A New Method for Computing Topological Pressure

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Abstract

The topological pressure introduced by Ruelle and similar quantities describe dynamical multifractal properties of dynamical systems. These are important characteristics of mesoscopic systems in the classical regime. Original definition of these quantities are based on the symbolic description of the dynamics. It is hard or impossible to find symbolic description and generating partition to a general dynamical system, therefore these quantities are often not accessible for further studies. Here we present a new method by which the symbolic description can be omitted. We apply the method for a mixing and an intermittent system.

In recent years the application of the thermodynamic formalism [1] (TF) in analyzing dynamical multifractal properties [2–4] has been widely accepted. Besides giving an illuminating analogy with the statistical mechanics, it provides a deeper understanding of nonanalytic behavior in the scaling properties of trajectories in dynamical systems which can be interpreted as phase transitions [5]. The topological pressure (TP) plays a central role in the TF. The computation of the TP is difficult in general, since the knowledge of the *symbolic dynamics* [6] of the system and its generating partition is inevitable. There is no general theory at present, which can provide a systematic and numerically realizable method to construct a generating partition to a general dynamical system [7]. Therefore the TP has been computed mostly for low dimensional maps and billiard systems, where the symbolic dynamics is accessible, or has been computed with averaging the generalized Lyapunov exponents [8]. In *mesoscopic* devices like dots, antidots and wells Hamiltonian dynamical systems with smooth potentials play significant role. It would be very enlighting to apply TF for such non-trivial systems. In this Letter we introduce a new technique based on a suitably defined correlation function [9] to measure the TP. The advantage of the method is that the detailed knowledge of the system is not required and the calculation can be made with low computational demand for either hyperbolic or mixing systems. The new method makes possible to measure TP by following one single trajectory of an ergodic system.

The TP for maps can be defined as the logarithm $P(q) = log z_0(q)$ of the leading zero $z_0(q)$ of the Ruelle zeta function $1/\zeta(z,q) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|^q}\right)$, where the infinite product goes for the non-repeating or primitive periodic orbits of the system, Λ_p is the largest eigenvalue of its stability matrix $\mathbf{J_p}$ and n_p is its period. This definition has been extended for continuous flows in Refs. [1,10]. The TP is the leading zero $P(q) = s_0(q)$ of the zeta function $1/\zeta(s,q) = \prod_p \left(1 - \frac{e^{sT_p}}{|\Lambda_p|^q}\right)$, where T_p is the time period of the periodic orbit. The TP is equivalent with the free energy [11] $qF(q)$ and also with the Rényi entropies [3] in hyperbolic systems which are defined in a slightly different way in the literature. In the definitions above, the problem to find the symbolic dynamics is hidden, since it is necessary to locate the periodic orbits in the system. At certain values of q the TP has a special meaning [12]. For $q = 0$ the quantity $-P(0)$ is the topological entropy per unit time. For $q = 1$ it is zero for bound systems, it is the escape rate for open systems and its derivative gives the metric entropy per unit time. The parameter q where the TP is zero ($P(q) = 0$) yields the fractional part of the fractal dimension of the repeller in scattering systems. The time $-1/P(1/2)$ gives a lower bound for the quantum resonance lifetimes in semiclassical approximation for open systems [13–15].

Zeta functions can be related to transfer operators. In Ref. [9] we have introduced a transfer operator in arbitrary finite dimensions, whose largest eigenvalue is related to the TP. Here we briefly repeat the main idea.

The dynamical system has to be extended to the tangent space of the flow, where stability of the orbits is multiplicative. We adjoin the d-dimensional transverse tangent space $\xi \in TU_x$, $\xi(x) \cdot \mathbf{v}(x) = 0$, to the $(d+1)$-dimensional dynamical evolution space $x \in U \subset \mathbf{R}^{d+1}$. The dynamics in the $(x,\xi) \in U \times TU_x$ space is governed by the system of equations of variations [17]:

$$\dot{x} = \mathbf{v}(x)\,, \quad \dot{\xi} = \mathbf{D}\mathbf{v}(x)\xi\,.$$

Here $\mathbf{Dv}(x)$ is the transverse derivative matrix of the flow. We write the solution as

$$x(t) = f^t(x_0)\,, \quad \xi(t) = \mathbf{J}^t(x_0)\cdot\xi_0\,, \tag{1}$$

with the tangent space vector ξ transported by the transverse stability matrix $\mathbf{J}^t(x_0) = \partial x(t)/\partial x_0$. In order to determine the length of the vector ξ we introduce a *signed norm*. An example is the function

$$g : TU_x \to \mathbf{R}, \quad g\begin{pmatrix}\xi_1\\ \xi_2\\ \cdots\\ \xi_d\end{pmatrix} = \xi_d\,. \tag{2}$$

