Trace formula for noise corrections to trace formulas

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We consider an evolution operator for a discrete Langevin equation with a strongly hyperbolic classical dynamics and Gaussian noise. Using an integral representation of the evolution operator $\mathcal{L}$ we investigate the high order corrections to the trace of $\mathcal{L}^n$. The asymptotic behaviour is found to be controlled by sub-dominant saddle points previously neglected in the perturbative expansion. We show that a trace formula can be derived to describe the high order noise corrections.

In the statistical theory of dynamical systems the development of the densities of particles is governed by a corresponding evolution operator. For a repeller, the leading eigenvalue of this operator $\mathcal{L}$ yields a physically measurable property of the dynamical system, the escape rate from the repeller. In the case of deterministic flows, the periodic orbit theory [1,2] yields explicit and numerically efficient formulas for the spectrum of $\mathcal{L}$ as zeros of its spectral determinant [3].

On all dynamical evolutions in nature stochastic processes of various strength have an influence. In a series of papers [4–7] the effects of noise on measurable properties such as dynamical averages in classical chaotic dynamical systems were systematically accounted. The theory developed is closely related to the semi-classical $\hbar$ expansions [8–10] based on Gutzwiller's formula for the trace in terms of classical periodic orbits [11] in that both are perturbative theories in the noise strength or $\hbar$, derived from saddle-point expansions of a path integral containing a dense set of unstable stationary points. The analogy with quantum mechanics and field theory is made explicit in [4] where Feynman diagrams are used to find the lowest nontrivial noise corrections to the escape rate.

In [6] we developed an explicit matrix representation of the stochastic evolution operator. The numerical implementation made it possible to reach up to order eight in expansion order, and the corrections to the escape rate were found to be a divergent series in the noise expansion parameter. This reflects that the corrections were calculated (using the so called cumulant expansion) from other divergent quantities, the traces of the evolution operator $\mathcal{L}^n$ [6].

In [7] the focus was on the high order noise corrections for the special case of the first trace, $\mathrm{Tr}\mathcal{L}$. The asymptotics of the trace of the evolution operator were governed by sub-dominant saddles previously neglected in the expansion.

In this paper we show that the high order noise corrections of $\mathrm{Tr}\mathcal{L}^n$ are also dominated by sub-dominant saddles. These sub-dominant saddles can be treated as generalised periodic orbits of the system and we associate them with periodic orbits of corresponding discrete Newtonian equations of motion. Our key result is (40) where the high order noise corrections are converted into a trace formula. We give as a numerical example the quartic map considered in [4–7].

First we introduce the noisy repeller and its evolution operator. An individual trajectory in presence of additive noise is generated by iterating

$$x_{n+1} = f(x_n) + \sigma\xi_n\,, \tag{1}$$

where $f(x)$ is a map, ξ_n a random variable with the normalised distribution $p(\xi)$, and σ parametrises the noise strength. In what follows we shall assume that the mapping $f(x)$ is one-dimensional and expanding, and that the ξ_n are uncorrelated. A density of trajectories $\phi(x)$ evolves with time on the average as

$$\phi_{n+1}(y) = (\mathcal{L}\circ\phi_n)(y) = \int dx\,\mathcal{L}(y,x)\phi_n(x) \tag{2}$$

where the $\mathcal{L}$ evolution operator has the general form

$$\begin{aligned}\mathcal{L}(y,x) &= \delta_\sigma(y-f(x)),\\ \delta_\sigma(x) &= \int \delta(x-\sigma\xi)p(\xi)d\xi \;=\; \frac{1}{\sigma}p\left(\frac{x}{\sigma}\right).\end{aligned} \tag{3}$$

For the calculations in this paper Gaussian weak noise is assumed. In the perturbative limit, $\sigma\to 0$, the evolution operator becomes

$$\mathcal{L}(x,y) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(y-f(x))^2}{2\sigma^2}}. \tag{4}$$

