Some semi-classical issues in boundary sine-Gordon model

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Abstract

The semi-classical quantisation of the two lowest energy static solutions of boundary sine-Gordon model is considered. A relation between the Lagrangian and bootstrap parameters is established by comparing their quantum corrected energy difference and the exact one. This relation is also confirmed by studying the semi-classical limit of soliton reflections on the boundary.

1 Introduction

The sine-Gordon model is one of the most extensively studied quantum field theories. The interest stems partly from the wide range of applications that extend from particle physics to condensed matter systems and partly from the fact that many of the interesting physical quantities can be computed exactly due to its integrability. All these properties are inherited by the boundary sine-Gordon model (BSG) obtained by restricting the ordinary one to the negative half line by imposing appropriate, integrability preserving, boundary conditions at $x = 0$ [1], [2].

The novel feature of BSG is the complicated spectrum of boundary bound states manifesting themselves as appropriate poles in the various reflection amplitudes [2]-[6]. These exact amplitudes are obtained from solving the boundary versions of the Yang-Baxter, unitarity and crossing equations [2] in the bootstrap program [4], [3], [5], [6]. Therefore in the general case the reflection factors and the spectrum of bound states depend on two 'bootstrap' or 'infrared' parameters that characterize the solutions of these equations. These parameters should be determined somehow by the two 'ultraviolet' or 'Lagrangian' boundary parameters appearing in the boundary potential enforcing the boundary condition. This question leads then to the problem of establishing a relation between the exact algebraic solution of the quantum theory and the classical Lagrangian. A semi-classical quantisation of the classical theory may provide the necessary link.

The quest for the relation connecting the two sets of parameters (also called UV-IR relation below) has a long history. For Dirichlet boundary conditions, when only one bootstrap and one Lagrangian parameters survive, it was obtained already in [2]. A general expression was given by Al.B. Zamolodchikov [7] obtained from describing the BSG model as a bulk and boundary perturbed conformal field theory, but unfortunately these results remained unpublished. Recently some arguments were presented for the general form of the $UV-IR$ relation in [6] by comparing the parameter dependencies of some patterns (such as global symmetries and ground state sequences) in the bootstrap solution and in the classical theory. While this general form is consistent with Zamolodchikov's solution, it leaves the coupling constant dependency of a crucial coefficient undetermined. A TCSA study of the spectrum of BSG in finite volume [8] confirmed that Zamolodchikov's constant has the correct β dependency. In contrast in the boundary sinh-Gordon model the $UV-IR$ relation was determined by Corrigan and Taormina by comparing the WKB and bootstrap spectra of breathers [9]. It turns out after analytically continuing this relation to the sine-Gordon model, that its general form is the expected one, but its coefficient depends on β in a different way.

Motivated by the above we consider in this paper two problems in boundary sine-Gordon model, where the semi-classical approximation can be determined starting from the classical Lagrangian, and the results can be compared to the appropriate limits of the exact solution. We choose these problems to involve in one way or other the solitons in BSG, as they have no analogues in sinh-Gordon theory, thus the results cannot be obtained or predicted by a simple analytic continuation.

The first problem we investigate is the semi-classically corrected energy difference of

the two lowest energy static solutions in boundary sine-Gordon model. These classical solutions are in fact given by a static bulk soliton/antisoliton 'standing at the right place' [10] [6], thus their semi-classical quantisation amounts to the adaptation of the soliton quantisation [11] to the boundary problem. On the other hand these solutions may be thought of as the classical analogues of the exact ground state $|\rangle$, and the first excited boundary state $|0\rangle$ respectively [6], thus the semi-classically corrected energy difference should be compared to the limit of these two exact energies. This leads then to a relation between the Lagrangian and the bootstrap parameters.

The second problem we investigate is the semi-classical soliton reflection on the boundary at $x = 0$. The idea to compare the semi-classical phase shift of this process - obtained from the classical time delay - and the limit of the exact amplitude coming from the algebraic solution was suggested by Saleur, Skorik and Warner [10]. Although they determined the classical time delay in the general case (for ground state boundary at least), they made the comparison for Dirichlet boundary conditions only. Here we show that the comparison in the general case leads to the same UV-IR relation we obtained from the first problem.

The paper is organized as follows: the semi-classical quantisation of the static solutions is carried out in sect. 2. The results are compared to the limit of the exact solution in section 3. Section 4 is reserved for the investigation of the soliton reflection and we make our conclusions in sect. 5.

2 Semi-Classical quantisation of the static solutions

In this section we carry out the semi-classical quantisation of two static solutions in boundary sine-Gordon model and compute the semi-classical quantum correction to the difference between their classical energies. We start by summarizing some known facts about this theory and the classical solutions in question.

The boundary version of sine-Gordon model is defined by the action [2]:

$$S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \mathcal{L}_{\text{SG}} - \int_{-\infty}^{\infty} dt V_{\text{B}}(\Phi_B), \qquad \mathcal{L}_{\text{SG}} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{m^2}{\beta^2}(1 - \cos(\beta\Phi)), \ (2.1)$$

where $\Phi(x, t)$ is a scalar field, β is a real dimensionless coupling and $\Phi_B(t) = \Phi(x, t)|_{x=0}$. To preserve the integrability of the bulk theory the boundary potential is chosen as

$$V_{\text{B}}(\Phi_B) = M_0 \left(1 - \cos\left(\frac{\beta}{2}(\Phi_B - \phi_0) \right) \right),$$

where M_0 and ϕ_0 are free parameters. As a result the scalar field satisfies the boundary condition:

$$\partial_x \Phi|_{x=0} = -M_0 \frac{\beta}{2} \sin\left(\frac{\beta}{2}(\Phi_B - \phi_0) \right). \tag{2.2}$$

Collecting all the possible equivalences between the boundary parameters their fundamental domain turns out to be [5] [6]:

$$0 \leq M_0 \leq \infty \quad ; \qquad 0 \leq \phi_0 \leq \frac{\pi}{\beta}.$$

In the classical theory the two static solutions with lowest energy are given by a static bulk soliton/antisoliton 'standing at the right place' [10] [6]: i.e. by choosing $\Phi \equiv \Phi_s(x, a^+)$ or $\Phi \equiv \Phi_{\bar{s}}(x, a^-)$ for $x \leq 0$, where

$$\Phi_s(x, a^+) = \frac{4}{\beta}\mathrm{arctg}(e^{m(x-a^+)}), \qquad \Phi_{\bar{s}}(x, a^-) = \frac{2\pi}{\beta} - \Phi_s(x, a^-),$$

and $a^{\pm}$ are determined by the boundary condition, eq.(2.2):

$$\sinh(ma^{\pm}) = \frac{\frac{4m}{M_0\beta^2} \pm \cos(\frac{\beta}{2}\phi_0)}{\sin(\frac{\beta}{2}\phi_0)}.$$

(a^+ and a^- are obtained from each other by $\phi_0 \leftrightarrow \frac{2\pi}{\beta} - \phi_0$). The energies of these two solutions can be written as

$$\begin{aligned}
E_s(M_0, \phi_0) &\equiv E_{\mathrm{bulk}} + V_{\mathrm{B}} = \frac{4m}{\beta^2} + M_0 - M_0 R(+), \\
E_{\bar{s}}(M_0, \phi_0) &= \frac{4m}{\beta^2} + M_0 - M_0 R(-) = E_s(M_0, \frac{2\pi}{\beta} - \phi_0),
\end{aligned} \qquad (2.3)$$

where we introduced

$$R(\pm) = \left[1 \pm 2A\cos(\alpha) + A^2\right]^{1/2}, \qquad A = \frac{4m}{M_0\beta^2}, \qquad \alpha = \frac{\beta}{2}\phi_0.$$

The difference between these two energies, which is called below the 'classical energy difference',

$$\Delta E_{\mathrm{cl}} \equiv E_{\bar{s}}(M_0, \phi_0) - E_s(M_0, \phi_0) = M_0(R(+) - R(-)),$$

is positive for $\alpha \in [0, \frac{\pi}{2})$, $M_0 > 0$ showing that in this range the soliton generates the ground state and the antisoliton the first excited one. From eq.(2.3) it follows that for $\phi_0 \to 0^{+1}$

$$E_s = 0, \qquad E_{\bar{s}} = \begin{cases} 2M_0 & M_0 < \frac{4m}{\beta^2} \\ \frac{8m}{\beta^2} & M_0 > \frac{4m}{\beta^2} \end{cases}. \qquad (2.4)$$

In the process of semi-classical quantisation the oscillators associated to the linearized fluctuations around the static solutions $\Phi(x, t) = \Phi_{s,\bar{s}} + e^{i\omega t}\xi_{\pm}(x)$ are quantised [11]. The equations of motion of these fluctuations can be written:

$$\left[-\frac{d^2}{dx^2} + m^2 - \frac{2m^2}{\cosh^2(m[x - a^{\pm}])}\right]\xi_{\pm}(x) = \omega^2 \xi_{\pm}(x); \qquad x < 0, \qquad (2.5)$$

and $\xi_{\pm}(x)$ must satisfy also the linearized version of the boundary condition (2.2):

$$\xi'_{\pm}(x)|_{x=0} = -\frac{M_0\beta^2}{4}\frac{1 \pm A\cos\alpha}{R(\pm)}\xi_{\pm}(0). \qquad (2.6)$$

These eigenvalue problems can be solved exactly by mapping eq.(2.5) to a hypergeometric differential equation [12].

