## Null vectors of the $WBC_2$ algebra

Z. Bajnok\*

Institute for Theoretical Physics Roland Eötvös University H-1088 Budapest, Puskin u. 5-7, Hungary

March 20, 2018

## Abstract

Using the fusion principle of Bauer et al. we give explicit expressions for some null vectors in the highest weight representations of the  $WBC_2$  algebra in two different forms. These null vectors are the generalization of the Virasoro ones described by Benoit and Saint-Aubin and analogues of the  $W_3$  ones constructed by Bowcock and Watts. We find connection between quantum Toda models and the fusion method.

<sup>\*</sup>E-mail: bajnok@ludens.elte.hu

Conformal field theories describe the statistical mechanical systems at their critical point. Their infinite dimensional symmetry algebra, the Virasoro algebra or its extension, plays an important role in analyzing them. In most cases these models correspond to degenerate representations of these algebras. Such representations are crucial since on one hand they possess a closed operator algebra with only a finite number of primary fields, while on the other hand they contain null vectors. These null vectors lead to differential equations for the correlation functions, and in this sense, they can be regarded as equations of motion governing the dynamics of the theory. Although this application needs explicit expressions for the null vectors such formulas were not given in general until the last few years.

Recently there has been great interest to describe these null vectors [1-6]. First Benoit and Saint Aubin constructed very compact expressions for a subclass of the Virasoro null vectors then Bauer et al. rewrote this formula using the fusion method. They showed how to produce null vectors in the generic case by fusing explicitly known null vectors with primary fields. Later Bowcock and Watts extended this method to W algebras. They found that the fusions of the null vectors and of their descendants should be investigated simultaneously.

In this paper we use their method to produce null vectors in the  $WBC_2$  algebra. Explicit expressions for the simplest null vectors are given. In the fusion method we deal with only those primary fields which belong to completely degenerate representations and have two independent null vectors at level one and two. It turns out that there is a special null vector among the descendants ones. This null vector is nothing but the quantum equation of motion of the appropriate Toda model and in order to produce null vectors one has to study the fusion of this descendant null vector. We start with the representation theory of the  $WBC_2$  algebra. Then we describe how to handle the action of a spin j chiral current in the fusion procedure. Finally, using this result we study two different cases namely the case of the  $B_2$  and  $C_2$  Toda models, respectively.

The  $WBC_2$  algebra is one of the simplest W algebras. It is an extension of the Virasoro algebra generated by the Virasoro field, L(z), and a spin 4 chiral current, W(z) [7]. The highest weight (hw.) representation of the algebra contains a hw. vector,  $|hw\rangle$ , which satisfies:

$$L_n|hw\rangle = \delta_{n,0}h|hw\rangle \quad ; \quad W_n|hw\rangle = \delta_{n,0}w|hw\rangle \quad ; \quad n \ge 0 \tag{1}$$

In order to analyze the determinant formula [8] we use the following reparametrisation of the W weights :

$$h(x,y) = \frac{Q}{4}(x^2 + 2xy + 2y^2) - \frac{1}{4}(5Q + \frac{10}{Q} - 14)$$

$$w(x,y) = \bar{B}\left(4Q(Q-3)(27Q-32)y^3(y+2x) - Q(3Q-2)(16Q-27)x^3(x+4y) + \frac{(Q^2-2)}{Q}\left(14Q(x^2+2xy+2y^2-6Qx^2y^2) + Q - 6 + \frac{2}{Q}\right)\right)$$
(2)

where

$$\bar{B}^{-1} = \frac{32}{Q^2} \sqrt{(4Q-5)(5Q-8)(7Q-6)(3Q-7)(3Q-2)(Q-3)(75Q^2-226Q+150))}$$

Here we used Q instead of the central charge (C). They are related to each other as

C = 86 - 60/Q - 30Q. Now the determinant formula dictates that if

$$x = a - c \frac{2}{Q} \quad \text{or} \quad y = b - d \frac{1}{Q} \tag{3}$$

then the representation is degenerate and there is a null vector level ac or bd. If both conditions hold at the same time then the representation is completely degenerate and contains two independent null vectors. We restrict ourselves to only such representations. The hw. vector in this case is denoted by  $|ab; cd\rangle$  and the two independent null vectors by  $\mathcal{O}_{a;c}|ab; cd\rangle$  and  $\mathcal{O}_{b;d}|ab; cd\rangle$ , respectively.

