A_2 Toda theory in reduced WZNW framework and the representations of the W algebra

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Abstract

Using the reduced WZNW formulation we analyse the classical W orbit content of the space of classical solutions of the A_2 Toda theory. We define the quantized Toda field as a periodic primary field of the W algebra satifying the quantized equations of motion. We show that this local operator can be constructed consistently only in a Hilbert space consisting of the representations corresponding to the minimal models of the W algebra.

1. Introduction

The Toda field theories (TT) associated to various simple Lie algebras $\mathcal{G}$ have received some interest recently, partly because the simplest of them, when the Lie algebra is just A_1, coincides with the Liouville theory. It has been known for a long time that these theories are conformally invariant [1,2] in addition to being exactly integrable [3]. Several methods have been suggested [1,2] to quantize them, and all of these methods showed convincingly that the quantized versions are bona fide conformal field theories (CFT). In a recent paper [4] it was shown that tuning the coupling constant of these theories carefully one can reproduce the central charges and highest weights of the various 'minimal' or 'coset' models.

Bilal and Gervais were the first to point out that through the Poisson brackets the TT associated to $\mathcal{G}$ provide a realisation of the $W\mathcal{G}$ algebras [5]. The concept of W algebras (i.e. extensions of the Virasoro algebra by higher, (half)integer spin currents) was introduced in the study of CFT a few years ago [6]. The $W\mathcal{G}$ algebras provide a set of W algebras where the spins of the currents (W_i), generating $W\mathcal{G}$, are determined by the exponents (h_i) of $\mathcal{G}$: $s_i = h_i + 1$. Using an essentially free field quantization it was shown in [2] that the quantized TT provide a systematic framework to construct the CFT-s that admit the $W\mathcal{G}$ algebras as symmetries.

It has been discovered recently [7] that the classical TT-s can naturally be viewed as Hamiltonian reductions of the WZNW theories. This reduction is achieved by imposing certain first class, conformally invariant constraints on the Kac Moody (KM) currents. These constraints reduce the chiral KM phase spaces to phase spaces carrying the chiral $W\mathcal{G}$ algebras as their Poisson bracket structures. The advantage of this reduced WZNW description is that it yields only a restricted set of relevant degrees of freedom but with a rich algebraic structure as well as giving a new way to describe the space of classical solutions. A natural way to quantize these theories is to promote only the relevant degrees of freedom to operators, trying to preserve – as much as possible – the boundary conditions and the algebraic structure.

Recently we carried out this program for the Liouville theory [8] and – contrary to our expectation – we found that this quantization becomes consistent only in the 'deep quantum' domain, but not in the region which is – at least naively – smoothly connected to the classical theory.

The aim of this paper is to show what we can gain both classically and in the quantum theory from using the WZNW framework to describe the A_2 TT (which is the next simplest one after the Liouville theory). In the classical theory we demonstrate that it enables us to gather information about the 'classical WA_2 algebra (W for short) representation' (classical W orbit) content of the space of classical solutions. In particular we shall be able to identify W orbits that are classical analogues of the quantum highest weight representations (h.w.r.) both in the singular and in the non singular sectors of the A_2 TT.

In the quantum case we show that promoting only the generators of the W symmetry and a single Toda field to operators is in a certain sense a minimal quantization. This means that we require only the definition of the quantum equivalent of this Toda field be periodic, and be consistent with W transformation properties, the equation of motion and locality, but we do not ask for the presence of any closing operator algebra or any quantum group structure. Yet we show that when these requirements are supplemented by having a positive central charge as well as a discrete spectrum of W highest weights in the Hilbert space $\mathcal{H}$, where our operators act, then we are inevitably lead to the conclusion that $\mathcal{H}$ must consists of W algebra representations corresponding to the (not necessarily unitary) minimal models [9], that have no smooth semiclassical limit. Since the presence of a discrete rather then a continuos spectrum of W highest weights in $\mathcal{H}$ corresponds to the singular sector of the A_2 TT we can say that quantizing it in the reduced WZNW framework works nicely for the singular sector in the deep quantum domain.

The paper is organized as follows: In section 2. we review the description of the classical A_2 TT in the WZNW framework. Using this in section 3. we investigate the classical W representation content of the space of classical solutions. In section 4. we derive the quantum equation of motion for the Toda field and determine the general form of the Hilbert space $\mathcal{H}$, where it may act irreducibly. We construct the local Toda field and obtain the precise form of $\mathcal{H}$ in section 5. We make our conclusions in section 6. The three appendices, A, B, C, contain some details about the way we determined the orbits corresponding to the classical highest weights, the way we computed the various matrix elements of W_n and the way we obtained the $x \to x^{-1}$ transformation rule of the generalized hypergeometric functions respectively.

2. Classical A_2 Toda theory in WZNW framework

The A_2 Toda theory describes the interaction of two real, periodic scalar fields $\Phi^a(x^0, x^1) = \Phi^a(x^0, x^1 + 2\pi)$; $a = 1, 2$ in two dimensions. Introducing light cone coordinates $x^\pm = (x^0 \pm x^1)$ their equations of motion have the form:

$$\partial_+\partial_-\Phi^1 + 2e^{\Phi^1 - \frac{1}{2}\Phi^2} = 0 \tag{1.1}$$

$$\partial_+\partial_-\Phi^2 + 2e^{\Phi^2 - \frac{1}{2}\Phi^1} = 0 \tag{1.2}$$

The corresponding Lagrangean is

$$\mathcal{L} = \sum_{a,b} \frac{1}{2} K_{ab} \partial_+\Phi^a \partial_-\Phi^b - 2\sum_a \exp(\frac{1}{2}K_{ab}\Phi^b)$$

where K_{ab} denotes the Cartan matrix of A_2. It has been known for a long time that this theory is conformally invariant; a property shared by all the other Toda theories (TT). The conformal invariance can be seen from the Feigin Fuchs form of the improved energy momentum tensor:

$$T_{\pm\pm} = \sum_{a,b} \frac{1}{2} K_{ab} \partial_\pm\Phi^a \partial_\pm\Phi^b - 2\sum_a \partial_\pm^2 \Phi^a$$

with $T_{+-} = 0$. As a consequence $T_{++} = L$ and $T_{--} = \bar{L}$ satisfy $\partial_- L = 0$, $\partial_+ \bar{L} = 0$ on shell. From ref.[5] we know that in the case of the classical A_2 TT we have two (commuting) copies of the WA_2 algebra (W algebra for short) generated by the spin two $L(x^+) = W_1$ and by a spin three current $W(x^+) = W_2$, together with their right moving counterparts $\bar{L}(x^-)$, $\bar{W}(x^-)$. (The $W(x^+)$ ($\bar{W}(x^-)$) quantities appearing here are somewhat complicated third order polynomials made of $\partial_+\Phi^a$, $\partial_+^2\Phi^a$, $\partial_+^3\Phi^a$, (resp. $\partial_-\Phi^a$, $\partial_-^2\Phi^a$, $\partial_-^3\Phi^a$) [5,10] but in the following we shall not need their actual form.)

Recently a unified description of classical W algebras associated to the TT-s was given [7] by exploiting the connection between TT and constrained WZNW models. In this WZNW description the constraints, that select the TT in the space of WZNW currents, generate gauge transformations (left moving upper and right moving lower triangular Kac Moody (KM) transformations) and the $W\mathcal{G}$ algebra is nothing, but the algebra of gauge invariant polynomials made of the constrained KM current and its derivatives. One advantage of this approach lies in the fact, that the brackets between the the W_i-s – which are induced by the canonical Poisson brackets of the original currents of the WZNW model – can be computed readily using some appropriate KM transformations that preserve the form of the constrained current [7]. In the A_2 case, when this "form" preserved by the special KM transformation was the "highest weight" one [7] rather than the more familiar "Wronskian" one we found that

$$\delta L = \delta W_1 = \left[a_1(W_1)^{'} + 2a_1^{'}W_1 - 2a_1^{'''}\right] + \left[2a_2(W_2)^{'} + 3a_2^{'}W_2\right] \tag{2}$$

$$\begin{aligned}\delta W = \delta W_2 =& \left[a_1(W_2)^{'} + 3a_1^{'}W_2\right] \\ &+ \Big[a_2\big(-\frac{1}{6}(W_1)^{'''} + \frac{2}{3}W_1(W_1)^{'}\big) + a_2^{'}\big(-\frac{3}{4}(W_1)^{''} + \frac{2}{3}(W_1)^2\big) \\ &- \frac{5}{4}a_2^{''}(W_1)^{'} - \frac{5}{6}a_2^{'''}W_1 + \frac{1}{6}a_2^{(V)}\Big]\end{aligned} \tag{3}$$

where $a_{1,2}(x^+)$ are the infinitesimal functions characterising the 'pure conformal' and 'pure W' parts of the complete W transformations. Eq.(3) shows that $W(x^+)$ transforms as a primary field of weight 3 under conformal transformations while its change under the pure W transformation depends only on the energy momentum tensor $L = W_1$. Eq.s (2,3) can be converted to the brackets between the W_i-s by

$$\delta W_i = \sum_j \int dy^1 a_j(y)\{W_i(x), W_j(y)\}|_{x^0=y^0} \tag{4}$$

In the case of A_2 the reduced WZNW framework also associates to the solutions of the TT an $SL(3)$ valued WZNW field, g, of rather restricted form, containing all the information. This approach identifies the fundamental and natural variables of the A_2 TT as the lower right corner element, $u(x^0, x^1) = u_2(x^0, x^1) = \exp(-\frac{1}{2}\Phi^2(x^0, x^1))$, of this matrix g, plus the (chiral) W_i ($\bar{W}_i$) generators of the W algebra, since the entire g field can be described in their terms. The explicit form of g is:

$$g = \begin{pmatrix} \partial_-^2\partial_+^2 u + H & \partial_-^2\partial_+ u - \frac{1}{2}L\partial_+ u & \partial_-^2 u - \frac{1}{2}Lu \\ \partial_-\partial_+^2 u - \frac{1}{2}\bar{L}\partial_- u & \partial_-\partial_+ u & \partial_- u \\ \partial_+^2 u - \frac{1}{2}\bar{L}u & \partial_+ u & u \end{pmatrix} \tag{5}$$

where $H = -\frac{1}{2}L\partial_+^2 u - \frac{1}{2}\bar{L}\partial_-^2 u + \frac{1}{4}\bar{L}Lu$. $u_1 = \exp(-\frac{1}{2}\Phi^1)$ is given as the lower right subdeterminant of g and this definition is equivalent to eq.(1.2). On the other hand, $\det g = 1$ – which is an integral of the (*linear*) equations of motion for the $u(x^0, x^1)$ field

$$Du = \partial_+^3 u - L(x^+)\partial_+ u - \big(W(x^+) + \frac{1}{2}L(x^+)'\big)u = 0 \tag{6}$$

