Solving topological defects via fusion

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Abstract

Integrable defects in two-dimensional integrable models are purely transmitting thus topological. By fusing them to integrable boundaries new integrable boundary conditions can be generated, and, from the comparison of the two solved boundary theories, explicit solutions of defect models can be extracted. This idea is used to determine the transmission factors and defect energies of topological defects in sinh-Gordon and Lee-Yang models. The transmission factors are checked in Lagrangian perturbation theory in the sinh-Gordon case, while the defect energies are checked against defect thermodynamic Bethe ansatz equations derived to describe the ground-state energy of diagonal defect systems on a cylinder. Defect bootstrap equations are also analyzed and are closed by determining the spectrum of defect bound-states in the Lee-Yang model.

1 Introduction

Recently, there has been an increasing interest in integrable quantum field theories including defects or impurities. This is motivated both by the realistic physical applications in statistical and solid state physics and also by the need of theoretical understanding of this so-far unexplored field.

The community of integrable systems have not payed much attention to defect theories at the beginning due to the no-go theorem formulated by Delfino, Mussardo and Simonetti in [1, 2]. The theorem, formulated originally for diagonal theories and extended later for a large class of non-diagonal ones in [3], states that a relativistically invariant theory with a non-free integrable interaction in the bulk can allow only two types of integrable defects: the purely reflecting and the purely transmitting ones. (Although some effort has been made to overcome this obstacle by giving up Lorentz invariance, see for instance [4] and references therein, in the present paper we restrict ourselves to the relativistically invariant case.)

The analysis of boundary integrable theories was initiated in [5] by formulating, in an axiomatic way, the properties of the reflection matrix: unitarity, boundary crossing unitarity and boundary bootstrap equation. The boundary bootstrap framework was completed by introducing boundary Coleman-Thun mechanism [6] and the bulk bootstrap equations [7]. Later this framework got a sound basis by developing boundary quantum field theories from first principles in [8, 9]. The success of the boundary bootstrap approach resulted in a large class of closed bootstrap theories in which the boundary reflection factors together with the spectrum of boundary excited states were determined [5, 6, 10, 11, 12, 13, 14]. The solutions, obtained by the bootstrap method, are not connected, however, to other formulations of the model such as to the classical field theory or to the perturbed conformal field theory (both defined by a Lagrangian). To connect the different descriptions one either has to check the reflection factors perturbatively, like in [15], or solve the theories in finite volume. The boundary thermodynamic Bethe ansatz (BTBA), developed in [16], systematically sums up the finite size corrections by taking into account the scatterings and reflections. By analyzing its small volume limit the needed link between the bootstrap and perturbed conformal field theoretical descriptions can be established.

As the no-go theorem showed non-free integrable defect theories are purely transmitting. This fact kept back the researchers for some time to analyze these models until new life was put into the subject due to their explicit Lagrangian realizations [17]. Following the original idea many integrable defect theories were constructed at the classical level [18, 19, 20]. The basis for the quantum formulation of defect theories is provided by the folding trick [21] by which one can map any defect theory into a boundary one. As a consequence defect unitarity, defect crossing symmetry and defect bootstrap equations together with defect Coleman-Thun mechanism are derived. Despite of these results the explicitly solved relativistically invariant defect quantum field theories are quite rare, containing basically the sine-Gordon and affine Toda field theories [22, 23, 24] and even in these cases the explicit relation to their Lagrangian have not been worked out yet.

One may think that purely transmitting theories are too simple and there is no point to analyze them, but we would like to argue that they carry very important information about an integrable quantum field theory, without which our knowledge cannot be complete. Purely transitivity implies the conservation of momentum from which the topological nature of the defect follows. Thus, such defects can freely be transported in space without affecting the physics of the theory. We can either move them close to each other or move them to integrable boundary conditions and, as a result, new integrable boundary conditions can be generated. These ideas were successfully applied in conformal field theories [25, 26, 27], in integrable lattice models [28, 29] and the aim of the present paper is to exploit it in solving integrable defects in the sinh-Gordon and Lee-Yang theories.

The paper is organized as follows: In section 2, on the example of the sinh-Gordon theory, we show how new boundary conditions can be obtained by fusing integrable defects to boundaries at the classical level. We focus on two cases in detail: fusing the integrable defect to Dirichlet boundary condition (DBC) the perturbed Neumann boundary condition (PNBC) can be obtained, while fusing it to the Neumann BC a new class of time-dependent integrable BCs can be generated. Section 3 recalls the quantum version of the fusion method together with the properties of transmission factors. By comparing the already known reflection factors and boundary energies of the DBC to those of the PNBC solutions for the defect transmission factor and defect energy can be extracted. The same method is then used to determine defect energies and transmission factors of the scaling Lee-Yang model. This method not only provides the explicit solutions of the sinh-Gordon defect theory but also relates its parameter to that of the Lagrangian. Since the relation obtained here is different from the suggestion of [23] we perform a perturbative analysis at one-loop level in section 4. (In the subsequent paper [24] the same authors raised the possibility of the quantum renormalization of the transmission parameter, which is confirmed at one-loop level here). In section 5 the calculated defect energies are subject to another consistency check. For this we derive a TBA equation to describe the groundstate energy of a diagonal defect system on a cylinder. From a careful ultra-violet (UV) analysis defect energies are extracted and the previous results are verified. By completing the defect bootstrap program we analyze the singularity structure of the transmission factors in section 6. Since the sinh-Gordon transmission factor does not contain any singularity in the physical strip we analyze the Lee-Yang model only. For each pole of the transmission factor in the physical strip we associate either a defect boundstate or a defect Coleman-Thun diagram and calculate the excited transmission factors in the former case from the defect bootstrap equation. Finally, we conclude in section 7 and give directions for further research.

2 Fusion method at the Lagrangian level

In this section we demonstrate, on the example of the sinh-Gordon (ShG) theory, how new integrable boundary conditions can be obtained by the fusion method at the Lagrangian or classical level [30].

The ShG theory is defined on the whole line by the following Lagrangian

$$\mathcal{L}_{\text{ShG}}(\Phi)=\frac{1}{2}(\partial_t\Phi)^2-\frac{1}{2}(\partial_x\Phi)^2-\frac{m_{\text{cl}}^2}{b^2}\cosh b\Phi \quad , \tag{1}$$

It describes an integrable field theory and by restricting the theory to the half line this property can be only maintained, whenever the following boundary potential is introduced [5]

$$\mathcal{L}_{\text{BShG}}=\Theta(-x)\mathcal{L}_{\text{ShG}}(\Phi)-\delta(x)B(\Phi) \quad ; \qquad B(\Phi)=M_0\cosh\frac{b}{2}(\Phi-\varphi_0) \tag{2}$$

By varying M_0 from 0 to ∞ the arising boundary condition interpolates between the Neumann $\partial_x\Phi|_{x=0}=0$ and the Dirichlet $\Phi(0,t)=\varphi_0$ ones.

The most general integrable defect condition can be obtained by the analytical continuation of the sine-Gordon result [23]. The Lagrangian in the ShG case reads as

$$\mathcal{L}_{\text{DShG}}=\Theta(-x)\mathcal{L}_{\text{ShG}}(\Phi_-)-\delta(x)D(\Phi_-,\Phi_+)+\Theta(x)\mathcal{L}_{\text{ShG}}(\Phi_+) \tag{3}$$

where the defect potential contains just one single parameter:

$$2D(\Phi_-,\Phi_+) \;=\; \Phi_+\dot{\Phi}_- - \Phi_-\dot{\Phi}_+ + M_{\text{cl}}e^{\mu}\cosh\frac{b}{2}(\Phi_+ + \Phi_-) + M_{\text{cl}}e^{-\mu}\cosh\frac{b}{2}(\Phi_+ - \Phi_-)$$

Here $\Phi_\mp$ are the fields living on the left/right half-line, respectively and $M_{\text{cl}}=\frac{4m_{\text{cl}}}{b^2}$.

The fusion idea is based on the integrability of the defect: Integrability guaranties the existence of an infinite number of commuting conserved charges which results in the possibility of shifting the trajectories of particles, without changing the amplitude of any scattering process. The shifting of all the trajectories can alternatively be described by shifting the location of the defect, which then, does not alter the physics.