Any vector $\xi \in TU_x$ can now be represented by the product $\xi = \Lambda\mathbf{u}$, where $\mathbf{u}$ is a unit vector in the signed norm: $g(\mathbf{u}) = 1$, and the factor

$$\Lambda^t(x_0, \mathbf{u}_0) = g(\xi(t)) = g(\mathbf{J}^t(x_0)\cdot\mathbf{u}_0) \tag{3}$$

is the multiplicative "stretching" factor

$$\Lambda^{t'+t}(x_0, \mathbf{u}_0) = \Lambda^{t'}(x(t), \mathbf{u}(t))\,\Lambda^t(x_0, \mathbf{u}_0).$$

The $\mathbf{u}$ evolution constrained to $ET_{g,x}$, the space of unit tangent vectors transverse to the flow $\mathbf{v}$, is given by rescaling of (1):

$$\mathbf{u}' = R^t(x, \mathbf{u}) = \frac{1}{\Lambda^t(x, \mathbf{u})}\mathbf{J}^t(x)\cdot\mathbf{u}\,. \tag{4}$$

Eqs. (1), (3) and (4) enable us to define a *multiplicative* evolution operator on the extended space $U \times ET_{g,x}$

$$\mathcal{L}^t(x', \mathbf{u}'; x, \mathbf{u}) = \delta(x' - f^t(x))\frac{\delta(\mathbf{u}' - R^t(x, \mathbf{u}))}{|\Lambda^t(x, \mathbf{u})|^{q-1}}\,. \tag{5}$$

This operator is the generalization of the well-known transfer operator $\hat{L}_q(x, y) =\mid f'(y)\mid^{1-q}$ $\delta(x - f(y))$, for arbitrary dimensions and continuous time. In analogy with the one dimensional problem, we would like to determine its leading eigenvalue in order to get the TP.

Our new method uses the relation of the generalized transfer operators introduced above and the correlation decay in chaotic systems. Let's take two arbitrary smooth observables $A(x, \mathbf{u})$ and $B(x, \mathbf{u})$ depending on the real variables x and on the tangent space variables $\mathbf{u}$. We can define their generalized correlation function as

$$C^q_{AB}(t) = \langle A(x, \mathbf{u}) \mid \Lambda^t(x, \mathbf{u}) \mid^{1-q} B(f^t(x), R^t(x, \mathbf{u}))\rangle, \tag{6}$$

where the average is uniform in the extended space

$$\langle A(x, \mathbf{u})\rangle = \int d\mathbf{u}dxA(x, \mathbf{u}). \tag{7}$$

With the help of the operator (5), the correlation function can be rewritten as

$$C^q_{AB}(t) = \langle A(x,\mathbf{u}) \int d\mathbf{u}' dx' \mathcal{L}^t(x,\mathbf{u};x',\mathbf{u}') B(x',\mathbf{u}') \rangle. \tag{8}$$

We can introduce the eigenvalues $-s_n(q)$ and eigenfunctions (which are distributions in general) $\psi_n(x,\mathbf{u})$ of the stationary problem of the operator (5)

$$e^{-s_i(q)t}\psi_n(x,\mathbf{u}) = \int d\mathbf{u}' dx' \mathcal{L}^t(x,\mathbf{u};x',\mathbf{u}')\psi_n(x',\mathbf{u}'). \tag{9}$$

For *Axiom A systems* [18] one can expand $B(x',\mathbf{u}')$ on the eigenbasis:

$$B(x',\mathbf{u}') = \sum_n b_n \psi_n(x',\mathbf{u}'). \tag{10}$$

The correlation function then becomes

$$C^q_{AB}(t) = \sum_n e^{-s_i(q)t} b_n \langle A(x,\mathbf{u})\psi_n(x,\mathbf{u})\rangle. \tag{11}$$

For large t this sum is dominated by the term corresponding to the largest eigenvalue $-s_0(q)$

$$C^q_{AB}(t) \sim e^{-s_0(q)t} C, \tag{12}$$

where the constant is $C = b_0\langle A(x,\mathbf{u})\psi_0(x,\mathbf{u}))\rangle$. For general, non-Axiom A systems, the decomposition (10) is not valid, since the eigenfunctions (distributions) not necessarily span the full function space. Nevertheless, at long times, the decay is always dominated by the largest eigenvalue in accordance with (12).