To produce null vectors we follow the method of Bauer et al.. They considered the fusion of a primary field,  $\phi_{h_0w_0}(z)$ , belonging to a degenerate representation, with another primary field,  $|hw\rangle = \phi_{hw}(0)$ . Setting the null vector equal to zero and demanding its consistency implies recursive equations for the descendant fields appearing in the OPE. Solving these equations iteratively explicit expression can be obtained for the null states in the corresponding representations. Everything one needs to know is the action of generators after the fusion. Using contour deformation technique for an arbitrary spin j generator we have:

$$\{W_{-n}\phi_{h_0w_0}(z)\}|hw\rangle = \left\{\frac{(-1)^{n+j}}{z^n} \left[\binom{n-1}{n-j}w_0 - \sum_{k=1}^{j-1} z^k f_{k,n,j}(1)W_{-k}\right]\phi_{h_0w_0}(z)\right\}|hw\rangle + \left\{\frac{(-1)^{n+j}}{z^n} \sum_{k=1}^{j-1} z^k f_{k,n,j}(q)W_{-k} + \sum_{l=0}^{\infty} \binom{n+l-j}{l}(pz)^l W_{-n-l}\right\}\phi_{h_0w_0}(z)|hw\rangle$$

$$(4)$$

where we have used  $f_{k,n,j}(q) = \sum_{i=1}^{k} {\binom{n-i-1}{n-j}} {\binom{j-i-1}{k-i}} q^{k-i}$ . The fusion point, the argument of the resulting field, is qz and p = 1 - q. For j = 2 the formula yields the result of [2] if we use  $L_{-1}\phi(z) = \partial_z\phi(z)$ . Unfortunately in the  $WBC_2$ , j = 4, case we have unknown modes:  $W_{-1}$ ;  $W_{-2}$ ;  $W_{-3}$ . However following the idea of Bowcock and Watts [3] we are able to solve this problem. In the  $WA_2$  case they could eliminate this kind of modes considering not only the two independent null vectors but also their descendants simultaneously.

For simplicity we deal with the following case: completely degenerate representations having null vectors at level one and two. First we analyze the  $u = |21; 11\rangle$  field. The h, w of this field are  $\Delta, \omega$ , respectively:

$$\Delta = \frac{1}{4}(5Q - 6)$$

$$\omega = -\frac{Q}{8}\Delta \sqrt{\frac{(4Q - 5)(\frac{8}{Q} - 5)(\frac{2}{Q} - 3)(\frac{6}{Q} - 7)}{(226 - 75Q - \frac{150}{Q})(Q - 3)(3Q - 7)}}$$
(5)

Analyzing the  $C_2$  Toda model in WZNW framework we obtained [9] that the only relevant operator, u, has a null state at level four. Studying the covariance of this quantum equation of motion we concluded that u must belong to a completely degenerate representation and so it has the following null states. At level one the null vector is:

$$\mathcal{O}_{-1}u = W_{-1}u + \beta^1 L_{-1}u = 0 \tag{6}$$

where

$$\beta^1 = -2(4Q - 5)(8/Q - 5)(7Q - 6)(3Q - 2)N$$

and

$$N = \frac{\omega}{\Delta} \frac{1}{(4Q-5)(8/Q-5)(7Q-6)(3Q-2)}$$

At level two this completely degenerate representation has an independent null vector. However it is more convenient to use the following null state:

$$\mathcal{O}_{-2}u = W_{-2}u + \beta^2 L_{-2}u + \beta^{11} L_{-1}^2 u = 0 \tag{7}$$

$$\beta^2 = 2(23 - 10Q)(7Q - 6)(3Q - 2)N \quad ; \quad \beta^{11} = 4(13Q - 25)(7Q - 6)(3Q - 2)N$$