(plus a similar one, $\bar{D}u = 0$, in the other light cone variable with $L \to \bar{L}$, $W \to \bar{W}$) – implies eq.(1.1). Regarding u, L, W, and $\bar{L}$, $\bar{W}$ as fundamental variables places the 'singular Toda solutions' (when u_1 and u_2 may have some zeroes) and the 'non singular' ones (when u_1 and u_2 have no zeroes) on an equal footing: both of them are described by a globally well defined and regular g matrix if $L(x^+)$, $W(x^+)$ ($\bar{L}(x^-)$, $\bar{W}(x^-)$) are non singular, periodic functions [7]. Using the previously mentioned form preserving KM transformation to implement the infinitesimal W transformations it is easy to see that $u(x^0, x^1)$ is a primary field of the W algebra since

$$\begin{aligned} \delta u =& a_1(x^+)\partial_+ u - a_1' u \\ &+ a_2(x^+)(\partial_+^2 u - \frac{2}{3}L(x^+)u) - \frac{1}{2}a_2'\partial_+ u + \frac{1}{6}a_2'' u \end{aligned} \tag{7}$$

(u transforms in an entirely analogous way under the right moving algebra generated by $\bar{L}$, $\bar{W}$.) If some non singular, periodic L, W ($\bar{L}$, $\bar{W}$) are given, then the u

field can be constructed from the solutions of the eq.(6) and its chiral partner as

$$u(x^0, x^1) = \sum_{k=1}^{3} \psi_k(x^+)\chi_k(x^-) \tag{8}$$

Here $\psi_k(x^+)$ ($\chi_k(x^-)$) stand for the three linearly independent solutions of $Du = 0$ ($\bar{D}u = 0$) normalized by

$$1 = \begin{vmatrix} \partial^2\phi_1 & \partial^2\phi_2 & \partial^2\phi_3 \\ \partial\phi_1 & \partial\phi_2 & \partial\phi_3 \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix}, \quad \phi = \psi,\ \chi, \quad \partial = \partial_+,\ \partial_-$$

3. Classical representations of the W algebra

Treating $u(x^0, x^1)$ and the currents of the W algebra as fundamental variables opens up a new possibility to analyze the space of classical solutions of A_2 TT. As eq.(2,3) and (7) were obtained from a KM transformation preserving the form of the constrained current, the transformed quantities, $u + \delta u$, $L + \delta L$, $W + \delta W$ will also solve eq.(6), i.e. the W algebra transforms classical solutions of A_2 TT into another solutions. Therefore the basic object we need is the family of solutions connected by W transformations: the so called *orbit* of the W algebra. (In more mathematical terms these W orbits are nothing but the simplectic leafs of the second Gelfand Dikii bracket [11], which is equivalent to eq.(4) [12].) Clearly these orbits may be viewed as the classical representations of the W algebra, and to say something about the representation content of the classical solution space one has to find the invariants characterizing the orbits. According to a recent study [12] there are just two types of invariants for the W orbits: a continuous one, the monodromy matrix M, and a discrete one, describing the homotopy classes of certain non degenerate curves associated to the solutions of $D\psi = 0$. The appearence of the monodromy matrix can be understood in the following way: though $u(x^0, x^1)$ must be periodic for $x^1 \to x^1 + 2\pi$ the solutions of the chiral d.e., (6), may be quasiperiodic

$$\psi_k(z + 2\pi) = M_{kl}\psi_l(z) \tag{9}$$

if the left and right monodromy matrices are not independent of each other. Furthermore eq.(7) shows that the W transformations act linearly on ψ thus they obviously preserve M. Of the homotopy classes it was shown [12] that in the most general case there are just three of them – in marked contrast to the Liouville case, when the discrete invariant could take infinitely many different values counting the (conserved) number of zeroes of the Liouville analogue of the u field [13,14].

Once we can characterize the orbits – the classical representations of the W algebra – the next question is to determine which of them may correspond to highest weight representations (h.w.r.). In the (quantum) h.w.r. the expectation value of the energy operator is bounded below and it attains its minimum value for the highest weight state, which is a simultaneous eigenvector of both L_0 and W_0 [6]. Therefore it is natural to expect that a W orbit would correspond to a h.w.r. if the total energy, $\int\limits_0^{2\pi} L(z)dz$, stays bounded below as we move along the orbit. Furthermore we also expect, that it also contains a solution of eq.(6) (the "classical h.w." vector) with *constant* L, W, such that the total energy has at least a local minimum there, i.e. $\int\limits_0^{2\pi} L(z)dz$ increases if we move away from this solution along the orbit.

To investigate the representation content of the classical solution space and in particular to see what parts of it may correspond to h.w.r. we adopted the following procedure [11]: first we picked a monodromy matrix M, and looked for such ψ_k-s that satisfy eq.(9) and would give constant L_0-s and W_0-s through eq.(6). (Technically we determined L_0 and W_0 using only two ψ_k-s and found the third one from the normalization condition.) Then, in the second step, by iterating the transformation leading to eq.(2,3) we determined if $\Delta L = E(a_1, a_2) - L_0$ – where $E(a_1, a_2) = \int\limits_0^{2\pi} L(z)dz$ – is positive for all (periodic) a_1 and a_2 or not. We call a W orbit a potential classical h.w.r. if ΔL is positive for all $a_{1,2}$ (for details see Appendix A).

So far we analysed only orbits with diagonalizable monodromy matrices in the generic case, i.e. when all the parameters appearing are different and nonvanishing.

Since $M \in SL(3)$, its eigenvalues are either all real or it has a complex conjugate pair of them and a real one. In the former case a large class of M-s can be described by

$$M = \mathrm{diag}\big(e^{\Lambda 2\pi},\ e^{m2\pi},\ e^{-(\Lambda+m)2\pi}\big) \qquad \Lambda \neq m \tag{10}$$

where Λ and m are arbitrary real parameters. The $\psi_k(x^+)$ satisfying eq.(9) with this M and yielding the constant energy and W densities

$$L_0 = \Lambda^2 + \Lambda m + m^2 \qquad W_0 = -m\Lambda(m+\Lambda) \tag{11}$$

are

$$\psi_1(x^+) = Ne^{\Lambda x^+}; \quad \psi_2(x^+) = Ne^{mx^+}; \quad \psi_3(x^+) = Ne^{-(\Lambda+m)x^+};$$
$$N = \big[(m-\Lambda)\{m\Lambda + 2(m+\Lambda)^2\}\big]^{-1/3} \tag{12}$$

Since the curve associated to these ψ_k-s has at most two zeroes this solution is in the 'non oscillatory' homotopy class in the classification of [12]. It is important to notice, that $L_0 > 0$ for all non vanishing Λ and m. From the analysis of $E(a_1, a_2)$ around this solution we concluded that this type of orbits can be classical h.w.r. for *all values* of Λ and m. The right moving sector can be obtained from eq.(11,12) by some trivial substitutions if the monodromy matrix there has the same form as eq.(10) but with $\Lambda \to \hat{\Lambda}$ and $m \to \hat{m}$. Using these chiral solutions in eq.(8) we see that $u(x^0, x^1)$ will be periodic if $\hat{\Lambda} = \Lambda$ and $\hat{m} = m$, and then

$$u(x^0, x^1) = N\hat{N}\big(e^{2\Lambda x^0} + e^{2mx^0} + e^{-(2\Lambda+m)x^0}\big)$$

i.e. the A_2 Toda sector corresponding to these orbits is the non singular one.

A large class of monodromy matrices having a real eigenvalue as well as a complex conjugate pair can be described by

$$M(\Lambda,\rho) = \begin{pmatrix} e^{\Lambda 2\pi}\cos(\rho\pi) & e^{\Lambda 2\pi}\sin(\rho\pi) & 0 \\ -e^{\Lambda 2\pi}\sin(\rho\pi) & e^{\Lambda 2\pi}\cos(\rho\pi) & 0 \\ 0 & 0 & e^{-2\Lambda 2\pi} \end{pmatrix} \tag{13}$$

where Λ and $\rho > 0$ are real parameters. We note that $M(\Lambda, \rho + 2K) = M(\Lambda, \rho)$, ($K$ integer), thus the domain of ρ containing only inequivalent M-s is $0 < \rho < 2$.

Furthermore if ρ is integer ($\neq 0$), then M has three real eigenvalues (in general a doubly degenerate one a non degenerate one) thus some of these cases correspond to the $\Lambda \to m$ limit of eq.(10). We also note, that for $\Lambda = 0$, $\rho = 2K$, M becomes the identity matrix.

The ψ_k-s satisfying eq.(9) with this M and yielding constant energy and W densities now have the following form:

$$\psi_1(x^+) = \tilde{N} e^{\Lambda x^+} \sin\frac{\rho x^+}{2}; \quad \psi_2(x^+) = \tilde{N} e^{\Lambda x^+} \cos\frac{\rho x^+}{2}; \quad \psi_3(x^+) = \tilde{N} e^{-2\Lambda x^+}$$

$$\tilde{N} = \left[-\rho(\frac{9}{2}\Lambda^2 + \frac{\rho^2}{8})\right]^{-1/3} \tag{14}$$

while the L_0 and W_0 densities are

$$L_0 = 3\Lambda^2 - \frac{\rho^2}{4}; \qquad W_0 = -2\Lambda(\Lambda^2 + \frac{\rho^2}{4}) \tag{15}$$

The solution of eq.(6) given by eq.(14,15) is more interesting than the one described by eq.(11,12). First of all we note that now – unlike in the previous case – the energy density may be negative, $L_0 < 0$, if $|\Lambda| < \rho/2\sqrt{3}$. In the $0 < \rho < 2$ domain the curve associated to this solution is again in the "non oscillatory" homotopy class, but the possibility of keeping M fixed while shifting ρ by an even integer may correspond to describing solutions with the same M but belonging to the 'higher' homotopy classes. Precisely this happens for $\Lambda = 0$ when $\rho = 2,\ 4,\ 6,$ since in these cases eq.(14,15) give the three representative solutions of the three homotopy classes belonging to $M = \mathrm{Id}$ as discussed in [12].