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[1]This limit is not smooth, see our remark later.

2.1   Discrete spectrum

In case of the *discrete spectrum* it is convenient to write $\omega^2 = m^2(1-\epsilon^2)$. The normalizable solutions of eq.(2.5) must vanish at $x \to -\infty$, and assuming ϵ to be positive, they are given by:

$$\xi_\pm(x) = Ne^{m\epsilon(x-a^\pm)}(\epsilon - \tanh[m(x-a^\pm)]).$$

The boundary conditions, eq.(2.6), determine the possible values of ϵ as

$$\epsilon^2 + \epsilon\frac{R(\pm)}{A} \pm \frac{\cos\alpha}{A} = 0.$$

It is easy to show, that for the solitonic ground state there is no positive solution of this equation, while for the antisolitonic 'exited' state one of the roots, namely

$$\epsilon = \frac{R(+) - R(-)}{2A}, \tag{2.7}$$

is positive. In fact a simple (numerical) study shows that for all positive A-s and $\alpha \in [0, \frac{\pi}{2})$

$$0 \leq \frac{R(+) - R(-)}{2A} \leq 1, \quad \text{and} \quad \frac{R(+) - R(-)}{2A} = 1 \quad \text{iff } \alpha = 0, \text{ and } A < 1.$$

In the framework of semi-classical quantisation these findings imply, that there are no boundary bound states for the ground state, described by Φ_s, while for the state, described by $\Phi_{\bar{s}}$, there is such a boundary bound state. The semi-classical energy of this bound state,

$$\omega_0 = m\sqrt{1 - \left(\frac{R(+) - R(-)}{2A}\right)^2}, \tag{2.8}$$

is real, $\omega_0 \geq 0$, and it vanishes only for $\alpha = 0$ and $A < 1$. In contrast to the traditional zero modes this vanishing ω_0 has nothing to do with $\Phi_{\bar{s}}$ not being invariant under a continuous symmetry of the Lagrangian, and it indicates some sort of instability of the state described by $\Phi_{\bar{s}}$. Indeed with this α and A values (2.4) gives an energy difference which is precisely the mass of the bulk soliton, and since topological charge is not conserved in the boundary theory, the higher energy state can decay into the lower one by emitting a standing soliton.

At this point it is worth comparing the stability analysis of this $\alpha \to 0$ situation and the one when $\alpha = 0$ is set from the start, to emphasize the non smooth nature of the limit. In the latter case the two classical solutions become $\Phi_1 \equiv \frac{2\pi}{\beta}$ and $\Phi_2 \equiv 0$. Repeating the stability analysis reveals that there are no normalizable bound state solutions of the fluctuation equations for the ground state, Φ_2, while for the 'excited' state, Φ_1, there is a normalizable solution with $\omega^2 = m^2(1 - A^{-2})$. When $A > 1$ this solution signals the existence of a boundary state, while for $A < 1$, when this ω^2 becomes negative, it indicates the instability of Φ_1. The instabilities found both in the $\alpha \to 0$ and in the $\alpha \equiv 0$ cases are consistent with the results of the bootstrap solution [5] [6] showing no excited boundary states in this range of parameters.

2.2 Continuous spectrum

In case of the *continuous spectrum* it is convenient to put $\omega^2 = m^2 + q^2$ (with $q \geq 0$). Then the solutions of eq.(2.5), which asymptotically become plane waves, can be written as

$$\xi_\pm(x) = \tilde{A}_\pm e^{-iq(x-a^\pm)} \frac{iq + m\tanh(m[x - a^\pm])}{iq + m} + \tilde{B}_\pm e^{iq(x-a^\pm)} \frac{iq - m\tanh(m[x - a^\pm])}{iq - m}.$$

The ratio $\tilde{A}_\pm / \tilde{B}_\pm$ is determined by the boundary condition eq.(2.6) at $x = 0$. Using this value the asymptotic $(x \to -\infty)$ form of the fluctuations can be written as

$$\xi_\pm(x) \to C_\pm(e^{ixq} + e^{-ixq}e^{i\delta^\pm(q)}),$$

where the classical reflection factor is

$$e^{i\delta^\pm(q)} = \frac{m - iq}{m + iq} \frac{\pm A^{-1}\cos\alpha - \frac{q^2}{m^2} + i\frac{q}{m}\frac{R(\pm)}{A}}{\mp A^{-1}\cos\alpha + \frac{q^2}{m^2} + i\frac{q}{m}\frac{R(\pm)}{A}} . \tag{2.9}$$

To handle the infinite volume limit it is convenient to confine the fluctuations to a box of size L, (i.e. to limit x to the section $(-L, 0)$), and impose Neumann boundary conditions at $x = -L$: $\xi'(-L) = 0$. This condition then determines the possible values of the momenta:

$$q_n^\pm 2L + \delta^\pm(q_n^\pm) = 2n\pi, \qquad n \quad \text{integer.} \tag{2.10}$$

The semi-classical correction to the classical energy difference, ΔE_{cl}, is given by the difference between the sums of the zero point energies of the fluctuations around $\Phi_{\bar{s}}$ and Φ_s:

$$\Delta E_{\text{semi}} = \Delta E_{\text{cl}} + \Delta E_{\text{cor}} = \Delta E_{\text{cl}} + \frac{\omega_0}{2} + \frac{1}{2}\sum_n \left(\sqrt{m^2 + (q_n^-)^2} - \sqrt{m^2 + (q_n^+)^2}\right) .$$

Replacing – as usual – the sum over n by an appropriate integral in the $L \to \infty$ limit, exploiting the

$$q_n^- = q_n^+ + \frac{\delta^+ - \delta^-}{2L}$$

consequence of eq.(2.10), and dropping all terms vanishing for $L \to \infty$ gives:

$$\begin{aligned}\Delta E_{\text{semi}} = & \Delta E_{\text{cl}} + \frac{\omega_0}{2} - \frac{M_0\beta^2}{8\pi}\left(R(+) - R(-)\right) + \frac{1}{2\pi}\left[\frac{m}{A}\left(R(-) - R(+)\right)I_1\right.\\ & \left. - \frac{m\cos\alpha}{A^2}\left(R(+) + R(-)\right)I_2\right] ,\end{aligned} \tag{2.11}$$

where

$$I_1 = \int_0^\infty dy \frac{y^2\sqrt{1 + y^2}}{D}, \qquad I_2 = \int_0^\infty dy \frac{\sqrt{1 + y^2}}{D},$$

$$D = y^4 + (1 + A^{-2})y^2 + A^{-2}\cos^2\alpha .$$

6

2.3 Renormalization

The first integral in eq.(2.11) is logarithmically divergent, showing the need of regularization and renormalization. This is hardly surprising since neither the bulk nor the boundary potentials are normal ordered, and already in the classic paper [13] it is shown on the example of the bulk soliton's mass correction, that this naive procedure leads to logarithmic divergences even in mass differences. The proper way to deal with these infinities [13] [11] is to use the counterterms, that account for the difference between the normal ordered and non ordered potentials.