At the level of the Lagrangian this observation can be formulated in the following way: The spectrum of the system, which contains a defect in the origin, $x=0$, in front of a boundary, located at $x=a$,

$$\mathcal{L}_{\text{DBShG}}=\Theta(-x)\mathcal{L}_{\text{ShG}}(\Phi_-)-\delta(x)D(\Phi_-,\Phi_+)+\Theta(x)\Theta(a-x)\mathcal{L}_{\text{ShG}}(\Phi_+)-\delta(x-a)B(\Phi_+)$$

does not actually depend on a. Thus we can perform the $a \to 0$ limit and represent the same system as a boundary one, but with a different (dressed) boundary condition:

$$\mathcal{L}_{\mathrm{DBShG}} = \Theta(-x)\mathcal{L}_{\mathrm{ShG}}(\Phi_-) - \delta(x)B'(\Phi_-,\Phi_+) \quad ; \qquad B'(\Phi_-,\Phi_+) = D(\Phi_-,\Phi_+) + B(\Phi_+)$$

The field Φ_+ lives on the boundary only and can be thought naively to be a boundary degree of freedom. It does not have, however, any kinetic term so it merely implements a new time-dependent integrable boundary condition. Let us specify these findings in two concrete examples that will be used later on.

If the original boundary condition is the Dirichlet one with $\Phi(a,t) = \phi_0$, then the arising dressed boundary condition is

$$B'(\Phi_-,\Phi_+) = D(\Phi_-,\varphi_0) = \frac{M_{\mathrm{cl}}e^{\mu}}{2}\cosh\frac{b}{2}(\Phi_- + \phi_0) + \frac{M_{\mathrm{cl}}e^{-\mu}}{2}\cosh\frac{b}{2}(\Phi_- - \phi_0) \qquad (4)$$

where we dropped the total time derivatives. This BC is exactly of the form of (2) with parameters

$$M_{\mathrm{cl}}\cosh(\mu \pm \frac{b}{2}\phi_0) = M_0 e^{\mp\frac{b}{2}\varphi_0}$$

Thus by fusing the integrable defect to the DBC we can reconstruct the most general (two parameter family of) PNBCs. Interestingly ϕ_0 and μ are the classical analogues of the parameters in which the boundary reflection is factorized [45], see also (13,14) in section 3.

By fusing the defect to the NBC we obtain the boundary potential

$$B'(\Phi_-,\Phi_+) = D(\Phi_-,\Phi_+)$$

Variation of action provides BCs in the form:

$$\partial_t\Phi_-|_{x=0} = -\frac{\partial B'(\Phi_+,\Phi_-)}{\partial\Phi_+} \quad ; \qquad (\partial_t\Phi_+ - \partial_x\Phi_-)|_{x=0} = \frac{\partial B'(\Phi_+,\Phi_-)}{\partial\Phi_-} \qquad (5)$$

By expressing Φ_+ in terms of Φ_- and $\partial_t\Phi_-$ then plugging back to the second equation we obtain a highly non-trivial boundary condition for Φ_- containing its second time derivative, which is nevertheless integrable as it follows from the construction. Obviously, this solution was not covered by the two parameter family of (time-independent) integrable boundary conditions determined in [5], thus by the fusion method we were able to construct a new type of integrable BC. By fusing other integrable defects with new free parameters to this dressed boundary we can generate integrable BCs with as many parameters as we want. What is nice in the construction, that the solution of the defects transmission factor will provide, via the fusion method, solutions for these general integrable BCs, too, as we will show in the next section.

Finally, we note that similar construction can be used in the case of the Lee-Yang model, however, the explicit form of the integrable defect perturbation has not been identified at the Lagrangian level yet (for details see the next section).

3 Fusion method in the bootstrap

For simplicity we present the fusion idea in the case of an integrable diagonal scattering theory with one particle type of mass m. The general discussion can be found in [21].

In integrable bulk theories multi-particle scattering processes factorize into the product of two particle scatterings, $S(\theta_{12})$, where $\theta_{12} = \theta_1 - \theta_2$ is the rapidity difference of the scattering particles whose momenta are parametrized as $p_i = m \sinh \theta_i$. In relativistically invariant theories the two particle S-matrix satisfies unitarity and crossing symmetry

$$S(-\theta) = S^{-1}(\theta) \quad ; \qquad S(i\pi - \theta) = S(\theta)$$

Once boundaries are introduced the basic process is the multi-particle reflection. Integrability ensures its factorization into pairwise scatterings $S(\theta_{ij})$ and individual reflections $R(\theta_i)$, where θ_i is the rapidity of the reflected particle. The reflection matrix satisfies unitarity and boundary crossing unitarity [5]

$$R(-\theta) = R^{-1}(\theta) \quad ; \qquad R(\frac{i\pi}{2} - \theta) = S(2\theta) R(\frac{i\pi}{2} + \theta)$$

Integrable non-free defects are severely restricted: they are either purely reflecting (thus boundaries, like above) or purely transmitting. The latter case can be described by the two, left ($-$) and right ($+$), transmission matrices $T_-(\theta)$ and $T_+(-\theta)$. We parametrize T_+ such a way that for its physical domain ($\theta < 0$) its argument is always positive. Transmission factors satisfy unitarity and defect crossing symmetry [21]

$$T_+(-\theta) = T_-^{-1}(\theta) \quad ; \qquad T_-(\theta) = T_+(i\pi - \theta) \tag{6}$$

If we place a defect with transmission matrices $T_\pm(\theta)$ in front of a boundary with reflection matrix $R(\theta)$ then the fused boundary will also be integrable and have reflection factor $R^{'}(\theta)$:

$$R^{'}(\theta) = T_+(\theta) R(\theta) T_-(\theta) \tag{7}$$

The correspondence (7) between the original $R(\theta)$ and the fused $R^{'}(\theta)$ reflection factors can be used either to generate new BCs or, if the two BCs are already known, to solve defect transmission factors. This will be illustrated in the next subsections for the sinh-Gordon and Lee-Yang models.

3.1 Solution of defect sinh-Gordon theory

The spectrum of ShG theory defined by (1) consists of one particle type with mass [32]

$$m = \frac{4\sqrt{\pi}}{\Gamma(\frac{1-B}{2})\Gamma(1+\frac{B}{2})} \left(\frac{-\pi m_{\text{cl}}^2 \Gamma(1+b^2)}{b^2 \Gamma(-b^2)} \right)^{\frac{1}{2+2b^2}} \qquad ; \qquad B = \frac{b^2}{8\pi + b^2} \tag{8}$$

The two particle scattering matrix is given by

$$S = \frac{\sinh\theta - i \sin B\pi}{\sinh\theta + i \sin B\pi} = -(-B)(1+B) \quad , \qquad (x) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2})}$$

It has no poles in the physical strip: $0 \leq \theta < i\pi$ and is invariant under the weak-strong duality, $\frac{b^2}{8\pi} \to \frac{8\pi}{b^2}$. The bulk energy density turns out to be [33]

$$\epsilon_{\text{bulk}} = \frac{m^2}{8 \sinh \pi B} \tag{9}$$

Integrable boundary conditions can be either Dirichlet type with $\Phi(0,t)=\phi_0$ or PN type (2). The corresponding reflection factors can be obtained from the analytical continuation of the sine-Gordon's first breather's one [5, 34]. In the Dirichlet case it reads as

$$R_{\rm Dir}(\theta,\eta_{\rm Dir})=\frac{\left(\frac{1}{2}\right)\left(1-\frac{B}{2}\right)}{\left(\frac{3}{2}-\frac{B}{2}\right)}\frac{\left(\frac{iB\eta_{\rm Dir}}{\pi}-\frac{1}{2}\right)}{\left(\frac{iB\eta_{\rm Dir}}{\pi}+\frac{1}{2}\right)} \tag{10}$$

where the reflection parameter $\eta_{\rm Dir}$ is related to ϕ_0 as

$$\eta_{\rm Dir}=\frac{4\pi}{b}\phi_0 \tag{11}$$

The boundary energy has been also calculated [16]

$$\epsilon_{\rm bdry}^{\rm Dir}(\eta_{\rm Dir})=\frac{m}{4\sin B\pi}\left(2\cosh B\eta_{\rm Dir}-\sin\frac{\pi B}{2}-\cos\frac{\pi B}{2}-1\right) \tag{12}$$