Therefore the TP can be read off from the semi-logarithmic plot of the correlation function for sufficiently large time t.

For an ergodic system the phase space average (7) can be computed as a suitable time average for a sufficiently long ergodic trajectory. If, for the simplicity, we choose the special observables $A(x,\mathbf{u}) = 1$ and $B(x,\mathbf{u}) = \varrho(x)$, where $\varrho(x)$ is the equilibrium probability distribution of the system, the correlation function becomes

$$C^q_{1\varrho}(t) = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt' \mid \Lambda^{t'+t}(x_0,\mathbf{u}_0) \mid^{1-q} \tag{13}$$

where x_0 and $\mathbf{u}_0$ represent an initial condition for a long ergodic trajectory.

To implement the technique for a two dimensional classical problem, we need a convenient choice of the $g(\xi)$ function. For 2-dimensional Hamiltonian dynamics let the 2-dimensional Poincaré section return map be $x_{i+1} = f(x_i)$. The stability matrix of cycle p is a product of the 2×2 stability matrices

$$\mathbf{J}_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix},$$

where $A_j = \partial f_1(x_j)/\partial x_1$, and so on. Assume the signed norm (2) and multiply an initial unit vector by the first stability matrix in the product. The resulting vector can be written as

$$\mathbf{J}_1 \begin{pmatrix} \kappa_1 \\ 1 \end{pmatrix} = (C_1\kappa_1 + D_1) \begin{pmatrix} \frac{A_1\kappa_1+B_1}{C_1\kappa_1+D_1} \\ 1 \end{pmatrix} .$$

Hence the dynamics acts on the unit vectors as a rational fraction transformation

$$\kappa_{k+1} = R(x_k, \kappa_k) = \frac{A_k\kappa_k + B_k}{C_k\kappa_k + D_k}, \tag{14}$$

with the signed norm (3) of the iterated vector given by

$$\Lambda^n(x_1, \kappa_1) = \prod_{i=1}^{n} (C_i\kappa_i + D_i) .$$

In the case of 2-dimensional billiards, κ_n is the Bunimovich-Sinai curvature [21]. The Bunimovich-Sinai curvature $\kappa(t)$ is the local curvature of the front (horocycle) formed around a central trajectory by nearby orbits started from a common point of origin with the same energy. For a periodic orbit $\kappa_{n_p} = \kappa_0$, the unit vector is an eigenvector of the stability matrix, and the corresponding eigenvalue is $\Lambda_p = \prod(C_i\kappa_i + D_i)$. The κ can be defined also for continuous time [22]. Instead of the rational fractional transformation (14), we get a differential equation for $\kappa(t)$. The "stretching factor" becomes the integral

$$\Lambda^t(x_0, \kappa_0) = \exp\left(\int_0^t \kappa(\kappa_0, x_0, t')dt'\right). \tag{15}$$

The time evolution of $\kappa(t)$ for 2-dimensional Hamiltonian systems with hamiltonian $H = \frac{1}{2}(p_x^2 + p_y^2) + U(x, y)$ can be derived easily: An infinitesimal configuration space volume V_0, formed around a central trajectory by nearby orbits started from a common point of origin with the same energy, is stretched after time t with the factor $\Lambda^t(x_0, \kappa_0)$ orthogonal to the central trajectory and with $\sqrt{\frac{2(E-U(x(t))}{2(E-U(x(0))}}$ along the trajectory, due to the change of the velocity along the trajectory. The change of the volume in unit time can be written as $\frac{dV_t}{dt} = \lambda(t)V_t$, where the expansion rate is the sum of the orthogonal and the parallel stretching rates:

