

Breather boundary form factors in sine-Gordon theory

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Abstract

A previously conjectured set of exact form factors of boundary exponential operators in the sinh-Gordon model is tested against numerical results from boundary truncated conformal space approach in boundary sine-Gordon theory, related by analytic continuation to sinh-Gordon model. We find that the numerical data strongly support the validity of the form factors themselves; however, we also report a discrepancy in the case of diagonal matrix elements, which remains unresolved for the time being.

1 Introduction

The investigation of integrable boundary quantum field theories started with the seminal work of Ghoshal and Zamolodchikov [1], who set up the boundary R-matrix bootstrap, which makes possible the determination of the reflection matrices and provides complete description of the theory on the mass shell.

For the calculation of correlation functions, matrix elements of local operators between asymptotic states have to be computed. In a boundary quantum field theory there are two types of operators, the bulk and the boundary operators, where their names indicate their localization point. The boundary form factor program for calculating the matrix elements of local boundary operators between asymptotic states was initiated in [2]. The validity of form factor solutions was checked in the case of the boundary scaling Lee-Yang model by calculating the two-point function using a spectral sum and comparing it to

the prediction of conformal perturbation theory. In [3] the spectrum of independent form factor solutions in the scaling Lee-Yang model and the sinh-Gordon model was compared to the boundary operator content of the ultraviolet boundary conformal field theory and a complete agreement was found. It is also possible to compare form factors to matrix elements of local operators evaluated directly from the boundary quantum field theory in a non-perturbative framework. For periodic boundary conditions, this was developed in [4, 5]; the extension to boundary form factors was obtained in [6] and used to verify the results of the form factor bootstrap in the scaling Lee-Yang model by comparison to boundary truncated conformal space approach.

Further solutions of the boundary form factor axioms were constructed and their structure was analyzed for the sinh-Gordon theory at the self-dual point in [7], and for the A_2 affine Toda field theory in [8]. One of the present authors constructed a solution for boundary exponential operators in sinh-Gordon theory [9], and the solution was checked by computing the conformal dimensions and vacuum expectation values of the fields in a cumulant expansion ordered by powers of the coupling constant, which can be compared to known exact results. However, the conformal dimension essentially tests only the part constructed out of the bulk form factors, and the expectation value was checked only to the lowest nontrivial order in the coupling constant.

Our aim in this paper is to provide a detailed non-perturbative verification of the solution presented in [9]: using the ideas of [6] we aim to compare the form factors to numerically computed finite volume matrix elements. However, this cannot be performed directly in the sinh-Gordon theory as we have no way to construct a truncated conformal space in this case. Fortunately, it is easy to argue that (at least to all order of perturbation theory) an analytic continuation to sine-Gordon model should work, and for this model working truncated conformal space program was developed in [10, 11]. This forms the basis of the present work.

The paper is structured as follows. In Section 2 the boundary form factors of exponential operators in the sinh-Gordon model are recalled (in doing so some unfortunate typos in the paper [9] are also fixed). In Section 3 these form factors are analytically continued to obtain breather form factors in sine-Gordon theory, and summarize briefly the necessary ingredients to obtain predictions finite volume matrix elements. Section 4 contains our numerical analysis, and we present our conclusions in section 5.

2 Boundary form factors in the sinh-Gordon model

2.1 Boundary sinh-Gordon model

The sinh-Gordon theory in the bulk is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{m^2}{b^2}(\cosh b\Phi - 1)$$

It can be considered as the analytic continuation of the sine-Gordon model for imaginary coupling. The S-matrix of the model is

$$S(\theta) = - \left(1 + \frac{B}{2}\right)_\theta \left(-\frac{B}{2}\right)_\theta = \left[-\frac{B}{2}\right]_\theta \quad ; \quad B = \frac{2b^2}{8\pi + b^2} \quad (2.1)$$

where

$$(x)_\theta = \frac{\sinh \frac{1}{2}(\theta + i\pi x)}{\sinh \frac{1}{2}(\theta - i\pi x)} \quad , \quad [x]_\theta = -(x)_\theta(1-x)_\theta = \frac{\sinh \theta + i \sin \pi x}{\sinh \theta - i \sin \pi x}$$

The minimal bulk two-particle form factor belonging to this S-matrix is [12]

$$f(\theta) = \mathcal{N} \exp \left[8 \int_0^\infty \frac{dx}{x} \sin^2 \left(\frac{x(i\pi - \theta)}{2\pi} \right) \frac{\sinh \frac{xB}{4} \sinh(1 - \frac{B}{2})\frac{x}{2} \sinh \frac{x}{2}}{\sinh^2 x} \right] \quad (2.2)$$

where

$$\mathcal{N} = \exp \left[-4 \int_0^\infty \frac{dx}{x} \frac{\sinh \frac{xB}{4} \sinh(1 - \frac{B}{2})\frac{x}{2} \sinh \frac{x}{2}}{\sinh^2 x} \right] \quad (2.3)$$

It satisfies $f(\theta, B) \rightarrow 1$ as $\theta \rightarrow \infty$, and approaches its asymptotic value exponentially fast.

Sinh-Gordon theory can be restricted to the negative half-line with the following action

$$\begin{aligned} \mathcal{A} = & \int_{-\infty}^\infty dt \int_{-\infty}^0 dx \left[\frac{1}{2}(\partial_\mu \Phi)^2 - \frac{m^2}{b^2}(\cosh b\Phi - 1) \right] \\ & + \int_{-\infty}^\infty dt M_0 \left[\cosh \left(\frac{b}{2}(\Phi(0, t) - \Phi_0) \right) - 1 \right] \end{aligned} \quad (2.4)$$

which maintains integrability [1]. The corresponding reflection factor depends on two continuous parameters and can be written as [14]

$$R(\theta) = \left(\frac{1}{2}\right)_\theta \left(\frac{1}{2} + \frac{B}{4}\right)_\theta \left(1 - \frac{B}{4}\right)_\theta \left[\frac{E-1}{2}\right]_\theta \left[\frac{F-1}{2}\right]_\theta \quad (2.5)$$

It can be obtained as the analytic continuation of the first breather reflection factor in boundary sine-Gordon model which was calculated by Ghoshal in [13]. The relation of the bootstrap parameters E and F to the parameters of the Lagrangian is known both from a semi-classical calculation [14, 15] and also in an exact form in the perturbed boundary conformal field theory framework [16].

2.2 Boundary form factors in sinh-Gordon theory

Here we recall the results of [9], but with a few typos corrected in the expression of the form factor polynomials in (2.12) and (2.16). For a local operator $\mathcal{O}(t)$ localized at the

boundary (located at $x = 0$, and parametrized by the time coordinate t) the form factors are defined as

$${}_{out}\langle\theta'_1, \theta'_2, \dots, \theta'_m | \mathcal{O}(t) | \theta_1, \theta_2, \dots, \theta_n \rangle_{in} = F_{mn}^{\mathcal{O}}(\theta'_1, \theta'_2, \dots, \theta'_m; \theta_1, \theta_2, \dots, \theta_n) e^{-imt(\sum \cosh \theta_i - \sum \cosh \theta'_j)}$$

for $\theta_1 > \theta_2 > \dots > \theta_n > 0$ and $\theta'_1 < \theta'_2 < \dots < \theta'_m < 0$, using the asymptotic *in/out* state formalism introduced in [17]. They can be extended analytically to other values of rapidities. With the help of the crossing relations derived in [2] all form factors can be expressed in terms of the elementary form factors

$${}_{out}\langle 0 | \mathcal{O}(0) | \theta_1, \theta_2, \dots, \theta_n \rangle_{in} = F_n^{\mathcal{O}}(\theta_1, \theta_2, \dots, \theta_n)$$