Analysing the behaviour of $E(a_1, a_2)$ around this solution we concluded that this type of orbits can be classical h.w.r. for all values of Λ if $\rho < 1$. This is surprising since it implies that the orbit containing the 'classical SL_2 invariant vacuum' ($\Lambda = 0$, $\rho = 2$) cannot be a highest weight one. This result is important as it implies that the quantum theory may have no smooth semiclassical limit if it contains the SL_2 invariant vacuum in a (quantum) highest weight representation. (One can see in the following way that eq.(14,15) with $\Lambda = 0$, $\rho = 2$ indeed describe the invariant classical vacuum : computing the brackets between the T_n, W_n Fourier components of $L(x^+)$ and $W(x^+)$ from eq.(2-4) one finds that the central term in $\{T_n, T_m\}$ is of the form $\frac{c}{12}n^3\delta_{n,-m}$ with $c = 24$. To convert it into the canonical $\frac{c}{12}n(n^2-1)\delta_{n,-m}$ form we have to make a shift in $T_0(\equiv L_0)$ by $c/24 = 1$, and after this shift the solution with $\Lambda = 0$, $\rho = 2$ will be the one of vanishing energy and W density. This argument also shows that all the orbits characterized by M-s in the form of eq.(10) will have an energy density bounded below by 1.)

The right moving $\chi_k(x^-)$ solutions can again be obtained by some obvious substitutions from eq.(14,15) if we assume that the right moving monodromy matrix differs from eq.(13) only in the parameter replacments $\Lambda \to \hat{\Lambda}$, $\rho \to \hat{\rho}$. Using these ψ_k-s and χ_k-s in eq.(8) to construct $u(x^0, x^1)$ we conclude that u will be periodic if $\hat{\Lambda} = \Lambda$ and $\rho + \hat{\rho} = 2J$ with J integer. From the actual form of u

$$u(x^0, x^1) = \tilde{N}\hat{\tilde{N}}\left(e^{2\Lambda x^0}\cos[\frac{\rho-\hat{\rho}}{2}x^0 + \frac{\rho+\hat{\rho}}{2}x^1] + e^{-4\Lambda x^0}\right)$$

we see that if $\Lambda \neq 0$ then – depending on the sign of Λ – it has zeroes either for $x^0 > 0$ or for $x^0 < 0$. This means that the number of zeroes of u may change in time, but nevertheless their mere existence implies that this type of orbits are in the 'singular sector' of the solution space of the A_2 TT.

Clearly for orbits characterized by M-s having the form of eq.(13) ρ is a kind of angular variable, thus we expect that in the quantum theory its eigenvalues would be discrete. Through eq.(15) this would imply that the Hilbert space corresponding to these orbits contains a discrete spectrum of W algebra highest weights.

In passing we emphasize that it is a rather special property of the orbits described so far that they contain representatives (the $\psi_k(x^+)$-s) yielding constant L_0 and W_0. When we changed M in eq.(10) slightly

$$M = \mathrm{diag}\left(-e^{\Lambda 2\pi},\ -e^{m2\pi},\ e^{-(\Lambda+m)2\pi}\right) \qquad \Lambda \neq m$$

we could construct only $\psi_k(x^+)$-s giving periodic and singularity free $L(x^+)$ and $W(x^+)$ (provided $|\Lambda - m| < 1$) but we were unable to find $\psi_k(x^+)$-s giving constant

L_0 and W_0. The same remark applies to orbits with monodromy matrices in the form of eq.(10) but belonging to the higher homotopy classes. Based on these we conjecture that these orbits would correspond to W representations which are neither highest nor lowest weight ones. Finally we remark that we did not inquire the orbits described by non diagonalizable M-s the reason being that the analogous case for the Liouville theory proved to be rather uninteresting [14].

4. The quantum equation of motion and the representation space for the Toda field

Motivated by the succes we gained from using the WZNW framework in describing the solution space of A_2 TT we envisage a quantization procedure that promotes only the relevant, natural degrees of freedom u, L, W, $\hat{L}$, $\hat{W}$ to operators. This seems to be the main difference between the earlier approaches [1,2] devoted to quantizing the (A_2)TT and the present one. Certainly our u operator is equivalent to some of the vertex operators constructed in [2] applying a modified free field quantization, but our framework is different. We are not going to use free fields thus we shall impose the quantized equation of motion – whose parameters we determine from its covariance – to define our Toda field, while the equivalent of this equation was verified in [2] for the particular vertex operator. Furthermore we are mainly interested in quantizing the A_2 TT in a domain which would correspond to the singular sector of the classical theory. In our approach we intend to maintain both the algebraic structure and the boundary conditions found classically. Technically we shall use short distance operator product expansions (and complexified coordinates) which are closer to the spirit of CFT than the method of canonical quantization.

The Hilbert space where our operators act is a big, reducible representation of the direct product of the left and right (quantum) W algebras $\mathcal{H} = \mathcal{W}_L \otimes \mathcal{W}_R$. $\mathcal{W}_L$ ($\mathcal{W}_R$) – which are supposed to contain h.w.representations only – are spanned by the Laurent coefficients of the currents $L(z)$, $W(z)$:

$$L(z) = W_1(z) = \sum_n L_n z^{-n-2} \qquad W(z) = W_2(z) = \sum_n W_n z^{-n-3} \tag{16}$$

($\bar{L}_n$, $\bar{W}_n$ are defined in an analogous way, from now on we shall give the formulae for the left moving sector only if it can lead to no confusion.) If $\phi(z,\bar{z})$ is any local field from the operator algebra then the W_n^j ($W_n^1 = L_n$, $W_n^2 = W_n$) operators act on it according to [6]

$$W_n^j \phi(z,\bar{z}) = \oint_z \frac{d\zeta}{2\pi} (\zeta - z)^{n+j} W_j(\zeta)\phi(z,\bar{z}) \tag{17}$$

L_n, W_n satisfy the quantum version of the W algebra [6]:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m} \\
[L_n, W_m] &= (2n-m)W_{n+m} \\
[W_n, W_m] &= \frac{c}{3\cdot 5!}(n^2-4)(n^2-1)n\delta_{n+m} + b^2(n-m)\Lambda_{n+m} \\
&\quad + (n-m)\Big(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\Big)L_{n+m}
\end{aligned} \tag{18}$$

where the central charge, c, is a free parameter, Λ_n is the composite operator built from the L_n-s

$$\Lambda_n = \sum_{k=-\infty}^{+\infty} : L_k L_{n-k} : + \frac{1}{5} x_n L_n$$

$$x_{2l} = (1+l)(1-l) \qquad x_{2l+1} = (l+2)(1-l)$$

and $b^2 = 16/(22+5c)$. The algebra given by eq.(18) in terms of commutators has the same structure as the one obtained from eq.(2-4) on the level of Poisson brackets of T_n and W_m; the only difference being that some of the constants got changed as a result of quantization. Indeed from eq.(2-4) we found $b^2_{\mathrm{class}} = 16/(5c)$ with $c = 24$ and $x_{2l}^{\mathrm{class}} = x_{2l+1}^{\mathrm{class}} = 2$, after rescaling the *classical* W_n by $\sqrt{5/2}$ to guarantee that the ratio of the central terms in $\{T_n, T_m\}$ and $\{W_n, W_m\}$ is the same as in eq.(18).

Of the $u(z,\bar z)$ we assume that it is a (periodic) primary field of the left (and right) W algebra(s):

$$L_n u(z,\bar z) = 0 \quad n>0 \qquad L_0 u(z,\bar z) = \Delta u(z,\bar z)$$

$$W_n u(z,\bar z) = 0 \quad n>0 \qquad W_0 u(z,\bar z) = \omega u(z,\bar z) \tag{19}$$

Please note that here Δ and ω may differ from their classical values encoded in eq.(7), but we assume that $u(z,\bar z)$ is a spinless field $\Delta=\bar\Delta$. The crucial assumption about $u(z,\bar z)$ is that it satisfies the 'quantized version' of the equation of motion, eq.(6) (plus its chiral counterpart). This quantized version differs from the classical one in two respects: first, since we are dealing with opeators now, all the products appearing in eq.(6) should be normal ordered, and in addition, as a result of renormalization, even the coefficients of the various terms may be different from their classical values. Interpreting the normal ordered products $:L(z)u(z,\bar z):$, $:W(z)u(z,\bar z):$ etc. as subtracting the singular terms from the ordinary ones plus using eq.(19,17) we finally get that $u(z,\bar z)$ should satisfy:

$$\kappa L_{-1}^3 u - L_{-2}L_{-1}u - \alpha W_{-3}u - \beta L_{-3}u = 0 \tag{20}$$

where the κ, α and β parameters are yet to be determined. The motivation to assume that the quantization we are considering keeps the form of the classical equation of motion and changes only the various coefficients comes from two sources: we saw that this happened with the defining relations of the W algebra in eq.(18), and this was found in the case of complete, unrestricted WZNW theory also in ref.[15].

Eq.(20) clearly has the form of a null vector. The requirement, that fixes the Δ, ω, κ, α and β parameters, is that this grade 3 null vector should be *covariant* under the W algebra i.e. denoting the left hand side of eq.(20) as χ, χ should be annihilated by all L_n, W_n, for $n>0$. Because of the commutation relations, eq.(18), for this it is sufficient if $L_1\chi = W_1\chi = L_2\chi = 0$. Analysing these conditions we found that they lead to a consistent system of equations for the parameters only if $u(z,\bar z)$ generates two independent null vectors, one on grade one:

$$2\Delta W_{-1}u - 3\omega L_{-1}u = 0 \tag{21.1}$$

and one on grade two:

$$AL_{-1}^2 u + BL_{-2}u + CW_{-2}u = 0 \tag{21.2}$$

where A/C and B/C are somewhat complicated functions of Δ and ω:

$$A/C = -\frac{3}{2(1-\Delta)}\big[-\frac{\omega}{4\Delta}+\frac{\Delta}{6\omega}\big];\ B/C = \frac{3}{2(1-\Delta)}\big[\omega - \frac{3\omega}{2\Delta}+\frac{\Delta(2\Delta+1)}{9\omega}\big].$$

The consistency of these two null vectors with eq.(18,19) (i.e. their covariance) determines Δ and ω as functions of c. In describing these functions (and the rest of the parameters) we found it extremely useful to introduce a new real parameter, Q, in place of c: $c = 2(3-4/Q)(3-4Q)$; then Δ and ω become:

$$\Delta = \frac{4Q}{3} - 1 \qquad \omega_{\pm} = \pm\frac{\Delta}{3}\sqrt{\frac{2}{3}}\sqrt{\frac{5Q-3}{5-3Q}} \tag{22}$$

This means that for any Q we get two u fields with the same conformal weight but opposite ω values; we shall denote by $u(z,\bar z)$ ($\tilde u(z,\bar z)$) the field with ω_+ (resp. ω_-). Eq.(22) also implies that u is a $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ field in the classification of ref.[9]. Using these parameters in the equations expressing the covariance of eq.(20) we got

$$\kappa = Q^{-1} \quad \alpha_{\pm} = \pm\frac{1}{\sqrt{6}}\sqrt{(5Q-3)(5-3Q)} \quad \beta = \frac{1}{2}\big(\frac{Q}{3}+1\big) \tag{23}$$

Since c is invariant under the substitution $Q \to Q^{-1}$ we get two new solutions from eq.(22,23) by making this change there; thus the total number of u fields belonging to a fixed c is four.

