Area law violation for the mutual information in a nonequilibrium steady state

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We study the nonequilibrium steady state of an infinite chain of free fermions, resulting from an initial state where the two sides of the system are prepared at different temperatures. The mutual information is calculated between two adjacent segments of the chain and is found to scale logarithmically in the subsystem size. This provides the first example of the violation of the area law in a quantum many-body system outside a zero temperature regime. The prefactor of the logarithm is obtained analytically and, furthermore, the same prefactor is shown to govern the logarithmic increase of mutual information in time, before the system relaxes locally to the steady state.

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I.   INTRODUCTION

In recent years, studies on correlations between sub-systems in many-body states have attracted great attention. At the heart of these investigations is the realization that for naturally occurring states, the correlations are most often restricted by an *area law* [1]. Historically this topic arose from black-hole physics, where the entropy of a black hole, scaling with the area of the event horizon, was interpreted to emerge from a general holographic principle [2–4]. Later it turned out that similar bounds on quantum correlations, measured by the entanglement entropy, also hold for ground states of local quantum many-body systems [5–7]. This insight helped, among other things, to understand the power of numerical methods capturing the structure of ground-state correlations [8, 9] and also led to the development of new types of trial states [10, 11]. The only relevant exceptions from a strict area law are quantum critical systems at zero temperature, where logarithmic violations may be found [12, 13]. These are particularly well understood for one-dimensional quantum systems with the help of conformal field theory (CFT) [14], but they also persist in higher dimensions for free-fermion ground states [15–17].

At finite temperatures the situation is more involved, since for mixed states no unique measure of quantum correlations exists. Nevertheless, one can quantify the amount of *total* (quantum and classical) correlations between two disjoint subsystems by the *mutual information*. Remarkably, this particular measure of correlations fulfills an area law for nonzero temperatures in great generality. Namely, for any Gibbs state of a lattice system defined by a short-range Hamiltonian, the mutual information between neighboring subsets is proportional to the area of the common boundary [18]. For free-fermion systems, the factor of proportionality can even be bounded by the logarithm of the inverse temperature [19]. The mutual information was also investigated numerically for Gibbs states of more general quantum [20, 21] and classical lattice systems [22, 23], with a focus on the temperature dependence and subleading scaling behavior.

The question naturally emerges whether such a strict area law persists if the system is driven out of equilibrium by preparing an initial state where two parts of the system are thermalized at different temperatures. Particularly interesting is the case of integrable one-dimensional quantum systems which, due to the large number of conserved quantities, do not thermalize in the usual sense and the steady state is given by a generalized Gibbs ensemble (GGE) instead [24–26]. For models close to an integrable point, GGE was found to be relevant in the description of the prethermalized state [27] which was also demonstrated in recent cold-atom experiments [28]. However, the implications of GGE with respect to the area law for the mutual information has not yet been addressed.

Here we demonstrate that the GGE steady state of a one-dimensional chain of non-interacting fermions can lead to a logarithmic violation of the area law. Due to the slow algebraic decay of the coefficients associated with the conserved quantities in the GGE, the steady state becomes effectively a thermal state of a long-range Hamiltonian and thus the arguments of Ref. [18] do not apply. From a mathematical point of view, the logarithmic growth of mutual information with the subsystem size can be attributed to a jump singularity in the spectral function, i.e., in the symbol of the Toeplitz matrix describing the fermionic correlators. The prefactor of the logarithm will be calculated analytically using the Fisher-Hartwig conjecture, by a generalization of the method in Ref. [29]. Beside determining the steady state behavior, we also study how the mutual information is built up in time. It turns out that the steady-state value is reached after a logarithmic growth in time, the prefactor of which is given by the same one found for the steady state.

The mutual information is defined as

$$I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \qquad (1)$$

where ρ_α with $\alpha = A, B, AB$ is the *reduced* density matrix of subsystem α and $S(\rho_\alpha) = -\text{Tr}\,\rho_\alpha \ln \rho_\alpha$ is the corresponding von Neumann entropy. The full state is defined on an infinite chain with site indices $m \in \mathbb{Z}$ and ρ_α is given by the partial trace over sites $\mathbb{Z}\backslash\alpha$. Throughout

the paper we will consider the subsystems to be neighboring segments of length L with $A = [-L+1, 0]$ and $B = [1, L]$.