We use the following descendant null vector of the two independent null vectors at level three:

$$\mathcal{O}_{-3}u = W_{-3}u + \beta^3 L_{-3}u + \beta^{12} L_{-1}L_{-2}u + \beta^{111}L_{-1}^3u = 0$$
(8)

where

$$\beta^{111} = \frac{16}{Q} (75Q^2 - 226Q + 150)N$$
  
$$\beta^3 = -6(5Q - 22)(4Q - 5)(Q - 2)N \quad ; \quad \beta^{12} = -24(34Q^2 - 113Q + 82)N$$

Finally u satisfies the quantum Toda equation of motion, which is again a descendant null vector:

$$\mathcal{O}_{-4}u = W_{-4}u + \beta^4 L_{-4}u + \beta^{22} L_{-2}^2 u + \beta^{13} L_{-1} L_{-3}u + \beta^{112} L_{-1}^2 L_{-2}u + \beta^{1111} L_{-1}^4 u = 0$$
(9)

where

$$\begin{split} \beta^4 &= -\frac{2}{Q}(3Q-2)(10Q^3-73Q^2+312Q-360)N\\ \beta^{13} &= -8(27Q^3-263Q^2+548Q-300)N \quad ; \quad \beta^{22} = 4(3Q-2)(16Q-27)N\\ \beta^{1111} &= Q\beta^{111} \quad ; \quad \beta^{1112} = -\beta^{111} \end{split}$$

The first three null vectors carry enough information to eliminate the unknown W modes while we can use the Toda equation to fuse with another primary field. This equation indicates which representations can occur after the fusion. We did this analysis in [9] and found that the non vanishing matrix elements of u are:

$$|21;11\rangle \times |ab;cd\rangle \rightarrow \begin{cases} |a+1 \ b;c \ d\rangle \\ |a-1 \ b;c \ d\rangle \\ |a+1 \ b-1;c \ d\rangle \\ |a-1 \ b+1;c \ d\rangle \end{cases}$$
(10)

For producing null vectors we consider the following fusion:

$$|21;11\rangle \times |0r;ss'\rangle \to |1r;ss'\rangle \tag{11}$$

Using the method of [2] we expand the two OPE as

$$u(z)|0r;ss'\rangle = \sum_{n\geq 0} z^{n-y} f_n \quad ; \quad \{\mathcal{O}_{-4}u(z)\}|0r;ss'\rangle = \sum_{n\geq 0} z^{n-y-4}\tilde{f}_n \tag{12}$$

where  $y = \Delta + h_{0r;ss'} - h_{1r;ss'}$ . Now using (4) and choosing the fusion point at p = 1; q = 0 we get the following recursive equation:

$$\tilde{f}_n = \alpha_n f_n + \sum_{j \ge 1} \left\{ W_{-j} + \gamma_{n,j} L_{-j} + \beta^{22} \Lambda_{-j} \right\}$$
(13)

where

$$\alpha_n = N(75Q^2 - 226Q + 150)\frac{16}{Q^2}n(n-s)(n+rQ - s - s')(n+rQ - 2s - s')$$

$$\gamma_{n,j} = \beta^{112}(n-y-2)(n-y-3) + \beta^{13}(n-y-3)(j-2) + \beta^4(j-2)(j-3)/2 + 3(\beta^2 + \beta^{12}(n-y-2) + \beta^3(j-2))$$
(14)

and  $\Lambda_{-j} = \sum_{i\geq 1}^{j-1} L_{-i}L_{-j+i}$ . The vanishing of  $\alpha_n$  at n = 0 and n = s explicitly shows that the fusion is allowed and that there is a null state at level s, respectively. Solving these equations iteratively we find for the null state at level s:

$$\mathcal{O}_{1;s}|1r;ss'\rangle = \sum_{n_i:\sum_{i=1}^p n_i=s} \frac{\prod_{i=1}^p (W_{-n_i} + \gamma_{N_i,n_i} L_{-n_i} + \beta^{22} \Lambda_{-n_i})}{\prod_{i=1}^{p-1} \alpha_{N_i}} |1r;ss'\rangle$$
(15)

where  $N_i = \sum_{j=1}^{i} n_j$ . Here and from now on we use the ordering from left to the right. Alternative expressions can be obtained by choosing the fusion point in a different way. The result is very elegant if we use q = 1. In this case :