In the PN case the reflection factor turns out to be

$$R_{\rm PN}(\theta,\eta,\vartheta)=\frac{\left(\frac{1}{2}\right)\left(1-\frac{B}{2}\right)}{\left(\frac{3}{2}-\frac{B}{2}\right)}\frac{\left(\frac{iB\eta}{\pi}-\frac{1}{2}\right)}{\left(\frac{iB\eta}{\pi}+\frac{1}{2}\right)}\frac{\left(\frac{iB\vartheta}{\pi}-\frac{1}{2}\right)}{\left(\frac{iB\vartheta}{\pi}+\frac{1}{2}\right)} \tag{13}$$

while the relation of η,ϑ to the parameters of the Lagrangian (UV-IR relation) is [45]

$$M\cosh\frac{b^2}{8\pi}(\eta\pm\vartheta)=M_0e^{\mp\frac{b\varphi_0}{2}}\quad;\qquad M=m_{\rm cl}\sqrt{\frac{2}{b^2\sin(b^2/8)}} \tag{14}$$

The boundary energy has been also determined [31, 45] as

$$\epsilon_{\rm bdry}^{\rm PN}(\eta,\vartheta)=\frac{m}{4\sin B\pi}\left(2\cosh B\eta+2\cosh B\vartheta-\sin\frac{\pi B}{2}-\cos\frac{\pi B}{2}-1\right) \tag{15}$$

We note that the results - both for the UV-IR relation and for the boundary energy - were obtained in the framework of perturbed BCFT in which the perturbing operator is normal-ordered to have a definite scaling dimension.

The integrable defect potential for the sinh-Gordon model can be written as in (3). Let us denote the transmission factors by $T_\pm(\theta,\mu)$. Fusing classically this defect to a DBC a PNBC can be obtained (4). The quantum analogue of this statement in view of (7) is

$$R_{\rm PN}(\theta,\eta,\vartheta)=T_+(\theta,\mu)R_{\rm Dir}(\theta,\eta_{\rm Dir})T_-(\theta,\mu)$$

Comparing the reflection factor of the PNBC (13) to that of the Dirichlet one (10) and taking into account defect unitarity and defect crossing symmetry (6) we can extract the transmission factors for the defect. The simplest possible solution corresponds to $\eta=\eta_{\rm Dir}$ and

$$T_-(\theta)=-i\frac{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{4}+\frac{B\vartheta}{2}\right)}{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{4}+\frac{B\vartheta}{2}\right)}\quad;\qquad T_+(\theta)=i\frac{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{4}-\frac{B\vartheta}{2}\right)}{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{4}-\frac{B\vartheta}{2}\right)} \tag{16}$$

All other solutions contain additional CDD type factors satisfying (6). Actually the solution (16) itself is a CDD factor, therefore it is the simplest non-trivial solution of (6).

To find the correspondence between the parameter of the quantum transmission factor ϑ and the Lagrangian parameter μ we follow the following strategy: Since the boundary results are derived in the perturbed BCFT normalization we allow not only the parameter μ but also $M_{\rm cl}$ to renormalize. Their renormalized quantum values are determined from the requirement that when fusing the defect to the DBC with (11) we obtain the PNBC with (14). The unique solution turns out to be

$$Me^{\pm\frac{b^2\vartheta}{8\pi}}=M_{\rm cl}e^{\pm\mu} \tag{17}$$

The renormalization of $M_{\rm cl}$ may depend on the scheme in which the quantum potential is defined. The $b\to 0$ limit, in which $M\to M_{\rm cl}$, shows that ϑ is the quantum renormalized version of μ.

Also the defect energy can be extracted as the difference of the boundary energies corresponding to the PN (15) and to the Dirichlet (12) one:

$$\epsilon_{\rm Def}(\vartheta)=\epsilon_{\rm bdry}^{\rm PN}(\eta,\vartheta)-\epsilon_{\rm bdry}^{\rm Dir}(\eta)=\frac{m\cosh B\vartheta}{2\sin B\pi} \tag{18}$$

Summarizing, by the fusion method we were able to solve the defect theory defined by the Lagrangian (3): The transmission factors are (16), the defect energy is (18), and the bootstrap parameter ϑ parametrizes the Lagrangian as (17). We spend the next section to provide consistency checks of this solution.

Once the defect theory is solved we can use it to generate new integrable BCs from known ones. In the example presented in section 2 the defect with parameter μ was fused to the NBC to generate a more general integrable BC (5). The quantum version of this fusion dresses up the Neumann reflection factor

$$R_{\rm N}(\theta)=\frac{\left(\frac{1}{2}\right)\left(1-\frac{B}{2}\right)\left(\frac{1}{2}-\frac{B}{2}\right)}{\left(\frac{3}{2}-\frac{B}{2}\right)\left(\frac{1}{2}+\frac{B}{2}\right)} \tag{19}$$

to the reflection factor

$$R(\theta,\vartheta)=T_{+}(\theta,\mu(\vartheta))R_{\rm N}(\theta)T_{-}(\theta,\mu(\vartheta))=\frac{\left(\frac{1}{2}\right)\left(1-\frac{B}{2}\right)\left(\frac{1}{2}-\frac{B}{2}\right)\left(\frac{iB\vartheta}{\pi}-\frac{1}{2}\right)}{\left(\frac{3}{2}-\frac{B}{2}\right)\left(\frac{1}{2}+\frac{B}{2}\right)\left(\frac{iB\vartheta}{\pi}+\frac{1}{2}\right)} \tag{20}$$

Thus we solved the more general integrable BC defined by (5) without doing any serious calculation. It is important to note, that the extra factor appearing in (20) compared to (19) is a CDD factor. Consequently, we have determined the physical meaning of the CDD factors appearing in the reflection factors: they represent integrable defects of the form of (5) standing in front of integrable boundaries. In principle, we can fuse as many integrable defects with various parameters as we want, the resulting theory can be solved and its reflection factor contains the corresponding boundary CDD factors.

Finally, we note that placing two defects with parameters $\vartheta_{\pm}=\pm i(1-\frac{\pi}{2B})$ after each other both the scattering matrix and the energy of a standing particle can be reproduced. Thus the defect with imaginary parameter can be considered as a 'half' particle. Similar phenomena was observed in [23] at the classical level.

3.2 Solution of defect scaling Lee-Yang model

The scaling Lee-Yang model can be defined as the perturbation of the $\mathcal{M}_{(2,5)}$ conformal minimal model with central charge $c=-\frac{22}{5}$. It contains two modules of the Virasoro algebra corresponding to the Id and the $\varphi(z,\bar{z})$ primary fields with weight $(0,0)$ and $(-\frac{1}{5},-\frac{1}{5})$, respectively. The only relevant perturbation by the field φ results in the simplest scattering theory with one neutral particle of mass m and scattering matrix [35]

$$S(\theta)=\frac{\sinh\theta+i\sin\frac{\pi}{3}}{\sinh\theta-i\sin\frac{\pi}{3}}=-\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$$

The pole at $\theta=\frac{i\pi}{3}$ shows that the particle can form a bound-state. The relation

$$S(\theta+i\frac{\pi}{3})S(\theta-i\frac{\pi}{3})=S(\theta)$$

however, implies that the bound-state is the original particle itself and the bulk bootstrap is closed. The bulk energy constant is given by $\epsilon_{\mathrm{bulk}}=-\frac{1}{4\sqrt{3}}m^2$.

We can impose two conformal invariant boundary conditions in the model [36, 37]. They can be labeled by $\mathbb{I}$ and Φ and correspond to the highest weight representations of a single copy of the Virasoro algebra with weight 0 and $-\frac{1}{5}$, respectively. Introducing the integrable bulk perturbations with the $\mathbb{I}$ conformal invariant boundary condition the integrability is maintained and the reflection factor of the particle can be written as

$$R_{\mathbb{I}}(\theta)=\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(-\frac{2}{3}\right)$$

while the boundary energy is given by $\epsilon_{\mathrm{bdry}}^{\mathbb{I}}=\frac{m}{2}\left(\sqrt{3}-1\right)$. If the conformal invariant boundary condition corresponds to Φ then additionally to the bulk perturbation we can introduce a one-parameter family of integrable boundary perturbations and the corresponding reflection factor turns out to be

$$R_b(\theta)=\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(-\frac{2}{3}\right)\left(\frac{b-1}{6}\right)\left(\frac{b+1}{6}\right)\left(\frac{5-b}{6}\right)\left(\frac{-5-b}{6}\right)$$

while the boundary energy is $\epsilon_{\mathrm{bdry}}=\frac{m}{2}\left(\sqrt{3}-1+2\sin\frac{b\pi}{6}\right)$. The boundary bound-states were analyzed in [6] where the boundary bootstrap program was carried out.

