0!

Next, one studies the multiplet structure of the model in presence of the symmetry breaking condensate. We restricted our investigation to the case of a single component condensate, v_{n^2-1}. There is no mixing between the modes belonging to the different generators. The tree level masses in the large N limit are given by the following formulae in the 'S'- and 'P'-sectors:

$$\begin{aligned}
(m_S^2)_{ab} - \mu^2\delta_{ab} = v_{n^2-1}^2 \Bigg(& g_1(\delta_{ab} + 2\delta_{n^2-1,a}\delta_{n^2-1,b}) \\
& + \frac{g_2}{2}\sum_c (d_{abc} d_{n^2-1,n^2-1,c} + 2 d_{a,n^2-1,c} d_{b,n^2-1,c})\Bigg), \\
(m_P^2)_{ab} - \mu^2\delta_{ab} = v_{n^2-1}^2 \Bigg(& \left(g_1 + 2\frac{g_2}{N}\right)\delta_{ab} - 2\frac{g_2}{N}\delta_{a,n^2-1}\delta_{b,n^2-1} \\
& + \frac{g_2}{2}\sum_c (3 d_{abc} d_{n^2-1,n^2-1,c} - 2 d_{a,n^2-1,c} d_{b,n^2-1,c})\Bigg).
\end{aligned} \tag{88}$$

The expressions for the masses, which can be calculated using the structure constants of Table 1, are given in Table 2 together with their multiplicities. The multiplicities take into account the twofold degeneracy of the "(ij)" modes, making in this way their sum N^2-1.

In the generic case, when n and $N-n$ are both $\mathcal{O}(N)$, one has three different multiplets of multiplicity $\mathcal{O}(N^2)$, which give finite tadpole contribution in the large N limit to the gap equations. In this case one has enough number of renormalisation conditions that one can consider the matrix form of the renormalisation conditions without projection. The other case is when either n, or $N-n$ are $\mathcal{O}(N^0)$. Then only a single multiplet of multiplicity $\mathcal{O}(N^2)$ is formed whose tadpole contributes to the gap equations of the different fields. In this case one has to consider appropriate traces in equations (57) and (81). One should note that the mass expressions listed in Table 2 have different large N values in the two cases: $a \sim \mathcal{O}(N)$ or $a \sim \mathcal{O}(N^0)$.

Let us start to discuss this last case first. The modes of the single contributing multiplet can be completed to the full set with no effect on the large N asymptotics. In the resulting gap equations we denote its exact mass by M_{min}:

$$(M_{S/P}^2)_{ab} = (m_{S/P}^2)_{ab} + 4(F_{abcc} + H_{ab,cc})T_F(M_{min}). \tag{89}$$

In the large N limit the partial trace of the coupling tensors reduces to

$$F_{abcc} = \frac{g_1}{4}N^2\delta_{ab} + \frac{g_2}{4}d_{acm}d_{bcm}, \qquad H_{ab,cc} = \frac{g_1}{4}N^2\delta_{ab} - \frac{g_2}{4}d_{acm}d_{bcm}. \tag{90}$$

$(m_S^2-\mu^2)_{ab}/v^2_{n^2-1}$	$(m_P^2-\mu^2)_{ab}/v^2_{n^2-1}$	Index $a=b=l^2-1$	Multiplicity
$g_1+g_2\big(\frac{3}{l(l-1)}-\frac{1}{N}\big)$	$g_1+g_2\big(\frac{1}{l(l-1)}-\frac{1}{N}\big)$	$l>n$	$N-n$
$3\Big[g_1+g_2\big(\frac{l^2-3l+3}{l(l-1)}-\frac{1}{N}\big)\Big]$	$g_1+g_2\big(\frac{l^2-3l+3}{l(l-1)}-\frac{1}{N}\big)$	$l=n$	1
$g_1+g_2\big(\frac{3}{n(n-1)}-\frac{1}{N}\big)$	$g_1+g_2\big(\frac{1}{n(n-1)}-\frac{1}{N}\big)$	$l<n$	$n-2$
		Index $a=b=(ij)$	
$g_1-g_2\frac{1}{N}$	$g_1-g_2\frac{1}{N}$	$n<i<j$	$(N-n)(N-n-1)$
$g_1+g_2\left(\frac{i-1}{i}-\frac{1}{N}\right)$	$g_1+g_2\left(\frac{i-1}{i}-\frac{1}{N}\right)$	$n=i<j$	$2(N-n)$
$g_1+g_2\big(\frac{1}{n(n-1)}-\frac{1}{N}\big)$	$g_1+g_2\big(\frac{1}{n(n-1)}-\frac{1}{N}\big)$	$i<n<j$	$2(N-n)(n-1)$
$g_1+g_2\big(\frac{j^2-3j+3}{j(j-1)}-\frac{1}{N}\big)$	$g_1+g_2\big(\frac{j^2-j+1}{j(j-1)}-\frac{1}{N}\big)$	$i<j=n$	$2(n-1)$
$g_1+g_2\big(\frac{3}{n(n-1)}-\frac{1}{N}\big)$	$g_1+g_2\big(\frac{1}{n(n-1)}-\frac{1}{N}\big)$	$i<j<n$	$(n-1)(n-2)$