$$\mathcal{O}_{1;s}|1r;ss'\rangle = \sum_{n_i=1,2,3,4:\sum_{i=1}^p n_i=s} \frac{\prod_{i=1}^p \Gamma_{n_i}^{N_i}}{\prod_{i=1}^{p-1} \alpha_{N_i}} |1r;ss'\rangle$$
(16)

where

$$\Gamma_{1}^{n} = \mathcal{O}_{-1} + \left( (n-y-1)(6\beta^{11}+9\beta^{111}(n-y-2)+4\beta^{1111}(n-y-2)(n-y-3)) + 3\beta^{21}(h+y+2-n) + 2\beta^{112}(n-y-3)(h+y-n) - \beta^{13}(2h+y+1-n) \right) L_{-1} \\
\Gamma_{2}^{n} = 3\mathcal{O}_{-2} + \left( (n-y-2)(3\beta^{12}+\beta^{112}(n-y-3)) + 2\beta^{22}(h+y+2-n) \right) L_{-2} + \left( 9\beta^{111}(n-y-2) + 6\beta^{1111}(n-y-2)(n-y-3) + \beta^{112}(h+y+2-n) \right) L_{-1}^{2} \\
\Gamma_{3}^{n} = 3\mathcal{O}_{-3} + (n-y-3)(\beta^{13}L_{-3}+2\beta^{112}L_{-1}L_{-2}+4\beta^{1111}L_{-1}^{3}) \\
\Gamma_{4}^{n} = \mathcal{O}_{-4}$$
(17)

We remark that the computations for the  $|11;12\rangle$  state proceed in the same way as in the  $|21;11\rangle$  case; the final result can be obtained form eq. (15)-(17) by the  $Q \rightarrow \frac{2}{Q}$  substitution. The null vector produced in this manner is in the  $|ss';r1\rangle$  representation at level s.

There are two other cases when the representation contains two independent null vectors at level one and two. The first can be described by  $\tilde{u} = |12;11\rangle$  while the other one can be obtained from this one by the  $Q \rightarrow \frac{2}{Q}$  substitution. Both correspond to the Toda field of the  $B_2$  Toda model. The h, w weights of  $\tilde{u}$  are:

$$\tilde{\Delta} = 2Q - 2$$

$$\tilde{\omega} = \frac{1}{2} \tilde{\Delta} \sqrt{\frac{(4Q - 5)(\frac{2}{Q} - 3)(\frac{6}{Q} - 7)(Q - 3)}{(226 - 75Q - \frac{150}{Q})(\frac{8}{Q} - 5)(3Q - 7)}}$$
(18)

As before, we analyze the leading null vectors of  $\tilde{u}$  and their descendants simultaneously. The  $\tilde{u}$  field satisfies equations entirely analogous to eq.(6)-(8) the only difference is that we have to substitute every  $\beta$  with the following  $\tilde{\beta}$ : At level one we have:

$$\tilde{\beta}^1 = -2(4Q-5)(Q-3)(7Q-6)(3Q-2)(Q+1)\tilde{N}$$

where

$$\tilde{N} = \frac{\tilde{\omega}}{\tilde{\Delta}} \frac{1}{(4Q-5)(Q-3)(7Q-6)(3Q-2)(Q+1)}$$

At level two we use again instead of the independent null states their convenient combination whose coefficients are:

$$\tilde{\beta}^2 = 4Q(Q-1)(16Q-27)(Q+1)(3Q-2)\tilde{N} \quad ; \quad \tilde{\beta}^{11} = -(59Q^2-199Q+50)(Q+1)(3Q-2)\tilde{N}$$

The coefficients of the null states at level three are:  $\tilde{\beta}^{12} = 6Q(41Q^2 - 113Q + 68)(Q + 1)\tilde{N}$ 

$$\tilde{\beta}^3 = 6Q(7Q-6)(Q-3)^2(Q+1)\tilde{N}$$
;  $\tilde{\beta}^{111} = \frac{Q(Q+1)}{8}\beta^{111}$ 