The Lee-Yang model has two types of conformal defects [38], but only one of them admits relevant chiral defect fields. They have weights $(-\frac{1}{5},0)$, $(0,-\frac{1}{5})$. We conjecture that perturbing in the bulk and with a certain combination of these defect fields we can maintain integrability and arrive at a purely transmitting theory. We plan to analyze this issue systematically in a forthcoming publication. Let us denote the transmission factors of this integrable defect by $T_\pm(\theta)$. Using that the fusion of the defect to the $\mathbb{I}$ boundary results in the perturbed Φ boundary we have

$$R_b(\theta)=T_+(\theta,b)R_{\mathbb{I}}(\theta)T_-(\theta,b)$$

This is supported by the fact that fusing the conformal defect to the $\mathbb{I}$ boundary we obtain the Φ boundary. Since the particle appears as a bound-state in the two particle scattering process the transmission matrix satisfies the defect bootstrap equation [3]:

$$T_-(\theta+\frac{i\pi}{3})T_-(\theta-\frac{i\pi}{3})=T_-(\theta) \tag{21}$$

Using this relation together with the defect unitarity and defect crossing symmetry (6) we can fix the transmission factor as

$$T_-(\theta)=[b+1][b-1]\quad;\qquad [x]=i\frac{\sinh(\frac{\theta}{2}+i\frac{\pi x}{12})}{\sinh(\frac{\theta}{2}+i\frac{\pi x}{12}-i\frac{\pi}{2})} \tag{22}$$

(Actually the inverse of the solution is also a solution but the two are related by the $b \to 6+b$ transformation). The defect energy, as in the sinh-Gordon case, can be obtained as

$$\epsilon_{\rm def} = \epsilon_{\rm bdry} - \epsilon^{\mathbb{I}}_{\rm bdry} = m \sin \frac{b\pi}{6}$$

We are going to recover this expression from the UV analysis of defect TBA in section 5. We also note that the defect with parameter $b = 3$ behaves as a standing particle both from the energy and from the scattering point of view.

4 Peturbative calculations

In this section we check the exact solution of the ShG defect system in the free/classical ($b \to 0$) limit, and develop a systematic perturbative expansion. The parameter b^2 plays the same role as $\hbar$ which can be seen by scaling it out from the Lagrangian via $\Phi \to b\Phi$. Since the classical groundstate $\Phi = 0$ is invariant under this scaling, the $b^2 \to 0$ limit corresponds both to the free and also to the classical limit. Moreover, the perturbative expansion in b^2 is equivalent both to the loop expansion and to the semi-classical approximation.

4.1 Classical/free limit

As a first step we identify the $b \to 0$ limit of the defect Lagrangian (3) as:

$$\begin{aligned} \mathcal{L} &= \Theta(-x)\left[\frac{1}{2}(\partial_\mu \Phi_-)^2 - \frac{m_{\rm cl}^2}{2}\Phi_-^2\right] + \Theta(x)\left[\frac{1}{2}(\partial_\mu \Phi_+)^2 - \frac{m_{\rm cl}^2}{2}\Phi_+^2\right] \\ &\quad -\frac{\delta(x)}{2}\left(\Phi_+\dot{\Phi}_- - \Phi_-\dot{\Phi}_+ + m_{\rm cl}\left[\cosh\mu\left(\Phi_+^2 + \Phi_-^2\right) + 2\sinh\mu\,\Phi_+\Phi_-\right]\right) \end{aligned}$$

Then we expand the fields on the two sides of the defect in terms of creation/annihilation operators

$$\Phi_\pm(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{1}{2\omega(k)}\left(a_\pm(k)e^{ikx - i\omega(k)t} + a_\pm^+(k)e^{-ikx+i\omega(k)t}\right) \quad ; \quad \omega(k) = \sqrt{k^2 + m_{\rm cl}^2}$$

where the a, a^+ operators are adjoint of each other with commutators:

$$[a_\pm(k), a_\pm^+(k')] = 2\pi 2\omega(k)\delta(k - k')$$

Imposing the defect condition (obtained by varying the action) at the origin

$$\pm\partial_t\Phi_\pm \mp \partial_x\Phi_\mp = m_{\rm cl}(\sinh\mu\,\Phi_\pm + \cosh\mu\,\Phi_\mp)$$

we can connect the creation/annihilation operators as

$$a_\pm(\pm k) = T_\mp(k)a_\mp(\pm k) \quad ; \quad T_\mp(k) = -\frac{m_{\rm cl}\sinh\mu \mp i\omega(k)}{m_{\rm cl}\cosh\mu - ik} \quad ; \quad k > 0$$

This shows that the defect is purely transmitting, that is we do not have any reflected wave. The transmission factor in the rapidity parametrization ($k = m_{\rm cl}\sinh\theta$) can be written also in the following form:

$$T_-(\theta) = -i\frac{\sinh(\frac{\theta}{2} - \frac{i\pi}{4} + \frac{\mu}{2})}{\sinh(\frac{\theta}{2} + \frac{i\pi}{4} + \frac{\mu}{2})}$$

Clearly it has exactly the same form as the exact quantum one (16) except the $B\vartheta \leftrightarrow \mu$ replacement. Having observed this coincidence the authors in [23] suggested that they might be the same $B\vartheta = \mu$. Using our defect UV-IR relation (17) we can perform the expansion:

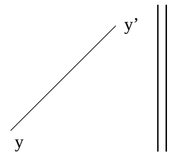
$$B\vartheta = \mu(1 - \frac{b^2}{8\pi} + \ldots) \tag{23}$$

The term of first order shows that our exact solution is correct in the classical limit, i.e. for $b \to 0$. The term of second order shows that the $B\vartheta = \mu$ relation suggested in [23] is not valid: $B\vartheta$ acquires nontrivial quantum correction. Since the renormalization of the parameter μ is crucial to decide about the two proposals we perform a perturbative check at order b^2.

4.2 Perturbation theory

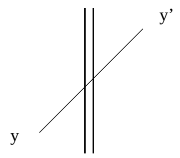
As a first step we collect the free propagators. If the fields are on the same side of the defect we have

$$_0\langle 0|T\left(\Phi_\pm(x,t)\Phi_\pm(x^{'},t^{'})\right)|0\rangle_0 = \int \tfrac{d^2q}{(2\pi)^2}\tfrac{i}{q^2-m_{\rm cl}^2+i\epsilon}e^{iq(y-y^{'})} = G_\pm^\pm(y,y')$$



where $q=(k,\omega)$ and $y=(x,t)$. The absence of an $e^{ik(x+x^{'})}$ term shows the absence of reflection. The other two point functions are

$$_0\langle 0|T\left(\Phi_\mp(x,t)\Phi_\pm(x^{'},t^{'})\right)|0\rangle_0 = \int \tfrac{d^2q}{(2\pi)^2}\tfrac{i}{q^2-m_{\rm cl}^2+i\epsilon}T_\pm(\omega,k)e^{iq(y-y^{'})} = G_\mp^\pm(y,y')$$



where

$$T_\pm(\omega,k) = -\frac{m_{\rm cl}\sinh\mu \pm i\omega}{m_{\rm cl}\cosh\mu - ik}$$

In the final equations we used the fact that the ω contour can be closed on the upper/lower half plane. As it was shown in [8] the reflection factor can be read off from the $\langle T(\Phi_\pm\Phi_\pm)\rangle$propagator of the fields. The defect/boundary equivalence [21] then implies that the transmission factor can be read off from the $\langle T(\Phi_\mp\Phi_\pm)\rangle$ propagator.