In the boundary sine-Gordon model we use the same counterterm for the bulk potential as in the bulk theory:

$$V_{\text{count}}[\Phi] = -\frac{\delta m^2}{\beta^2} \int\limits_{-\infty}^{0} dx \, (1 - \cos(\beta\Phi)) \, ; \qquad \delta m^2 = -\frac{m^2\beta^2}{4\pi} \int\limits_{0}^{\Lambda} \frac{dk}{\sqrt{k^2 + m^2}} \, ;$$

but the integral is over the $x \leq 0$ half space only. The argument for this choice is based on its local nature: as such it should be independent of the presence of the boundary. For the boundary potential we *assume* that its counterterm has an analogous form

$$V_{\text{B count}}[\Phi] = -\delta M_0 \left(1 - \cos(\frac{\beta}{2}(\Phi_{\text{B}} - \phi_0)) \right) \, ,$$

with δM_0 being some parameter. The total contribution of counterterms to the energy difference

$$CT = V_{\text{count}}[\Phi_{\bar{s}}] + V_{\text{B count}}[\Phi_{\bar{s}}] - V_{\text{count}}[\Phi_s] - V_{\text{B count}}[\Phi_s]$$

may remove the logarithmic divergence in eq.(2.11), if it is proportional to $R(+) - R(-)$. This condition determines δM_0:

$$\delta M_0 = -\frac{M_0\beta^2}{4 \cdot 2\pi} \int\limits_{0}^{\Lambda} \frac{dk}{\sqrt{k^2 + m^2}} \, ,$$

and with this choice CT becomes

$$CT = \frac{m}{2\pi A} \left(R(+) - R(-) \right) \int\limits_{0}^{\Lambda/m} \frac{dy}{\sqrt{y^2 + 1}} \, .$$

Since the overall magnitude of CT is fixed by δm^2 there are no more free parameters. Thus the fact that adding CT to ΔE_{semi} *does* remove the divergence gives a partial justification of the renormalization procedure used.[2] In the renormalized energy difference

$$\Delta E_{\text{semi}}^{\text{ren}} = \Delta E_{\text{semi}} + CT$$

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[2]By setting up a systematic perturbation theory in boundary sine-Gordon model treating simultaneously both the bulk and the boundary interactions one can confirm the correctness of both δm^2 and δM_0 [14].

only the term containing I_1 gets modified and is replaced by

$$\frac{m}{2A\pi}\left(R(-) - R(+)\right)I_1 \quad \rightarrow \quad \frac{m}{2A^3\pi}\left(R(+) - R(-)\right)\tilde{I}_1,$$

with

$$\tilde{I}_1 = \int\limits_0^\infty \frac{dy}{\sqrt{1+y^2}}\frac{y^2 + \cos^2\alpha}{D}.$$

The convergent integrals $\tilde{I}_1$, and I_2 can be computed symbolically with the aid of Maple. For this it is helpful to write $D = (y^2 + a)(y^2 + b)$ with

$$a = \left(\frac{R(+) + R(-)}{2A}\right)^2 \geq 1, \qquad b = \left(\frac{R(+) - R(-)}{2A}\right)^2, \quad 0 \leq b \leq 1,$$

and tell Maple the range of these parameters. Using the explicit form of these integrals, after some algebra, the renormalized energy difference is obtained as

$$\begin{aligned}
\Delta E_{\text{semi}}^{\text{ren}} =\, & M_0(R(+) - R(-)) + \frac{m}{2}\sqrt{1 - \left(\frac{R(+) - R(-)}{2A}\right)^2} - \frac{M_0\beta^2}{8\pi}\left(R(+) - R(-)\right) \\
& - \frac{m}{\pi}\sqrt{1 - \left(\frac{R(+) - R(-)}{2A}\right)^2}\arccos\left(\frac{R(+) - R(-)}{2A}\right).
\end{aligned}$$

(2.12)

It is a remarkable feature of this expression, that it depends only on the difference $(R(+) - R(-))/(2A)$.

3 Comparison to the exact results

In this section the main results of the previous semi-classical quantisation, namely the (non) existence of semi-classical bound states, the classical reflection factors and the semi-classically corrected energy difference are compared to the results obtained from the exact (bootstrap) solution.

In this process the sine-Gordon field is assumed to correspond to the semi-classical limit of the first breather, while the exact ground state $|\rangle$ and the first excited boundary state $|0\rangle$ are identified as the quantum analogues of the classical states (solutions) Φ_s, $\Phi_{\bar{s}}$. This latter identification was suggested in [6] on the basis of the existence of a (Z_2 reflection type) transformation that changes the roles of these two states in the same way as the classical $\Phi \leftrightarrow \frac{2\pi}{\beta} - \Phi$, $\phi_0 \leftrightarrow \frac{2\pi}{\beta} - \phi_0$ changes Φ_s and $\Phi_{\bar{s}}$ into each other.

In the exact solution of the boundary sine-Gordon model [2], [5], [6], [4] the coupling constant β appears through

$$\lambda = \frac{8\pi}{\beta^2} - 1,$$

while the dependence on the boundary condition appears in the form of two real parameters, η and ϑ, the fundamental ranges of which are [6]

$$0 \le \eta \le \frac{\pi}{2}(\lambda + 1), \qquad 0 \le \vartheta \le \infty .$$

Boundary bound states appear in the exact solution as poles in the various reflection amplitudes at purely imaginary rapidity $u = -i\theta$. The location of these poles depends on the η parameter only and is given by appropriate combinations of

$$\nu_n = \frac{\eta}{\lambda} - (2n+1)\frac{\pi}{2\lambda}, \qquad w_k = \frac{\bar\eta}{\lambda} - (2k+1)\frac{\pi}{2\lambda}, \qquad \bar\eta = \pi(\lambda + 1) - \eta .$$

Though the semi-classical quantisation is non perturbative, its validity is restricted to weak coupling [11], which in our case means to $\beta \to 0$. Therefore it is the $\lambda \to \infty$ limit of the exact solution that should be compared to the semi-classical results. The η parameter should be scaled to obtain a non trivial spectrum in this limit, and we propose to write

$$\eta = c\frac{\pi}{2}(\lambda + 1), \qquad 0 \le c \le 1,$$

and keep c fixed.

3.1 Boundary states

The reflection factor of the first breather, B^1, on the ground state boundary is given by [4]

$$R^{(1)}(\theta) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2\lambda} + 1\right)\left(\frac{\eta}{\pi\lambda} - \frac{1}{2}\right)\left(\frac{i\vartheta}{\pi\lambda} - \frac{1}{2}\right)}{\left(\frac{1}{2\lambda} + \frac{3}{2}\right)\left(\frac{\eta}{\pi\lambda} + \frac{1}{2}\right)\left(\frac{i\vartheta}{\pi\lambda} + \frac{1}{2}\right)} \quad , \quad (x) = \frac{\sinh\left(\frac{\theta}{2} + i\frac{\pi x}{2}\right)}{\sinh\left(\frac{\theta}{2} - i\frac{\pi x}{2}\right)} . \tag{3.1}$$

(θ is the rapidity of B^1). B^1's reflection factor on $|0\rangle$, $R^{(1)}_{|0\rangle}(\theta)$, is obtained from this expression by the substitution $\eta \to \bar\eta = \pi(\lambda + 1) - \eta$ [6] (see also [5]). The only pole of $R^{(1)}(\theta)$ which may describe a boundary state is at

$$\frac{\eta}{\lambda} - \frac{\pi}{2} = \frac{1}{2}(\nu_0 - w_1) .$$

This corresponds to a bound state if it is in the physical strip, i.e. if $0 \le \frac{1}{2}(\nu_0 - w_1) \le \frac{\pi}{2}$. In the semi-classical ($\lambda \to \infty$) limit, keeping c fixed,

$$\frac{1}{2}(\nu_0 - w_1) = (c-1)\frac{\pi}{2} + \frac{c\pi}{2\lambda} \sim (c-1)\frac{\pi}{2} ,$$

and since this is negative we conclude that B^1 can not create a bound state on $|\rangle$. On the other hand, $R^{(1)}_{|0\rangle}(\theta)$ has a pole at

$$\frac{\pi}{\lambda} - \frac{\eta}{\lambda} + \frac{\pi}{2} = \frac{1}{2}(w_0 - \nu_1) ,$$

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which may describe a bound state if it is in the physical strip. Since in the semi-classical limit

$$\frac{1}{2}(w_0 - \nu_1) = (1-c)\frac{\pi}{2} + \frac{(2-c)\pi}{2\lambda} \sim (1-c)\frac{\pi}{2}$$

is in the physical strip we conclude that B^1 can create a bound state (in fact it is the state $|1\rangle$) when reflecting on $|0\rangle$. Recalling, that semi-classically B^1 should correspond to the sine-Gordon field, we see that these findings fit nicely with the semi-classical results and strengthen the association $(\Phi_s, \Phi_{\bar{s}}) \leftrightarrow (|\rangle, |0\rangle)$.