The general form factor solution can be written as [2]

$$F_n(\theta_1, \theta_2, \dots, \theta_n) = H_n \frac{Q_n(y_1, y_2, \dots, y_n)}{\prod_i y_i \prod_{i < j} (y_i + y_j)} \prod_{i=1}^n r(\theta_i) \prod_{i < j} f(\theta_i - \theta_j) f(\theta_i + \theta_j) \quad (2.6)$$

where

$$y = 2 \cosh \theta \quad (2.7)$$

The Q_n are symmetric polynomials of its variables, and the minimal one-particle boundary form factor is given by

$$r(\theta) = \frac{i \sinh \theta}{(\sinh \theta - i \sin \gamma)(\sinh \theta - i \sin \gamma')} u(\theta, B) \quad , \quad \gamma = \frac{\pi}{2}(E - 1) \quad \gamma' = \frac{\pi}{2}(F - 1) \quad (2.8)$$

where

$$u(\theta) = \exp \left\{ \int_0^\infty \frac{dt}{t} \left[\frac{1}{\sinh \frac{t}{2}} - 2 \cosh \frac{t}{2} \cos \left[\left(\frac{i\pi}{2} - \theta \right) \frac{t}{\pi} \right] \right] \times \frac{\sinh \frac{tB}{4} + \sinh \left(1 - \frac{B}{2} \right) \frac{t}{2} + \sinh \frac{t}{2}}{\sinh^2 t} \right\} \quad (2.9)$$

and

$$H_n = \left(\frac{4 \sin \pi B / 2}{f(i\pi)} \right)^{n/2} \quad (2.10)$$

is a convenient normalization factor. The polynomials Q_n satisfy the following recursion relations:

$$\begin{aligned} \mathcal{K} : \quad & Q_2(-y, y) = 0 \\ & Q_{n+2}(-y, y, y_1, \dots, y_n) = \\ & (y^2 - 4 \cos^2 \gamma)(y^2 - 4 \cos^2 \gamma') P_n(y | y_1, \dots, y_n) Q_n(y_1, \dots, y_n) \quad \text{for } n > 0 \\ \mathcal{B} : \quad & Q_1(0) = 0 \\ & Q_{n+1}(0, y_1, \dots, y_n) = \\ & 4 \cos \gamma \cos \gamma' B_n(y_1, \dots, y_n) Q_n(y_1, \dots, y_n) \quad \text{for } n > 0 \end{aligned} \quad (2.11)$$

where

$$\begin{aligned}
B_n(y_1, \dots, y_n) &= \frac{1}{4 \sin \frac{\pi B}{2}} \left(\prod_{i=1}^n \left(y_i - 2 \sin \frac{\pi B}{2} \right) - \prod_{i=1}^n \left(y_i + 2 \sin \frac{\pi B}{2} \right) \right) \\
&= - \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(2 \sin \frac{\pi B}{2} \right)^{2l} \sigma_{n-1-2l}
\end{aligned} \tag{2.12}$$

and

$$P_n(y|y_1, \dots, y_n) = \frac{1}{2(y_+ - y_-)} \left[\prod_{i=1}^n (y_i - y_-)(y_i + y_+) - \prod_{i=1}^n (y_i + y_-)(y_i - y_+) \right] \tag{2.13}$$

with the notations

$$\begin{aligned}
y_+ &= \omega z + \omega^{-1} z^{-1} \\
y_- &= \omega^{-1} z + \omega z^{-1} \quad , \quad \omega = e^{i\pi \frac{B}{2}}
\end{aligned} \tag{2.14}$$

with the auxiliary variable z defined as a solution of $y = z + z^{-1}$; i.e. from (2.7) one has $z = e^\theta$.

Let us define the elementary symmetric polynomials by

$$\begin{aligned}
\prod_{i=1}^n (x + x_i) &= \sum_{l=1}^n x^{n-l} \sigma_l^{(n)}(x_1, \dots, x_n) \\
\sigma_l^{(n)} &\equiv 0 \quad \text{if } l < 0 \text{ or } l > n
\end{aligned}$$

The upper index will be omitted in the sequel, as the number of variables will always be clear from the context. Let us also denote

$$[n] = \frac{\omega^n - \omega^{-n}}{\omega - \omega^{-1}} = \frac{\sin \frac{n\pi B}{2}}{\sin \frac{\pi B}{2}}$$

Let us also introduce the polynomials $P_n^{(k)}$:

$$\begin{aligned}
P_1^{(k)} &= [k] \\
P_n^{(k)} &= [k] \det M^{(n)}(k) \quad n > 1 \\
M_{ij}^{(n)}(k) &= [i - j + k] \sigma_{2i-j}(x_1, x_2, \dots, x_n) \quad i, j = 1, \dots, n-1
\end{aligned}$$

which are ingredients of the bulk form factor solution [18].

In terms of these definitions, the form factor solution can be written as

$$Q_n^{(k)} = \epsilon_1 \epsilon_2 B_{n-1} Q_{n-1}^{(k)} + \sigma_n P_n^{(k)} + \sigma_n A_n^{(k)} \tag{2.15}$$

where $A_n^{(k)}$ is a linear combination of products of σ_l with total degree strictly less than $n(n-1)/2$, and the first term is understood with the replacement

$$\sigma_l^{(n-1)} \rightarrow \sigma_l^{(n)}$$

and the notation

$$\epsilon_1 = 2 \cos \gamma \quad , \quad \epsilon_2 = 2 \cos \gamma'$$

was introduced. The A -polynomials are given by

$$\begin{aligned} A_2^{(k)} &= 0 \\ A_3^{(k)} &= [k](\epsilon_1^2 + \epsilon_2^2 + [k]\epsilon_1\epsilon_2)\sigma_1 \\ A_4^{(k)} &= 4 \sin^2 \frac{\pi B}{2} [k]^2 \sigma_1 \sigma_3 + [k]^2 (\epsilon_1^2 + \epsilon_2^2 + [k]\epsilon_1\epsilon_2) \sigma_1^2 \left(\sigma_2 + 4 \sin^2 \frac{\pi B}{2} \right) \end{aligned} \quad (2.16)$$

up to 4-particle level, and it can easily be extended to higher levels using any symbolic algebra software.

These form factors correspond to the field

$$\frac{1}{\langle e^{k \frac{b}{2} \Phi(0,t)} \rangle} e^{k \frac{b}{2} \Phi(0,t)}$$

which is normalized to have unite vacuum expectation value.

3 Breather boundary form factors in the sine-Gordon model

3.1 The boundary form factors of multi- B_1 states

The theory is defined by the following action

$$\mathcal{A}_{\text{bsG}} = \int d^2x \left[\frac{1}{4\pi} (\partial_t \phi)^2 - \frac{1}{4\pi} (\partial_x \phi)^2 - 2\mu \cos 2\beta \phi \right] - 2\mu_B \int dt \cos \beta (\phi(t,0) - \phi_0) \quad (3.1)$$

The bulk spectrum of this theory consists of a soliton-antisoliton doublet of mass M , with a number of breathers (depending on the coupling β) and their S -matrices are known [19]. For the full boundary spectrum and associated reflection factors the interested reader is referred to [20] (and references therein). For the purposes of the present work only the consideration of the first breather B_1 is needed.

In this model, the exact expectation values of boundary exponential operators are known [21, 11]:

$$\langle 0 | e^{ia\phi(t,0)} | 0 \rangle = \left(\frac{\pi\mu\Gamma(1-\beta^2)}{\Gamma(\beta^2)} \right)^{\frac{a^2}{2(1-\beta^2)}} g_0(a, \beta) g_S(a, \beta, z, \bar{z}) g_A(a, \beta, z, \bar{z}) \quad (3.2)$$

where

$$\begin{aligned}
g_0(a, \beta) &= \exp \left\{ \int_0^\infty \frac{dt}{t} \left[\frac{2 \sinh^2(a\beta t) \left(e^{(1-\beta^2)t/2} \cosh(t/2) \cosh(\beta^2 t/2) - 1 \right)}{\sinh(\beta^2 t) \sinh(t) \sinh((1-\beta^2)t)} - a^2 e^{-t} \right] \right\} \\
g_S(a, \beta, z, \bar{z}) &= \exp \left\{ \int_0^\infty \frac{dt}{t} \frac{\sinh^2(a\beta t) (2 - \cos(2zt) - \cos(2\bar{z}t))}{2 \sinh(\beta^2 t) \sinh(t) \sinh((1-\beta^2)t)} \right\} \\
g_A(a, \beta, z, \bar{z}) &= \exp \left\{ \int_0^\infty \frac{dt}{t} \frac{\sinh(2a\beta t) (\cos(2zt) - \cos(2\bar{z}t))}{\sinh(\beta^2 t) \sinh(t) \cosh((1-\beta^2)t)} \right\}
\end{aligned}$$