The map considered here is the same as in our previous papers, a quartic map on the $(0,1)$ interval given by

$$f(x) = 20\left[\frac{1}{16} - \left(\frac{1}{2}-x\right)^4\right]. \tag{5}$$

Throughout the theory developed in previous works [4–7] , the periodic orbits of the system played a major role. A periodic orbit of length n was defined simply by

$$x_{j+1} = f(x_j), \qquad j = 1, ..., n \tag{6}$$
$$x_{n+1} = x_1. \tag{7}$$

For a repeller the leading eigenvalue of the evolution operator yields a physically measurable property of the dynamical system, the escape rate from the repeller. In the case of deterministic flows, the periodic orbit theory yields explicit formulas for the spectrum of $\mathcal{L}$ as zeros of its spectral determinant [3]. One of the most important goals of the theory related to stochastic evolution operators is to explore the dependence of the eigenvalues ν of $\mathcal{L}$ on the noise strength parameter σ. The eigenvalues are determined by the eigenvalue condition

$$F(\sigma, \nu(\sigma)) = \det(1 - \mathcal{L}/\nu(\sigma)) = 0 \tag{8}$$

where $F(\sigma, 1/z) = \det(1 - z\mathcal{L})$ is the spectral determinant of the evolution operator $\mathcal{L}$, which can be expressed as

$$\det(1 - z\mathcal{L}) = \exp\left(-\sum_{n}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{L}^n\right). \tag{9}$$

Equation (9) shows that noise dependence of the eigenvalues of the evolution operator are very closely related to the noise dependence of the trace of $\mathcal{L}^n$, which shall be the object of study from now on.

The trace of $\mathcal{L}^n$ can be expressed as

$$\mathrm{Tr}\mathcal{L}^n = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int dx_1 dx_2 ... dx_n \mathrm{e}^{-\frac{S}{\sigma^2}}, \tag{10}$$

where

$$S = \frac{1}{2} \sum_{j=1}^{n} (x_{j+1} - f(x_j))^2, \tag{11}$$
$$x_{n+1} = x_1. \tag{12}$$

In order to give a deeper insight on the forthcoming calculations we draw a correspondence between discrete Hamiltonian mechanics and our system, with the S defined above playing the role of the classical action. According to (11), the least action principle requires

$$x_j - f(x_{j-1}) - f'(x_j)(x_{j+1} - f(x_j)) = 0. \tag{13}$$

We define

$$p_j := x_j - f(x_{j-1}), \tag{14}$$

the quantity corresponding to the momentum in the classical mechanics. From (13) we obtain

$$x_{j+1} = f(x_j) + p_{j+1}, \tag{15}$$
$$p_{j+1} = \frac{p_j}{f'(x_j)}, \tag{16}$$

which are the equations corresponding to the classical Newtonian equations of motion. The generalised periodic orbits of length n are those orbits, which obey these equations and $x_{n+1} = x_1$, $p_{n+1} = p_n$. Those generalised periodic orbits which have non-zero momentum will control the asymptotic behaviour of the corrections to $\mathrm{Tr}\mathcal{L}^n$ as we shall demonstrate later. The original periodic orbits defined by (6),(7) are those with zero momentum. The generalised periodic orbits with non-zero momentum and the original periodic orbits proliferate with growing n as suggested by Fig 1.

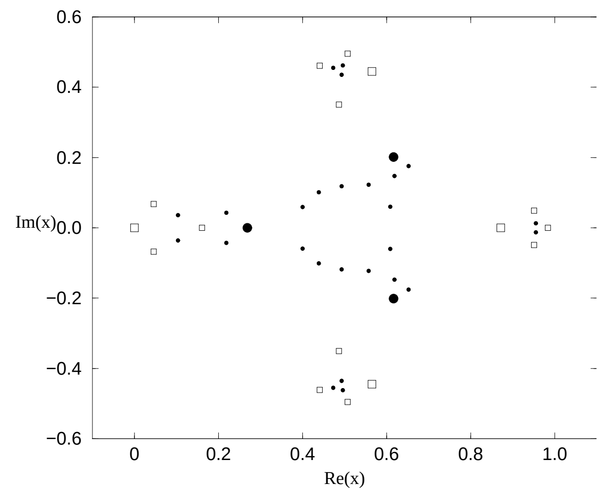

FIG. 1. The sets of original and generalised periodic orbits. Squares indicate original periodic orbits, dots indicate generalised periodic orbits, large symbols indicate orbits of length one, small symbols indicate orbits of length two.