II. THE MODEL

We are interested in the nonequilibrium dynamics of a free-fermion system, resulting from an initial state given by the density matrix

$$\rho_0 = \frac{1}{Z_\ell} e^{-\beta_\ell \mathcal{H}_\ell} \otimes \frac{1}{Z_r} e^{-\beta_r \mathcal{H}_r}, \qquad (2)$$

which describes two disconnected reservoirs of fermions thermalized at inverse temperatures β_ℓ and β_r, with $\beta_\ell > \beta_r$. The chemical potentials are set to zero, corresponding to half-filling. The respective Hamiltonians on the left and right hand side are given by

$$\mathcal{H}_\ell = -\frac{1}{2} \sum_{m=-\infty}^{-1} \left(c_m^\dagger c_{m+1} + c_{m+1}^\dagger c_m \right),$$
$$\mathcal{H}_r = -\frac{1}{2} \sum_{m=1}^{\infty} \left(c_m^\dagger c_{m+1} + c_{m+1}^\dagger c_m \right), \qquad (3)$$

where $c_m^\dagger$ is a fermionic creation operator at site m. At time $t = 0$ the two semi-infinite chains are connected and the unitary time evolution $\rho_t = e^{-i\mathcal{H}t} \rho_0 e^{i\mathcal{H}t}$ of the full system is governed by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{m=-\infty}^{\infty} \left(c_m^\dagger c_{m+1} + c_{m+1}^\dagger c_m \right). \qquad (4)$$

In particular, one is interested in the asymptotic behavior of the system. The steady state ρ_∞ exists if, for *any* local observable $\mathcal{O}_S$ supported on a *finite* set of sites S, the expectation values can be given as

$$\lim_{t \to \infty} \mathrm{Tr}\, (\rho_t \mathcal{O}_S) = \mathrm{Tr}\, (\rho_\infty \mathcal{O}_S). \qquad (5)$$

In fact, for the system at hand this steady state can be uniquely constructed [30–32] and reads

$$\rho_\infty = \frac{1}{Z} e^{-\beta \mathcal{H}_{\mathrm{eff}}}, \quad \mathcal{H}_{\mathrm{eff}} = \sum_{n=0}^{\infty} \left(\mu_n^+ Q_n^+ + \mu_n^- Q_n^- \right), \quad (6)$$

where $\beta = (\beta_\ell + \beta_r)/2$ and the effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}$ involves two infinite sets of conserved quantities [33]

$$Q_n^+ = -\frac{1}{2} \sum_{m=-\infty}^{\infty} \left(c_m^\dagger c_{m+n} + c_{m+n}^\dagger c_m \right),$$
$$Q_n^- = -\frac{i}{2} \sum_{m=-\infty}^{\infty} \left(c_m^\dagger c_{m+n} - c_{m+n}^\dagger c_m \right). \qquad (7)$$

In particular, one has $Q_1^+ = \mathcal{H}$ and Q_1^- and Q_2^- are, up to a factor, the operators of the particle and energy current,

respectively. Thus, the steady state in (6) has exactly the form of a GGE [24] with the Lagrange multipliers associated to the conserved charges given by [31]

$$\mu_n^+ = \delta_{n,1}, \quad \mu_n^- = \begin{cases} \frac{4}{\pi} \frac{\beta_\ell - \beta_r}{\beta_\ell + \beta_r} \frac{n}{n^2-1} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases} \quad (8)$$

Note, that the μ_n^- coefficients decay asymptotically as $1/n$ and, since Q_n^- contains hopping terms over n sites, the resulting $\mathcal{H}_{\mathrm{eff}}$ is long range.

The consequences of the long-range GGE form of the steady state can also be traced on the form of the fermionic correlation functions

$$C_{mn} = \mathrm{Tr}\, (\rho_\infty c_m^\dagger c_n) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iq(m-n)} F(q), \quad (9)$$

that are given by the elements of a Toeplitz matrix with a discontinuous symbol [31]

$$F(q) = \begin{cases} \frac{1}{e^{\beta_r \omega_q}+1} & q \in (-\pi, 0) \\ \frac{1}{e^{\beta_\ell \omega_q}+1} & q \in (0, \pi) \end{cases} \quad (10)$$

where $\omega_q = -\cos q$ is the singe-particle dispersion of free fermions. The symbol $F(q)$ has a simple interpretation in this particle picture. Namely, the particles with positive momenta $q > 0$, initially located on the left-hand side and propagating to the right, are described by a Fermi distribution with β_ℓ. Similarly, the particles with $q < 0$ are emitted from the right-hand side reservoir and are thus thermalized at β_r.

III. STEADY-STATE MUTUAL INFORMATION

The symbol in Eq. (10) has a jump singularity at $q = 0$ between the values $a = F(0^-)$ and $b = F(0^+)$ and there is a second jump from $1 - b = F(\pi)$ to $1 - a = F(-\pi)$ at the ends of the spectrum. Therefore the strict proof of the area law, worked out in Ref. [19] for free-fermion states with smooth symbols, cannot be applied to this case. On the contrary, such a Fisher-Hartwig type singularity was shown to lead to the logarithmic scaling of the entropy in the zero temperature case, where the symbol jumps from 0 to 1 [29].