and

$$\cosh^2 \pi z = e^{-2i\beta\phi_0} \frac{\mu_B^2 \sin \pi \beta^2}{\mu} \quad , \quad \cosh^2 \pi \bar{z} = e^{2i\beta\phi_0} \frac{\mu_B^2 \sin \pi \beta^2}{\mu} \quad (3.3)$$

provided the operators are normalized as

$$\begin{aligned}
e^{2ia\phi(x)} e^{-2ia\phi(x)} &\sim \frac{1}{|x-y|^{4a^2}} + \dots \\
e^{ia\phi(t,0)} e^{-ia\phi(t',0)} &\sim \frac{1}{|t-t'|^{2a^2}} + \dots
\end{aligned}$$

which also defines the normalization of the couplings μ and μ_B . The coupling μ can be related to the mass M of the soliton as follows [22]:

$$\mu = \frac{\Gamma(\beta^2)}{\pi \Gamma(1-\beta^2)} \left[M \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2(1-\beta^2)}\right)}{2 \Gamma\left(\frac{\beta^2}{2(1-\beta^2)}\right)} \right]^{2-2\beta^2}$$

The analytic continuation between the sinh-Gordon and sine-Gordon theory in the bulk is defined by

$$\beta = \frac{ib}{\sqrt{8\pi}}$$

Under this substitution the sinh-Gordon particle's S -matrix (2.1) becomes the S -matrix of the first breather in sine-Gordon theory:

$$\begin{aligned}
S_{B_1 B_1}(\theta) &= [\xi]_\theta = \frac{\sinh \theta + i \sin \pi \xi}{\sinh \theta - i \sin \pi \xi} \\
\xi &= -\frac{B}{2} = \frac{\beta^2}{1-\beta^2}
\end{aligned}$$

The mass of the first breather is related to the soliton mass M as

$$m_1 = 2M \sin \frac{\pi \xi}{2}$$

The identification of the two models can be completed by relating the boundary parameters. Introducing the Ghoshal-Zamolodchikov parameters [1]

$$z = \frac{\beta^2}{\pi}(\vartheta - i\eta) \quad , \quad \bar{z} = \frac{\beta^2}{\pi}(\vartheta + i\eta)$$

the reflection factor of the first breather can be written in the form

$$R_{B_1}^{(\alpha)}(\vartheta) = \left(\frac{1}{2}\right)_\theta \left(\frac{1}{2} - \frac{\xi}{2}\right)_\theta \left(1 + \frac{\xi}{2}\right)_\theta \left[\frac{\xi\eta}{\pi} - \frac{1}{2}\right]_\theta \left[\frac{i\xi\vartheta}{\pi} - \frac{1}{2}\right]_\theta \quad (3.4)$$

where α denotes the boundary condition parametrized by η and ϑ . Under this identification the relation (3.3) coincides with the relation between the GZ parameters and the Lagrangian boundary parameters derived by Alyosha Zamolodchikov [16, 11].

Comparing this to (2.5) one obtains the identification

$$E = \frac{2\xi\eta}{\pi} \quad , \quad F = \frac{2i\xi\vartheta}{\pi}$$

As a result, the form factors of the boundary operator

$$\mathcal{O}_k = e^{ik\beta\phi(0,t)}$$

can be written in terms of the sinh-Gordon form factor solution (presented in subsection 2.2) as follows

$$\underbrace{F_{\underbrace{1 \dots 1}_n}^{(k)}}(\theta_1, \dots, \theta_n) = \mathcal{G}_k H_n \frac{Q_n^{(k)}(y_1, y_2, \dots, y_n)}{\prod_i y_i \prod_{i < j} (y_i + y_j)} \prod_{i=1}^n r(\theta_i) \prod_{i < j} f(\theta_i - \theta_j) f(\theta_i + \theta_j)$$

with $B \rightarrow -2\xi \quad , \quad \gamma = \xi\eta - \frac{\pi}{2} \quad , \quad \gamma' = i\xi\vartheta - \frac{\pi}{2}$ (3.5)

and

$$\mathcal{G}_k = \langle 0 | e^{ik\beta\phi(t,0)} | 0 \rangle$$

is the exact vacuum expectation value (3.2).

3.2 Finite volume form factors

We now briefly recall the formalism developed in [6] for the description of matrix elements of boundary operators in finite volume, specialized for the levels that consist of first breathers.

3.2.1 Finite volume energy levels

Introduce the bulk phase-shifts

$$S_{B_1 B_1}(\theta) = e^{i\delta(\theta)}$$

and their boundary counterparts

$$R_{B_1}^{(\alpha)}(\theta) = e^{i\delta^{(\alpha)}(\theta)} \quad (3.6)$$

where α denotes the boundary condition. Putting the theory on a strip of width L with boundary conditions α and β , the finite volume levels can be obtained by solving the Bethe-Yang equations [23]:

$$Q_j(\theta_1, \dots, \theta_n) = 2\pi I_j \quad (3.7)$$

where the phases describing the wave function monodromy are given by

$$Q_j(\theta_1, \dots, \theta_n) = 2m_1 L \sinh \theta_j + \sum_{k \neq j} \{\delta(\theta_j - \theta_k) + \delta(\theta_j + \theta_k)\} + \delta^{(\alpha)}(\theta_j) + \delta^{(\beta)}(\theta_j)$$

Here all rapidities θ_j (and accordingly all quantum numbers I_j) are taken to be positive. The quantum numbers can be taken ordered as $I_1 < \dots < I_n$; they must all be different due to the exclusion principle ($S(0) = -1$). The corresponding multi-particle state is denoted by

$$|\{I_1, \dots, I_n\}\rangle_L$$

and its energy (relative to the ground state) is given by

$$E_{I_1 \dots I_n}(L) = \sum_{j=1}^n m_1 \cosh \tilde{\theta}_j + O(e^{-\mu L})$$

where $\{\tilde{\theta}_j\}_{j=1, \dots, n}$ is the solution of eqns. (3.7) at the given volume L . The Bethe-Yang equations gives the energy of the multi-particle states to all order in $1/L$, neglecting only finite size effects decaying exponentially with L (where μ is some finite mass scale, dependent on the details of the spectrum).

3.2.2 Non-diagonal matrix elements

For non-diagonal matrix elements, it was shown in [6] that

$$\begin{aligned} & |\langle \{I'_1, \dots, I'_m\} | \mathcal{O}(0) | \{I_1, \dots, I_n\} \rangle_L | = \\ & \frac{\left| F_{\underbrace{1 \dots 1}_{n+m}}^{\mathcal{O}}(\tilde{\theta}'_m + i\pi, \dots, \tilde{\theta}'_1 + i\pi, \tilde{\theta}_1, \dots, \tilde{\theta}_n) \right|}{\sqrt{\rho(\tilde{\theta}_1, \dots, \tilde{\theta}_n) \rho(\tilde{\theta}'_1, \dots, \tilde{\theta}'_m)}} + O(e^{-\mu L}) \end{aligned} \quad (3.8)$$

where

$$\rho(\tilde{\theta}_1, \dots, \tilde{\theta}_n) = \det \left\{ \frac{\partial Q_k(\theta_1, \dots, \theta_n)}{\partial \theta_l} \right\}_{k,l=1, \dots, n} \quad (3.9)$$

is the finite volume density of states, which is the Jacobi determinant of the mapping between the space of quantum numbers and the space of rapidities given by the Bethe-Yang equations (3.7). In general, the phase conventions used for the exact form factors and the finite volume matrix elements differ, so only the absolute values can be compared. Evaluating the above expansion requires analytic continuation of the form factors to complex values of θ which can be accomplished using the formulae given in Appendix A.