The energy of this bound state above $E_{|0\rangle}$ is determined by the location of the pole

$$E - E_{|0\rangle} = m_1 \cos\left((1-c)\frac{\pi}{2} + \frac{(2-c)\pi}{2\lambda}\right) , \tag{3.2}$$

where $m_1 = 2M \sin\left(\frac{\pi}{2\lambda}\right)$ is the mass of the B^1 and M is the soliton mass. Using the semi-classical expression $M = \frac{8m}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)$ one finds from (3.2) for $\lambda \to \infty$ ($\beta \to 0$)

$$E - E_{|0\rangle} \sim m \sin\left(\frac{c\pi}{2}\right) .$$

Identifying this limiting energy difference with the energy of the semi-classical bound state ω_0, eq.(2.8), determines the (limiting value of the) 'infrared' (bootstrap) parameter η in terms of the 'ultraviolet' (Lagrangian) M_0 and ϕ_0:

$$\sin\left(\frac{c\pi}{2}\right) = \sqrt{1 - \left(\frac{R(+) - R(-)}{2A}\right)^2}. \tag{3.3}$$

3.2 The limit of the reflection factors

The next step is to establish a relation between the (semi)classical limits of $R^{(1)}(\theta)$ and $R_{|0\rangle}^{(1)}(\theta)$, and the classical reflection factors $e^{i\delta^{\pm}(q)}$. Since the exact quantum reflection factors eq.(3.1) depend also on the ϑ parameter, for a non trivial limit we have to scale also this parameter. In analogy with the η parameter we propose to write

$$\vartheta = \vartheta_{\rm cl}(\lambda + 1), \qquad 0 \leq \vartheta_{\rm cl} \leq \infty.$$

This way, keeping only the leading constant terms in the $\lambda \to \infty$ limit, one obtains:

$$R^{(1)}(\theta) \to \frac{i\sinh\theta - 1}{i\sinh\theta + 1} \frac{\cos\left(\frac{c\pi}{2}\right)\cosh\vartheta_{\rm cl} - \sinh^2\theta + i\sinh\theta\left(\cos\left(\frac{c\pi}{2}\right) + \cosh\vartheta_{\rm cl}\right)}{\cos\left(\frac{c\pi}{2}\right)\cosh\vartheta_{\rm cl} - \sinh^2\theta - i\sinh\theta\left(\cos\left(\frac{c\pi}{2}\right) + \cosh\vartheta_{\rm cl}\right)}. \tag{3.4}$$

The expression for the limiting value of $R_{|0\rangle}^{(1)}(\theta)$ is obtained by making the substitution $c \to \bar{c} = 2 - c$, (which amounts to changing the sign of $\cos\left(\frac{c\pi}{2}\right)$) in eq.(3.4). Identifying

these limiting $R^{(1)}(\theta)$ and $R^{(1)}_{|0\rangle}(\theta)$ with $e^{i\delta^{\pm}(q)}$, eq.(2.9), using $\sinh\theta = \frac{q}{m}$, determines the bootstrap parameters $\frac{c\pi}{2}$ and ϑ_{cl} as

$$
\begin{aligned}
\cos\left(\frac{c\pi}{2}\right) + \cosh\vartheta_{\text{cl}} &= \frac{R(+)}{A}, \\
\cosh\vartheta_{\text{cl}} - \cos\left(\frac{c\pi}{2}\right) &= \frac{R(-)}{A},
\end{aligned}
\tag{3.5}
$$

together with

$$
\cos\left(\frac{c\pi}{2}\right)\cosh\vartheta_{\text{cl}} = \frac{\cos\alpha}{A}.
\tag{3.6}
$$

The algebraic solution of eq.(3.5)

$$
\cos\left(\frac{c\pi}{2}\right) = \frac{R(+) - R(-)}{2A}, \qquad \cosh\vartheta_{\text{cl}} = \frac{R(+) + R(-)}{2A},
\tag{3.7}
$$

satisfies eq.(3.6) and is also consistent with eq.(3.3).

3.3 The limit of $E_{|0\rangle} - E_{|\rangle}$ and the UV-IR relation

According to the bootstrap solution [5] [6] the energy difference between the lowest excited boundary state and the ground state is given by

$$
\Delta E_{\text{bst}} \equiv E_{|0\rangle} - E_{|\rangle} = M\cos\nu_0 = M\cos\left(\frac{\eta}{\lambda} - \frac{\pi}{2\lambda}\right),
$$

where M is the soliton mass. In the semi-classical limit, using the appropriately scaled η parameter, this can be written as

$$
\Delta E_{\text{bst}} = M\cos\left(\frac{c\pi}{2}\right) - M\sin\left(\frac{c\pi}{2}\right)\frac{\beta^2}{8\pi}\left(\frac{c\pi}{2} - \frac{\pi}{2}\right) + M\mathcal{O}(\beta^4).
\tag{3.8}
$$

Now it is easy to show, using the complete semi-classical expression, $M = \frac{8m}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)$, in the first term, the leading $M = \frac{8m}{\beta^2}$ in the (higher order) second one, together with the actual value of $\cos\left(\frac{c\pi}{2}\right)$ in (3.7), that the first four terms of ΔE_{bst} coincide term by term with the expression of $\Delta E^{\text{ren}}_{\text{semi}}$ eq.(2.12).

Now we can understand the importance of the fact that in spite of the intermediate stages the dependency on $(R(+) + R(-))/(2A)$ cancels in the final form of the semi-classical $\Delta E^{\text{ren}}_{\text{semi}}$. This should happen since ΔE_{bst}, just as the whole spectrum of boundary states predicted by the bootstrap solution, is also independent of ϑ thus in the semi-classical limit it should depend only on $\frac{c\pi}{2}$ but should be independent of ϑ_{cl}.

The nice matching between $\Delta E^{\text{ren}}_{\text{semi}}$ and ΔE_{bst} confirms the relation between the bootstrap and Lagrangian parameters eq.(3.7). This relation makes it possible to determine the (semi-classical limit of the) only free parameter in the so called UV-IR relation.

On general grounds the generic form of the relation between the bootstrap and Lagrangian parameters of boundary sine-Gordon model (i.e of the UV-IR relation) is

$$\cos\left(\frac{\eta}{\lambda+1}\right)\cosh\left(\frac{\vartheta}{\lambda+1}\right) = \frac{M_0}{M_{\mathrm{crit}}}\cos\alpha\,,$$
$$\sin\left(\frac{\eta}{\lambda+1}\right)\sinh\left(\frac{\vartheta}{\lambda+1}\right) = \frac{M_0}{M_{\mathrm{crit}}}\sin\alpha\,,$$

$$(3.9)$$

where the parameter M_{crit} (M_0/M_{crit}) may depend on β. Our aim is to say something on this parameter and on this dependence. First of all, $\frac{\eta}{\lambda+1}$ and $\frac{\vartheta}{\lambda+1}$ are nothing but $c\pi/2$ and ϑ_{cl} in the way they were introduced, thus eq.(3.9) determines in fact these parameters *for all values of* λ. Making this identification explicit in eq.(3.9) and comparing to eq.(3.6) gives, that in the semi-classical limit

$$\frac{M_0}{M_{\mathrm{crit}}} = \frac{1}{A}, \qquad \text{i.e.} \qquad M_{\mathrm{crit}} = \frac{4m}{\beta^2}\,. \qquad (3.10)$$

Note that this is the same value as the classical one appearing in eq.(2.4).

There are several points that should be stressed about M_{crit} in general and its actual value in particular. The first point to mention is that M_0/M_{crit} appearing in eq.(3.9) may depend on the regularization scheme used to define the quantum theory and the value in (3.10) is in the 'semi-classical scheme'. In a recent paper Corrigan and Taormina obtained the UV-IR relation in sinh-Gordon model by semi-classically quantising the (periodic) boundary breathers [9]. Analytically continuing their results in β (and accounting for the differences between the parameters) one can show, that their M_{crit} is identical to eq.(3.10). In this respect it is worth emphasizing that the analogues of the static solutions Φ_s and $\Phi_{\bar{s}}$ just like the states $|\rangle$ and $|0\rangle$, upon which our investigation is based, are *absent* in the sinh-Gordon theory, thus the results of this paper give an independent confirmation of the M_{crit} obtained in [9].