We introduce an integral representation of the noisy kernel, which will be of great use in the later calculations:

$$\mathcal{L}(x, y) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(y - f(x))^2}{2\sigma^2}} = \frac{1}{2\pi} \int dk \mathrm{e}^{-\frac{\sigma^2 k^2}{2} + ik(y - f(x))}. \tag{17}$$

Using this new integral representation,

$$\mathrm{Tr}\mathcal{L}^n = \frac{1}{(2\pi)^n} \int dk^n dx^n \mathrm{e}^{-\frac{\sigma^2}{2}\sum_{j=1}^n k_j^2 + i\sum_{j=1}^n k_j(x_{j+1} - f(x_j))}, \tag{18}$$

or equivalently

$$\mathrm{Tr}\mathcal{L}^n = \frac{1}{(2\pi)^n} \int dk^n \int dp^n J(p) \mathrm{e}^{-\frac{\sigma^2}{2}\sum_{j=1}^n k_j^2 + i\sum_{j=1}^n k_j p_j}, \tag{19}$$

where $J(p) = D(x)/D(p)$ denotes the Jacobian. Since

$$\frac{1}{(2\pi)^n} \int dk^n \mathrm{e}^{i\sum_{j=1}^n k_j p_j} = \prod_{j=1}^{n} \delta(p_j), \tag{20}$$

we can reduce (19) to

$$\mathrm{Tr}\mathcal{L}^n = \int dp^n J(p) \mathrm{e}^{\frac{\sigma^2}{2}\Delta} \prod_{j=1}^{n} \delta(p_j) =$$
$$\mathrm{e}^{\frac{\sigma^2}{2}\Delta} J(p)\Big|_{p_j=0}, \qquad (21)$$

where Δ denotes the Laplacian

$$\Delta = \frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + ... + \frac{\partial^2}{\partial_n^2}. \qquad (22)$$

Our object of study is the Taylor expansion of (21) in the noise parameter:

$$\mathrm{Tr}\mathcal{L}^n = \sum_{N=0}^{\infty} \left(\mathrm{Tr}\mathcal{L}^n\right)_N \sigma^{2N}, \qquad (23)$$

$$\left(\mathrm{Tr}\mathcal{L}^n\right)_N = \frac{1}{2^N} \frac{\Delta^N}{N!} J(p)\Big|_{p_j=0}. \qquad (24)$$

The N-th power of the Laplacian in the equation above can be written as

$$\Delta^N = \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{N!}{j_1!...j_n!} \frac{\partial^{2j_1}}{\partial p_1^{2j_1}} ... \frac{\partial^{2j_n}}{\partial p_n^{2j_n}} \delta_{N,\sum_{k=1}^n j_k}, \qquad (25)$$

where δ_{jl} is the Kronecker-delta. With the help of the multidimensional residue formula from complex calculus [12]

$$\frac{\partial^{n_1+\ldots+n_k} f(z)}{\partial z_1^{n_1}...\partial z_k^{n_k}} =$$
$$\frac{n_1!...n_k!}{(2\pi i)^k} \oint_{c_1} ... \oint_{c_k} \frac{f(\xi) d\xi_1...d\xi_k}{(\xi_1 - z_1)^{n_1+1}...(\xi_k - z_k)^{n_k+1}}, \qquad (26)$$

we obtain

$$\left(\mathrm{Tr}\mathcal{L}^n\right)_N = \frac{1}{(2\pi i)^n 2^N} \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{(2j_1)!}{j_1!} ... \frac{(2j_n))!}{j_n!}$$
$$\times \delta_{N,\sum_{k=1}^n j_k} \oint_{c_1} ... \oint_{c_n} \frac{J(p) dp_1...dp_n}{p_1^{2j_1+1}...p_n^{2j_n+1}}. \qquad (27)$$

The contours are around the $p_j = 0$ points. The integrals can be transformed back to contour integrals in the original x_j variables, and the contours will be placed around the original periodic orbits of the system defined by $(6-7)$, since it is these orbits which fulfil the $p_j = 0$ conditions. From now on we shall restrict our calculations to the asymptotic large N limit. We will replace the summations in (27) by integrals and then use the saddle-point method to get a compact formula for $(\mathrm{Tr}\mathcal{L}^n)_N$. We approximate the factorials via the Stirling-formula [13] as

$$\frac{(2j_k)!}{j_k!} \simeq \frac{\left(\frac{2j_k}{e}\right)^{2j_k} \sqrt{4\pi j_k}}{\left(\frac{j_k}{e}\right)^{j_k} \sqrt{2\pi j_k}} = 2^{2j_k+1/2} j_k^{j_k} \mathrm{e}^{-j_k} =$$
$$2^{1/2} \mathrm{e}^{2(\ln 2)j_k + j_k \ln j_k - j_k}. \qquad (28)$$