The calculation can be generalized to obtain the steady-state mutual information $I(A : B)$. The entropies $S_\alpha \equiv S(\rho_\alpha)$ of subsystems $\alpha = A, B, AB$ can be written as $S_\alpha = \sum_k s(\lambda_{\alpha,k})$, where

$$s(\lambda) = -\lambda \ln \lambda - (1 - \lambda) \ln(1 - \lambda), \quad (11)$$

and $\lambda_{\alpha,k}$ are the eigenvalues of the reduced correlation matrix $\mathbf{C}_\alpha$ with the elements in Eq. (9) restricted to $m, n \in \alpha$ [12]. This formula makes it possible to evaluate $I(A : B)$ numerically for large system sizes. The analytic treatment, however, requires an integral representation of the entropy [29]

$$S_\alpha = \frac{1}{2\pi i} \oint_\Gamma d\lambda\, s(\lambda) \frac{d \ln D_\alpha(\lambda)}{d\lambda}, \quad (12)$$
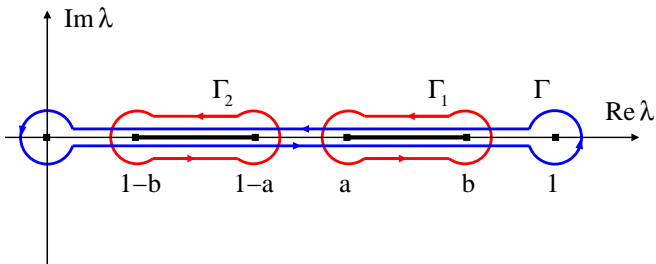
FIG. 1. (color online) Contour of the integral yielding the entropies S_α in the complex λ plane. The large (blue) contour Γ is used in Eq. (12), while the two small (red) contours Γ_1 and Γ_2 are associated to the jump-symbols, responsible for the $\ln L$ contributions. PSfrag replacements
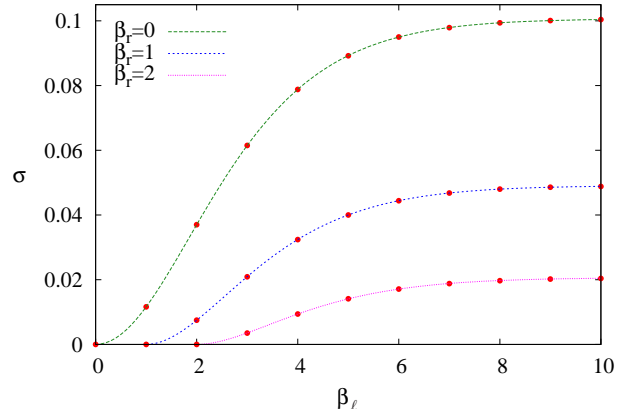

FIG. 2. (color online) Prefactor σ of the logarithmic term in the steady-state mutual information $I(A:B)$ as a function of β_ℓ and for various β_r. The lines represent the analytic result in Eq. (15), while the dots are the results of fitting the numerical data.

where $D_\alpha(\lambda) = \det(\lambda\mathbf{1} - \mathbf{C}_\alpha)$ is a Toeplitz determinant constructed from the reduced correlation matrix $\mathbf{C}_\alpha$. The contour of the integration Γ, depicted on Fig. 1, encircles the eigenvalues $\lambda_{\alpha,k}$ of $\mathbf{C}_\alpha$ on the real line and, through the logarithmic derivative of $D_\alpha(\lambda)$, gives a pole contribution at each eigenvalue.

In order to obtain asymptotic expressions for S_α, one has to invoke the Fisher-Hartwig conjecture [34] for the determinants $D_\alpha(\lambda)$ of Toeplitz matrices with symbol $\phi(q) = \lambda - F(q)$. First, the symbol is written in the factorized form $\phi(q) = \psi(q) t_{\beta_1,0}(q) t_{\beta_2,\pi}(q)$ where $t_{\beta_1,0}(q)$ and $t_{\beta_2,\pi}(q)$ describe the jumps at $q = 0$ and $q = \pi$, respectively, while $\psi(q)$ is a smooth function of q. The canonical expressions of the jump-singularities involve the auxiliary functions β_1 and β_2 of the variable λ (see Appendix for definitions) with cuts over the intervals $[a, b]$ and $[1-b, 1-a]$, respectively. The asymptotics of the $L \times L$ Toeplitz determinant is then given by

$$D_L = (\mathcal{F}[\psi])^L \, L^{-(\beta_1^2 + \beta_2^2)} \mathcal{E}[\psi, \beta_1, \beta_2], \qquad (13)$$

where $\mathcal{F}$, which yields the extensive part of the entropy, is a functional of ψ given by the Szegő limit theorem [35], while $\mathcal{E}$ is a functional of ψ, β_1, β_2 and independent of L.

The asymptotics of S_α with $\alpha = A$ or B is thus evaluated through the expression in Eq. (13), while the entropy for the joint subsystem $\alpha = AB$ involves the determinant D_{2L}. It is then straightforward to see that the extensive parts cancel out in $I(A:B)$ and the leading behavior is given by

$$I(A:B) = \sigma \ln L + \text{const.} \qquad (14)$$

The prefactor σ is entirely determined by the singular parts of the symbol, described by the functions β_1 and β_2, and thus the contour of the integration can be reduced to the loops Γ_1 and Γ_2 encircling the cuts, see Fig. 1. The calculation is rather lengthy and is presented in the Appendix. The result for the prefactor reads

$$\sigma = \frac{1}{\pi^2} \left[a \operatorname{Li}_2\left(\frac{a-b}{a}\right) + (1-a)\operatorname{Li}_2\left(\frac{b-a}{1-a}\right) \right.$$
$$\left. + b \operatorname{Li}_2\left(\frac{b-a}{b}\right) + (1-b)\operatorname{Li}_2\left(\frac{a-b}{1-b}\right) \right], \quad (15)$$

where $\operatorname{Li}_2(x)$ denotes the dilogarithm function.