In [9] it is conjectured that this result for M_{crit} may be exact. To support this conjecture we note that our results make it possible to check that M_{crit} receives no $\mathcal{O}(\beta^0)$ correction:

$$M_{\mathrm{crit}} = \frac{4m}{\beta^2}\left(1 + \mathcal{O}(\beta^4)\right)\,.$$

To show this denote the (β dependent) M_0/M_{crit} as H and determine $\cos\left(\frac{c\pi}{2}\right)$ from eq.(3.9)

$$\cos\left(\frac{c\pi}{2}\right) = \frac{H}{2}\left(\sqrt{1+H^{-2}+2H^{-1}\cos\alpha} - \sqrt{1+H^{-2}-2H^{-1}\cos\alpha}\right)\,,$$

and finally write $H = \frac{1}{A}\left(1 + \delta H\frac{\beta^2}{8\pi}\right)$. Now plugging this expression for $\cos\left(\frac{c\pi}{2}\right)$ (and the equivalent one for $c\pi/2$) into (3.8) reveals that the only choice that guarantees the agreement between eq.(3.8) and eq.(2.12) is $\delta H = 0$.[3]

---

[3]Since the $M\mathcal{O}(\beta^4)$ terms are not calculated we cannot say anything about the higher order corrections.

Perturbed conformal field theory is another useful scheme to describe the boundary sine-Gordon model. In this description BSG is viewed as a $c = 1$ boundary CFT perturbed by the (relevant) vertex operators constituting the bulk and boundary potentials [17]:

$$S = S_{c=1} + \frac{\mu}{2} \int\limits_{-\infty}^{\infty} dt \int\limits_{-\infty}^{0} dx (V_\beta[\Phi] + V_{-\beta}[\Phi]) + \frac{\tilde{\mu}}{2} \int\limits_{-\infty}^{\infty} dt (\Psi_{\beta/2}[\Phi]\mathrm{e}^{-i\alpha} + \Psi_{-\beta/2}[\Phi]\mathrm{e}^{i\alpha}) \,,$$

where

$$V_\beta[\Phi] = n(z, \bar{z}) : \mathrm{e}^{i\beta\Phi(x,t)} : \,, \qquad \Psi_{\beta/2}[\Phi] =: \mathrm{e}^{i\frac{\beta}{2}\Phi(0,t)} : \,,$$

and $n(z, \bar{z})$ denotes the appropriate normal ordering function. The μ and $\tilde{\mu}$ parameters play the role of m and M_0 respectively and have non trivial dimensions:

$$[\mu] = \mathrm{mass}^{\,2-\frac{\beta^2}{4\pi}}, \qquad [\tilde{\mu}] = \mathrm{mass}^{\,1-\frac{\beta^2}{8\pi}}.$$

The relation between μ and the soliton mass M is known from a TBA study of the bulk sine-Gordon model [15]

$$\mu = \kappa(\beta)M^{2-2\Delta}, \qquad \kappa(\beta) = \frac{2\Gamma(\Delta)}{\pi\Gamma(1-\Delta)} \left(\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2-2\Delta}\right)}{2\Gamma\left(\frac{\Delta}{2-2\Delta}\right)} \right)^{2-2\Delta}, \quad \Delta = \frac{\beta^2}{8\pi}. \qquad (3.11)$$

In this scheme the UV-IR relation takes the form of eq.(3.9) with the replacement

$$\frac{M_0}{M_{\mathrm{crit}}} \to \frac{\tilde{\mu}}{\mu_{\mathrm{crit}}}, \qquad \mu_{\mathrm{crit}} = \sqrt{\frac{2\mu}{\sin\frac{\beta^2}{8}}}. \qquad (3.12)$$

This relation was obtained by Al.B. Zamolodchikov [7] and has recently been verified by a TCSA study of the spectrum of boundary sine-Gordon model [8].

Thus the β dependence of the constant on the right hand side of eq.(3.9) is different in the semi-classical and in the perturbed CFT schemes. Nevertheless in the semi-classical limit the two results coincide. In the perturbed CFT scheme the limiting values of $c\pi/2$ and ϑ_{cl} should be obtained from eq.(3.9) with $\frac{\tilde{\mu}}{\mu} = H$. Furthermore, for the comparison, the μ, $\tilde{\mu}$ and the m, M_0 parameters of the two schemes should be related to each other. Using the semi-classical expression for M in the $\beta \to 0$ limit of eq.(3.11) gives $\mu \to \frac{m^2}{\beta^2}$ and matching the leading (classical) term of eq.(3.8) to the scheme independent ΔE_{cl} fixes $\tilde{\mu} \to M_0$; thus $\mu_{\mathrm{crit}} \to \frac{4m}{\beta^2} = M_{\mathrm{crit}}$ indeed.

4 Semi-Classical soliton reflections

In this section the semi-classical limits of soliton/antisoliton reflection amplitudes on the boundary at $x = 0$ are studied. The relevant classical solutions are time dependent - as opposed to the static ones considered in section 2 - but just like the static ones are

specific to sine-Gordon and have no analogues in sinh-Gordon theory. A long time ago a completely general expression for the semi-classical phase shift was given in terms of the classical time delay and of the number of semi-classical bound states by Jackiw and Woo [16]. The idea to compare in boundary sine-Gordon model this expression and the semi-classical limit of the exact reflection amplitudes (obtained from the bootstrap) as a consistency check and to gain information on the relation between the Lagrangian and the bootstrap parameters was put forward by Saleur, Skorik and Warner (SSW) in [10]. SSW determined the classical time delay in case of soliton/antisoliton reflections on ground state boundary for the general boundary conditions, but only for Dirichlet boundary conditions made the comparison with the exact results. In this section the comparison is made in case of ground state boundaries with general boundary conditions and also for the lowest excited boundary in case of Neumann boundary condition.

4.1   Neumann boundary condition

The expression given in [16] for the semi-classical phase shift $e^{i\delta(E)}$ is

$$\delta(E) = n_B \pi + \int\limits_{E_{\text{th}}}^{E} dE' \Delta t(E'), \tag{4.1}$$

where n_B is the number of the (semi-classical) bound states and $\Delta t(E')$ is the classical time delay. As an illustration consider the (anti)solitons reflecting on a ground state Neumann boundary, i.e. when $\partial_x \Phi|_{x=0} = 0$ (corresponding to $M_0 = 0$)[4]. Then there are classical solutions only for solitons reflecting as antisolitons (and vice versa) but not for solitons reflecting as solitons. Furthermore, the classical solution describing an asymptotic soliton with velocity v heading to and reflecting from the boundary at $x = 0$ can be obtained by restricting to the $x \leq 0$ half line a special solution of the bulk theory, that describes a soliton with velocity v scattering on an antisoliton with velocity $-v$ [10], [17]. Therefore the classical time delay of the soliton reflecting on the Neumann boundary is identical to the time delay in the soliton antisoliton scattering in the bulk theory:

$$\Delta t = \frac{2\ln v}{m\gamma v}, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}.$$

The number of bound states, i.e. the number of boundary breathers with Neumann b.c. were obtained in [17] by semi-classically quantising the classical boundary breathers with the result that $n_B = \left[\frac{\lambda}{2}\right]$. In the semi-classical limit $\lambda \to \infty$ thus $n_B \sim \frac{\lambda}{2} = \frac{4\pi}{\beta^2}$. Since the energy of the reflecting soliton is $E = \frac{M}{\sqrt{1-v^2}} = M\cosh(\theta) = \frac{8m}{\beta^2\sqrt{1-v^2}}$, eq.(4.1) yields in this

---

[4]Since the vanishing M_0 makes α a redundant parameter, and the bootstrap parameters take fixed values (η becomes the maximally allowed $\frac{\pi}{2}(\lambda + 1)$ and ϑ vanishes) this illustration may serve only as a consistency check.

case

$$\delta(E) = \frac{4\pi^2}{\beta^2} + \frac{16}{\beta^2} \int\limits_0^{\tanh\theta} dv' \frac{\ln v'}{1 - v'^2} \,.$$