Using (28) and an integral representation of the delta function we get

$$\left(\mathrm{Tr}\mathcal{L}^n\right)_N \simeq \frac{2^{\frac{n}{2}-N}}{(2\pi i)^n 2\pi} \sum_{j_1,\ldots,j_n=0}^{\infty} \int dt \oint_{c_1} ... \oint_{c_n} dx_1...dx_n$$
$$\times \exp\left[it(N - \sum_{k=1}^n j_k) + (2\ln 2 - 1)\sum_{k=1}^n j_k + \sum_{k=1}^n j_k \ln j_k \right.$$
$$\left. + \sum_{k=1}^n \ln(x_k - f(x_{k-1}))(2j_k + 1) \right]. \qquad (29)$$

Now we replace j_k with the new variables $y_k = \frac{j_k}{N}$ and in the asymptotic (N large) limit approximate the summations by y_k with integrals by y_k as

$$\left(\mathrm{Tr}\mathcal{L}^n\right)_N \simeq$$
$$\frac{2^{\frac{n}{2}-N} N^n}{(2\pi i)^n 2\pi} \int_0^\infty dy_1 ... \int_0^\infty dy_n \int dt \oint_{c_1} ... \oint_{c_n} dx_1...dx_n$$
$$\times \exp\left[it(N - N\sum_{k=1}^n y_k) + N(2\ln 2 - 1)\sum_{k=1}^n y_k \right.$$
$$\left. + N\sum_{k=1}^n y_k \ln(N y_k) + \sum_{k=1}^n \ln(x_k - f(x_{k-1}))(2N y_k + 1) \right].$$
$$(30)$$

We evaluate the y integrals with the saddle point method to get

$$\left(\mathrm{Tr}\mathcal{L}^n\right)_N \simeq \frac{2^{-N+\frac{n}{2}}}{(2\pi)^{\frac{n}{2}} i^n 2\pi} \int dt \oint_{c_1} ... \oint_{c_n} dx_1...dx_n$$
$$\exp\left[it\left(N + \frac{n}{2}\right) - \mathrm{e}^{it} \frac{S}{2} \right]. \qquad (31)$$

Next we implement the saddle point method to the integral in t as well, asymptotically resulting in

$$\left(\mathrm{Tr}\mathcal{L}^n\right)_N \simeq \frac{N^{\frac{n-1}{2}}}{2^{2N+\frac{1}{2}} (2\pi)^{\frac{n+1}{2}} i^{n+1}} \frac{(2N)!}{N!}$$
$$\times \int dx^n \mathrm{e}^{-\left(N+\frac{n}{2}\right)\log(S)}. \qquad (32)$$

The last step is to evaluate the contour integrals in the x_k variables. We deform the contours, until the saddle points are reached so the contours run along the routes of the steepest descent. The leading contribution comes from the saddle points, which fulfil the following equation

$$\frac{1}{S}\left[x_j^* - f(x_{j-1}^*) - (x_{j+1}^* - f(x_j^*)) f'(x_j^*) \right] = 0. \qquad (33)$$

By comparing (33) and (13) one can see that the saddle-points are all generalised periodic orbits of the system. Since the contours ran originally around the orbits with zero momentum, these do not come into account as saddle points. The second derivative matrix is

$$-\left(N+\frac{n}{2}\right)\frac{1}{S}D^2S, \qquad (34)$$

where D^2S denotes the second derivative matrix of S

$$(D^2S)_{ij} = \frac{\partial^2 S}{\partial x_i \partial x_j}. \qquad (35)$$

This would be the matrix to deal with if we would have taken the saddle point approximation of (10) directly. We reorganise the prefactor in (32) with the use of the Stirling formula [13] and the result of the saddle point integration is written as

$$(\mathrm{Tr}\mathcal{L}^n)_N \simeq \sum_{s.p.} \frac{N^{\frac{n-1}{2}}}{2\pi i} \frac{\Gamma(N+\frac{1}{2})}{\left(N+\frac{n}{2}\right)^{\frac{n}{2}}} \frac{S_p^{-N}}{\sqrt{\det D^2 S_p}}, \qquad (36)$$

which is our main result. For $n = 1$ this formula gives back the result of [7] as it should.