We can understand the appearence of the fields, u and $\tilde u$, degenerate in Δ but having opposite ω-s in the following way: The algebra described by eq.(18) is left invariant by the transformation $L_n \to L_n$, $W_n \to -W_n$. Denoting by $\mathcal{M}$ the operator implementing this automorphism: $\mathcal{M}L_n\mathcal{M}^{-1} = L_n$; $\mathcal{M}W_n\mathcal{M}^{-1} = -W_n$, we find from eq.(19) that $L_0\mathcal{M}u\mathcal{M}^{-1} = \Delta\mathcal{M}u\mathcal{M}^{-1}$; $W_0\mathcal{M}u\mathcal{M}^{-1} = -\omega\mathcal{M}u\mathcal{M}^{-1}$. Therefore we can write $\tilde{u}(z,\bar{z}) = \mathcal{M}u(z,\bar{z})\mathcal{M}^{-1}$ expressing the fact that u and $\tilde{u}$ provide a representation of the automorphism. Therefore in the following we shall treat u and $\tilde{u}$ on an equal footing.

Looking only at the central charge and the conformal weights of the solutions described by eq.(22,23) the obvious classical limit ($c \to \infty$, $\Delta \to -1$) would be $Q \to_- 0$. However the whole $Q < 0$ ($c > 98$) domain is ruled out if we insist on having *real* ω and α, since this restricts Q to $3/5 \leq Q \leq 5/3$ (which even shrinks to $3/4 \leq Q \leq 4/3$ if we demand $c > 0$). Though it may seem surprising that this entirely *chiral* condition forces us into the 'deep quantum' domain, $0 < c < 2$, it is in fact in accord with the Kac determinant for the W algebra [16]: from the latter one also finds that in the $c > 98$ domain a $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ field with a conformal weight given by eq.(22) can be degenerate only for purely imaginary ω-s. Therefore, with real ω, the quantization we propose, can be carried out only for $0 < c < 2$.

The $u(z,\bar{z})$ ($\tilde{u}(z,\bar{z})$) operator acting in $\mathcal{H}$ is known if we know its matrix elements. Since we assumed that $\mathcal{W}_L$ ($\mathcal{W}_R$) consist of h.w.r. only it is enough if we know the matrix elements of u between highest weight states $\left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle$:

$$\begin{matrix} W^j_n \\ \bar{W}^j_n \end{matrix} \left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle = 0 \qquad n > 0 \quad j = 1,2 \tag{24}$$

$$\begin{matrix} L_0 \\ \bar{L}_0 \end{matrix} \left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle = \begin{matrix} h \\ \bar{h} \end{matrix} \left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle \qquad \begin{matrix} W_0 \\ \bar{W}_0 \end{matrix} \left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle = \begin{matrix} w \\ \bar{w} \end{matrix} \left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle$$

From conformal symmetry alone it follows that

$$\left\langle \begin{matrix} H & \bar{H} \\ \Omega & \bar{\Omega} \end{matrix} \right| u(z,\bar{z}) \left| \begin{matrix} h & \bar{h} \\ w & \bar{w} \end{matrix} \right\rangle = G(H, h, \ldots) z^{H-h-\Delta} \bar{z}^{\bar{H}-\bar{h}-\bar{\Delta}}$$

where the constant amplitude, G, that depends on all the parameters characterizing the h.w. states and the u field is left undetermined. However the equation of motion, eq.(20), together with the W_0 part of eq.(19) restrict G; indeed sandwiching eq.(20) and the W_0 part of eq.(19) between h.w. states and using the freedom to deform the contour in eq.(17) together with eq.(21,24) we found after a somewhat lengthy computation that G vanishes unless $y = h + \Delta - H$ and Ω satisfy (for details see Appendix B):

$$
\begin{aligned}
-\kappa y(y+1)(y+2) + (y+2)(y+h) - \alpha\big\{w + \omega\big[\frac{3}{2\Delta}y - 2\big(y(y+1)\beta^{-1-1} \\
+ (y+h)\beta^{-2}\big)\big]\big\} + (\beta-1)(y+2h) = 0
\end{aligned}
\tag{25}
$$

$$
\Omega = -w - \omega + \omega\big\{\frac{3}{\Delta}y - \big(y(y+1)\beta^{-1-1} + (y+h)\beta^{-2}\big)\big\} \tag{26}
$$

where $\omega\beta^{-1-1}$ ($\omega\beta^{-2}$) denote the A (B) coefficients in eq.(21.2) when C is scaled to -1. These eqations become tractable if instead of h and w characterizing the h.w. states we introduce two new parameters a and b:

$$
\begin{aligned}
h(a,b) =& \frac{Q}{3}(a^2+b^2+ab) - \frac{(Q-1)^2}{Q} \\
w(a,b) =& \frac{1}{9}\sqrt{\frac{2}{3}}\frac{Q^2(b-a)(2b+a)(2a+b)}{\sqrt{(5Q-3)(5-3Q)}}
\end{aligned}
\tag{27}
$$

The vacuum state is described by $a_{\rm vac} = b_{\rm vac} = \pm(1-Q^{-1})$ while the a, b parameters of the u field are $a_u^{(1)} = -b_u^{(2)} = 1 - Q^{-1}$, $b_u^{(1)} = -a_u^{(2)} = 2 - Q^{-1}$ (the parameters of the $\tilde{u}$ field are obtained from these expressions by interchanging a and b). Substituting eq.(27) into eq.(25,26) one gets that the $u(z,\bar{z})$, $\tilde{u}(z,\bar{z})$ fields have nonvanishing transitions only if the A, B parameters of the final state and the a, b parameters of the initial one are related as

$$
A,\ B = \begin{matrix} b-1, & a+1 \\ b+1, & a \\ b, & a-1 \end{matrix} \qquad\qquad A,\ B = \begin{matrix} b+1, & a-1 \\ b-1, & a \\ b, & a+1 \end{matrix} \tag{28}
$$

respectively. To understand the meaning of these selection rules it is important to realise that the scalar product between the (chiral) h.w. states we are using is $\langle d, c|a, b\rangle \sim \delta_{ac}\delta_{bd}$. Therefore eq.(28) can be interpreted as saying that u ($\tilde{u}$) maps the state $|a, b\rangle$ to $|a', b'\rangle$ where $a' = B$ and $b' = A$. This means that by acting on a (h.w.) state with u ($\tilde{u}$) we can shift a and b (in appropriate combinations) by ±1. Interestingly, if a and b were integers characterizing the Dynkin labels of an $SL(3)$ irrep $[a, b]$, then the a' and b' obtained from eq.(28) in the case of the u field would have the same form as the Dynkin labels of irreps appearing in the tensor product $3 \otimes [a, b]$ ($\bar{3} \otimes [a, b]$ for the $\tilde{u}$ field). This is the quantum equivalent of the classical property, that the u field was a specific component of an $SL(3)$ triplet.

From the chiral partner of the equation of motion, eq.(20), one finds an entirely analogous selection rule for the $\bar{a}, \bar{b}$ ($\bar{A}, \bar{B}$) parameters characterizing the transformation properties of the h.w.s. under the right moving W algebra. A connection between a, b and $\bar{a}, \bar{b}$ parameters can be established by requiring $u(z, \bar{z})$, $\tilde{u}(z, \bar{z})$ to be periodic. Indeed looking at the 'diagonal' transitions ($\left\langle \begin{matrix} b+1 & a \\ \bar{b}+1 & \bar{a} \end{matrix} \right| u(z,\bar{z}) \left| \begin{matrix} a & b \\ \bar{a} & \bar{b} \end{matrix} \right\rangle$ etc.) one immediately obtaines that u and $\tilde{u}$ can be periodic only if

$$\bar{a} = a - (2N + M)Q^{-1}, \qquad \bar{b} = b + (N - M)Q^{-1} \tag{29}$$

where N and M are integers. It is also easy to see that the periodicity of u and $\tilde{u}$ in the 'non – diagonal' transitions ($\left\langle \begin{matrix} b+1 & a \\ \bar{b} & \bar{a}-1 \end{matrix} \right| u(z,\bar{z}) \left| \begin{matrix} a & b \\ \bar{a} & \bar{b} \end{matrix} \right\rangle$ etc.) together with the diagonal ones would imply that Qa and Qb are integers. However, as we shall see later, this possibility is unacceptable.

Therefore, putting everything together, in the following we choose the Hilbert space, where our u and $\tilde{u}$ operators act irreducibly as

$$\mathcal{H} = \sum_{k,l} \mathcal{W}_{a_0+k, b_0+l} \otimes \bar{\mathcal{W}}_{a_0+k, b_0+l} \tag{30}$$

where $\mathcal{W}_{a_0+k,b_0+l}$ ($\bar{\mathcal{W}}$) is the full left (right) Verma modul corresponding to the h.w.s. $\left| \begin{matrix} a_0+k & b_0+l \\ a_0+k & b_0+l \end{matrix} \right\rangle$. (Choosing $N = M = 0$ in in eq.(29) guarantees the absence of 'non – diagonal' transitions and this choice will be forced upon us if — eventually — we want to represent the other two operators – whose Δ and ω were obtained by the $Q \to Q^{-1}$ substitution from eq.(22) – as periodic fields in the same Hilbert space.) The summation over the integers k, l in eq.(30) is either infinite or restricted to a subset, but in any case the Hilbert space, (30), contains at most a discrete infinity of h.w. modules. We emphasize, that this choice is a very natural one in view of the selection rules, eq.(28), but is not the only possibility, since we could start with a Hilbert space containing a continuum spectrum of W

algebra highest weights. We chose eq.(30) since it naturally corresponds to the set of singular solutions described in sect.3 and may contain the SL_2 invariant vacuum $\left|\begin{matrix} a_{vac} & b_{vac} \\ a_{vac} & b_{vac}\end{matrix}\right\rangle$.