In the exact solution of BSG with Neumann b.c. there are two amplitudes that describe the reflections of solitons and antisolitons on the ground state boundary: $P(\theta)$ describes the 'diagonal' scattering, i.e. when solitons reflect as solitons and antisolitons as antisolitons, while $Q(\theta)$ describes the 'non - diagonal' scattering, when solitons reflect as antisolitons (and vice versa). In [17] simple integral representations were given for them:

$$P(\theta) = \frac{\sin(\frac{\lambda\pi}{2})}{\sin\left(\frac{\lambda\pi}{2} + i\lambda\theta\right)} \mathrm{e}^{-iI(\lambda,\theta)}, \qquad Q(\theta) = -i\frac{\sinh(\lambda\theta)}{\sin\left(\frac{\lambda\pi}{2} + i\lambda\theta\right)} \mathrm{e}^{-iI(\lambda,\theta)}\,,$$

$$I(\lambda,\theta) = \int_0^\infty \frac{dt}{t} t \sin\left(\frac{2\theta t}{\pi}\right) \left[\frac{2\sinh\left(\frac{3t}{2}\right)\sinh\left(\frac{\lambda-1}{2\lambda}t\right)}{\sinh\left(\frac{t}{2\lambda}\right)\sinh(2t)} + \frac{\sinh(t/\lambda) - \sinh(t)}{\cosh(t)\sinh(t/\lambda)}\right]\,.$$

In the semi-classical limit $P(\theta) \sim \mathrm{e}^{-\lambda\theta}\mathrm{e}^{-iI(\lambda,\theta)} \to 0$, which is consistent with the absence of diagonal classical reflection. On the other hand

$$Q \to \mathrm{e}^{i\frac{\lambda\pi}{2}}\mathrm{e}^{-iI_1(\lambda,\theta)}, \quad I_1(\lambda,\theta) = \lim_{\lambda\to\infty} I(\lambda,\theta) = \lambda \int\limits_0^\infty \frac{dt}{t^2} \sin\left(\frac{2\theta t}{\pi}\right) \tanh\left(\frac{t}{2}\right) + \mathcal{O}(\lambda^0)\,, \quad (4.2)$$

where we neglected all $\mathcal{O}(\lambda^0)$ terms in the exponents. The integral $\partial_\theta I_1$ can be found in Gradstein Ryzhikh, [19], thus

$$I_1 = -\frac{2\lambda}{\pi} \int_0^\theta dv \ln\tanh v = -\frac{2\lambda}{\pi} \int_0^{\tanh\theta} dv' \frac{\ln v'}{1 - v'^2}\,.$$

Using finally the semi-classical relation $\lambda \sim \frac{8\pi}{\beta^2}$ in eq.(4.2) reproduces the semi-classical phase shift indeed.

4.1.1 Excited Neumann boundary

The exact soliton/antisoliton reflection amplitudes are known also when the Neumann boundary is in its excited states $|n\rangle$ $n = 1, \ldots, \left[\frac{\lambda}{2}\right]$[5]. The P, Q reflection factors on the lowest excited state $|1\rangle$ change as [17]

$$P \to \tilde{P} = P(\theta)B(\lambda,\theta), \qquad Q \to \tilde{Q} = Q(\theta)B(\lambda,\theta),$$

---

[5]For Neumann boundary condition the pole described by ν_0 is at $\theta = i\frac{\pi}{2}$, and it corresponds to the emission of a soliton/antisoliton by the boundary [2] rather than to a bound state. Alternatively one can say that $|0\rangle$ becomes identical to the ground state $|\rangle$, as not only their energies but also the $P(\theta)$ and $Q(\theta)$ reflection factors on them become identical [17].

$$B(\lambda, \theta) = \tan\left[\frac{u}{2} + \frac{\pi}{2}\left(\frac{1}{\lambda} + \frac{1}{2}\right)\right] \tan\left[\frac{u}{2} - \frac{\pi}{2}\left(\frac{1}{\lambda} - \frac{1}{2}\right)\right] \tan^2\left(\frac{u}{2} + \frac{\pi}{4}\right), \quad u = -i\theta.$$

In the semi-classical limit

$$\lim_{\lambda \to \infty} B(\lambda, \theta) = \frac{1 - i\sinh\theta}{1 + i\sinh\theta} \tan^2\left(\frac{-i\theta}{2} + \frac{\pi}{4}\right),$$

which gives only an $\mathcal{O}(\lambda^0)$ correction in the exponent of $\tilde{Q}$. Thus the leading term in the exponent, i.e. the semi-classical phase shift, is identical to what was found for the ground state boundary.

With Neumann b.c. the state $|1\rangle$ may be thought of classically as a (classical) breather bound to the boundary at $x = 0$ [17]. Thus the classical reflection process may be described as a soliton antisoliton pair reflecting on the breather at $x = 0$, and the classical time delay should be obtained from this picture. The relevant classical solution is constructed by the τ function method [10] [18] in two steps. First a 4 soliton solution describing two pairs of solitons and antisolitons is determined and the relevant time delays are obtained. Then we continue the parameters of one of the pairs to purely imaginary values to describe the breather and make the necessary changes in the expression of the time delay.

In the τ function method each soliton and antisoliton is characterized by its velocity, by its 'rapidity type' parameter and by its 'position type' parameter. In the solution below the following parameters are used: the soliton of the first (second) pair moves with velocity u (v), its rapidity type parameter is denoted by k (p) and its position type parameter by a_1 (b_1); for the antisoliton of the first (second) pair the corresponding quantities are $-u$ $(-v)$, $1/k$ $(1/p)$, and a_2 (b_2) respectively. (These quantities give a redundant characterization as u and k -alternatively v and p - can be expressed in terms of the θ_1 and θ_2 rapidities of the first and second solitons: $u = \tanh\theta_1$, $k = e^{\theta_1}$; $v = \tanh\theta_2$, $p = e^{\theta_2}$). Then, using also the

$$\gamma = \frac{1}{\sqrt{1 - u^2}}, \qquad \tilde{\gamma} = \frac{1}{\sqrt{1 - v^2}}$$

quantities, in the centre of mass system the τ function of the solution may be written as

$\tau = 1 + e^{-2\gamma x}e^{-a_1 - a_2}u^2 - e^{-2\tilde{\gamma}x}e^{-b_1 - b_2}u^2$

$- e^{-\gamma(x+ut)}e^{-\tilde{\gamma}(x+vt)}e^{-a_1 - b_1}\left(\frac{k-p}{k+p}\right)^2 + e^{-\gamma(x+ut)}e^{-\tilde{\gamma}(x-vt)}e^{-a_1 - b_2}\left(\frac{k-\frac{1}{p}}{k+\frac{1}{p}}\right)^2$

$+ e^{-\gamma(x-ut)}e^{-\tilde{\gamma}(x+vt)}e^{-a_2 - b_1}\left(\frac{\frac{1}{k}-p}{\frac{1}{k}+p}\right)^2 - e^{-\gamma(x-ut)}e^{-\tilde{\gamma}(x-vt)}e^{-a_2 - b_2}\left(\frac{\frac{1}{k}-\frac{1}{p}}{\frac{1}{k}+\frac{1}{p}}\right)^2$

$+ e^{-2\gamma x}e^{-2\tilde{\gamma}x}e^{-a_1 - a_2 - b_1 - b_2}u^2 v^2 \left(\frac{k-p}{k+p}\right)^2\left(\frac{k-\frac{1}{p}}{k+\frac{1}{p}}\right)^2\left(\frac{\frac{1}{k}-p}{\frac{1}{k}+p}\right)^2\left(\frac{\frac{1}{k}-\frac{1}{p}}{\frac{1}{k}+\frac{1}{p}}\right)^2$

$+ i[e^{-\gamma(x+ut)}e^{-a_1} - e^{-\gamma(x-ut)}e^{-a_2} + e^{-\tilde{\gamma}(x+vt)}e^{-b_1} - e^{-\tilde{\gamma}(x-vt)}e^{-b_2}$

$+ e^{-2\gamma x}e^{-\tilde{\gamma}(x+vt)}e^{-a_1 - a_2 - b_1}u^2 \left(\frac{k-p}{k+p}\right)^2\left(\frac{\frac{1}{k}-p}{\frac{1}{k}+p}\right)^2 - e^{-2\gamma x}e^{-\tilde{\gamma}(x-vt)}e^{-a_1 - a_2 - b_2}u^2 \left(\frac{k-\frac{1}{p}}{k+\frac{1}{p}}\right)^2\left(\frac{\frac{1}{k}-\frac{1}{p}}{\frac{1}{k}+\frac{1}{p}}\right)^2$