However, contrary to the previous case, this representation has no null states of the form of (9) at level four. Analyzing the  $B_2$  Toda model we learn that  $\tilde{u}$  has to satisfy the following equation, which shows the existence of a descendant null state at level five:

$$\mathcal{O}_{-5}\tilde{u} = \tilde{\beta}^{14}L_{-1}L_{-4}u - \frac{2}{(Q+1)}\tilde{\beta}L_{-1}W_{-4}u + \tilde{\beta}W_{-5}u + \tilde{\beta}^{122}L_{-1}L_{-2}^2u + \tilde{\beta}^{113}L_{-1}^2L_{-3}u + \tilde{\beta}^{1112}L_{-1}^3L_{-2}u + \tilde{\beta}^{11111}L_{-1}^5u + \tilde{\beta}^{23}L_{-2}L_{-3}u + \tilde{\beta}^{5}L_{-5}u = 0$$
(19)

where

$$\begin{split} \tilde{\beta}^5 &= 4(Q+1)(Q-3)(3Q^3-2Q^2+48Q-60)\tilde{N} \quad ; \quad \tilde{\beta}^{1112} = -\frac{Q}{4}\beta^{111} \\ \tilde{\beta}^{14} &= -6(Q-3)(3Q^3-2Q^2+48Q-60)\tilde{N} \quad ; \quad \tilde{\beta}^{113} = -6(Q-3)(7Q^2-54Q+50)\tilde{N} \\ \tilde{\beta}^{122} &= 2Q(Q-3)(27Q-32)\tilde{N} \quad ; \quad \tilde{\beta}^{23} = -2Q(Q-3)(Q+1)(27Q-32)\tilde{N} \quad ; \quad \tilde{\beta}^{11111} = \frac{1}{8}\beta^{1111} \\ \end{array}$$

Eq.(19) is the quantum equation of motion of the  $B_2$  Toda model. From this equation we conclude that the possible fusions of  $\tilde{u}$  are the following:

$$|a \ b + 1; c \ d\rangle$$

$$|a - 2 \ b + 1; c \ d\rangle$$

$$|a - 2 \ b + 1; c \ d\rangle$$

$$|a \ b; c \ d\rangle \rightarrow |a \ b; c \ d\rangle$$

$$|a \ b - 1; c \ d\rangle$$

$$|a + 2 \ b - 1; c \ d\rangle$$

$$(20)$$

Considering the

$$|12;11\rangle \times |r0;ss'\rangle \rightarrow |r1;ss'\rangle$$
 (21)

fusion and using the same technique as before: i.e. expanding the two OPE as

$$u(z)|r0;ss'\rangle = \sum_{n\geq 0} z^{n-y} f_n \quad ; \quad \{\mathcal{O}_{-5}u(z)\}|r0;ss'\rangle = \sum_{n\geq 0} z^{n-y-5}\tilde{f}_n \tag{22}$$

with  $y = \tilde{\Delta} + h_{r0;ss'} - h_{r1;ss'}$  and using (4) we obtain

$$\tilde{f}_{n} = \tilde{\alpha}_{n} f_{n} + \sum_{j \ge 1} \left\{ \tilde{N} (j - 4 - \frac{2(n - y - 4)}{Q + 1}) W_{-j} + \tilde{\gamma}_{n,j} L_{-j} + \tilde{\beta}^{122} (n - y - 4) \Lambda_{-j} + \tilde{\beta}^{23} \Lambda_{-j}' \right\}$$
(23)

where

$$\tilde{\alpha}_{n} = \tilde{N} \frac{2}{Q} (75Q^{2} - 226Q + 150)n(n - s')(n + \frac{r+1}{2}Q - s - s')(n + rQ - 2s - s')(n + rQ - 2s - 2s')$$

and

$$\tilde{\gamma}_{n,j} = (n-y-4) \left( \tilde{\beta}^{114} {\binom{j-2}{2}} + (n-y-3) \left( \tilde{\beta}^{1112} (n-y-4) + \tilde{\beta}^{113} (j-2) \right) \right) - \tilde{\beta}^{5} {\binom{j-2}{3}} + 2 \tilde{\beta}^{122} (h+y+1-n) + \tilde{\beta}^{23} ((h+3+y-n)(j-2)+n-y-2h) - \frac{2(n-y-4)}{Q+1} (3\tilde{\beta}^{2} + \tilde{\beta}^{12} (x-2) + 3\tilde{\beta}^{3} (j-2)) - (8\tilde{\beta}^{2} + \tilde{\beta}^{12} (x-2) + 6\tilde{\beta}^{3} (j-2)); \quad (24)$$