Note that the result does not depend on the details of the symbol $F(q)$ except for the values $a = (e^{-\beta_r} + 1)^{-1}$ and $b = (e^{-\beta_\ell} + 1)^{-1}$ at the $q = 0$ discontinuity, and is manifestly symmetric under the simultaneous exchange $a \to 1 - a$ and $b \to 1 - b$. Comparing with the numerical results, obtained from logarithmic fits on data sets up to segment sizes $L = 200$, one finds an excellent agreement with a precision up to four digits, as shown in Fig. 2.

IV. DYNAMICS OF MUTUAL INFORMATION

The next question we address is how the steady-state value of the mutual information is reached in the course of the time evolution, after the two sides of the system are connected. This can be considered as a generalization of the local quench setup at zero temperature where the time evolution of entanglement entropy was studied [36, 37]. Since the initial state in Eq. (2) is factorized, one clearly has $I_0(A:B) = 0$. To study the growth of the mutual information, one needs the time dependent fermionic correlations [36]

$$C_{mn}(t) = i^{n-m} \sum_{k,l \in \mathbb{Z}} i^{k-l} J_{m-k}(t) J_{n-l}(t) C_{kl}(0), \quad (16)$$

where $J_m(t)$ are Bessel functions and the initial correlation matrix is given by $C_{kl}(0) = \operatorname{Tr}(\rho_0 c_k^\dagger c_l)$. Due to exponentially vanishing contributions from terms with $|m - k| \gg t$ and $|n - l| \gg t$, the infinite sums in Eq. (16) can be truncated and the matrix elements $C_{mn}(t)$ can be evaluated numerically. The mutual information $I_t(A:B)$ can then be extracted from a formula analogous to Eq. (1) by diagonalizing the reduced correlation matrices $\mathbf{C}_\alpha(t)$ and using $S_\alpha(t) = \sum_k s(\lambda_{\alpha,k}(t))$.
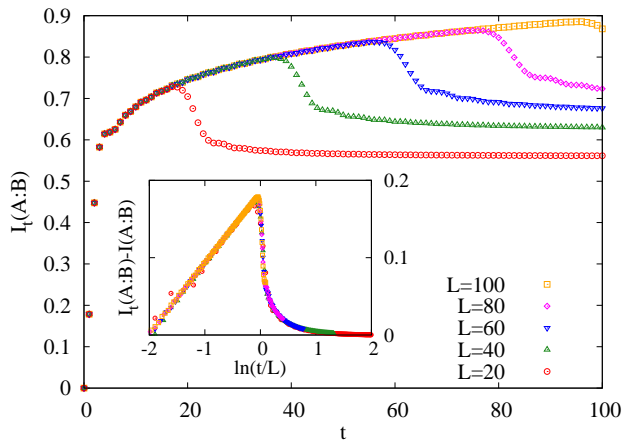
FIG. 3. (color online) Time evolution of the mutual information $I_t(A : B)$ for $\beta_\ell = 5$, $\beta_r = 0$ and various segment sizes L, after connecting the chains at $t = 0$. The inset shows the difference from the steady state value $I(A : B)$, plotted against $\ln(t/L)$.

The resulting $I_t(A : B)$ is shown on Fig. 3 for a range of segment sizes L with $\beta_\ell = 5$ and $\beta_r = 0$ fixed. After an initial logarithmic increase, the mutual information drops sharply around $t = L$ and converges slowly towards its steady-state value $I(A : B)$. Considering the distance from this asymptotic value, the curves for different L can be scaled together using the variable $\ln(t/L)$, which is shown in the inset of Fig. 3. One can see a cusp emerging between the growth and relaxation parts of the scaling function, with the former showing a pure logarithmic behavior. The prefactor of the logarithm was fitted for various values of β_ℓ and β_r and, with a good precision, we recover the steady-state prefactor in Eq. (15).

The appearance of the same prefactor governing the time evolution as well as the steady state behavior is reminiscent of the situation for the entanglement entropy in a local quench at zero temperature. In the latter case, for $t \ll L$, one has $S \sim 1/3 \ln(t)$ and thus the equilibrium scaling appears with t and L interchanged [36, 37]. However, for intermediate times one has additional terms in the entropy, obtained from a CFT calculation and scaling as $\ln(L \pm t)$ [37], which are not present for $I_t(A : B)$. We have also checked the dynamics of the mutual information on a finite chain of length $2L$ where the same logarithmic growth of $I_t(A : B)$ persists up to $t \approx 2L$, in contrast to the zero temperature case, where the entropy reaches a maximum at $t = L$ [38].