$$\lambda(t) = \kappa(t) + \frac{d}{dt}\frac{1}{2}\log(2(E - U(x(t)))).$$

Computation of $\kappa(t)$ is then reduced to the problem of finding the expansion rate of volumes. This can be recovered by investigating the evolution of trajectories, which deviate infinitesimally $\delta x(t)$ from the central trajectory $x(t)$. These are described by the linearized Newton equation $\delta\ddot{x}(t) = -\mathbf{D^2}U(x(t))\delta x(t)$, where $\mathbf{D^2}U(x(t))$ is the second derivative matrix of the potential. Since the trajectories are restricted to the same energy surface as the central trajectory, we get the additional constraint: $\dot{x}(t) \cdot \delta\dot{x}(t) + \nabla U(x(t))\delta x(t) = 0$. The evolution in two dimensions can be reformulated to the linear form

$$\delta\dot{x}(t) = \mathbf{M}(t)\delta x(t), \tag{16}$$

where the four elements of the two-by-two matrix $\mathbf{M}(t)$ can be constructed from the four vector components of $\delta\dot{x}(t)$ and $\delta x(t)$. The evolution of the matrix $\mathbf{M}(t)$ can be determined from the linearized Newton equation: $\dot{\mathbf{M}}(t) = -\mathbf{M}^2(t) - \mathbf{D^2}U(x(t))$, and the energy conservation constraint yields: $\dot{x}(t)\mathbf{M}(t) + \nabla U(x(t)) = 0$. The expansion rate of the configuration

space volume is the divergence of the velocity field (16), which is the trace of the matrix $\mathbf{M}(t)$: $\lambda(t) = Tr\mathbf{M}(t)$.

For the $\kappa(t)$ we get:

$$\dot{\kappa} = -\kappa^2 - 3\frac{(p_y\partial_x U - p_x\partial_y U)^2}{(p_x^2+p_y^2)^2} - \frac{p_y^2\partial_{xx}U - 2p_xp_y\partial_{xy}U + p_x^2\partial_{yy}U}{p_x^2+p_y^2} \tag{17}$$

The numerical calculations have confirmed the applicability of the method. For concrete calculations we have chosen the Hamiltonian of the Anisotropic Kepler Problem (AKP) [19,20]

$$H = \frac{1}{2}(p_x^2+p_y^2) - \frac{1}{(x^2+\alpha y^2)^{1/2}},$$

which describes a Bloch electron with anisotropic mass tensor in a Coulomb potential and the Hamiltonian

$$H = \frac{1}{2}(p_x^2+p_y^2) + \frac{1}{2}x^2y^2$$

which can be considered as a model of a soft wall mesoscopic cross junction [23]. We have measured the correlation function (13) for the AKP at parameter $\alpha = 5.0$ and for the $\frac{1}{2}x^2y^2$ potential at energy $E = -0.5$ up to $T = 13000$ and $E = +0.5$ up to $T = 30000$ respectively, for several initial conditions while we integrated the equations with $dt = 0.001$ using the fourth order Runge-Kutta [24] method with adaptive step-size control. Both systems can be considered practically ergodic, since regular islands cover less than 0.005 percent of their phase space [25]. The main difference between these systems is that the AKP is a mixing one at the chosen value of α, while the other one is intermittent.

For the AKP the correlation function (13) reached the asymptotic behavior very quickly for all q (Fig. 1b), while for the intermittent system one can see the effect of the critical slowing down [26] at $q > 1$ (Fig. 2b). The measured TP functions are depicted on Fig. 1a and 2a.

We hope, that the method presented here can be applied for many dynamical systems, where other methods have failed so far.