 $\Lambda_{-j}$  is the same as before and  $\Lambda'_{-j} = \sum_{i\geq 1}^{j-1} (i-2) L_{-i} L_{-j+i}$ . One can prove that in the  $|r1; ss'\rangle$  representation there is a null state at level s'. An explicit expression for this null vector can be obtained by solving the recursion:

$$\begin{aligned} \mathcal{O}_{1;s'}|r1;ss'\rangle &= \\ &= \sum_{n_i:\sum_{i=1}^p n_i=s} \frac{\prod_{i=1}^p \left\{ \tilde{N}(j-4-\frac{2(n-y-4)}{Q+1})W_{-j} + \tilde{\gamma}_{n,j}L_{-j} + \tilde{\beta}^{122}(n-y-4)\Lambda_{-j} + \tilde{\beta}^{23}\Lambda'_{-j} \right\}}{\prod_{i=1}^{p-1} \tilde{\alpha}_{N_i}} |r1;ss'\rangle \end{aligned}$$

where  $N_i = \sum_{j=1}^{i} n_j$ . An alternative expression can be obtained by choosing the fusion point at q = 1, however this straightforward formula looks complicated so we omit it here.

Summarizing we find that the fusion rules of the Toda fields  $-u, \tilde{u}$  – are the same as the fusion rules of the (10), (01) representations of the corresponding Lie algebra. However

the fusions of the various Toda fields are determined by the quantum equations of motion of the corresponding Toda models and these quantum equations of motion take the form of particular descendant null vectors. So we conclude that for producing null vectors via the fusion method one has to consider these descendant null vectors. Furthermore we conjecture how to generate null vectors in the most general case, i.e. in the highest weight representation of the  $WBC_2$  algebra corresponding to the  $|rr', ss'\rangle$  hw. vector. One has to consider the WZNW model associated not to the defining but to a higher dimensional representation, labelled by (r, 0), of the appropriate group and use the Drinfeld-Sokolov reduction [10] to obtain the corresponding Toda model. Fusing its  $|r1, 11\rangle$  type special descendant null vector with a  $|0r', ss'\rangle$  primary field a null vector of the above mentioned representation is hoped to be obtained.

I would like to thank L. Palla and P. Bántay for the many useful discussions and comments and L. P. for the careful reading of the manuscript.

## References

- [1] L. Benoit and Y. Saint Aubin Phys. Lett. **215B** (1988) 517
- M. Bauer, P. di Francesco, C. Itzykson and J.-B. Zuber Nucl. Phys. B362 (1991) 515 Phys. Lett.
   260B (1991) 323
- [3] P. Bowcock and G.M.T. Watts Phys. Lett. **297B** (1992) 282
- [4] G.M.T. Watts Null Vectors of the Superconformal Algebra: The Ramond Sector Cambridge University Preprint DAMPT 93-14, hep-th 9306034
- [5] L. Benoit and Y. Saint Aubin Int. J. Mod. Phys. A13 (1992) 3023, Letters in Math. Phys. 23 (1992) 117
- [6] Adrian Kent Phys. Lett. 273B (1991) 56, Phys. Lett. 278B (1992) 443
- Bouwknegt Phys. Lett. 207B (1988) 295 K.-J. Hamada and M. Takao Phys. Lett. 209B (1988) 247 R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen Nucl. Phys. B361 (1991) 255
- [8] H.G.Kausch and G.M.T.Watts Int. J. Mod. Phys. A7 (1992) 4175, Duality in Quantum Toda theory and W-algebras Cambridge University Preprint DAMPT 92-07
- [9] Z. Bajnok C<sub>2</sub> Toda Theory in Reduced WZNW Framework ITP Budapest Report No.501 hep-th 9304019 submitted to Int.J.Mod.Phys. A
- [10] P. Forgács, A. Wipf, J. Balog, L. Fehér and L. O'Raifeartaigh: Phys. Lett. B227 (1989) 214;
   B244 (1990) 435; Ann. Phys. (N. Y.) 203 (1990) 76.