The perturbation at order b^2 follows from (1):

$$\begin{aligned}\delta\mathcal{L} &= -\Theta(-\zeta)\left[\frac{m_{\rm cl}^2 b^2}{4!}\Phi_-^4\right] - \Theta(\zeta)\left[\frac{m_{\rm cl}^2 b^2}{4!}\Phi_+^4\right] \\ &\quad -\delta(\zeta)\frac{m_{\rm cl}b^2}{4\cdot 4!}\left[\cosh\mu\left(\Phi_+^4+6\Phi_+^2\Phi_-^2+\Phi_-^4\right)+4\sinh\mu\,\Phi_+\Phi_-\left(\Phi_+^2+\Phi_-^2\right)\right]\end{aligned}$$

We calculate the propagators upto first order in b^2:

$$\langle 0|T\left(\Phi_\mp(x,t)\Phi_\pm(x^{'},t^{'})\right)|0\rangle = {}_0\langle 0|T\left(\Phi_\mp(x,t)\Phi_\pm(x^{'},t^{'})(1-i\int d\zeta\int d\tau\,\delta\mathcal{L}+\ldots)\right)|0\rangle_0$$

Using Wick's theorem we obtain the contribution of the following diagrams:

We have two bulk diagrams presented on Figure 1:

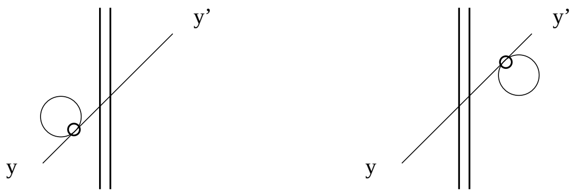


Figure 1: The two bulk diagrams with prefactor $\frac{m_{\rm cl}^2 b^2}{2}$

where the bulk interaction point, denoted by an empty circle, represents $z = (\zeta, \tau)$ and we have to integrate over the whole left/right space-time. Thus the contribution of the first diagram is

$$\frac{m_{\rm cl}^2 b^2}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{0} dz\, G_-^-(y,z) G_-^-(z,z) G_-^+(z,y')$$

Clearly $G_-^-(z,z)$ is divergent and we have to regularize it by introducing a cutoff Λ:

$$G_-^-(z,z) = \int \frac{d^2q}{(2\pi)^2} \frac{i}{q^2 - m_{\rm cl}^2 + i\epsilon} = \int_0^{\Lambda} \frac{1}{\sqrt{k^2 + m_{\rm cl}^2}} \frac{dk}{2\pi} = \Delta(m_{\rm cl})$$

Its contribution can be absorbed into the renormalization of the mass parameter $m_{\rm cl}^2 \to m_{\rm cl}^2 - m_{\rm cl}^2 \frac{b^2}{2} \Delta(m_{\rm cl})$ which results in extra counter term diagrams presented on Figure 2:

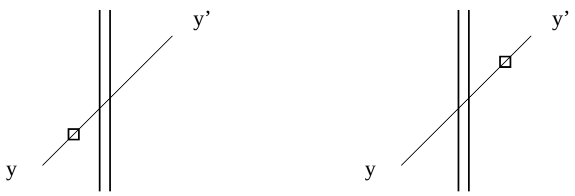


Figure 2: Bulk counter-term diagrams with prefactor $-m_{\rm cl}^2 \frac{b^2}{2} \Delta(m_{\rm cl})$

The contributions from the defect terms can be grouped in two sets of diagrams. The first contains the same divergent loop integral and consists of those on Figure 3:

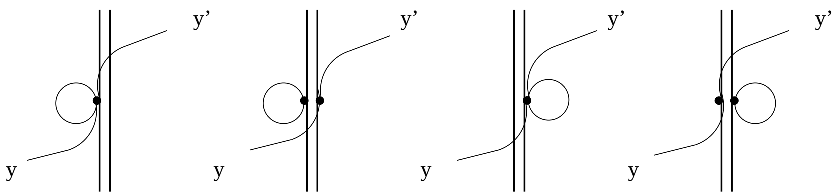


Figure 3: Divergent even defect loop diagrams with prefactor $\frac{m_{\rm cl} b^2}{8} \cosh \mu$

where the interaction vertex is even together with the odd diagrams presented on Figure 4:

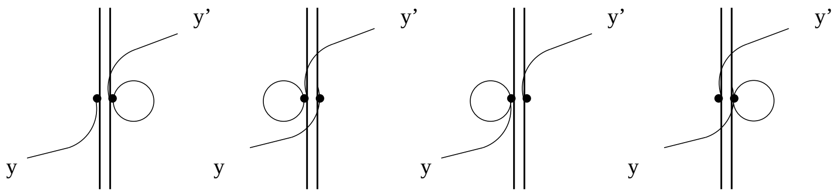


Figure 4: Divergent odd defect loop diagrams with prefactor $\frac{m_{\rm cl} b^2}{8} \sinh \mu$

We have to integrate in time τ over the real axis and the left/right part of the defect represents the contraction with the operators Φ_- and Φ_+, respectively. They all contain the divergent and regularized $\Delta(m_{\rm cl})$ loop integral which can be absorbed into the renormalization of the defect parameter $m_{\rm cl} \to m_{\rm cl} - m_{\rm cl} \frac{b^2}{4} \Delta(m_{\rm cl})$, which is consistent with the bulk renormalization. The resulting counter-terms produce the diagrams on Figure 5:

Figure 5: Defect counter terms with prefactors $-m_{\rm cl} \frac{b^2}{4} \Delta(m_{\rm cl}) \cosh \mu$ and $-m_{\rm cl} \frac{b^2}{4} \Delta(m_{\rm cl}) \sinh \mu$, respectively

The fact that all these singularities can be absorbed into the renormalization of $m_{\rm cl}$ is a nontrivial statement, since we have eight divergent diagrams having different propagators on the outer legs those we canceled just by renormalizing one single parameter in the original Lagrangian. The form of the renormalized Lagrangian is the same as the original one thus the integrable/topological nature of the defect is not spoiled by quantum effects, there is no anomaly. Observe also that the bulk mass term $m_{\rm cl}^2$ and the boundary term $m_{\rm cl}$ renormalizes the same way so the bulk is the square of the other.

The last group of the diagrams is the one which really contributes to the transmission factor. They are presented on Figure 6:

Figure 6: Defect diagrams contributing to the transmission factor. They have prefactors $\frac{m_{\rm cl} b^2}{4} \sinh \mu$ and $\frac{m_{\rm cl} b^2}{4} \cosh \mu$, respectively

The contribution of the first diagram is

$$\frac{m_{\rm cl}b^2}{4}\sinh\mu\int_{-\infty}^{\infty}d\tau\,G_-^-(y,z)G_-^+(z,z)G_-^+(z,y')$$

Each of the terms on Figure 6 contains the finite contribution of the propagator

$$G_\pm^\mp(z,z)=\int\frac{d^2q}{(2\pi)^2}\frac{i}{q^2-m_{\rm cl}^2+i\epsilon}T_\pm(\omega,k)=-\frac{\mu}{2\pi}$$

This term together with the prefactor can be interpreted as the finite renormalization of the parameter $\mu\to\mu+\delta\mu$, where $\delta\mu=-\frac{\mu b^2}{8\pi}$. Summing up all the contributions and taking into account the different transmission factor dependent contributions on the outer legs we obtain the correction of order b^2 to the transmission factor as

$$T_-(\theta,B\vartheta(\mu))=T_-(\theta,\mu)+\frac{\mu b^2}{8\pi}\frac{1}{1-i\sinh(\theta+\mu)}+\mathcal{O}(b^4)$$

which is in complete agreement with (23). Thus we confirmed the renormalization of the parameter μ, but we have not checked the renormalization of the parameter M which has not shown up at this order. We suspect, however, that in this perturbative scheme the boundary parameter $m_{\rm cl}$ renormalizes as the square-root of the $m_{\rm cl}^2$ term and only μ renormalizes as $B\vartheta$. It would be interesting to perform a two-loop perturbative calculation to decide about the renormalization of $M_{\rm cl}$ in the perturbative scheme.

We note that we have also performed a perturbative calculation of order b^2 of the reflection factor, which can be extracted from $G_+^+(y,y')$, and confirmed the absence of reflection at this order. In [39] the form of the sine-Gordon Lagrangian was fixed at order b^6 by demanding the absence of particle creation. Following a similar line it would be tempting to see how the absence of reflection restricts the form of the defect potential in perturbation theory.