3.2.3 Diagonal matrix elements

In the diagonal case, there are disconnected contributions which can be taken care of by regularizing the appropriate form factor as

$$F_{\underbrace{1 \dots 1}_{2n}}(\theta_n + i\pi + \epsilon_n, \dots, \theta_1 + i\pi + \epsilon_1, \theta_1, \dots, \theta_n) = \prod_{i=1}^n \frac{1}{\epsilon_i} \cdot \sum_{i_1=1}^n \dots \sum_{i_n=1}^n \mathcal{A}_{i_1 \dots i_n}(\theta_1, \dots, \theta_n) \epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n} + \dots \quad (3.10)$$

where $\mathcal{A}_{i_1 \dots i_n}$ is a completely symmetric tensor of rank n in the indices i_1, \dots, i_n , and the ellipsis denote terms that vanish when taking $\epsilon_i \rightarrow 0$ simultaneously. The connected matrix element can be defined as the ϵ_i independent part of eqn. (3.10), i.e. the part which does not diverge whenever any of the ϵ_i is taken to zero:

$$F^c(\theta_1, \theta_2, \dots, \theta_n) = n! \mathcal{A}_{1 \dots n}(\theta_1, \dots, \theta_n) \quad (3.11)$$

where the appearance of the factor $n!$ is simply due to the permutations of the ϵ_i . The formula for the diagonal matrix element reads

$$\langle \{I_1 \dots I_n\} | \mathcal{O}(0) | \{I_1 \dots I_n\} \rangle_L = \frac{1}{\rho(\tilde{\theta}_1, \dots, \tilde{\theta}_n)} \sum_{A \subset \{1, 2, \dots, n\}} F^c(\{\tilde{\theta}_k\}_{k \in A}) \tilde{\rho}_{a_1 \dots a_n}(\tilde{\theta}_1, \dots, \tilde{\theta}_n | A) + O(e^{-\mu L}) \quad (3.12)$$

The summation runs over all subsets A of $\{1, 2, \dots, n\}$. For any such subset the appropriate sub-determinant

$$\tilde{\rho}(\tilde{\theta}_1, \dots, \tilde{\theta}_n | A) = \det \mathcal{J}_A(\tilde{\theta}_1, \dots, \tilde{\theta}_n)$$

of the $n \times n$ Bethe-Yang Jacobi matrix

$$\mathcal{J}(\tilde{\theta}_1, \dots, \tilde{\theta}_n)_{kl} = \frac{\partial Q_k(\theta_1, \dots, \theta_n)}{\partial \theta_l} \quad (3.13)$$

can be obtained by deleting the rows and columns corresponding to the subset of indices A . The determinant of the empty sub-matrix (i.e. when $A = \{1, 2, \dots, n\}$) is defined to equal 1 by convention.

Note that in contrast to (3.8) in (3.12) it is not necessary to take the absolute value, as phase redefinitions drop out from a diagonal matrix element.

Eqns. (3.10) and (3.12) are expected to give the finite volume matrix elements to all order in $1/L$, neglecting only finite size effects decaying exponentially with L [4, 5, 6].

4 Numerical verification

The finite volume energy levels and matrix elements can be evaluated using the boundary truncated conformal space approach (BTCSA) [24] which is an extension of the method developed by Yurov and Zamolodchikov [25] to boundary quantum field theories. For the sine-Gordon model the program used here was developed in [10, 11], to which the interested reader is referred for details.

To perform a specific check of the boundary form factor solution (3.5) we choose the operator (corresponding to $k = 1$)

$$\mathcal{O}_1 = e^{i\beta\phi(0,t)}$$

and evaluate its matrix elements between the BTCSA eigenvectors numerically. As in [6], all energies and matrix elements are measured in units of (appropriate powers of) the characteristic mass scale (here given by the soliton mass M), and we also use the dimensionless volume variable $l = ML$. States can be identified by matching them with the energy levels predicted by the Bethe-Yang equations, and the appropriate matrix elements can then be compared to the predictions of (3.8) and (3.12), obtained by substituting the exact form factor solution (2.6). The typical BTCSA cutoff value was 16, resulting in a truncated Hilbert space with several thousand states. Only a small, but representative sample of our numerical results are presented; we took care to verify our results for numerous different values of the model parameters ξ, η, ϑ .

4.1 Ground state energy and vacuum expectation value

Before embarking on the evaluation of the form factors, the accuracy of BTCSA data can be tested by extracting the bulk and boundary vacuum energy constants. For each bulk coupling ξ we considered a number of different boundary conditions. In all cases, on one side of the strip ($x = L$) the boundary condition β is a pure Neumann one

$$\eta_0 = \eta_N = \frac{\pi(1 + \xi)}{2\xi} \quad \vartheta_0 = 0$$

and varied the boundary condition α on the other side $x = 0$; from now on we only give the parameters η, ϑ of this other boundary condition α which is also where the boundary operator \mathcal{O}_1 is located for the matrix element calculations. As discussed in [10], the energy of the ground state is predicted to be

$$E_0(L) = \mathcal{B}M^2L + (\mathcal{B}_b(\eta_N, 0) + \mathcal{B}_b(\eta, \vartheta))M + O(e^{-\mu L})$$

where

$$\begin{aligned} \mathcal{B} &= -\frac{1}{4} \tan \frac{\pi\xi}{2} \\ \mathcal{B}_b(\eta, \vartheta) &= -\frac{1}{2 \cos \frac{\pi\xi}{2}} \left(\cos(\xi\eta) + \cosh(\xi\vartheta) - \frac{1}{2} \cos\left(\frac{\pi\xi}{2}\right) + \frac{1}{2} \sin\left(\frac{\pi\xi}{2}\right) - \frac{1}{2} \right) \end{aligned}$$

ξ	η/η_N	ϑ	\mathcal{B} (exact)	\mathcal{B} (BTCSA)	\mathcal{B}_b (exact)	\mathcal{B}_b (BTCSA)
50/391	0.5	0	-0.050904	-0.05009	-0.33307	-0.33238
50/311	0.7	0	-0.064512	-0.06350	-0.16625	-0.16802
50/239	0.9	0	-0.085246	-0.08355	0.04476	0.03796
50/311	0.5	0.2	-0.064512	-0.06030	-0.33289	-0.33204
50/391	0.7	0.5	-0.050904	-0.05039	-0.17695	-0.17767

Table 4.1: Bulk and boundary energy constants from BTCSA compared to the exact predictions, with $\mathcal{B}_b = \mathcal{B}_b(\eta_N, 0) + \mathcal{B}_b(\eta, \vartheta)$.

are the bulk and boundary energy constants in units of the soliton mass M . The comparison between the BTCSA results and the above exact predictions is illustrated in table 4.1.

It is also possible to test the exact vacuum expectation value against the BTCSA. The finite volume expectation value is expected to behave as

$$\langle 0|e^{i\beta\phi(0,t)}|0\rangle_L = \mathcal{G}_1 + O(e^{-\mu L})$$

where the exact vacuum expectation value

$$\mathcal{G}_1 = \langle 0|e^{i\beta\phi(0,t)}|0\rangle$$

can be evaluated from eqn. (3.2). This is illustrated in figure 4.1; the deviations at large volume ($l \gtrsim 5$) are due to truncation effects. Note that agreement is better for smaller ξ , which agrees with the general trend observed from all the spectral data (masses, energy levels, bulk and boundary energy constants) that (B)TCSA converges better if the perturbing operator is more relevant (i.e. its conformal dimension is smaller)¹. It is also important to notice that the derivative of the boundary energy with respect the coupling constant μ_B is the expectation value of the boundary perturbation [11], and so the boundary parameter dependence of these expectation values have already been implicitly checked by the data in table 4.1.

4.2 Level identification

In order to evaluate matrix elements it is necessary to identify the finite volume levels. As illustrated in figure 4.2, this is performed by matching predictions from the Bethe-Yang equations (3.7) to the BTCSA spectrum. It is apparent that this is not an easy task, in fact, a magnitude harder than in the case of bulk theories. In contrast to the case of periodic boundary conditions [26], there is no conserved momentum, and so the Hilbert space cannot be split into sectors on the basis of momentum. As a result, the continuum (in infinite volume limit) starts at the one-particle threshold, as opposed to the bulk case. Since topological charge is not conserved either, another way of reducing dimensionality is

¹It also happens to be the case that for a given value of ξ convergence is better for smaller values of the boundary parameter η .