In the Hilbert space (30) the $u(z,\bar z)$, $\tilde u(z,\bar z)$ operators are characterized by three types of constant amplitudes $G_i(a,b)$ $(\tilde G_i(a,b))$ $i=1,..3$:

$$G_1(a,b) = \langle b, a-1|u(1,1)|a,b\rangle;\ G_2(a,b) = \langle b+1, a|u(1,1)|a,b\rangle$$

$$G_3(a,b) = \langle b-1, a+1|u(1,1)|a,b\rangle;\ \tilde G_1(a,b) = \langle b, a+1|\tilde u(1,1)|a,b\rangle \tag{31}$$

$$\tilde G_2(a,b) = \langle b-1, a|\tilde u(1,1)|a,b\rangle;\ \tilde G_3(a,b) = \langle b+1, a-1|\tilde u(1,1)|a,b\rangle$$

where $|a,b\rangle$ is a short notation for $\left|\begin{matrix} a & b \\ a & b\end{matrix}\right\rangle$. The automorphism, $\mathcal{M}$, transforming u and $\tilde u$ into each other relates the constant amplitudes of the $\tilde u$ field to those of u:

$$\tilde G_1(a,b) = G_2(b,a);\quad \tilde G_2(a,b) = G_1(b,a);\quad \tilde G_3(a,b) = G_3(b,a) \tag{32}$$

Exploiting the fact that $u(z,\bar z)$ is a real field reduces further the number of independent constant amplitudes since it implies

$$G_2(a,b) = G_1^*(b+1,a);\qquad G_3^*(a,b) = G_3(b-1,a+1) \tag{33}$$

From eq.(31-33) we see that both the $u(z,\bar z)$ and the $\tilde u(z,\bar z)$ fields are completely parametrized if we give the constant amplitudes $G_1(a,b)$, $G_3(a,b)$ for all a,b-s belonging to $\mathcal{H}$.

5. Construction of the local Toda fields

These constant amplitudes will be further restricted by requiring the u, $(\tilde u)$ operators to be mutually local. This can be studied by analysing the behaviour of the 4-point functions; i.e. the expectation values of the products of two field operators $u(z,\bar z)$ $u(\zeta,\bar\zeta)$ $(u(z,\bar z)$ $\tilde u(\zeta,\bar\zeta))$ between h.w. states. Conformal symmetry implies that these 4-point functions have the form:

$$\left\langle\begin{matrix} H & \bar H \\ \Omega & \bar\Omega\end{matrix}\right| u(z,\bar z) u(\zeta,\bar\zeta) \left|\begin{matrix} h & \bar h \\ w & \bar w\end{matrix}\right\rangle = (z\zeta)^{\lambda}(\bar z\bar\zeta)^{\bar\lambda} f_{uu}(x,\bar x)$$

where $\lambda = \frac{1}{2}(H-h)-\Delta$, $x=\zeta/z$, $\bar{x}=\bar{\zeta}/\bar{z}$. The $f_{\tilde{u}\tilde{u}}(x,\bar{x})$, $f_{u\tilde{u}}(x,\bar{x})$ and $f_{\tilde{u}u}(x,\bar{x})$ functions are defined in an analogous way. The locality of the u ($\tilde{u}$) operators requires that the functions describing the expectation values of the products of identical operators be symmetric under $x \to x^{-1}$: $f_{uu}(x,\bar{x}) = f_{uu}(x^{-1},\bar{x}^{-1})$ ($f_{\tilde{u}\tilde{u}}(x,\bar{x}) = f_{\tilde{u}\tilde{u}}(x^{-1},\bar{x}^{-1})$), while for the functions describing the expectation values of the products of different operators it means that they should go into each other under $x \to x^{-1}$: $f_{u\tilde{u}}(x,\bar{x}) = f_{\tilde{u}u}(x^{-1},\bar{x}^{-1})$. On the other hand eq.(20) implies that each of the $f(x,\bar{x})$ functions satisfies an – in general different – 3-rd order linear differential equation in both x and $\bar{x}$. The constant amplitudes determine the linear combination coefficients in the solutions of this d.e. through the boundary conditions at $x=\bar{x}=0$ ($z\to\infty$) where only the h.w. states contribute: Indeed inserting a complete system of states between the uu ($u\tilde{u}$) operators and taking the $z\to\infty$ ($x\to 0$) limit when the descendant states are suppressed we get schematically:

$$\langle AB|u(z,\bar{z})u(\zeta,\bar{\zeta})|ab\rangle \to \sum_{c,d}\langle AB|u(z,\bar{z})|cd\rangle\langle dc|u(\zeta,\bar{\zeta})|ab\rangle(1+\ldots) =$$

$$= (z\bar{z}\zeta\bar{\zeta})^{\lambda}\sum_{c,d} G(AB;cd)G(dc;ab)(x\bar{x})^{h(c,d)-\frac{1}{2}(h(A,B)+h(a,b))}(1+\ldots) \qquad (34)$$

where the summation runs over those highest weight states whose presence between $\langle AB|$ and $|ab\rangle$ is allowed by the selection rules, the dots stand for a polynomial of x, $\bar{x}$ representing the contribution of the descendant states, and $G(AB;cd)$ ($G(dc;ab)$) denotes the constant ampitude appropriate for the transition $|cd\rangle \to \langle AB|$ ($|ab\rangle \to \langle dc|$). Thus the requirement of locality can be translated into a system of equations for the constant amplitudes. As we shall see this system, when supplemented by some minor and very natural additional assumptions, determines them completely.

In the following we first derive the 3-rd order differential equations and analyze their general properties then we turn to a detailed investigation of the various transitions distinguished by the number of intermediate states in eq.(34). Because the automorphism $\mathcal{M}$ transforms u and $\tilde{u}$ into each other there are only two essentially independent f functions: $f_{uu}(x,\bar{x})$ and $f_{u\tilde{u}}(x,\bar{x})$ say. Applying the same method we described in Appendix B for the three point function we found that both $f_{uu}(x,\bar{x})$ and $f_{u\tilde{u}}(x,\bar{x})$ satisfy an equation of the form

$$\kappa(\mathrm{I}) - (\mathrm{II}) - (\beta-1)(\mathrm{III}) - \alpha(\mathrm{IV}) = 0 \tag{35}$$

where for both functions

$$(\mathrm{I}) = (\lambda-2)(\lambda-1)\lambda f - 3x(\lambda-2)(\lambda-1)f' + 3x^2(\lambda-2)f'' - x^3 f''' \tag{36}$$

$$\begin{aligned}(\mathrm{II}) =& -2f\big(\frac{\Delta}{(1-x)^3}+h\big) + \big(\frac{\Delta}{(1-x)^2}+h\big)[\lambda f - xf'] \\ &- \frac{1}{(1-x)^2}\big(\frac{\lambda}{x}f+f'\big) - (x-1)f' + \lambda(1+\frac{1}{x})f - x^2(1+\frac{1}{1-x})f'' \\ &+ xf'[2(\lambda-1) - \frac{1}{1-x}] + f\lambda[\frac{\lambda}{1-x} - (\lambda-1)]\end{aligned} \tag{37}$$

$$(\mathrm{III}) = -2f\big(\frac{\Delta}{(1-x)^3}+h\big) - \frac{1}{(1-x)^2}\big(\frac{\lambda}{x}f+f'\big) - (x-1)f' + \lambda(1+\frac{1}{x})f \tag{38}$$

and $h = h(a,b)$. The difference between the equations of f_{uu} and $f_{u\tilde{u}}$ comes from the matrix element of W_{-3} appearing in the fourth term of eq.(35): in the case of f_{uu} it is

$$\begin{aligned}(\mathrm{IV}) =& wf + \frac{\omega}{(1-x)^3}f + \frac{3\omega}{2\Delta}\big[\lambda f\big(\frac{x}{1-x}+\frac{1}{(1-x)^2}\big) + f'\big(\frac{1}{(1-x)^2}-(1-x)\big)\big] \\ &+ \omega\beta^{-1-1}\big[\lambda(\lambda-1)\big(\frac{1}{1-x}-2\big)f + x^2 f''\big(\frac{1}{1-x}-2\big) + 2\lambda\frac{x}{1-x}f' + \\ &4x(\lambda-1)f'\big] + \omega\beta^{-2}\Big[\Delta f\frac{x^2-2(1-x)}{(1-x)^3} + hf\big(\frac{1}{1-x}-2\big) + \big(\frac{1}{1-x} - \\ &3\big)[(x-1)f' - \lambda(1+\frac{1}{x})f] + \lambda f\frac{2-x^2-2/x}{(1-x)^2} + f'\frac{x^3-2(1-x)}{(1-x)^2}\Big]\end{aligned} \tag{39}$$

(here $w = w(a,b)$), while for $f_{u\tilde{u}}$ we got:

$$\begin{aligned}(\mathrm{IV}) =& wf - \frac{\omega}{(1-x)^3}f - \frac{3\omega}{2\Delta}\big[\lambda f\frac{3-3x+x^2}{(1-x)^2} + xf'\frac{1+x-x^2}{(1-x)^2}\big] \\ &-\omega\beta^{-1-1}\big[x^2f''\big(\frac{1}{1-x}+2\big) + 2xf'\big(\frac{\lambda}{1-x}-2(\lambda-1)\big) + \lambda(\lambda-1)f\frac{3-2x}{1-x}\big] \\ &-\omega\beta^{-2}\Big[\Delta f\big(\frac{1}{(1-x)^3}+\frac{1}{1-x}\big) + hf\big(\frac{1}{1-x}+2\big) + xf'\big(\frac{1}{(1-x)^2}+2\big) \\ &+ \lambda f\big(-1-\frac{x^2}{(1-x)^2}\big)\Big]\end{aligned} \tag{40}$$

(here $\omega = \omega_+$ in eq.(22). Once we obtained the equation for $f_{u\tilde{u}}$ for a given transition from eq.(35-38,40) we can get that of $f_{\tilde{u}u}$ for the same transition by simply changing the sign of the $\alpha w f$ term.)