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FIGURES

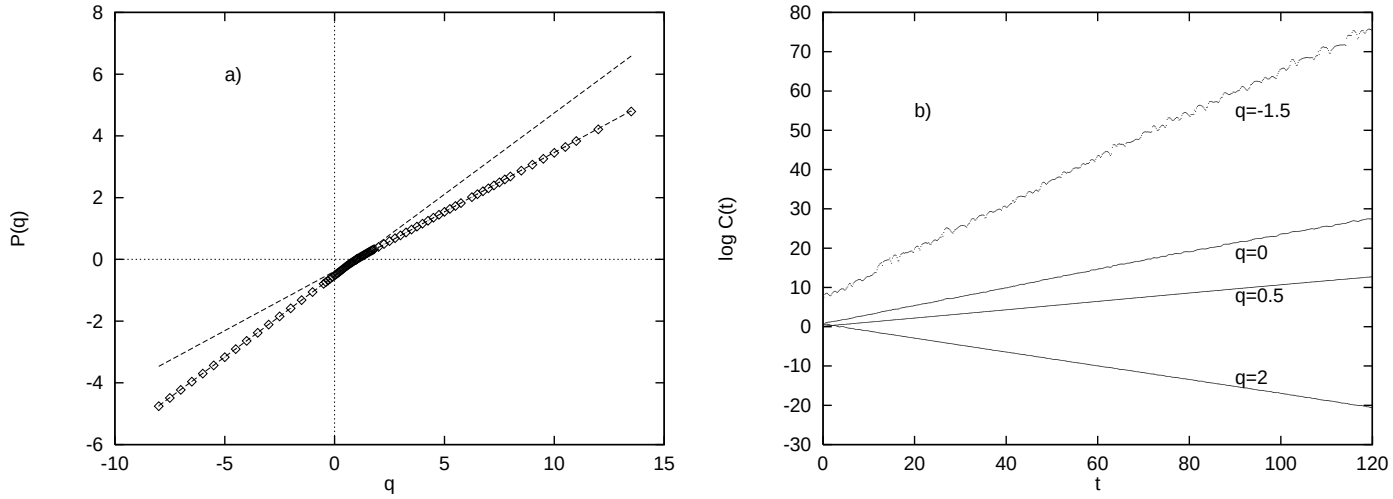

FIG. 1. a) The measured Topological Pressure (per unit time) function for the AKP as a function of q with the asymptotes. b) The common logarithm of the measured correlation function of the AKP as a function of time t. Different straight lines correspond to different values of q, due to the fast convergence to the asymptotic behavior.

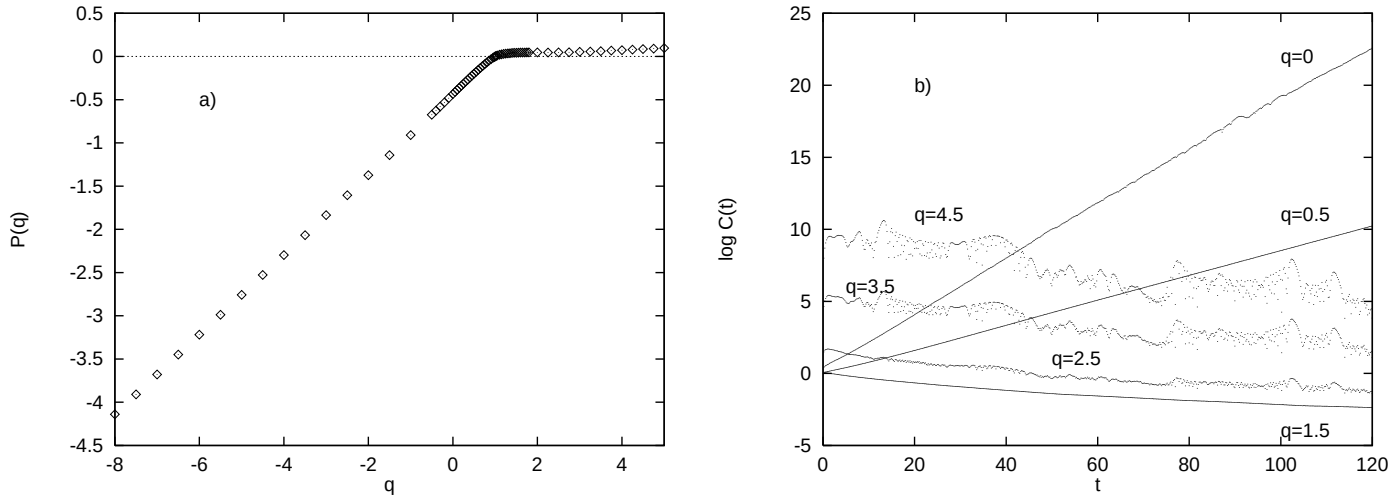

FIG. 2. a) The measured Topological Pressure (per unit time) function for the $\frac{1}{2}x^2y^2$ potential as a function of q. b) The common logarithm of the correlation function of the $\frac{1}{2}x^2y^2$ potential as a function of time t. One can observe the critical slowing down of the convergence as q becomes greater than 1