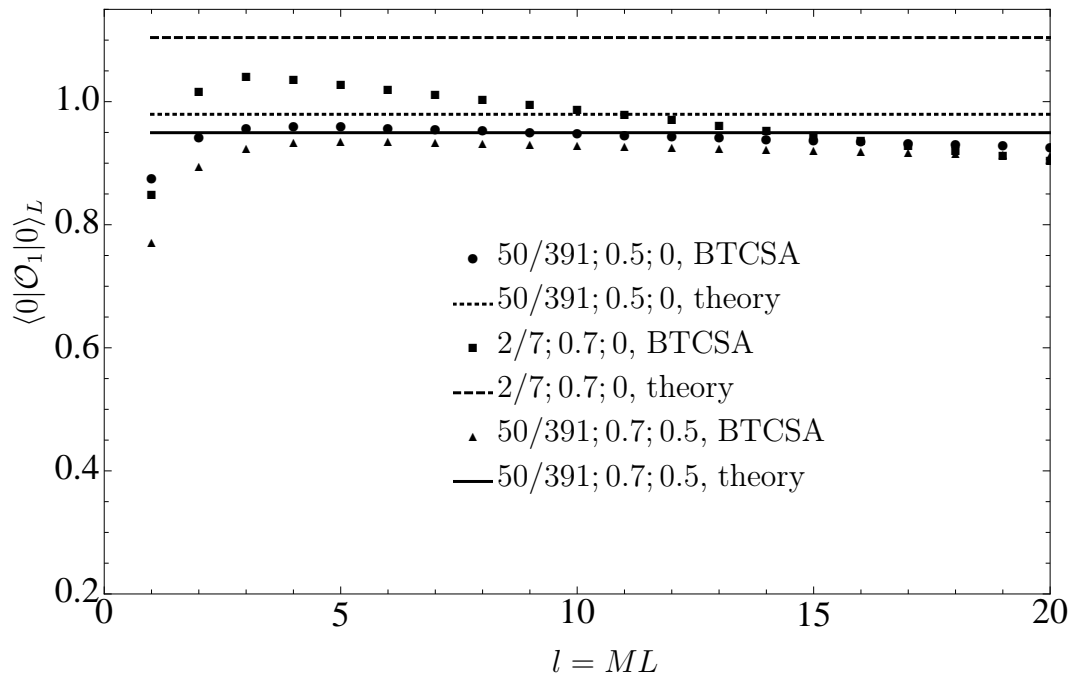


Figure 4.1: Vacuum expectation values: a comparison between the exact prediction and BTCSA. The model parameters are listed as $\xi; \eta/\eta_N; \vartheta$. All values are in units of the soliton mass M . The straight lines are the infinite volume vacuum expectation values from eqn. (3.2), the discrete dots are the BTCSA data.

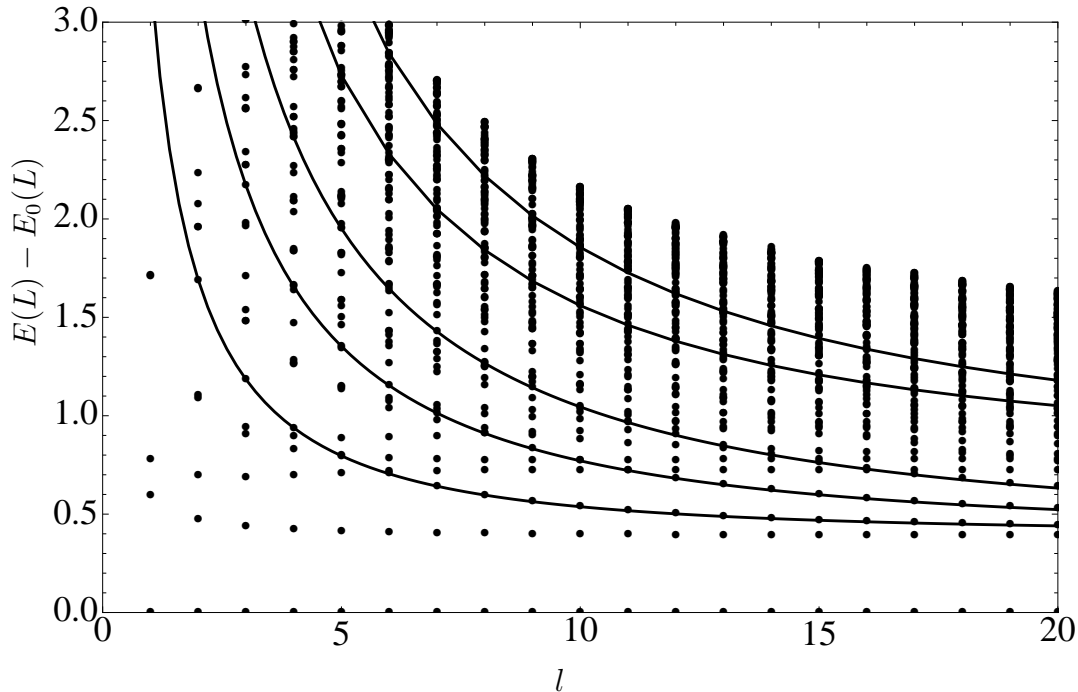


Figure 4.2: Level identification for $\xi = 50/391, \eta = 0.5\eta_N, \vartheta = 0$. The dots are the BTCSA energy levels (with the vacuum subtracted), the continuous lines are Bethe-Yang predictions for three one-particle levels $|\{1\}\rangle, |\{2\}\rangle, |\{3\}\rangle$ and two two-particle levels $|\{1, 2\}\rangle$ and $|\{1, 3\}\rangle$.

also lost, resulting in an extremely dense spectrum. Additional complexity arises from the complicated particle spectrum and the presence of boundary bound states [27, 20] (in fact, the lowest excited level in figure 4.2 is precisely such a state and the next two are also clearly visible in the spectrum). As a result, in contrast to the bulk theories investigated in [4, 5] there is no way of reliably identifying states with more than two particles. It is also apparent from the figure that the two-particle states are already in a very dense part of the spectrum, and their identification is made harder by the numerous level crossings characteristic of integrable models. At certain values of the volume L there can be more than one BTCSA candidates for a given Bethe-Yang solution; identification can be completed by selecting the candidate on the basis of one of the form factor measurements, which still leaves other matrix elements involving the state as cross-checks.

Level crossings also present a problem in numerical stability, since in their vicinity the state of interest is nearly degenerate to another one. Since the truncation effect can be considered as an additional perturbing operator, the level crossings are eventually lifted. However, such a near degeneracy greatly magnifies truncation effects on the eigenvectors and therefore the matrix elements [6].

4.3 The one-particle form factor

One of the most important ingredients of the boundary form factor bootstrap is the minimal boundary form factor (2.8) which can be obtained via measuring the one-particle matrix elements

$$\begin{aligned} F_1(\theta) &= \langle 0 | e^{i\beta\phi(0,t)} | B_1(\theta) \rangle \\ &= \mathcal{G}_1 H_1 r(\theta) \end{aligned}$$

(note that the polynomial $Q_1(y) = P_1(y) = [k]y = y$ for $k = 1$). Eqn. (3.8) can be used to express

$$\left| F_1(\tilde{\theta}) \right| = \sqrt{\rho_1(\tilde{\theta})} |\langle 0 | \mathcal{O}(0) | \{I\} \rangle_L| + O(e^{-\mu L})$$

where $\tilde{\theta}$ solves the Bethe-Yang equation

$$Q_1(\theta) = 2m_1 L \sinh \theta + \delta^{(\alpha)}(\theta) + \delta^{(\beta)}(\theta) = 2\pi I \quad (4.1)$$

and

$$\rho_1(\tilde{\theta}) = Q_1'(\tilde{\theta}) \quad (4.2)$$

Therefore it is possible to compare the values extracted from several different one-particle lines (distinguished by the value of I) on the same plot. The numerics and the theoretical prediction are in excellent agreement, as shown in figure 4.3. Note that the numerics deviate from the prediction for small θ (large L) due to truncation errors, while for large θ (small L) the exponential corrections show up. The advantage of lower I states is that the scaling regime corresponds to the low- θ domain, while higher I states are useful in scanning the large- θ behaviour of the form factor function.