It is no surprise that the differential equations for f_{uu} and $f_{u\tilde{u}}$ have three singular points at $x = 0$, $x = 1$ and $x = \infty$. First we discuss the properties of the singularity at $x = 1$. Since $x \to 1$ corresponds to $z \to \zeta$ we expect them to contain some information about the short distance behaviour of the uu ($u\tilde{u}$) operator products. Therefore they should depend only on the operators involved but should be independent of the external states ($|ab\rangle$, $\langle AB|$). In the case of f_{uu} from eq.(35-39) we found that the indices characterizing the solution around $x = 1$ ($f_{uu} \sim (1-x)^\nu$) are:

$$\nu_1 = 1 - \frac{4Q}{3}, \quad \nu_2 = \frac{2Q}{3}, \quad \nu_3 = 2 + \frac{2Q}{3} \tag{41}$$

These ν_i-s imply the appearance of three operators O_i $i = 1, .., 3$ with conformal dimensions

$$\Delta_1 = \frac{4Q}{3} - 1 = \Delta, \quad \Delta_2 = \frac{10Q}{3} - 2, \quad \Delta_3 = \frac{10Q}{3} \tag{42}$$

in the operator product expansion (OPE) of uu. The appearance of Δ among the Δ_i-s means that O_1 may correspond to either u or $\tilde{u}$. It is interesting to observe, that Δ_2 has also the form of $h(a,b)$ in eq.(27) with $a = \pm(1 - Q^{-1})$, $b = \pm(3 - Q^{-1})$, thus O_2 is a new W primary field propping up in the OPE of uu. On the other hand Δ_3 differs from Δ_2 by a positive integer indicating that the corresponding operator may be a (W) descendant of O_2. Repeating the same analysis for the $f_{u\tilde{u}}$ function we found that the indices are now given by:

$$\mu_1 = 2 - \frac{8Q}{3}, \quad \mu_2 = \frac{Q}{3}, \quad \mu_3 = 1 + \frac{Q}{3} \tag{43}$$

These indices imply that the three operators U_i, $i = 1, .., 3$ appearing in the $u\tilde{u}$ OPE have the following conformal dimensions:

$$\Delta_1 = 0, \quad \Delta_2 = 3Q - 2, \quad \Delta_3 = 3Q - 1. \tag{44}$$

It is natural to assume that U_1 is nothing but the identity operator. Δ_2 can again be written in the form of $h(a,b)$ in eq.(27) with $a = b = \pm(2 - Q^{-1})$, thus U_2 is again a new W primary field emerging in the $u\tilde{u}$ OPE, while U_3 can again be interpreted as a descendant of U_2. We note that one pair of indices is differing by an integer for both f_{uu} and $f_{u\tilde{u}}$ and this raises the danger of one of the fundamental solutions at $x = 1$ being logarithmic instead of polynomial [17]. We come back to this problem soon.

As we mentioned earlier the various transitions defining the various types of f_{uu} and $f_{u\tilde{u}}$ functions can be classified according to the number of intermediate states in eq.(34). In fact we can use eq.(34) together with the selection rules, eq.(28), to determine all the non vanishing 4-point functions built on the initial state $|ab\rangle$ and collect the transitions leading to the same final state $\langle AB|$. The six uu transitions (f_{uu} functions) belong to two groups: three of them – when A, B are $b, a-2$; $b+2, a$ and $b-2, a+2$ respectively – have just one intermediate state, while the other three – when A, B are $b, a+1$; $b-1, a$ and $b+1, a-1$ respectively – have two intermediate states. Of the seven $u\tilde{u}$ transitions the diagonal one, i.e. when $\langle AB| = \langle ba|$, is a class of its own by having three intermediate states, while all the others (with A, B being $b+1, a+1$; $b-2, a+1$; $b-1, a-1$; $b-1, a+2$; $b+2, a-1$ and $b+1, a-2$ respectively) have only one intermediate state.

Our strategy to determine the functions belonging to transitions with one and two intermediate states is the following: first we analyse the exponents of x ($\bar{x}$) appearing in eq.(34) then combining them with the known indices at $x = 1$ (eq.(41,43)) we construct some trial functions, whose validity we check on the computer using the symbolic formula manipulating program FORM [18]. Once we completed this we derive from the requirement of locality the equations for the constant amplitudes.

In case of the three f_{uu} functions with one intermediate state (IS) we found that the exponent of x ($\bar{x}$) in eq.(34) is just $-Q/3$, i.e. is independent of a, b. Combining this with the expression $(1-x)^{2Q/3}$ corresponding to ν_2 in eq.(41) we get a trial function

$$\left(x^{-1}(1-x)^2\right)^{Q/3}\left(\bar{x}^{-1}(1-\bar{x})^2\right)^{Q/3} \tag{45}$$

which, in addition to exhibiting the singular solutions at $x = 0$ and $x = 1$ is also symmetric under $x \to x^{-1}$. Using FORM to substitute this expression into the corresponding equations we checked that it really solves them. Thus when multiplied by the appropriate products of G-s, eq.(45) yields a complete solution to the three f_{uu} functions with one IS without any restriction on the constant amplitudes.

In case of the six $f_{u\tilde{u}}$ functions with one IS the exponents of x in eq.(34) ($-Q(1+b-a)/6$, $-Q(1+2a+b)/6$, $-Q(1-2b-a)/6$, each of them appearing twice) do depend on a, b. Furthermore computing the exponents for the $\tilde{\mathbf{u}}\mathbf{u}$ product between the <u>same</u> states we found that in each case they differ from the previous ones as a result of the different IS but only in replacing a and b by $-a$, $-b$ respectively. Therefore using the expression $(1-x)^{Q/3}$ corresponding to μ_2 in eq.(43) we get trial functions

$$\sigma_0(x)\sigma_0(\bar{x})(x\bar{x})^{-\frac{Q}{6}(b-a)}; \quad \sigma_0(x)\sigma_0(\bar{x})(x\bar{x})^{-\frac{Q}{6}(2a+b)}; \quad \sigma_0(x)\sigma_0(\bar{x})(x\bar{x})^{\frac{Q}{6}(2b+a)} \tag{46}$$

(where $\sigma_0(x) = \left(x^{-1}(1-x)^2\right)^{Q/6}$) that again exhibit the correct behaviour at $x = 0$ and $x = 1$. Furthermore for these trial functions the $x \to x^{-1}$ substitution amounts to the replacement $a \to -a$, $b \to -b$. Having checked that these trial functions do solve the corresponding equations we multiplied them with the appropriate combinations of constant amplitudes and found the following three independent equations

$$G_3(a+1,b)G_2(b,a) = G_2(b-1,a+1)G_3(a,b) \tag{47.a}$$

$$G_3(a,b-1)G_1(b,a) = G_1(b-1,a+1)G_3(a,b) \tag{47.b}$$

$$G_1(a,b-1)G_1(b,a) = G_1(b,a-1)G_1(a,b) \tag{47.c}$$

from the requirement of locality, $f_{u\tilde{u}}(x,\bar{x}) = f_{\tilde{u}u}(x^{-1},\bar{x}^{-1})$.

In case of the three uu transitions with two IS f_{uu} starts at $x = 0$ as a linear combination of two terms. Motivated by this we assumed, that at $x = 1$ it is also a linear combination of two terms, namely those, whose singular behaviour is given by ν_1 and ν_2 in eq.(41). Therefore computing the exponents in eq.(34) we constructed our trial functions as a sum of terms $(1-x)^{1-4Q/3}x^{\text{exponent}}F(\alpha,\beta,\gamma;x)$ where $F(\alpha,\beta,\gamma;x)$ is the usual hypergeometric function, analytic around $x = 0$. To every exponent we determined α, β and γ from demanding two things: first that the singularities of the sum at $x = 1$ be given by ν_1 and ν_2 and second that the members of the sum be transformed into each other's linear combination under $x \to x^{-1}$. Putting everything together the trial fuctions for the three f_{uu}-s with two IS can be written in the following compact form:

$$\begin{aligned}\langle b, a+1|uu|ab\rangle : \quad & \sigma_1(x)\sigma_1(\bar{x})\big[G_3(a,b+1)G_2(a,b)\psi_b(x)\psi_b(\bar{x}) \\ & + G_2(a+1,b-1)G_3(a,b)\psi_{-b}(x)\psi_{-b}(\bar{x})\big]\end{aligned} \tag{48.a}$$

$$\begin{aligned}\langle b-1, a|uu|ab\rangle : \quad & \sigma_1(x)\sigma_1(\bar{x})\big[G_1(a+1,b-1)G_3(a,b)\psi_a(x)\psi_a(\bar{x}) \\ & + G_3(a-1,b)G_1(a,b)\psi_{-a}(x)\psi_{-a}(\bar{x})\big]\end{aligned} \tag{48.b}$$

$$\begin{aligned}\langle b+1, a-1|uu|ab\rangle : \quad & \sigma_1(x)\sigma_1(\bar{x})\big[G_1(a,b+1)G_2(a,b)\psi_{a+b}(x)\psi_{a+b}(\bar{x}) \\ & + G_2(a-1,b)G_1(a,b)\psi_{-a-b}(x)\psi_{-a-b}(\bar{x})\big]\end{aligned} \tag{48.c}$$

where $\sigma_1(x) = \left(x^{-1}(1-x)^2\right)^{\frac{1}{2}-\frac{2Q}{3}}$ and

$$\psi_b(x) = x^{\frac{1}{2}[Q(b-1)+1]}F(Q[b-1]+1, 1-Q, 1+Qb; x).$$

The trick we used to check the validity of these expressions on the computer was to express the second and third derivatives of the hypergeometric functions F in terms of F' and F using the hypergeometric differential equation and to verify that the coefficients of F' and F vanish separately in eq.(35-39).

The well known $x \to x^{-1}$ transformation properties of the hypergeometric functions (see e.g. [19]) imply that:

$$\psi_b(x) = B_1(b)\frac{x^{Q(b-1)+1}}{(-x)^{Q(b-1)+1}}\psi_b(1/x) + B_2(b)\frac{x^{1-Q}}{(-x)^{1-Q}}\psi_{-b}(1/x) \tag{49}$$

where $B_1(b) = \frac{\Gamma(1+Qb)\Gamma(-Qb)}{\Gamma(1-Q)\Gamma(Q)}$; $B_2(b) = \frac{\Gamma(1+Qb)\Gamma(Qb)}{\Gamma(Q[b-1]+1)\Gamma(Q[b+1])}$. Using eq.(49) in the expressions in eq.(48) to implement the $x \to x^{-1}$ symmetry we found that for this

$$\frac{G_1(a+1,b-1)G_3(a,b)}{G_1(a,b)G_3(a-1,b)} = \phi(a); \quad \frac{G_3(a,b+1)G_1^*(b+1,a)}{G_1^*(b,a+1)G_3(a,b)} = \phi(b) \tag{50.a}$$

$$\frac{G_1(a,b+1)G_1^*(b+1,a)}{G_1(a,b)G_1^*(b+1,a-1)} = \phi(a+b) \tag{50.b}$$

must hold for the constant amplitudes. Here

$$\phi(b) = -\frac{\Gamma^2(-Qb)\Gamma(Q[b+1])\Gamma(1+Q[b-1])}{\Gamma^2(Qb)\Gamma(Q[1-b])\Gamma(1-Q[b+1])} = \frac{s(b+1)}{s(b-1)}\frac{\Gamma^2(-Qb)\Gamma^2(Q[b+1])}{\Gamma^2(Qb)\Gamma^2(Q[1-b])}$$

with $s(x) = \sin(\pi Q x)$.