$+ e^{-2\tilde{\gamma}x}e^{-\gamma(x+ut)}e^{-a_1 - b_1 - b_2}v^2 \left(\frac{k-p}{k+p}\right)^2\left(\frac{k-\frac{1}{p}}{k+\frac{1}{p}}\right)^2 - e^{-2\tilde{\gamma}x}e^{-\gamma(x-ut)}e^{-a_2 - b_1 - b_2}v^2 \left(\frac{\frac{1}{k}-p}{\frac{1}{k}+p}\right)^2\left(\frac{\frac{1}{k}-\frac{1}{p}}{\frac{1}{k}+\frac{1}{p}}\right)^2].$

16

(Here we use dimensionless x and t coordinates : $x \to mx$, $t \to mt$, thus the true time delay is obtained from the dimensionless one presented below by dividing it by m). Analyzing the $t \to \mp\infty$ limits of the solution and requiring that it should correspond to the sum of two non interacting soliton antisoliton pairs determines the a_i b_i $i = 1, 2$ parameters in terms of the initial $(t = t_0)$ soliton/antisoliton positions $(x_0^{is,\bar{s}})$ as well as the time delays: from the $t \to -\infty$ limit it is found

$$
\begin{aligned}
a_1 &= -\gamma(x_0^{1s} + ut_0), & a_2 &= -\gamma(x_0^{1\bar{s}} - ut_0) + 2\ln u + \ln\left(\frac{\frac{1}{k} - p}{\frac{1}{k} + p}\right)^2 + \ln\left(\frac{\frac{1}{k} - \frac{1}{p}}{\frac{1}{k} + \frac{1}{p}}\right)^2, \\
b_1 &= -\tilde{\gamma}(x_0^{2s} + vt_0) + \ln\left(\frac{k-p}{k+p}\right)^2, & b_2 &= -\tilde{\gamma}(x_0^{2\bar{s}} - vt_0) + 2\ln v + \ln\left(\frac{k - \frac{1}{p}}{k + \frac{1}{p}}\right)^2,
\end{aligned}
\tag{4.3}
$$

while the $t \to \infty$ limit yields the time delays of the two pairs

$$
\begin{aligned}
\Delta t_1 &= \frac{2\ln u + \ln\left(\frac{\frac{1}{k} - p}{\frac{1}{k} + p}\right)^2 + \ln\left(\frac{k-p}{k+p}\right)^2}{\gamma u}, \\
\Delta t_2 &= \frac{2\ln v + \ln\left(\frac{\frac{1}{k} - p}{\frac{1}{k} + p}\right)^2 - \ln\left(\frac{k-p}{k+p}\right)^2}{\tilde{\gamma} v}.
\end{aligned}
\tag{4.4}
$$

(The asymmetry in eq.(4.3-4.4) stems from assuming $u > v$). These expressions for the time delay have a simple interpretation: they give the sum of the time delays suffered in the various collisions. Indeed the first terms on the right hand sides of eq.(4.4) give the time delays of the solitons from the scattering on their own partners, while a simple Lorentz transformation shows, that the second and third terms are nothing but the contributions from the scattering on the two members of the other pair.

In the Neumann boundary problem the breather should be located at $x = 0$ and the soliton/antisoliton pair (representing the scattering soliton) should also come together at the boundary. To accomplish this the 4 soliton solution should be expressed in terms of the 'collision place' and 'collision time' of each pair instead of the initial positions. The collision place of each pair is trivially $x^{*1} = (x_0^{1s} + x_0^{1\bar{s}})/2$, $x^{*2} = (x_0^{2s} + x_0^{2\bar{s}})/2$. Assuming that the slower moving members of the inner pair collide first, the t^{*1}, t^{*2} collision times can be obtained from the addition rule of the time delays just shown, and the a_i, b_i can be expressed more symmetrically using these four quantities:

$$
\begin{aligned}
a_1 &= -\gamma(x^{*1} + ut^{*1}) + \ln u + \ln\left(\frac{\frac{1}{k} - p}{\frac{1}{k} + p}\right)^2, & a_2 &= -\gamma(x^{*1} - ut^{*1}) + \ln u + \ln\left(\frac{\frac{1}{k} - \frac{1}{p}}{\frac{1}{k} + \frac{1}{p}}\right)^2, \\
b_1 &= -\tilde{\gamma}(x^{*2} + vt^{*2}) + \ln v + \ln\left(\frac{k-p}{k+p}\right)^2, & b_2 &= -\tilde{\gamma}(x^{*2} - vt^{*2}) + \ln v + \ln\left(\frac{k - \frac{1}{p}}{k + \frac{1}{p}}\right)^2.
\end{aligned}
\tag{4.5}
$$

Now the parameters of the solution relevant for the Neumann problem are obtained as follows: assuming we use the second pair to describe the breather we set $x^{*2} = 0$ and continue v to purely imaginary values $v = iw$ (w real) and use eq.(4.5) to express the b parameters; however the a parameters are to be obtained from eq.(4.3) with $x_0^{1s} = -x_0^{1\bar{s}}$.

The reason behind this is that the first two equations in (4.5) were obtained by assuming that the soliton scatters on the individual members of the other pair, which is now replaced by the breather. The time delay of the soliton is independent of these parameters and is obtained from the first equation in (4.4), which gives a real value in spite of p being a complex number:

$$p = \sqrt{\frac{1+v}{1-v}} = \sqrt{\frac{1+iw}{1-iw}} = \frac{1+iw}{\sqrt{1+w^2}} = e^{i\arctan w} .$$

Using this time delay in the integral in the semi-classical expression (4.1) gives

$$\frac{16}{\beta^2} \int\limits_0^{\tanh\theta} dv' \frac{\ln v'}{1-v'^2} + \frac{8}{\beta^2} \int\limits_0^k \frac{dy}{y} \left(\ln\left(\frac{y-p}{y+p}\right)^2 + \ln\left(\frac{y^{-1}-p}{y^{-1}+p}\right)^2 \right) . \qquad (4.6)$$

The first integral reproduces what is obtained above for ground state boundary. In the second integral the p parameter of the breather is obtained by matching the classical and quantum expressions of its energy

$$M \sin\left(\frac{\pi}{2\lambda}\right) = \frac{M}{\sqrt{1+w^2}} .$$

Therefore in the semi-classical limit $p = i + \frac{\pi}{2\lambda}$; and using it in the second integral shows that it is only an $\mathcal{O}(\lambda^0)$ correction to the first one. Thus we verified the matching between eq.(4.1) and the limit of the exact amplitude also in case of solitons reflecting on excited Neumann boundary.

4.2   Ground state boundary with general boundary conditions

Finally we show that comparing the semi-classical limit of the exact soliton/antisoliton reflection amplitude on the ground state boundary with general boundary conditions and the semi-classical phase shift obtained from eq.(4.1) with the aid of the classical time delay derived by SSW in [10], one can confirm the UV-IR relation discussed in the previous section.

The most general reflection factor of the soliton antisoliton multiplet $|s, \bar{s}\rangle$ on the ground state boundary, satisfying the boundary versions of the Yang Baxter, unitarity and crossing equations was found by Ghoshal and Zamolodchikov [2] as:

$$
\begin{aligned}
R(\eta, \vartheta, \theta) &= \begin{pmatrix} P^+(\eta, \vartheta, \theta) & Q(\eta, \vartheta, \theta) \\ Q(\eta, \vartheta, \theta) & P^-(\eta, \vartheta, \theta) \end{pmatrix} \\
&= \begin{pmatrix} P_0^+(\eta, \vartheta, \theta) & Q_0(\theta) \\ Q_0(\theta) & P_0^-(\eta, \vartheta, \theta) \end{pmatrix} R_0(\theta) \frac{\sigma(\eta, \theta)}{\cos(\eta)} \frac{\sigma(i\vartheta, \theta)}{\cosh(\vartheta)} , \\
P_0^\pm(\eta, \vartheta, \theta) &= \cosh(\lambda\theta)\cos(\eta)\cosh(\vartheta) \pm i\sinh(\lambda\theta)\sin(\eta)\sinh(\vartheta) \\
Q_0(\theta) &= i\sinh(\lambda\theta)\cosh(\lambda\theta) .
\end{aligned}
$$