V. CONCLUSIONS AND OUTLOOK

In conclusion, we have shown that the area law for the mutual information breaks down in a simple nonequilibrium steady state of free fermions. Remarkably, all the previous examples of local Hamiltonians producing such logarithmic violations are essentially restricted to the zero temperature regime. This includes, on one hand, the ground [13, 14] and low-lying excited states [39, 40] of critical systems, as well as highly excited *pure* states of free fermions which, however, can be interpreted to be ground states of effective local Hamiltonians [41]. A simple example is given by the pure current-carrying steady state of the XX chain [42]. This formally corresponds to a GGE of Eq. (6) in the limit $\beta \to \infty$ with only two nonzero multipliers μ_1^+ and μ_2^-, defining the effective local Hamiltonian it is the ground state of.

In contrast, here we have pointed out the logarithmic scaling of the mutual information in a clearly nonzero temperature context, providing the first violation of mixed-state area laws [18, 19, 43]. The necessary condition for the violation is the slow algebraic decay of the Lagrange multipliers μ_n^- in the GGE which, however, does not seem to be a sufficient one. Indeed, a similar long-range behavior has recently been pointed out for a magnetic field quench in the transverse Ising chain, where the multipliers μ_n^+, associated to analogous conserved charges, were shown to decay as $1/n$ [44]. Nevertheless, the symbol of the respective (block-Toeplitz) correlation matrix does not show any jump singularities in this case, and thus the resulting extensive subsystem entropies do not involve logarithmic corrections. A similar conclusion was reached by a recent analytic calculation of the entropy after an interaction quench in a Bose gas where the subleading term evaluates to a constant [45]. Hence it is an interesting open problem whether a global quench without time-reversal symmetry breaking could produce a GGE with a mutual information asymptotics that violates the area law.

On the other hand, by following our result, various such violations may be found among the nonequilibrium steady states. In particular, the initial condition in Eq. (2) can be considered for the XY model, leading to a steady state where the spin-correlation matrices have a block-Toeplitz form with discontinuous symbols [32]. Presumably, this would lead again to logarithmic violations of $I(A : B)$ and the analytical calculation might even be generalized to this case. One could also address the question, whether long-range spin-correlations, that are also common features of nonequilibrium steady states via incoherent driving [46, 47], could alone be responsible for an area-law violation in general quantum chains. Finally, it would be interesting to see if the calculations can be carried through in the framework of nonequilibrium CFT, where the corresponding steady states have recently been constructed [48]. Such an approach might shed light to some universal aspects of the problem.

Appendix: Analytical Calculation of $I(A:B)$

In this Appendix, the complete analytical derivation of the logarithmic scaling of the mutual information $I(A : B)$ is presented, providing a closed form for the prefactor of the logarithm. In the calculation we generalize the method of Ref. [29].

As discussed in the main text, the von Neumann entropy of L consecutive spins in the translational invariant steady state ρ_∞ is given by $S_L = \sum_{k=1}^{L} s(\lambda_k)$, where $s(\lambda) = -\lambda \ln \lambda - (1-\lambda)\ln(1-\lambda)$. The λ_k are eigenvalues of a Toeplitz matrix corresponding to the symbol $F(q)$ defined in Eq. (10) and sketched on Fig. 4. The mutual information of two adjacent subsystems of length L is then given by $I(A : B) = 2S_L - S_{2L}$. Using the residue theorem, we can rewrite the entropy as

$$S_L = \sum_{k=1}^{L} s(\lambda_k) = \frac{1}{2\pi i} \oint_\Gamma d\lambda\, s(\lambda) \sum_{k=1}^{L} \frac{1}{\lambda - \lambda_k}$$
$$= \frac{1}{2\pi i} \oint_\Gamma d\lambda\, s(\lambda) \frac{d \ln D_L(\lambda)}{d\lambda}, \quad \text{(A.1)}$$

where the contour Γ is shown in Fig. 1 and $D_L(\lambda)$ is the determinant of the $L \times L$ Toeplitz matrix T_L with entries

$$(T_L)_{kl} = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iq(k-l)} \phi(q), \quad \text{(A.2)}$$

generated by the symbol $\phi(q) = \lambda - F(q)$.