4.4 Two-particle form factors

For the two-particle form factor, three independent tests can be performed:

1. Vacuum–two-particle matrix element:

$$\left| \langle 0 | e^{i\beta\phi(0,t)} | \{I_1, I_2\} \rangle_L \right| = \frac{|F_{11}(\tilde{\theta}_1, \tilde{\theta}_2)|}{\sqrt{\rho_{11}(\tilde{\theta}_1, \tilde{\theta}_2)}} + O(e^{-\mu L})$$

where $\tilde{\theta}_1, \tilde{\theta}_2$ are solutions of the two-particle Bethe-Yang equations, ρ_{11} is the corresponding Bethe-Yang Jacobian, F_{11} is given by the $n = 2$ case of (3.5), and the exact \mathcal{G} is inserted for proper normalization of the operator (figure 4.4).

2. Non-diagonal one-particle–one-particle matrix element:

$$\left| \langle \{I'\} | e^{i\beta\phi(0,t)} | \{I\} \rangle_L \right| = \frac{|F_{11}(i\pi + \tilde{\theta}', \tilde{\theta})|}{\sqrt{\rho_1(\tilde{\theta}')\rho_1(\tilde{\theta})}} + O(e^{-\mu L})$$

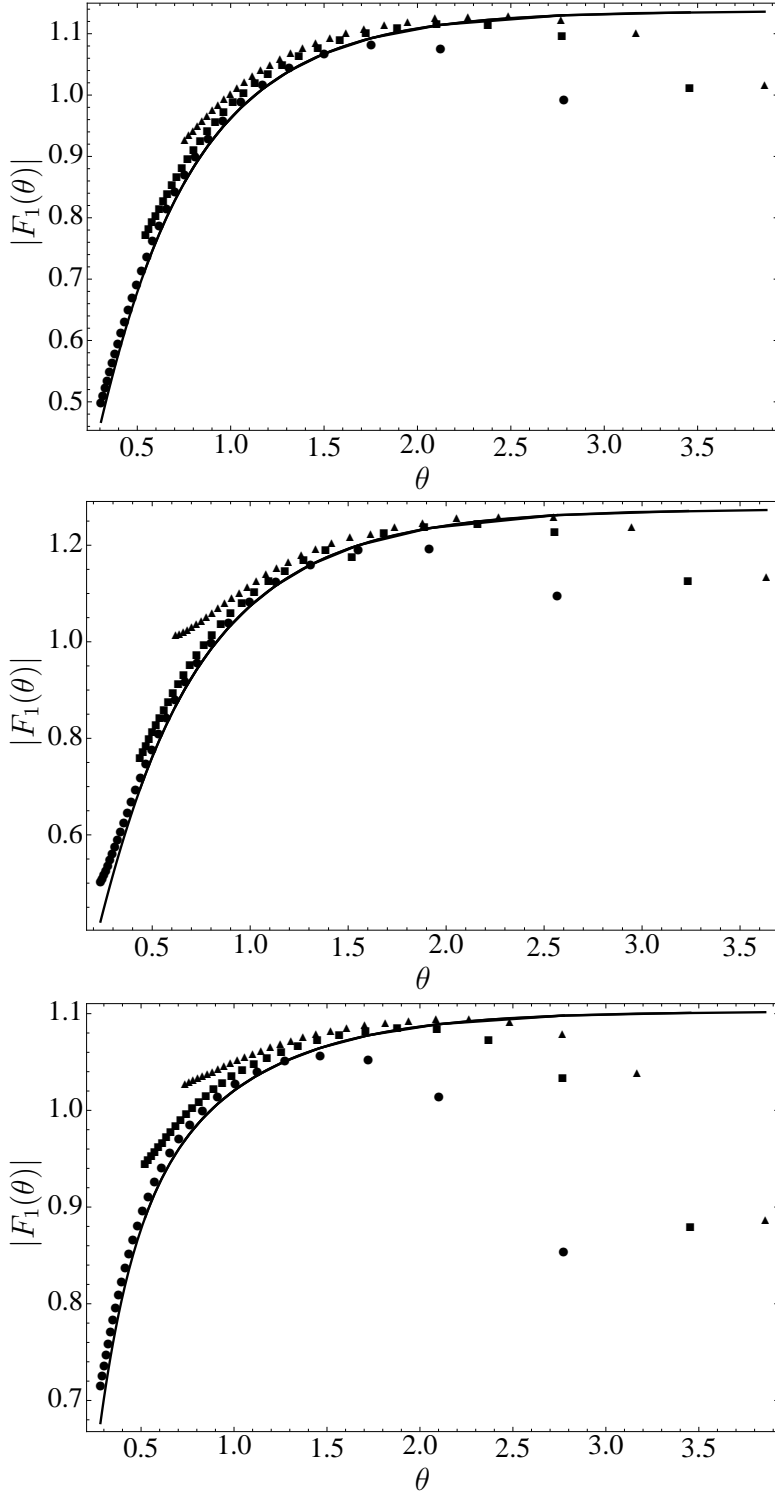


Figure 4.3: One-particle form factors extracted from BTCSA and compared to the bootstrap prediction. Continuous lines are the bootstrap predictions, while the circles, squares and triangles show data extracted using $I = 1, 2, 3$ one-particle lines respectively. The parameters $(\xi, \eta/\eta_N, \vartheta)$ for the three figures are: $(50/391, 0.5, 0)$, $(50/311, 0.5, 0.2)$ and $(50/391, 0.7, 0.5)$.

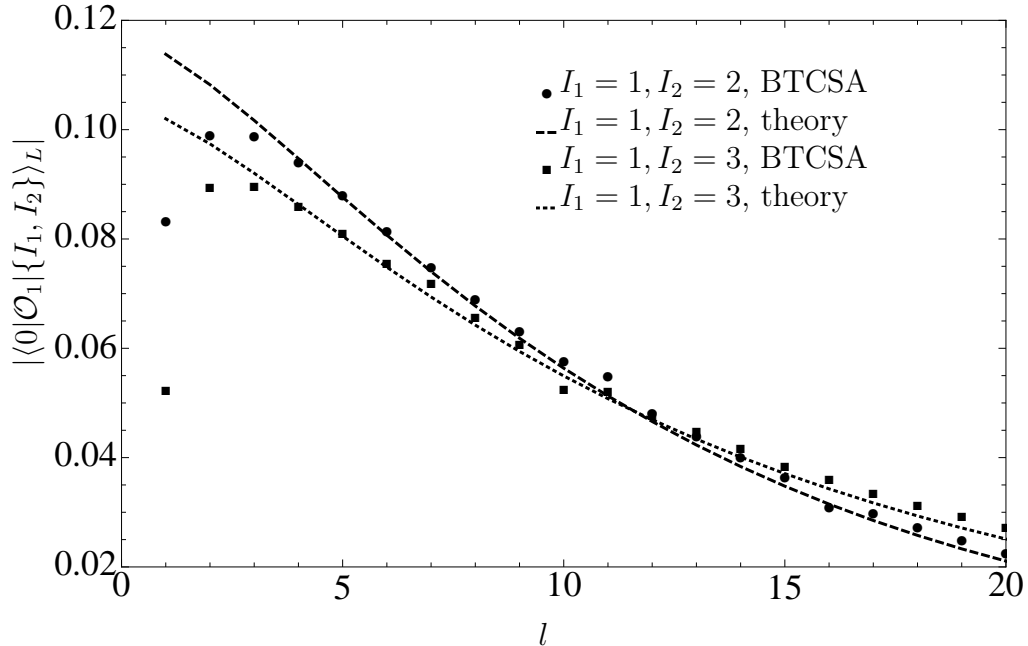


Figure 4.4: Vacuum-two-particle matrix elements for two different two-particle states, at $\xi = 50/311, \eta = 0.7\eta_N, \vartheta = 0$.

where $\tilde{\theta}$ and $\tilde{\theta}'$ are solutions of the one-particle Bethe-Yang equations (4.1) with quantum numbers I and I' and ρ_1 is the one-particle Bethe-Yang Jacobian (4.2) (figure 4.5).