Table 2: Tree level mass splittings of the $SU(N)\times SU(N)$ symmetric meson model due to the symmetry breaking condensate v_{n^2-1} with their respective degeneracies.

Since $d_{acm}d_{bcm}=(N^2-4)/N\delta_{ab}$, in view of the N-scaling of the couplings (87), in the coupling tensors above the terms proportional with g_2 are subleading relative to those with g_1. Only the $O(2N^2)$ symmetric coupling survives the $N\to\infty$ limit.

The large N form of the renormalisation conditions is obtained from (57) by taking the trace of the coefficients multiplying $O_{me}O_{ne}T_F(M_{min})$ in the indices m,n. For a consistent N-scaling of the coupling counterterms one has to assume that the counter couplings $\delta f_i,\delta h_i$ obey the same $\sim N^{-2}$ scaling like the renormalised couplings. The only entry of the multiplication table (78) which produces a factor $\sim N^2$ compensating the quadratic dependence on the couplings of the terms of (57) proportional to T_d is t^1*t^1. One arrives at the simplified conditions:

$$\begin{aligned}\delta f_1+4T_dN^2((f_1+\delta f_1)f_1+(h_1+\delta h_1)h_1)=0,\\ \delta h_1+4T_dN^2((f_1+\delta f_1)h_1+(h_1+\delta h_1)f_1)=0.\end{aligned} \tag{91}$$

Since at large N $f_1=h_1=g_1/4$, one can consistently choose $\delta h_1=\delta f_1\equiv\delta g_1$ and renormalise the large N gap equations of the $SU(N)$ symmetric scalar theory with a single counterterm (no counterterm has to be introduced to leading order to g_2). This is essentially the large N limit of broken symmetry phase of the $O(2N^2)$ symmetric model (compare to (25)):

$$\delta g_1=-\frac{2N^2T_dg_1^2}{1+2N^2T_dg_1}. \tag{92}$$

Also in the generic case with three $\mathcal{O}(N^2)$ size multiplets their gap equations will depend only on the coupling g_1. This is consistent with the fact that to leading order only terms proportional to t^1 appear in the renormalisation conditions. Therefore only g_1 will renormalise non-trivially, and exactly the same way as above. One can observe that the masses of the smaller multiplets depend also on g_2 through the tree level contribution. To leading order g_2 is renormalisation invariant, its counterterm will be $\mathcal{O}(N^{-3})$.

7 Conclusions

In this paper we succeeded to construct explicitly the counterterms of the 2PI-Hartree approximation to the quantum action of a rather general class of scalar field theories in a transparent single-step procedure. The allowed set of counterterms arises by considering all rank-4 invariant tensors of the actual internal symmetry group of the theory. Depending on the spectral degeneracy in the broken symmetry phase, the number of counterterms one actually needs might reduce. This circumstance is rather peculiar, since in the usual (non-resummed) perturbative renormalisation the counterterms are universal, that is independent on any infrared details, especially on the way the symmetry is broken. This circumstance is usually considered to be the consequence of truncating the 2PI effective action. In high enough order of the truncation one expects that a unique renormalisation scale dependence will be approached for all 4-point functions defined with functional derivatives of the effective action with respect to alternative combinations of fields and propagators.

We have applied this scheme of counterterms to a number of scalar field theories. For the $O(N)$ symmetric model, to leading order in the large N approximation, the scheme simplifies to a single quartic coupling. Its expression coincides with the exact large N asymptotics of the full theory (not truncated at the Hartree level). The asymptotic expressions for the counterterms of the $SU(N) \times SU(N)$ theory coincide with the expressions of the $O(2N^2)$ symmetric model.

We plan to investigate the relation of the presented renormalisation scheme, which requires multiple coupling counterterms with the symmetric and mass-independent renormalisation which in the "symmetryzed" approximation of [19] to the $O(N)$ model was proven to have the counterterm structure expected from the perturbation theory. Another interesting direction will be to see to what extent the proposed single-step procedure can be incorporated in the renormalisation of the momentum-dependent truncations of the 2PI approximation recently studied in [9, 1, 10, 31].

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