Finally we draw the attention to the close connection between the generalised periodic orbits of the system and D^2S. The stability matrix of a general periodic orbit is expressed as

$$J = J_1 \cdot J_2 \cdot J_3 ... \cdot J_n \qquad (37)$$

$$J_k = \begin{pmatrix} f'(x_k) - \frac{p_k}{(f'(x_k))^2} f''(x_k) & \frac{1}{f'(x_k)} \\ -\frac{p_k}{(f'(x_k))^2} f''(x_k) & \frac{1}{f'(x_k)} \end{pmatrix} \qquad (38)$$

The determinant of D^2S can be expressed with the help of the stability matrix as

$$\det D^2 S_p = \det(J_p - 1). \qquad (39)$$

This way we reformulate (36) as

$$(\mathrm{Tr}\mathcal{L}^n)_N \simeq \frac{N^{\frac{n-1}{2}}}{2\pi} \frac{\Gamma(N+\frac{1}{2})}{\left(N+\frac{n}{2}\right)^{\frac{n}{2}}} \sum_{p.o.} \frac{\mathrm{e}^{-N \log S_p}}{\sqrt{\det(1-J_p)}}, \qquad (40)$$

where the summation runs over generalised periodic orbits, with non-zero momentum. This is fully analogous to a trace formula and is our main result.

Finally we turn towards testing our result obtained so far. In [7] we developed a contour integral method to calculate high order noise corrections to the trace of $\mathcal{L}$. We showed that the agreement between the exact results and a formula which coincides with the (36) in the $n = 1$ case is very good. Now we step ahead and produce numerically high order noise corrections to the trace of $\mathcal{L}^2$. We shall start from (27) by transforming the integrals in p back to integrals in x as

$$(\mathrm{Tr}\mathcal{L}^n)_N = \frac{1}{(2\pi i)^n 2^N} \sum_{j_1,...,j_n=0}^{\infty} \frac{(2j_1)!}{j_1!} ... \frac{(2j_n))!}{j_n!}$$
$$\times \delta_{N,\sum_{k=1}^n j_k} \oint_{c_1} ... \oint_{c_n}$$
$$\frac{dx_1 ... dx_n}{(x_1 - f(x_n))^{2j_1+1} ... (x_n - f(x_{n-1}))^{2j_n+1}}. \qquad (41)$$

The contours at (27) were around the $p_j = 0$ points, so the contours above are placed around the original periodic orbits of the system, defined by (6),(7). These contour integrals can be evaluated numerically. The Fig. 2 shows the ratio of $(\mathrm{Tr}\mathcal{L}^2)_N$ obtained from (36) and evaluated via the procedure described above as a function of N.

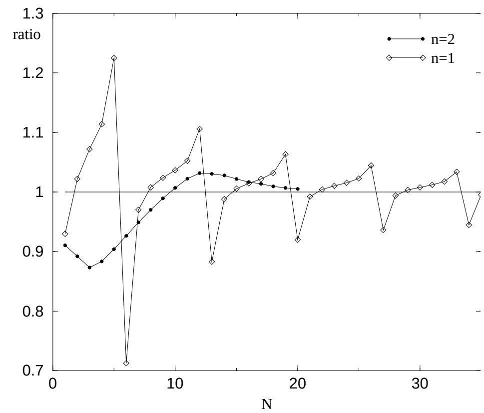

FIG. 2. The ratio of $(\mathrm{Tr}\mathcal{L}^2)_N$ calculated via the asymptotic formula (36) to its value computed by numerical integration

In summary we have studied the evolution operator for a discrete Langevin equation with a strongly hyperbolic classical dynamics and a Gaussian noise distribution. Using an integral representation of the evolution operator $\mathcal{L}$ we have revealed the asymptotic behaviour of the corrections to the trace of $\mathcal{L}^n$. This behaviour is governed by sub-dominant terms corresponding to terms previously neglected in the perturbative expansion, and a fully analogous trace formula can be derived for the late terms in the noise extension series of the trace of $\mathcal{L}^n$.

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