To calculate $D_L = \det(T_L)$, we use a simplified version of the Fisher-Hartwig conjecture [34, 49]: Suppose that $\phi(q)$ has the following factorization form

$$\phi(q) = \psi(q) \prod_{j=1}^{R} t_{\beta_j, q_j}(q), \quad \text{(A.3)}$$

where $\psi(q)$ is a continuously differentiable function and t_{β_j, q_j} describe jumps at positions $q = q_j$ in the form

$$t_{\beta_j, q_j}(q) = \exp[-i\beta_j(\pi - q + q_j)], \quad \text{(A.4)}$$

where the 2π periodic quasi-momenta q are taken from the interval $q_j < q < 2\pi + q_j$. Then the $L \to \infty$ asymptotics of the determinant is

$$D_L = (\mathcal{F}[\psi])^L \left(\prod_{j=1}^{R} L^{-\beta_j^2} \right) \mathcal{E}[\psi, \{\beta_j\}, \{q_j\}], \quad \text{(A.5)}$$

where $\mathcal{F}[\psi] = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \ln \psi(q) dq \right)$, and the $\mathcal{E}$ term does not depend on L.
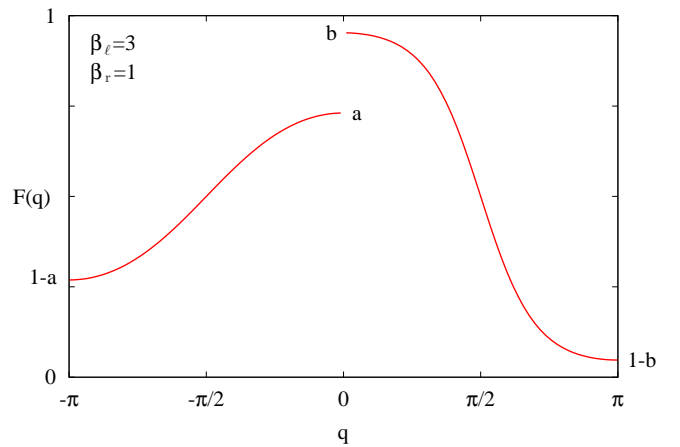

FIG. 4. Sketch of the symbol $F(q)$ for a given β_ℓ and β_r.

In our case, the symbol is $\phi(q) = \lambda - F(q)$ and there are two jumps, hence the canonical factorization reduces to $\phi(q) = \psi(q) t_{\beta_1, 0}(q) t_{\beta_2, \pi}(q)$, where

$$\beta_1(\lambda) = \frac{1}{2\pi i} \ln \left(\frac{\lambda - (e^{-\beta_r} + 1)^{-1}}{\lambda - (e^{-\beta_\ell} + 1)^{-1}} \right), \quad \text{(A.6)}$$

$$\beta_2(\lambda) = \frac{1}{2\pi i} \ln \left(\frac{\lambda - (e^{\beta_\ell} + 1)^{-1}}{\lambda - (e^{\beta_r} + 1)^{-1}} \right). \quad \text{(A.7)}$$

The logarithm of D_L reads

$$\ln D_L = L \ln \mathcal{F}[\psi] - (\beta_1^2(\lambda) + \beta_2^2(\lambda)) \ln L + \ln \mathcal{E}. \quad \text{(A.8)}$$

Since we are interested only in the leading behavior of $I(A : B)$, we drop the last term which gives a $\mathcal{O}(1)$ contribution. The derivative of $\ln D_L$ then reads

$$\frac{d \ln D_L(\lambda)}{d\lambda} = \frac{d \ln \mathcal{F}[\psi]}{d\lambda} L - \frac{a-b}{\pi i} \left[\frac{\beta_1(\lambda)}{(a-\lambda)(b-\lambda)} \right.$$
$$\left. + \frac{\beta_2(\lambda)}{(1-a-\lambda)(1-b-\lambda)} \right] \ln L, \quad \text{(A.9)}$$

where $a = (e^{-\beta_r} + 1)^{-1}$ and $b = (e^{-\beta_\ell} + 1)^{-1}$.

According to Eq. (A.1), this has to be integrated along the large contour Γ depicted on Fig. 1 of the main text, containing the interval $[0, 1]$. Let us here emphasize that the extensive (linear in L) part of S_L has indeed contributions from the entire contour Γ. However, it is easy to see that in $I(A : B)$ the term proportional to L drops out and thus only a part of the contour is of importance. We will show this using the fact that in the neighborhood of the real line one has

$$\beta_1(x + i0^\pm) = \frac{1}{2\pi i} \left[\ln \frac{a-x}{b-x} \mp i(\pi - 0^+) \right]$$
$$= \beta_1(x) \mp \left(\tfrac{1}{2} - 0^+ \right), \quad \text{(A.10)}$$

for $x \in (a, b)$ and similarly for β_2:

$$\beta_2(x + i0^\pm) = \frac{1}{2\pi i} \left[\ln \frac{(1-b)-x}{(1-a)-x} \mp i(\pi - 0^+) \right]$$
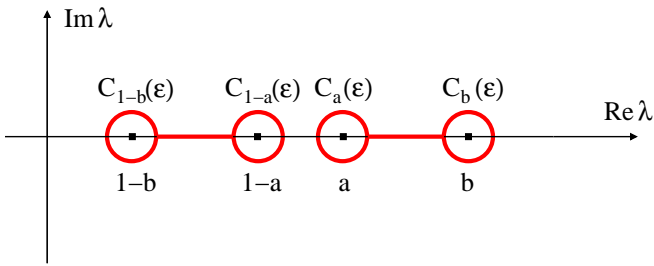$$= \beta_2(x) \mp \left(\tfrac{1}{2} - 0^+ \right). \quad \text{(A.11)}$$