3. Diagonal one-particle–one-particle matrix element ($I = I'$ case):

$$\langle \{I\} | e^{i\beta\phi(0,t)} | \{I\} \rangle_L = \frac{F_{11}(i\pi + \tilde{\theta}, \tilde{\theta})}{\rho_1(\tilde{\theta})} + \mathcal{G}_1 + O(e^{-\mu L})$$

(figure 4.6).

The first two tests show excellent agreement², while the third one reveals a striking discrepancy. Since this disagreement is only seen in diagonal matrix elements, we think that the exact form factors are correct, and the issue is with the finite size corrections in diagonal matrix elements. A detailed discussion is given in the conclusions.

²Even so, some points are slightly displaced. As discussed in subsection 4.2, this is due to the occurrence of line crossings at the particular value of the volume, where identification of the state becomes more difficult and BTCSA cutoff errors are also magnified.

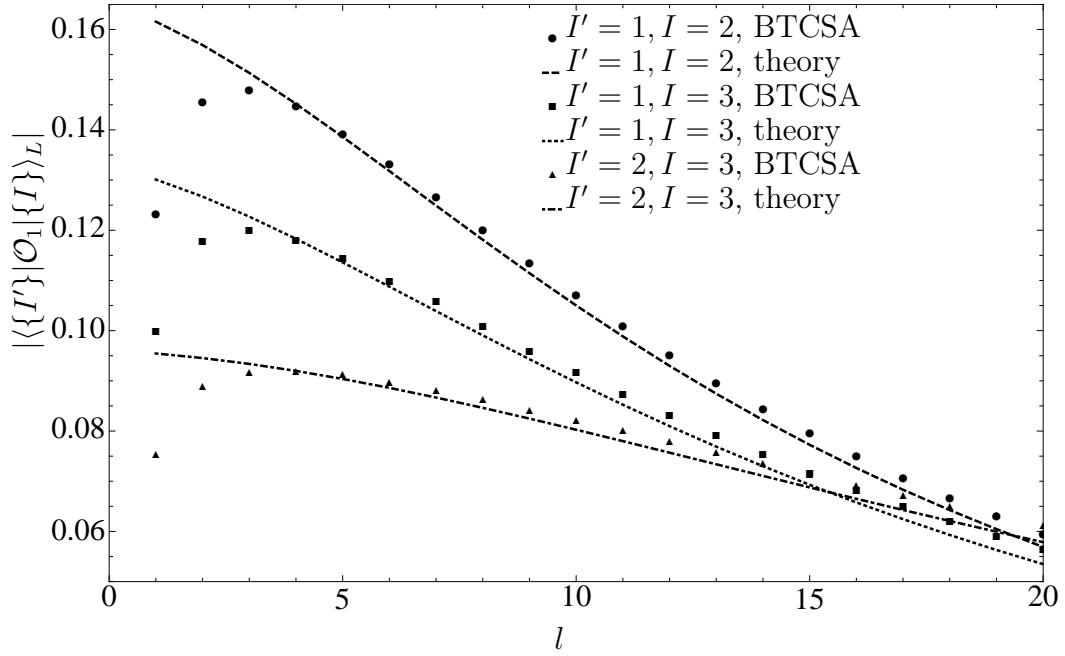


Figure 4.5: Non-diagonal one-particle–one-particle matrix elements for three different choices of the one-particle states, at $\xi = 50/391, \eta = 0.7\eta_N, \vartheta = 0.5$.

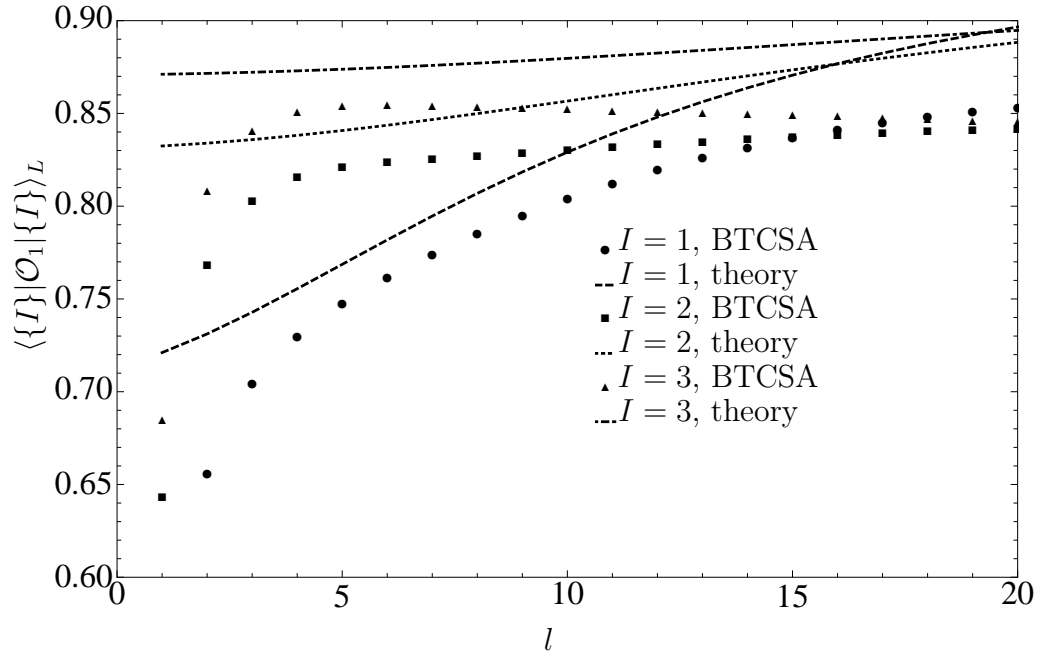


Figure 4.6: Diagonal one-particle–one-particle matrix elements for three different choices of the one-particle state, at $\xi = 50/391, \eta = 0.7\eta_N, \vartheta = 0.5$.

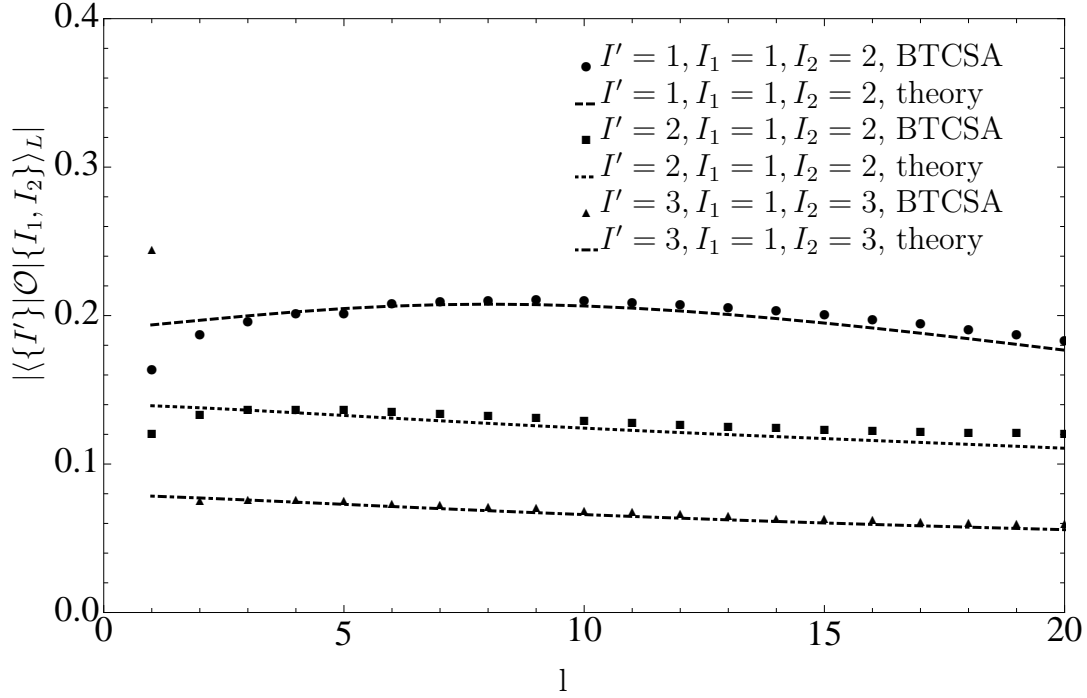


Figure 4.7: One-particle–two-particle matrix elements for $\xi = 50/391$, $\eta = 0.5\eta_N$, $\vartheta = 0$.