After some straightforward algebra one can show that the solution of eq.(33), (47) and (50) can be written as:

$$|G_3(a,b)|^2 = N\frac{\Gamma(Qb)\Gamma(-Q[b-1])\Gamma(Q[a+1])\Gamma(-Qa)}{\Gamma(1-Qb)\Gamma(1+Q[b-1])\Gamma(1-Q[a+1])\Gamma(1+Qa)} \tag{51}$$

$$|G_1(a,b)|^2 = M\frac{\Gamma(Q[a+b])\Gamma(-Q[a+b-1])\Gamma(-Q[a-1])\Gamma(Qa)}{\Gamma(1-Q[a+b])\Gamma(1+Q[a+b-1])\Gamma(1+Q[a-1])\Gamma(1-Qa)} \tag{52}$$

where $N = N(Q)$ and $M = M(Q)$ are undetermined functions of Q. (Very precisely they still could depend on a and b through such combinations that stay invariant under $a, b \to a \pm 1, b \pm 1$.)

The diagonal $u\tilde{u}$ transition (i.e. when the final state is $\langle ba|$) needs special care since now – unlike in the previous cases – all three singularities at $x = 1$ may contribute raising the danger of a logarithmic singularity. Therefore we determined the indices of the differential equation one gets from eq.(35-38,40) with $\lambda = -\Delta$ at $x = 0$ and $x = \infty$ first. At the origin we got

$$\nu_1^{(0)} = \frac{Q}{3}(1+b-a); \quad \nu_2^{(0)} = \frac{Q}{3}(1+b+2a); \quad \nu_3^{(0)} = \frac{Q}{3}(1-2b-a) \tag{53}$$

which nicely coincide with the exponents computed from eq.(34) – as is expected – while at infinity we found

$$\nu_1^{(\infty)} = \frac{Q}{3}(1+a-b); \quad \nu_2^{(\infty)} = \frac{Q}{3}(1+2b+a); \quad \nu_3^{(\infty)} = \frac{Q}{3}(1-b-2a) \tag{54}$$

Combining the indices in eq.(43), (53) and (54) we see that our differential equation is of the Fuchs type. At $x = 0$ ($x = \infty$) its solution will be free of logarithms – thus it may correspond to our boundary conditions, eq.(34) – if none of the index pairs is differing by an integer [17], i.e. if neither Qa nor Qb is an integer. If this is the case then we don't have to worry about the potential logarithmic singularity at $x = 1$, since in the lack of an additional branch point it must be absent. It is also encouraging to observe that Qa, Qb not being integers also guarantees that all the hypergeometric functions appearing in eq.(48) are indeed well defined power series.

We solved the differential equation for the diagonal $f_{u\tilde{u}}$ by realising that factoring out $(1-x)^{Q/3}x^{\nu_i^{(0)}}$ $i = 1,..3$ from $f_{u\tilde{u}}$ in eq.(35-38,40) one gets the differential equation

$$\Big[x\frac{d}{dx}\prod_{j=1}^{2}(x\frac{d}{dx}+\beta_j-1)-x\prod_{k=1}^{3}(x\frac{d}{dx}+\alpha_k)\Big]v=0 \tag{55}$$

satisfied by the generalized hypergeometric function ${}_3F_2 = v$ [19]:

$${}_3F_2\Big(\begin{matrix}\alpha_1 & \alpha_2 & \alpha_3\\ & \beta_1 & \beta_2\end{matrix}\Big|x\Big)=\sum_{n=0}^{\infty}\frac{\alpha_1^{(n)}\alpha_2^{(n)}\alpha_3^{(n)}}{\beta_1^{(n)}\beta_2^{(n)}}\frac{x^n}{n!}$$

where $\alpha_i^{(n)} = \frac{\Gamma(\alpha_i+n)}{\Gamma(\alpha_i)}$. To every exponent $\nu_i^{(0)}$ we determined the α_i, β_i parameters as functions of a, b and Q by matching the coefficients of the various terms we got from the computer to that of coming from eq.(55). Therefore the complete diagonal transition has the form:

$$\begin{aligned}\langle ba|u\tilde{u}|ab\rangle &= (z\bar{z}\zeta\bar{\zeta})^{-\Delta}\sigma_0(x)\sigma_0(\bar{x})\big[|G_1(a+1,b)|^2 I_1(a,b|x)I_1(a,b|\bar{x})\\ &+|G_3(b,a)|^2 I_2(a,b|x)I_2(a,b|\bar{x})+|G_1(b,a)|^2 I_3(a,b|x)I_3(a,b|\bar{x})\big]\end{aligned} \tag{56}$$

where

$$I_1(a,b|x)=x^{Q(\frac{1}{2}+\frac{1}{3}[2a+b])}\ {}_3F_2\Big(\begin{matrix}Q & Q(1+a+b) & Q(1+a)\\ & 1+Q(a+b) & 1+Qa\end{matrix}\Big|x\Big) \tag{57.a}$$

$$I_2(a,b|x)=x^{Q(\frac{1}{2}+\frac{1}{3}[b-a])}\ {}_3F_2\Big(\begin{matrix}Q & Q(1+b) & Q(1-a)\\ & 1+Qb & 1-Qa\end{matrix}\Big|x\Big) \tag{57.b}$$

$$I_3(a,b|x) = x^{Q(\frac{1}{2}-\frac{1}{3}[2b+a])}\ {}_3F_2\Big(\begin{matrix} Q & Q(1-a-b) & Q(1-b) \\ & 1-Q(a+b) & 1-Qb \end{matrix}\Big|x\Big). \qquad (57.c)$$

(The diagonal $f_{\tilde{u}u}$ function between the same states can be obtained from eq.(56), (57) by replacing a and b.) In Appendix C we derive the $x \to x^{-1}$ transformation rule for the generalized hypergeometric functions ${}_3F_2$; using them and the form of the constant amplitudes given in eq.(51), (52) after a somewhat lenghty calculation we found that $f_{u\tilde{u}}(x,\bar{x}) = f_{\tilde{u}u}(x^{-1},\bar{x}^{-1})$ is satisfied for the diagonal transition if $N(Q)$ and $M(Q)$ appearing in $G_3(a,b)$ (resp. $G_1(a,b)$) are equal $M(Q) = N(Q)$, and are indeed independent of a, b.

Clearly to determine the actual Q dependence of M we have to impose some sort of normalization in addition to $u(z,\bar{z})$, $\tilde{u}(z,\bar{z})$ being local operators. We may require that the dimension zero operator appearing at $x=1$ in the $u\tilde{u}$ product be the true identity operator, or, equivalently, that the operator with conformal dimension Δ emerging at $x=1$ in the uu product be the correctly normalized u or $\tilde{u}$. We chose the technically simpler second possibility. Comparing the initial and final states in eq.(48) to the selection rules it is clear that O_1 can be identified with $\tilde{u}$ and requiring that the residues of the $(1-x)^{\nu_1}$ singularity in the three expressions in eq.(48) be the properly normalized matrix elements of $\tilde{u}$ we found that

$$M(Q) = N(Q) = \Big(\frac{\Gamma(Q)\Gamma(2-2Q)}{\Gamma(1-Q)\Gamma(2Q-1)}\Big)^2 \qquad (58)$$

Thus we see that requiring u and $\tilde{u}$ to be local operators together with this normalization condition indeed completely determines the constant amplitudes. We may think of the identifications $U_1 \sim \mathrm{Id}$, $O_1 \sim \tilde{u}$ as the quantum equivalents of the classical conditions $\det g = 1$ and $\exp(-\frac{1}{2}\Phi^1)$ being the lower right subdeterminant of g respectively: just as in the classical case they are automatically satisfied as a consequence of the equations of motion, apart from an overall normalization. With this remark we end the constructuion of the quantized Toda fields u and $\tilde{u}$, and in the following we analyze the properties of this solution.

In the first step we rewrite $|G_1|^2$ and $|G_3|^2$ in a form more suitable for our purposes:

$$|G_3(a,b)|^2 = N\pi^4 S_1(Q,a,b)\big(\Gamma(Qb)\Gamma(-Q[b-1])\Gamma(Q[a+1])\Gamma(-Qa)\big)^2$$

$$|G_1(a,b)|^2 = M\pi^4 S_2(Q,a,b)\big(\Gamma(Q[a+b])\Gamma(-Q[a+b-1])\Gamma(-Q[a-1])\Gamma(Qa)\big)^2 \tag{59}$$

where

$$S_1(Q,a,b) = s(Qb)s(Q[b-1])s(Qa)s(Q[a+1])$$

$$S_2(Q,a,b) = s(Q[a+b])s(Q[a+b-1])s(Qa)s(Q[a-1]) \tag{60}$$

The expressions on the left hand side of eq.(59) should be non negative by definition. However, because of the sine factors, the expressions on the right hand side may change sign as a and b run through their domain in eq.(30). Of course our construction of the (local) u and $\tilde{u}$ operators makes sense only if this does not happen; i.e. if for no a, b belonging to $\mathcal{H}$ is either S_1 or S_2 negative. So our remaining task is to find out the values of Q and the domain of a, b guaranteeing this. We emphasize that the condition that the modulus squared of a complex number be non negative has nothing to do with the possible (non)unitarity of the W representation built on the h.w. state $|ab\rangle$.

Looking at eq.(59,60) we note that in the case of irrational Q-s starting from a state $|a_0 b_0\rangle$ (with $Qa_0 \neq$integer, $Qb_0 \neq$integer) we can never 'stop' again, i.e. applying u and $\tilde{u}$ sufficiently many times to $|a_0 b_0\rangle$ we can change the a, b parameters of the final state to differ from a_0 and b_0 by any integer without ever finding a vanishing G_1 or G_3. This clearly poses a problem since then the sine factors in eq.(59,60) will sooner or later change sign contradicting the positivity of $|G_1|^2$ and $|G_3|^2$.