5 Defect thermodynamic Bethe ansatz

In this section we would like to check the consistency between the transmission factors and defect energies. In doing so we derive a DTBA to describe the ground state energy of a purely transmitting diagonal integrable defect on the circle of perimeter L. In boundary and defect systems there are two inequivalent ways to derive TBA equations. We can either focus on the groundstate energy or on the so-called g-factors which is related to the finite volume normalization of defect/boundary states. Both BTBA equations have been analyzed in [16] although the g-function type required further refinement [40]. Some details of the g-function type DTBA can be found in [41] and references therein. Here we consider the groundstate DTBA in the periodic setting as opposed to the strip geometry analyzed in [21].

5.1 Derivation of DTBA

In order to derive the DTBA equation for the ground state energy we compactify the time-like direction with period R and calculate the partition function in two inequivalent ways by changing the role of the space and time coordinates.

In the original description the defect is located in space as drawn in Figure 7:

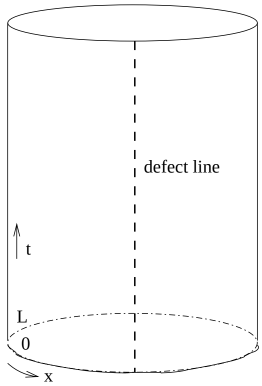


Figure 7: Defect is located in space

By taking the $R \to \infty$ limit the groundstate energy can be extracted from the partition function as

$$\lim_{R\to\infty} Z(L,R) = \lim_{R\to\infty} Tr\left(e^{-H(L)R}\right) = e^{-E_0(L)R} + \dots$$

In the alternative description when the role of time and space is exchanged as shown on Figure 8:

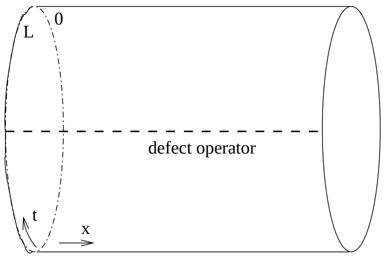


Figure 8: Defect is located in time acting as a defect operator

the defect becomes an operator of the form [21]

$$D = \exp\left\{\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}T_+(\frac{i\pi}{2}-\theta)a^+(\theta)a(\theta)\right\}$$

which acts on the Hilbert space of the periodic model, $\mathcal{H}$. The partition function can be calculated as

$$Z(L,R) = Tr\left(e^{-H(R)L}D\right) = \sum_{|n\rangle\in\mathcal{H}}\frac{\langle n|D|n\rangle e^{-E_n(R)L}}{\langle n|n\rangle} \tag{24}$$

The Hilbert space consists of multi-particle states

$$|\theta_1,\theta_2,\dots,\theta_n\rangle = a^+(\theta_1)a^+(\theta_2)\dots a^+(\theta_n)|0\rangle \quad ; \qquad \theta_1 > \theta_2 > \dots > \theta_n$$

on which the defect operator collects nontrivial diagonal matrix elements from

$$D|\theta_1,\theta_2,\dots,\theta_n\rangle = T_+(\frac{i\pi}{2}-\theta_1)T_+(\frac{i\pi}{2}-\theta_2)\dots T_+(\frac{i\pi}{2}-\theta_n)|\theta_1,\theta_2,\dots,\theta_n\rangle + \dots$$

while the energy operator acts as

$$H|\theta_1,\theta_2,\dots,\theta_n\rangle = (m\cosh(\theta_1)+m\cosh(\theta_2)+\dots+m\cosh(\theta_n))\,|\theta_1,\theta_2,\dots,\theta_n\rangle$$

We can introduce another energy operator via $\hat{H} = H - \frac{1}{L}\log D$ such that the partition function can be written as

$$Z(L,R) = Tr\left(e^{-H(R)L}D\right) = Tr\left(e^{-\hat{H}(R)L}\right) = \sum_{|n\rangle\in\mathcal{H}} e^{-\hat{E}_n(R)L}$$

This partition function can be calculated in the $R\to\infty$ limit by standard saddle point approximation taking into account the scattering of the particles. The calculation follows the usual route of TBA calculations, but now the kinetic term is shifted $m\cosh\theta \to m\cosh\theta - \frac{1}{L}\log T_+(\frac{i\pi}{2}-\theta)$. As a consequence we obtain the following DTBA equations for the pseudo energy

$$\tilde{\epsilon}(\theta) = mL\cosh\theta - \log T_+(\frac{i\pi}{2}-\theta) - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\phi(\theta-\theta')\log(1+e^{-\tilde{\epsilon}(\theta')}) \qquad (25)$$

where $\phi(\theta) = -i\frac{d}{d\theta}\log S(\theta)$. Once $\tilde{\epsilon}$ is known the ground state energy can be expressed as

$$E_0(L) = -m\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\,\log(1+e^{-\tilde{\epsilon}(\theta)})$$

Here we do not have to shift the $\cosh\theta$ term since it comes from the derivative of the momentum ($\sinh\theta$), which appears in the quantization condition. For the result in this generality see e.g. [42].

Alternatively, we can redefine the pseudo energy as $\tilde{\epsilon}(\theta) = \epsilon(\theta) - \log T_+(\frac{i\pi}{2}-\theta)$ to obtain

$$\epsilon(\theta) = mL\cosh\theta - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\phi(\theta-\theta')\log\left(1+T_+(\frac{i\pi}{2}-\theta')e^{-\epsilon(\theta')}\right)$$

from which the ground state energy turns out to be

$$E_0(L) = -m\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\,\log\left(1+T_+(\frac{i\pi}{2}-\theta)e^{-\epsilon(\theta)}\right)$$

The ground state energy is real which can be easily seen form (6) since $T_+^*(\frac{i\pi}{2}-\theta) = T_+(\frac{i\pi}{2}+\theta)$. As a simple consistency check we can see that the DTBA equation for the trivial defect, $T_+=1$, reduces to the periodic TBA equation [43]. We analyze the large and small volume limits separately in the next two subsections.

5.2 Lüscher type correction in defect systems

If the volume L is large then $\epsilon(\theta)\cong mL\cosh\theta$ is large and we can expand the logarithm to obtain

$$E_0(L) = -m\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\,T_+(\frac{i\pi}{2}-\theta)e^{-mL\cosh\theta} + O(e^{-2mL})$$

This result can be calculated directly from (24) by taking the large L limit there. In that case, however, only the one-particle transmission term contributes which is universal for any quantum field theory. The result obtained is the analogue of the boundary Lüscher type correction to the ground state energy [44] and is valid in any theory even in non-integrable ones.

5.3 Defect energy

Here we analyze the small volume behavior of the ground state energy. Its normalization depends on the scheme in which the quantum field theory is defined. If we would like to compare the DTBA normalization to that of a perturbed defect conformal field theory, in which the perturbing operators have dimensions h on the defect and (h,h) in the bulk, then we have:

$$E_0(L) = -\epsilon_{\text{def}} - \epsilon_{\text{bulk}} L + \frac{2\pi}{L}\sum_{n=0}^{\infty} c_n \, l^{n(1-h)} \quad ; \qquad l = mL$$

Only the perturbative terms, c_n, are present in a perturbed rational defect CFT. (In non-rational CFT-s, like the UV limit of the boundary sinh-Gordon theory, we expect terms with logarithmic behaviour, see [45] for the details). By calculating the small volume limit of $E_0(L)$ from DTBA ϵ_{def} and ϵ_{bulk} can be extracted exactly. The computation is analogous to the boundary one [31, 36], so we sketch only here. In the $L \to 0$ limit the solution for $\tilde{\epsilon}$ in (25) develops two kink regions around $\theta = \pm \log \frac{2}{l}$ and a breather region around the origin. The behaviour of the solutions are determined by the $\theta \to \pm\infty$ asymptotics of the integral kernel and defect source term:

$$\phi(\theta) = Ce^{-|\theta|} + O(e^{-2|\theta|}) \quad ; \quad \log(T_+(\frac{i\pi}{2} - \theta)) = A_\pm e^{\mp\theta} + O(e^{\mp 2\theta}) \qquad \text{as} \quad \theta \to \pm\infty$$

The two kink functions are responsible for the terms giving the central charge and the bulk energy constant, while the central/breather part gives the defect energy in the following form:

$$\epsilon_{\text{bulk}} = \frac{m^2}{2C} \quad ; \qquad \epsilon_{\text{def}} = -\frac{m(A_+ + A_-)}{2C} \tag{26}$$

We note that the kink type behaviour does not exists for the whole parameter range of C and $A_\pm$. The results is understood that we analytically continued it from a range where the calculation is reliable.