FIG. 5. The final integration contour decomposes into two intervals and four circular contours around the points a, b, $1 - a$ and $1 - b$.

with $x \in (1-b, 1-a)$. In other words the functions β_1 and β_2 have cuts along the intervals $[a, b]$ and $[1-b, 1-a]$, respectively, but they are analytic on the rest of $[0, 1]$. This means that we can reduce the contour integration along Γ to the contours Γ_1 and Γ_2 that encircle these cuts, see Fig. 1 in the main text. Thus, we obtain that $I(A : B) = \sigma \ln L + \text{const}$, with

$$\sigma = \frac{a-b}{2\pi^2} \left[\oint_{\Gamma_1} \mathrm{d}\lambda \frac{s(\lambda)\,\beta_1(\lambda)}{(a-\lambda)(b-\lambda)} + \oint_{\Gamma_2} \mathrm{d}\lambda \frac{s(\lambda)\,\beta_2(\lambda)}{(1-a-\lambda)(1-b-\lambda)} \right]. \quad (A.12)$$

The contours Γ_1 and Γ_2 can now be contracted and, using Eqs. (A.10) and (A.11), the integration has to be carried out on the intervals $[a+\epsilon, b-\epsilon]$ and $[1-b+\epsilon, 1-a-\epsilon]$ and along circular contours around the points a, b, $1-a$ and $1-b$, see Fig. 5. A further simplification occurs by observing the symmetry of the problem under the exchange of variables $\lambda \to 1 - \lambda$. Indeed, one has $\beta_2(1 - \lambda) = -\beta_1(\lambda)$ where the minus sign cancels out with the reversal of the direction $\Gamma_2 \to -\Gamma_1$ of the contours upon reflection. Thus the two contributions in Eq. (A.12) are equal and lead to the following sums of integrals

$$\sigma = \lim_{\epsilon \to 0} \frac{a-b}{\pi^2} \left[\int_{a+\epsilon}^{b-\epsilon} \frac{\mathrm{d}\lambda\, s(\lambda)}{(a - \lambda)(b - \lambda)} + \oint_{C_a(\epsilon)} \frac{\mathrm{d}\lambda\, s(\lambda)\beta_1(\lambda)}{(a - \lambda)(b - \lambda)} + \oint_{C_b(\epsilon)} \frac{\mathrm{d}\lambda\, s(\lambda)\beta_1(\lambda)}{(a - \lambda)(b - \lambda)} \right], \quad (A.13)$$

where $C_v(\epsilon)$ denotes a circular contour of radius ϵ with $v = a, b$ as center. Let us evaluate such a principal value integral around b. Substituting $\lambda = b + \epsilon e^{i\theta}$ one has

$$\lim_{\epsilon \to 0} \oint_{C_b(\epsilon)} \frac{\mathrm{d}\lambda}{2\pi i} s(\lambda) \frac{\ln(\lambda - a) - \ln(\lambda - b)}{(\lambda - a)(\lambda - b)} =$$
$$\lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} s(b) \frac{\ln(b - a) - \ln(\epsilon) - i\theta}{b - a} =$$
$$\lim_{\epsilon \to 0} \frac{s(b)}{b - a} \ln\left(\frac{b - a}{\epsilon}\right), \quad (A.14)$$

where in the second line we used $\mathrm{d}\lambda = i\epsilon e^{i\theta}\mathrm{d}\theta$ which cancels out the term $(\lambda - b)$ in the denominator. Note, that the result is divergent and one has to consider it as a limit. The integral for $C_a(\epsilon)$ is evaluated analogously with the substitution $\lambda = a - \epsilon e^{i\theta}$ and yields a similar result where $s(b)$ is replaced with $s(a)$.

For the line-integral, we get the following expression

$$\lim_{\epsilon \to 0} \int_{a+\epsilon}^{b-\epsilon} \frac{\mathrm{d}\lambda\, s(\lambda)}{(a - \lambda)(b - \lambda)} =$$
$$\frac{1}{a - b} \left[a \operatorname{Li}_2\left(\frac{a-b}{a}\right) + (1-a)\operatorname{Li}_2\left(\frac{b-a}{1-a}\right) \right.$$
$$\left. + b \operatorname{Li}_2\left(\frac{b-a}{b}\right) + (1-b)\operatorname{Li}_2\left(\frac{a-b}{1-b}\right) \right]$$
$$+ \lim_{\epsilon \to 0} \frac{s(a) + s(b)}{b - a} \ln\left(\frac{\epsilon}{b - a}\right), \quad (A.15)$$

where $\operatorname{Li}_2(x)$ is the dilogarithm function defined as

$$\operatorname{Li}_2(x) = -\int_0^x \mathrm{d}\lambda \frac{\ln(1 - \lambda)}{\lambda}. \quad (A.16)$$

Note, that the last term of Eq. (A.15) is again divergent. However, inserting it into Eq. (A.13) together with the result (A.14) for the circular contours, the divergences cancel out and one finally obtains the prefactor σ as given in Eq. (15) of the main text.