In [10] useful integral representations are given for $R_0(\theta)$ and $\sigma(x,\theta)$; for $R_0(\theta)$ we use this, while - by going back to the infinite product representation of [2] and [6] - we replace

$$\frac{\sigma(x,\theta)}{\cos x} = \frac{\Sigma(x,\theta)}{\cos(x + i\lambda\theta)}$$

with

$$\ln\Sigma(x,\theta) = i\int_0^\infty \frac{dy}{y} \frac{\sin(\frac{2\theta y}{\pi})}{\sinh(y/\lambda)} \frac{\sinh(y - \frac{2x}{\pi\lambda}y)}{\cosh(y)}\,,$$

as this gives a convergent integral in the entire range $0 \le \eta \le \frac{\pi}{2}(\lambda+1)$. Expressing η and ϑ in terms of c and $\vartheta_{\rm cl}$ as in section 3 and using the integral representations one obtains

$$R_0(\theta)\Sigma(\eta,\theta)\Sigma(i\vartheta,\theta) = {\rm e}^{i\hat\delta}{\rm e}^J, \qquad J = \int_0^\infty \frac{dy}{y}\frac{\sin\left(\frac{2y\theta}{\pi}\right)\sin\left(\frac{2y\vartheta_{\rm cl}}{\pi}(\lambda^{-1}+1)\right)}{\sinh(y/\lambda)}\,. \qquad (4.7)$$

In the semi-classical limit, neglecting the $\mathcal{O}(\lambda^0)$ terms in the exponent

$${\rm e}^J \to \begin{cases} {\rm e}^{\lambda\vartheta_{\rm cl}} & \theta > \vartheta_{\rm cl} \\ {\rm e}^{\lambda\theta} & \theta < \vartheta_{\rm cl} \end{cases}\,.$$

Therefore the three amplitudes, $P^\pm$ and Q, have rather different semi-classical limits depending on whether the rapidity of the incident particle is bigger or smaller than $\vartheta_{\rm cl}$:

$$\begin{aligned} \lim_{\lambda\to\infty} P^\pm &= {\rm e}^{\pm ic\frac{\pi}{2}\lambda}{\rm e}^{ic\frac{\pi}{2}\lambda}{\rm e}^{i\hat\delta}, & \lim_{\lambda\to\infty} Q &= 0, & \theta < \vartheta_{\rm cl} \\ \lim_{\lambda\to\infty} P^\pm &= 0, & \lim_{\lambda\to\infty} Q &= {\rm e}^{ic\frac{\pi}{2}\lambda}{\rm e}^{i\hat\delta}, & \theta > \vartheta_{\rm cl}\,. \end{aligned} \qquad (4.8)$$

This behaviour is consistent with the known facts, that classically, for Dirichlet boundary conditions ($\vartheta_{\rm cl} = \infty$) solitons reflect as solitons, while for Neumann boundary condition ($\vartheta_{\rm cl} = 0$) as antisolitons. Furthermore the classical solution found by SSW [10] shows the same critical behaviour as in eq.(4.8), so that $\vartheta_{\rm cl}$ may be identified with one of the parameters of that paper. To make the correspondence complete one has to compute the semi-classical limit of $i\hat\delta$ as well. Using the aforementioned integral representations, after some algebra, keeping only the leading terms, one finds:

$$\begin{aligned} \lim_{\lambda\to\infty} i\hat\delta &= -i\lambda\int_0^\infty \frac{dy}{y^2}\sin\left(\frac{2\theta y}{\pi}\right)\left(\tanh(\frac{y}{2}) + \frac{\sinh([c-1]y)}{\cosh y} + \tanh y - \tanh y\cos(\frac{2y\vartheta_{\rm cl}}{\pi})\right) \\ &= -i(I_1 + I_2 + I_3 + I_4)\,. \end{aligned}$$

All integrals I_j are computed by realizing that $\frac{\partial I_j}{\partial\theta}$ can be found in [19]. There is a subtlety with I_4, as,

$$\frac{\partial I_4}{\partial\theta} = \frac{\lambda}{\pi}\ln\left(\tanh\left[\frac{\theta + \vartheta_{\rm cl}}{2}\right]\tanh\left[\frac{|\theta - \vartheta_{\rm cl}|}{2}\right]\right),$$

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where $|\theta - \vartheta_{\rm cl}|$ is the modulus of $\theta - \vartheta_{\rm cl}$. Therefore the $\theta < \vartheta_{\rm cl}$ and the $\theta > \vartheta_{\rm cl}$ domains are separated by a logarithmic singularity, and this matches nicely with eq.(4.8). Finally

$$i\hat{\delta} = \frac{i\lambda}{\pi} \int\limits_{\theta_{\rm th}}^{\theta} dv \ln \frac{\tanh^2 v \tanh^2(v/2)}{\tanh\left(\frac{1}{2}(v + i\frac{c\pi}{2})\right) \tanh\left(\frac{1}{2}(v - i\frac{c\pi}{2})\right) \tanh\left[\frac{v+\vartheta_{\rm cl}}{2}\right] \tanh\left[\frac{|v-\vartheta_{\rm cl}|}{2}\right]} ,$$

where $\theta_{\rm th}$ is 0 in the $\theta < \vartheta_{\rm cl}$ domain, while it is $\vartheta_{\rm cl}$ in the $\theta > \vartheta_{\rm cl}$ one. Now we are in a position to compare this to the integral of the classical time delay derived in [10]. SSW used two parameters, ζ and η_{SSW} (which we denote by $\hat{\chi}$ to avoid confusion) in that paper to describe the dependence of the time delay on the Lagrangian parameters. These parameters are related to the Lagrangian parameters of this paper by

$$
\begin{aligned}
2\cosh\zeta \cos\hat{\chi} &= -\frac{M_0\beta^2}{2m}\cos\alpha \, , \\
2\sinh\zeta \sin\hat{\chi} &= -\frac{M_0\beta^2}{2m}\sin\alpha \, .
\end{aligned}
\tag{4.9}
$$

Now making the shift $\hat{\chi} = \pi + \chi$ and the identifications

$$\chi \to c\frac{\pi}{2}, \qquad \zeta \to \vartheta_{\rm cl},$$

converts on the one hand the integral of the classical time delay in [10] into $\hat{\delta}$, while on the other it maps eq.(4.9) to our previous UV-IR relation eq.(3.9-3.10).[6] Thus it is demonstrated that the UV-IR relation and $M_{\rm crit} = \frac{4m}{\beta^2}$ in particular are also consistent with the semi-classical soliton/antisoliton reflections.

5   Conclusions

In this paper two semi-classical issues of boundary sine-Gordon models are investigated to get a better understanding of the relation between the exact (algebraic) solution of the quantum theory and the classical Lagrangian.

First the semi-classical corrections to the energy difference of the two lowest energy static solutions were determined. In this procedure it turned out that one has to renormalize also the boundary potential just in the same way as the bulk one to obtain a finite result. Then we showed that comparing the main results of the semi-classical quantisation - which include in addition to the energy difference the semi-classical bound states and the classical reflection factor of the sine-Gordon field - and the semi-classical limit of the exact solution one can obtain a relation between the Lagrangian and bootstrap parameters provided we scale the bootstrap parameters in an appropriate way. After analytic continuation the form of this relation coincides with what was found by Corrigan and

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[6]Note that the $\zeta \to \vartheta_{\rm cl}$ identification is the same as the one obtained from comparing the critical behaviour of the classical solution [10] and the limit of the quantum amplitude mentioned above.

Taormina by semi-classically quantising the boundary breathers in sinh-Gordon theory [9]. Since our computation is done in a sector of sine-Gordon theory, which has no analogue in sinh-Gordon, this is an independent confirmation of the results in [9]. We also showed that in the semi-classical limit the UV-IR relation obtained from describing the boundary sine-Gordon model as a bulk and boundary perturbed conformal field theory [7] coincides with our result.

Finally we analyzed the semi-classical soliton reflections building on the ideas and results put forward by Saleur, Skorik and Warner [10]. As a consistency check we showed that the semi-classical phase shift determined from the classical time delay and the number of bound states agrees with the semi-classical limit of the exact reflection amplitudes both for ground state and for the first excited Neumann boundary. In the latter case we obtained the time delay from the analytic continuation of a special two soliton - two antisoliton solution of the bulk theory, that we constructed by the τ function method. Then we analyzed the semi-classical limit of soliton/antisoliton reflections on ground state boundary with general boundary conditions and confirmed the UV-IR relation connecting the Lagrangian and bootstrap parameters.

Acknowledgments

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