Let us concretes the result for the two cases in question. In the sinh-Gordon model

$$C = 4\sinh \pi B \quad ; \qquad A_\pm = -2e^{\mp B\vartheta}$$

so using (26) we recover (9) and (18).

In the Lee-Yang case we have

$$C = -2\sqrt{3} \quad ; \qquad A_\pm = \mp 2i(e^{\pm i\pi\frac{b+1}{6}} + e^{\pm i\pi\frac{b-1}{6}})$$

Plugging these expressions back to (26) the results confirms the bulk energy density and the defect energy.

We emphasize that the agreement obtained in the two cases confirm the solutions on one side and the DTBA equation on the other.

The other perturbative coefficients c_n can be calculated from the DTBA only numerically. In the Lee-Yang case, however, one can gain further analytical information. One has to define $Y(\theta) = e^{-\epsilon(\theta)}$ and to show from (21) that it satisfies the Lee-Yang Y-system relation

$$Y(\theta - \frac{i\pi}{3})Y(\theta + \frac{i\pi}{3}) = 1 + Y(\theta)$$

from which the $Y(\theta) = Y(\theta + \frac{5i\pi}{3})$ periodicity follows. Similarly to the boundary case this gives the exponent of the perturbative expansion to be $\frac{6}{5}$ showing that the dimension of the perturbing operator is $h = -\frac{1}{5}$.

6 Defect bound-states and bootstrap closure

In this section we analyze the analytic structure of the transmission factors for the whole range of their parameters. Since the sinh-Gordon transmission factors (16) never have poles in the physical strip we focus on the Lee-Yang model only. Recall that in our convention the physical strip of the transmission factors $T_{\mp}(\theta)$ are $\Im m\theta \in [0, \frac{\pi}{2}]$.

6.1 Pole analysis on the ground-state defect

We analyze the pole structure of both

$$T_-(\theta) = [b+1][b-1] \quad \text{and} \quad T_+(\theta) = [5-b][-5-b]$$

as the function of the parameter b, simultaneously. We note that by folding the theory to a boundary one (with two particles) we could analyze its bootstrap in the usual boundary formulation. Here, however, we present the results in the defect language since the diagrams are more clear-cut. (The whole procedure, going from the reflection factor to the defect transmission factor, can be interpreted as taking a sort of square root of the boundary theory. By closing the defect bootstrap we would like to show that such a theory is indeed sensible. Since on the $\mathbb{I}$ boundary there is no boundstate any pole of the reflection factor appears either in T_- or in T_+ so the bootstrap will be very similar to the boundary one [6]).

In determining the fundamental range of the parameter b we can see that $b \to b + 12$ is a symmetry. Moreover $b \leftrightarrow 6 - b$ exchanges $T_- \leftrightarrow T_+$ so we can restrict ourselves to the range $b \in [-3, 3]$. We will see by analyzing the defect excited states that the fundamental range is even smaller, only $b \in [-3, 2]$, as in the boundary case, since at $b = 2$ the role of the ground-state and the first excited state is exchanged.

The poles and zeros of the transmission factor $T_-(\theta)$ are

$$\text{poles of } T_- \text{ are at } \theta = -i\frac{\pi}{6}(b \pm 5) \quad ; \qquad \text{zeros of } T_- \text{ are at } \theta = -i\frac{\pi}{6}(b \pm 1)$$

The analogous expressions for T_+ are

$$\text{poles of } T_+ \text{ are at } \theta = i\frac{\pi}{6}(b \pm 1) \quad ; \qquad \text{zeros of } T_+ \text{ are at } \theta = i\frac{\pi}{6}(b \pm 5)$$

They can be drawn as the function of the parameter b as shown on Figure 9:

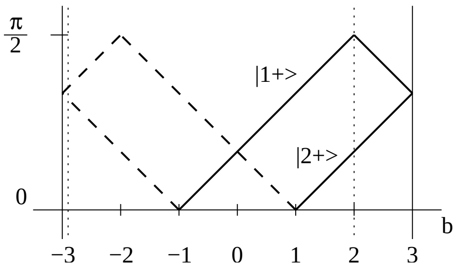


Figure 9: Poles and zeros of T_+ and T_- as function of b . Solid lines correspond to poles, while dashed ones to zeroes. The dotted lines show the fundamental range.

For $b \in [-1, 2]$ there is a pole in the transmission factor T_+ at $\theta = iu = i\frac{\pi}{6}(b + 1)$, for which we associate a defect boundstate and denote it by $|1+\rangle$. Its energy is $m \cos \frac{\pi}{6}(b + 1)$ and the corresponding excited transmission factor can be calculated from the defect bootstrap equation shown on Figure 10:

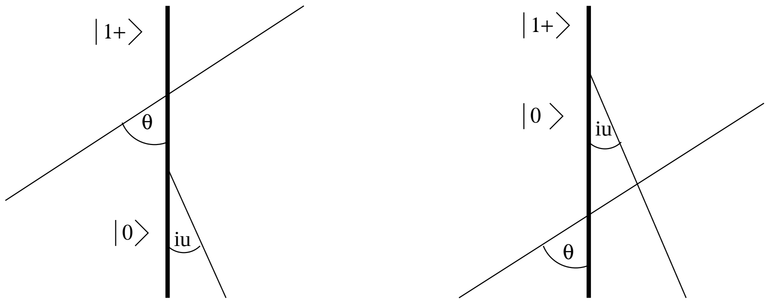


Figure 10: Defect bootstrap equations

$$T_-^{|1+\rangle}(\theta) = T_-(\theta)S(\theta + iu)$$

From the defect crossing symmetry (6) we can calculate $T_+^{|1+\rangle}(\theta)$ as

$$T_+^{|1+\rangle}(\theta) = T_-^{|1+\rangle}(i\pi - \theta) = T_-(i\pi - \theta)S(i\pi - \theta + iu) = T_+(\theta)S(\theta - iu)$$

which is consistent with the other bootstrap equation where the second particle arrives from the right. The resulting transmission factors are

$$T_-^{|1+\rangle}(\theta) = [b + 1][b + 3] \quad ; \qquad T_+^{|1+\rangle}(\theta) = [5 - b][3 - b]$$

They are related to the groundstate ones as $T_\pm^{|1+\rangle}(b \to 4 - b, \theta) = T_\mp(b, \theta)$. This symmetry together with the defect energies indicate that when b exceeds 2 the role of the ground-state and the excited state $|1+\rangle$ are exchanged. This confirms that the fundamental range is indeed $b \in [-3, 2]$.

In the range $b \in [1, 2]$ the transmission factor $T_+(\theta)$ has another pole at $\theta = i\frac{\pi}{6}(b - 1)$ for which we associate the defect boundstate $|2+\rangle$. It has energy $m \cos \frac{\pi}{6}(b - 1)$ and transmission factor

$$T_\pm^{|2+\rangle}(\theta) = T_\pm(\theta)S(\theta \mp i\frac{\pi}{6}(b - 1))$$

Now we turn to the pole analysis of excited defect states.

6.2 Pole analysis on the exited defect state $|1+\rangle$

The poles and zeros of the transmission factors on the state $|1+\rangle$ are indicated on Figure 11.

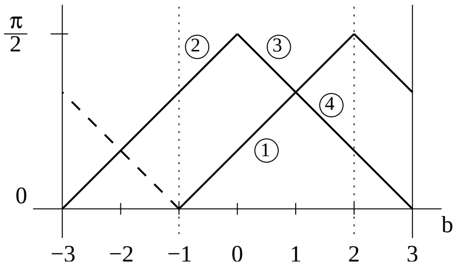


Figure 11: Poles and zeros on the $|1+\rangle$ defect excited state. Dotted line shows the range where the excited state $|1+\rangle$ exists

The state exist in the $b \in [-1, 2]$ domain so we have to explain the poles in this range only.