[1] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).
[2] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D **34**, 373 (1986).
[3] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).
[4] T. Nishioka, S. Ryu, and T. Takayanagi, J. Phys. A: Math. Theor. **42**, 504008 (2009).
[5] M. B. Plenio, J. Eisert, J. Dreissig, and M. Cramer, Phys. Rev. Lett. **94**, 060503 (2005).
[6] M. B. Hastings, J. Stat. Mech. (2007) P08024.
[7] F. G. S. L. Brandão and M. Horodecki, Nature Physics **9**, 721 (2013).
[8] I. Peschel, X. Wang, M. Kaulke, and K. Hallberg, eds., *Density-Matrix Renormalization*, Lecture Notes in Physics, Vol. 528 (Springer, Berlin, 1999).
[9] U. Schollwöck, Rev. Mod. Phys. **77**, 259 (2005).

[10] F. Verstraete, J. I. Cirac, and V. Murg, Adv. Phys. **57**, 143 (2008).

[11] J. I. Cirac and F. Verstraete, J. Phys. A: Math. Theor. **42**, 504004 (2009).

[12] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).

[13] J. I. Latorre and A. Riera, J. Phys A: Math. Theor **42**, 504002 (2009).

[14] P. Calabrese and J. Cardy, J. Phys. A: Math. Theor. **42**, 504005 (2009).

[15] M. M. Wolf, Phys. Rev. Lett. **96**, 010404 (2006).

[16] D. Gioev and I. Klich, Phys. Rev. Lett. **96**, 100503 (2006).

[17] S. Farkas and Z. Zimborás, J. Math. Phys. **48**, 102110 (2007).

[18] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac, Phys. Rev. Lett. **100**, 070502 (2008).

[19] H. Bernigau, M. J. Kastoryano, and J. Eisert, arXiv:1301.5646.

[20] R. G. Melko, A. B. Kallin, and M. B. Hastings, Phys. Rev. B **82**, 100409(R) (2010).

[21] R. R. P. Singh, M. B. Hastings, A. B. Kallin, and R. G. Melko, Phys. Rev. Lett. **106**, 135701 (2011).

[22] J. Wilms, M. Troyer, and F. Verstraete, J. Stat. Mech. (2011) P10011.

[23] J. Iaconis, S. Inglis, A. B. Kallin, and R. G. Melko, Phys. Rev. B **87**, 195134 (2013).

[24] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007).

[25] T. Barthel and U. Schollwöck, Phys. Rev. Lett. **100**, 100601 (2008).

[26] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).

[27] M. Kollar, F. A. Wolf, and M. Eckstein, Phys. Rev. B **84**, 054304 (2011).

[28] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. Adu Smith, E. Demler, and J. Schmiedmayer, Science **337**, 1318 (2012).

[29] B. Q. Jin and V. E. Korepin, J. Stat. Phys. **116**, 79 (2004).

[30] T. G. Ho and H. Araki, Proc. Steklov Inst. Math. **228**, 191 (2000).

[31] Y. Ogata, Phys. Rev. E **66**, 016135 (2002).

[32] W. H. Aschbacher and C.-A. Pillet, J. Stat. Phys. **112**, 1153 (2002).

[33] M. P. Grabowski and P. Mathieu, Ann. Phys. **243**, 299 (1995).

[34] E. L. Basor and C. A. Tracy, Physica A **177**, 167 (1991).

[35] A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators* (Springer-Verlag, Berlin, 1990).

[36] V. Eisler and I. Peschel, J. Stat. Mech. (2007) P06005.

[37] P. Calabrese and J. Cardy, J. Stat. Mech. (2007) P10004.

[38] J.-M. Stéphan and J. Dubail, J. Stat. Mech. (2011) P08019.

[39] L. Masanes, Phys. Rev. A **80**, 052104 (2009).

[40] F. C. Alcaraz, M. I. Berganza, and G. Sierra, Phys. Rev. Lett. **106**, 201601 (2011).

[41] V. Alba, M. Fagotti, and P. Calabrese, J. Stat. Mech. (2009) P10020.

[42] V. Eisler and Z. Zimborás, Phys. Rev. A **71**, 042318 (2005).

[43] M. J. Kastoryano and J. Eisert, J. Math. Phys. **54**, 102201 (2013).

[44] M. Fagotti and F. H. L. Essler, Phys. Rev. B **87**, 245107 (2013).

[45] M. Collura, M. Kormos, and P. Calabrese, J. Stat. Mech. (2014) P01009.

[46] T. Prosen and I. Pižorn, Phys. Rev. Lett. **101**, 105701 (2008).

[47] T. Prosen and M. Žnidaric, Phys. Rev. Lett. **105**, 060603 (2010).

[48] D. Bernard and B. Doyon, J. Phys. A: Math. Theor. **45**, 362001 (2012).

[49] E. L. Basor, Trans. Amer. Math. Soc. **239**, 33 (1978).