4.5 Higher form factors

The identified one- and two-particle states can be used to obtain tests of the three- and four-particle form factors from (3.5), according to the formulae

$$|\langle\{I'\}|e^{i\beta\phi(0,t)}|\{I_1, I_2\}\rangle_L| = \frac{|F_{111}(i\pi + \tilde{\theta}', \tilde{\theta}_1, \tilde{\theta}_2)|}{\sqrt{\rho_1(\tilde{\theta}')\rho_{11}(\tilde{\theta}_1, \tilde{\theta}_2)}} + O(e^{-\mu L})$$

and

$$|\langle\{I'_1, I'_2\}|e^{i\beta\phi(0,t)}|\{I_1, I_2\}\rangle_L| = \frac{|F_{1111}(i\pi + \tilde{\theta}'_2, i\pi + \tilde{\theta}'_1, \tilde{\theta}_1, \tilde{\theta}_2)|}{\sqrt{\rho_{11}(\tilde{\theta}'_1, \tilde{\theta}'_2)\rho_{11}(\tilde{\theta}_1, \tilde{\theta}_2)}} + O(e^{-\mu L})$$

(the latter is only valid for the non-diagonal case). These tests, illustrated by figures 4.7 and 4.8, also show a good agreement between numerical results and theoretical expectations. Using (3.12) diagonal two-particle–two-particle matrix elements can also be constructed, and they reveal the same disagreement as the one-particle ones in the previous subsection.

5 Conclusions

We have performed an extensive test of the form factors conjectured in [9] by comparing them to results obtained using the boundary truncated conformal space approach. For gen-

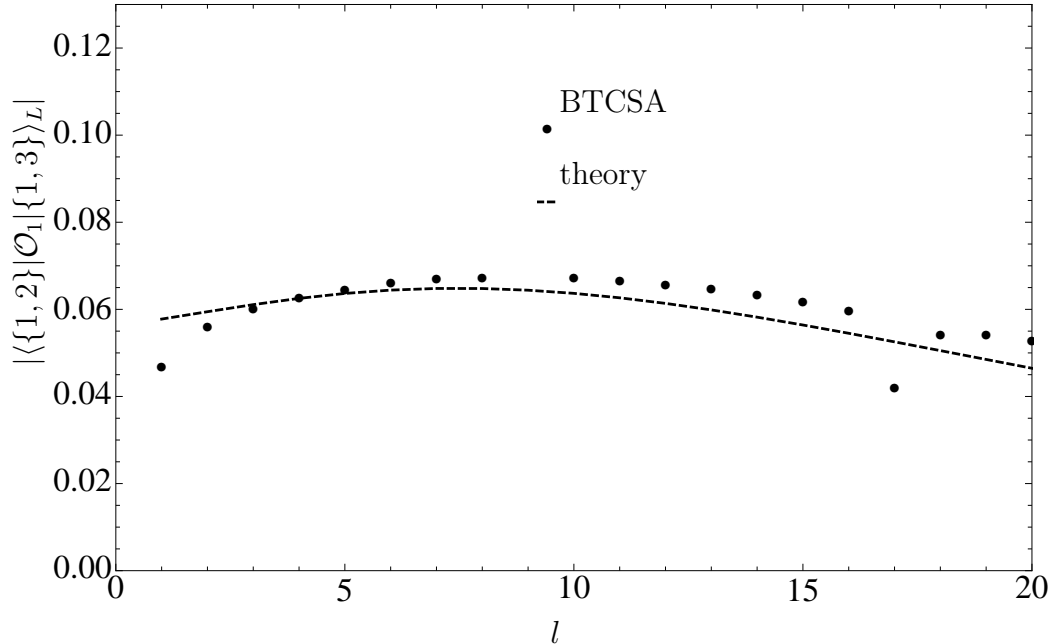


Figure 4.8: Off-diagonal two-particle–two-particle matrix elements for $\xi = 50/311, \eta = 0.5\eta_N, \vartheta = 0.2$.

eral matrix elements an excellent agreement was found. However, for the case of diagonal matrix elements theoretical expectations and numerical results clearly disagree.

The first point to make that the conjectured form factors are probably correct, as they survived all the tests with the exception of diagonal matrix elements. Our methods probed them up to four particles, at which level all of the ingredients of the form factor bootstrap (minimal form factors, recursion relations from bulk and boundary poles) are already heavily involved. Therefore we do not expect the problem to be related to the bootstrap.

The finite size description, on the other hand, have only been obtained for diagonal scattering theories and the formula (3.12) for the diagonal matrix elements is only an educated guess based on the bulk case. One issue is that sine-Gordon model is a theory with a non-diagonal scattering; however, the breather scattering is diagonal, and therefore it is unlikely that this is the cause of the discrepancy. The other issue is the validity of the conjectured form of disconnected terms for the diagonal case; however, this was thoroughly tested both in the bulk and the boundary cases [5, 6] and there is no reason to expect any modification for sine-Gordon model.

The most likely reason for the disagreement is the presence of so-called μ -terms which can be attributed to the fact that the first breather is a soliton-antisoliton bound states. Such effects were observed in [4, 5] and their detailed description was given in [28]. It is a very interesting fact that similar effects were found for bulk breather form factors (but not for solitonic ones!) in sine-Gordon theory with periodic boundary conditions [29], which

strengthens our suspicions about their origin. However, detailed investigation of this issue requires a knowledge of solitonic form factors of the exponential operators, which have not yet been constructed. We intend to investigate this issue further in the near future.

Acknowledgments

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A Analytic continuation of minimal form factors

The integral representations for the bulk (2.2) and boundary (2.8) minimal form factors only converge in a suitable strip around the real axis. To evaluate them at complex values with imaginary parts as large as π , as required in (3.8) and (3.12) they must be continued analytically by separating pole factors that fall inside the strip of interest. This can be accomplished using the identity

$$\begin{aligned} \mathcal{V}(\theta; a) &= \exp \left[\int_0^\infty \frac{dt}{t} \left(\frac{a}{2 \sinh \frac{t}{2}} - \frac{\cos \left(\frac{\theta t}{\pi} \right) \sinh at}{\sinh^2 t} \right) \right] \\ &= \prod_{k=1}^N \left(\frac{\pi^2 (2k - a)^2 + \theta^2}{\pi^2 (2k + a)^2 + \theta^2} \right) \\ &\quad \times \exp \left[\int_0^\infty \frac{dt}{t} \left(\frac{a}{2 \sinh \frac{t}{2}} - (N + 1 - Ne^{-2t}) e^{-2Nt} \frac{\cos \left(\frac{\theta t}{\pi} \right) \sinh at}{\sinh^2 t} \right) \right] \end{aligned}$$

where the natural number N is a regulatory parameter such that the value of the functions are independent of N , but the width of the convergence strip of the integral part grows with increasing N .

The bulk minimal form factor can be expressed as

$$f(\theta) = \mathcal{V}(\theta - i\pi; B/2 - 1) \mathcal{V}(\theta - i\pi; -B/2) \mathcal{V}(\theta - i\pi; 1) \quad (1.1)$$

and for the boundary minimal form factor function $u(\theta)$ (2.9) one can use

$$u(\theta) = \mathcal{U}(\theta; B/4) \mathcal{U}(\theta; 1/2 - B/4) \mathcal{U}(\theta; 1/2) \quad (1.2)$$

where

$$\begin{aligned} \mathcal{U}(\theta; a) &= \mathcal{V}(\theta; a) \mathcal{V}(\theta - i\pi; a) \\ &= \exp \left[\int_0^\infty \frac{dt}{t} \left(\frac{a}{\sinh \frac{t}{2}} - \frac{2 \cosh \frac{t}{2} \cos \left(\left(i\frac{\pi}{2} - \theta \right) \frac{t}{\pi} \right) \sinh at}{\sinh^2 t} \right) \right] \end{aligned}$$

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