The pole of $T_+^{|1+\rangle}$ labeled by 1 on Figure 11 is at the same location as the one which creates the excited state itself, namely at $\theta = i\frac{\pi}{6}(b+1)$ in the full range $b \in [-1, 2]$. It can be explained by the first of the defect Coleman-Thun diagrams on Figure 12

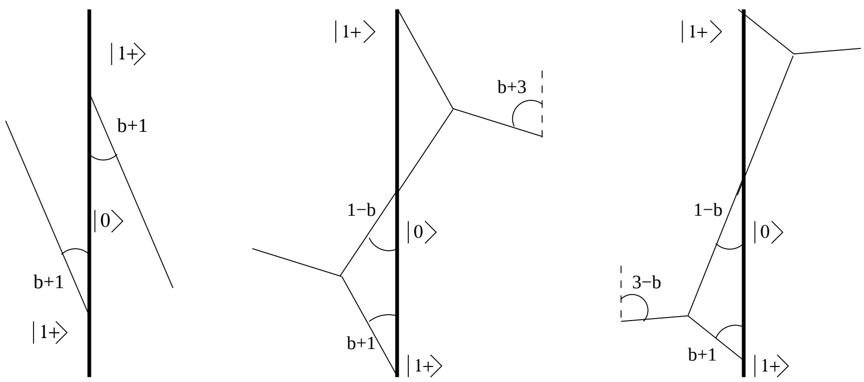


Figure 12: Defect Coleman-Thun diagrams for the $|1+\rangle$ state. The angles are measured in units of $\frac{i\pi}{6}$

The pole of $T_+^{|1+\rangle}$ labeled by 2 on Figure 11 is at $\theta = i\frac{\pi}{6}(b+3)$ and can be explained by the second diagram on Figure 12. Observe that by applying the Cutkosky rules [9] we would obtain a pole of second order but the transmission factor T_- has a first order zero at $\theta = i\frac{\pi}{6}(1-b)$ which, in this way, reduces the order of the pole to one.

The pole of $T_-^{|1+\rangle}$ labeled by 3 on Figure 11 is at $\theta = i\frac{\pi}{6}(3-b)$. In the range $b \in [0, 1]$ it can be explained by the third diagram on Figure 12. Since the transmission factor T_- on the ground state has a zero at $\theta = i\frac{\pi}{6}(1-b)$ the order of the diagram is reduced to one again. In order for the diagram to exist the particle has to travel towards the defect, that is $1-b>0$. This explains the pole in the range $b\in[0,1]$. In the range $b\in[1,2]$ the particle creates a defect boundstate which is nothing but $|2+\rangle$. This can be seen both from the energy of the excited state $m\cos\frac{\pi}{6}(b+1)+m\cos\frac{\pi}{6}(3-b)=m\cos\frac{\pi}{6}(b-1)$ and from the transmission factor. If the left particle creates a defect boundstate at rapidity $\theta=iu$ then the excited states transmission factors are $T_{\pm}^{ex}(\theta)=T_{\pm}(\theta)S(\theta\pm iu)$. Now we can see from the bulk bootstrap equation that

$$T_{\pm}^{|1+\rangle}(\theta)S(\theta\pm i\frac{\pi}{6}(3-b))=T_{\pm}(\theta)S(\theta\mp i\frac{\pi}{6}(b+1))S(\theta\pm i\frac{\pi}{6}(3-b))=T_{\pm}(\theta)S(\theta\mp i\frac{\pi}{6}(b-1))=T_{\pm}^{|2+\rangle}(\theta)$$

that is the transmission factors also supports the identification.

6.3 The pole analysis on the excited defect state $|2+\rangle$

The defect boundstate labeled by $|2+\rangle$ has transmission factor

$$T_{-}^{|2+\rangle}(\theta)=[b-1][b+1]^2[b+3]\quad;\qquad T_{+}^{|2+\rangle}(\theta)=[3-b][5-b]^2[7-b]$$

The singularity structure can be summarized as follows.

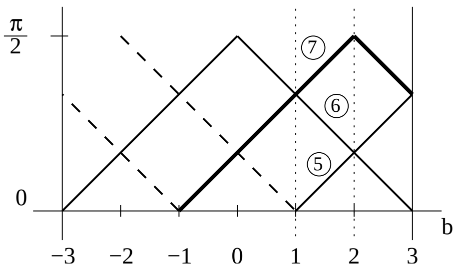


Figure 13: Singularity structure of the transmission factors on the $|2+\rangle$ state. Bold straight lines represent poles of second order. The relevant interval where the boundstate $|2+\rangle$ exists is indicated by dotted lines.

The pole labeled by 5 on Figure 13 is in $T_{+}^{|2+\rangle}(\theta)$ at $\theta=i\frac{\pi}{6}(b-1)$ and can be explained in the full range $b\in[1,2]$ by the first diagram on Figure 12, if we replace $|1+\rangle$ by $|2+\rangle$.

The pole labeled by 6 on Figure 13 is in $T_{-}^{|2+\rangle}(\theta)$ at $\theta=i\frac{\pi}{6}(3-b)$ and can be explained by a diagram similar to the third one of Figure 12 in which the $|1+\rangle$ state is replaced by $|2+\rangle$ and the vacuum $|0\rangle$ is replaced by $|1+\rangle$.

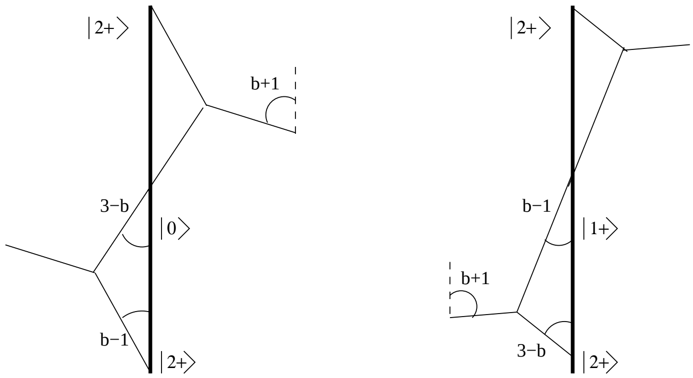


Figure 14: Defect Coleman-Thun diagrams for the excited state $|2+\rangle$

The pole labeled by 7 on Figure 13 is a second order one in $T_{+}^{|2+\rangle}(\theta)$ at $\theta = i\frac{\pi}{6}(b+1)$ and can be explained by the two diagrams on Figure 14. Clearly the transmission factor $T_{-}(\theta)$ does not have zeros neither at $\theta = i\frac{\pi}{6}(3-b)$ nor at $\theta = i\frac{\pi}{6}(b-1)$ so the pole is of second order.

By now we explained all the poles of all the transmission factors of the ground and excited defect states. We used either the creation of a new defect boundstate or presented the appropriate defect Coleman-Thun diagram which was responsible for the singularity. By finishing this procedure the spectrum become complete and we managed to define a sensible defect theory. It would be nice to check these findings by the defect truncated conformal space approach (TCSA).

7 Conclusions

We have demonstrated how the fusion idea can be used to solve topological defects in the sinh-Gordon and Lee-Yang models. In the sinh-Gordon case we determined the transmission factors and the defect energy as a function of a bootstrap parameter whose relation to the Lagrangian was also given. We checked these results in perturbation theory and against the newly derived DTBA.

In the Lee-Yang case we determined the transmission factors together with the defect energy and checked them in DTBA. For certain range of the parameter the transmission factor admits poles in the physical strip. We closed the defect bootstrap programme: we explained all poles either by associating new defect boundstates or by giving the appropriate defect Coleman-Thun mechanism both for the groundstate and for excited defect states.

The relation obtained between the transmission parameter and that of the Lagrangian in the sinh-Gordon theory can be analytically continued to describe the analogues relation in the sine-Gordon theory. This result also passes the test of first order perturbation theory and together with the transmission factors obtained in [22, 23] gives the complete solution of defect sine-Gordon model. We have checked this solution by performing the fusing procedure on the solitonic transmission factors. This is analogous to the dressing procedure in the XXZ spin chain developed in [46].

The perturbation theory developed here can also be used in higher rank affine Toda theories to connect the parameters of the transmission factor of the bootstrap solution [24] to the parameters of their Lagrangians [18].

The derivation of the DTBA generalizes to any diagonal scattering theory. A large and small volume analysis analogous to the one presented in the paper will provide the leading finite size correction to the groundstate energy and give the bulk/defect energies, respectively.

In the present paper we were concerned with the bootstrap (IR) description of our models. There is a need, however, to understand their UV behavior which probably can be described by perturbed defect CFTs. To connect these alternative descriptions we can use methods starting either from the IR side, like DTBA, or starting from the UV side, like defect TCSA. There are works in progress in both directions.

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