If Q is rational; $Q = r/s$ with $r, s > 0$ coprime integers, then it is conceivable that starting from a h.w. state $|a_0 b_0\rangle$, after applying several times u and $\tilde{u}$, we arrive at a final state for which some of the constant amplitudes vanish; i.e. in this case – at least in principle – we may be able to 'stop'. However this possibility raises

$$
\begin{tikzcd}
G_3(b,a) & G_1^*(b+1,a) & \\
G_1(a,b) & (a,b) \arrow[l] \arrow[r] \arrow[u, Rightarrow] \arrow[d, Rightarrow] \arrow[ul] \arrow[dr] & G_1^*(a+1,b) \\
& G_1(b,a) & G_3(a,b)
\end{tikzcd}
$$

$$\uparrow b \qquad \cdot \xrightarrow{a}$$

Fig.1

the danger of having a-s and b-s in $\mathcal{H}$ with the unacceptable property Qa =integer, Qb =integer. (In addition in this case we also have to worry about some of the Γ function's arguments becoming a negative integer.) We may resolve this problem if we can find a domain in the (a,b) plane such that the constant amplitudes that would correspond to transitions leading out of the domain vanish on its border, but inside (or on the border) there are no points for which Qa or Qb is integer. This may happen as the the six possible transitions from $|ab\rangle$ listed in eq.(28) are characterized by different values of G_1 and G_3. Pictorially they can be represented as on Fig.1. (This means e.g. that the transition keeping b fixed while increasing a by 1 is characterized by $G_1^*(a+1,b)$.)

The domains where all of our conditions are met are triangular ones with two sides being paralel to the a and b axis ($a \equiv a_0 = 1 - L(s/r)$; $b \equiv b_0 = 1 - K(s/r)$ where $K, L \geq 1$ are integers also satisfying $K + L \leq r - 1$) and the third one inclining at 135° to the positive a axis ($a + b + 1 = s(r - K - L)/r$). These domains are characterized by the two positive integers K, L with $K + L \leq r - 1$, and the set of a-s and b-s belonging to the domain have the form

$$a = 1 + l - L\frac{s}{r}; \qquad b = 1 + k - K\frac{s}{r}; \qquad 1 + l + 1 + k \leq s - 1 \tag{61}$$

where l and k are non negative integers. This means that for each $Q = r/s$ and K, L we get a Hilbert space $\mathcal{H}_{KL}$ where the local u and $\tilde{u}$ operators act irreducibly and are defined consistently if we take the sum in eq.(30) to run over the a, b-s in eq.(61). $\mathcal{H}_{11}$ is the Hilbert space containing the SL_2 invariant vacuum with $a_{\rm vac} = b_{\rm vac} = 1 - (s/r)$. The 'largest' Hilbert space where u and $\tilde{u}$ are defined consistently is the union of the irreducible $\mathcal{H}_{KL}$-s: $\mathcal{H} = \sum_{1}^{K+L\leq r-1} \oplus \mathcal{H}_{KL}$. Using eq.(61) and (27) it is easy to see that $\mathcal{H}$ consists of nothing but the W representations characterising the (not necessarily unitary) minimal models [9] belonging to $c(Q = r/s)$. We also remark that the set of a, b-s in eq.(61) and the maximal $\mathcal{H}$ are identical to the ones we obtain if we quantize the Toda fields with the other ($Q \to Q^{-1}$) choice for Δ and ω. More precisely using the a-s and b-s one gets from eq.(61) by keeping l and k fixed while leting K and L run in $K, L \geq 1$; $K + L \leq r - 1$, from eq.(30) we obtain a Hilbert space $\mathcal{H}_{kl}$ providing a representation for the other two Toda fields. The whole $\mathcal{H}$ is obtained if we insist on the simultaneous presence of both types of Toda fields.

6. Conclusions

In this paper we investigated the A_2 TT describing it in the reduced WZNW framework. In the classical theory working out this framework in the less familiar 'highest weight gauge' [7] we identified the relevant variables as a single Toda field, $u(z, \bar{z})$ and the generators of the classical W symmetry. Using them we showed that the space of classical solutions can be divided into classical representations of the W algebra, the W orbits, that are characterized by the monodromy matrix and a discrete invariant. We determined two types of monodromy matrices guaranteeing that the orbits belonging to them are of the classical highest weight type, in addition to lying in the singular and non singular sectors of the A_2 TT respectively. Surprisingly, we found that the orbit corresponding to the classical SL_2 invariant vacuum is not of the highest weight type.

In the quantum theory we promoted only the Toda field $u(z, \bar{z})$ and the generators of symmetries to operators. Working in a Hilbert space containing only at most a discrete infinity of W highest weight representations we defined $u(z, \bar{z})$ as a periodic primary field satisfying the quantized equation of motion. We constructed this $u(z,\bar z)$ operator – and its partner, $\tilde u(z,\bar z)$, generated from it by the automorhism of the algebra – in two steps: first by deriving the selection rules we determined the types of constant amplitudes parametrising them, then by imposing their locality through the 4-point functions we determined these constant amplitudes completely. As a result we learned that these local Toda fields can be defined consistently if the Q parameter determining the central charge as $c(Q) = 2(3-4/Q)(3-4Q)$ is rational and the Hilbert space is the collection of W representations corresponding to the minimal models. We find these results interesting as we arrived at them without ever demanding the presence of a closing operator algebra or any quantum group structure.

Summarizing we can say that the reduced WZNW framework gave new insights both in the classical and in the quantum versions of the A_2 TT. In the quantum case we also see that to go beyond the minimal models we have to drop some of our assumptions. The obvious possibilities are to replace the assumption about the representation content of the Hilbert space by something else and/or to drop one of the basic axioms of CFT, namely the equivalence between states and fields, that underlined our computations.

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Appendix A

Search for orbits of the classical highest weight type.

We argued in sect.3 that the classical analogues of the quantum highest weight states are solutions of eq.(6) with constant L and W such that the value of $\int_0^{2\pi} L(z)dz$ is bounded below along the orbit. Thus the classical highest weight state must correspond to at least a local minimum of this integral.

Let's define

$$l = \int_0^{2\pi} L(z)dz, \qquad w = \int_0^{2\pi} W(z)dz$$

If L and W are constant (resp. L_0, W_0) then the transformation rules of the classical WA_2 algebra (eq.(2-3)) simplify

$$\delta L = [2a_1' L_0 - 2a_1'''] + 3a_2' W_0 \tag{A.1}$$

$$\delta W = 3a_1' W_0 + [\frac{2}{3}a_2' W_0^2 - \frac{5}{6}a_2''' W_0 + \frac{1}{6}a_2^{(V)}] \tag{A.2}$$

Using these to compute the changes in l and w we see that $\delta l = 0$ as well as $\delta w = 0$ (since a_1 and a_2 are periodic) i.e. the points of constant L and W are stationary points of l and w along the orbit.

Being a classical highest weight state requires also

$$\delta\delta l \geq 0 \tag{A.3}$$

We call this the stability condition.

We can calculate the concrete formula for $\delta\delta l$ by iterating the WA_2 transformation laws (in the second step we have to use the full eq.(2-3) as after the first step L and W are no longer constants). Discarding total derivative terms we find

$$\delta\delta l = \int_0^{2\pi} (a_1' \delta L + a_2' \delta W) dz$$

which can be rewritten as

$$\delta\delta l = \int_0^{2\pi} (\, a_1' \quad a_2' \,) \begin{pmatrix} 2L_0 - 2\frac{d^2}{dz^2} & 3W_0 \\ 3W_0 & \frac{2}{3}L_0^2 - \frac{5}{6}\frac{d^2}{dz^2} + \frac{1}{6}\frac{d^4}{dz^4} \end{pmatrix} \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} dz \tag{A.4}$$

This is a quadratic form in terms of a_1' and a_2' and the stability condition amounts to its positive definity. We take an orthogonal basis in the space of (a_1',a_2') of the form

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \mathbf{q} e^{inz}, \qquad n \neq 0$$

In the subspace of given n the matrix appearing in eq.(A.4) takes the form

$$M(L_0, W_0) = \begin{pmatrix} 2L_0 + 2n^2 & 3W_0 \\ 3W_0 & \frac{2}{3}L_0^2 + \frac{5}{6}n^2 + \frac{1}{6}n^4 \end{pmatrix}$$

The positive definity means that the eigenvalues of this matrix

$$\lambda_{1,2}(L_0, W_0, n) = a + b \pm \sqrt{(a+b)^2 - 4ab + 9W_0^2},$$

where a and b are

$$a = L_0 + n^2, \quad b = \frac{1}{12}\big[(2L_0 + n^2)^2 + L_0 n^2\big],$$

must be positive for all n. Consequently

$$a + b > 0, \quad 4ab > 9W_0^2 \quad \text{for all} \quad |\, n \,| \geq 1$$

The first of these conditions is satisfied iff $L_0 > -1$.The second one is satisfied for all values of n iff it holds for $n = 1$. Therefore we have the following inequalities for stability

$$L_0 > -1, \quad (L_0 + 1)(4L_0^2 + 5L_0 + 1) > 9W_0^2 \tag{A.5}$$

Taking the solutions described by eq.(10-12) we obtain that they are stable for all possible values of Λ and m. In case of the solutions given in eq.(13-15) the second condition in (A.5) leads to the inequality

$$(1-y)\big[(3x)^2 + x(\frac{y}{2} + 2) + \frac{1}{(12)^2}(y-4)^2\big] > 0, \text{ where } \quad y = \rho^2, \quad x = \Lambda^2 \tag{A.6}$$

This implies that the values of Λ are not restricted and the first condition in eq.(A.5) is satisfied as well if $\rho < 1$. If $\rho \geq 1$ (A.6) does not hold for any value of Λ . This implies, as mentioned in sect.3, that the classical SL_2 invariant vacuum, which corresponds to $\Lambda = 0, \rho = 2$, cannot be a classical highest weight state.

Appendix B

W matrix elements

In this appendix we illustrate the method we used to compute the various matrix elements of W_{-3} on the example of $W_{-3}u$ between highest weight states:

$$\langle H\Omega | W_{-3}u(z) | hw \rangle = \lim_{z_1 \to \infty} \lim_{z_3 \to 0} z_1^{2H} \langle \Phi_H(z_1) W_{-3}u(z) \Phi_h(z_3) \rangle. \tag{B.1}$$

(We determined the matrix elements of L_n in the standard way [20].) In (B.1) $\Phi_H(z_1)$and $\Phi_h(z_3)$ denote two (chiral) W primary fields characterized by the L_0; W_0 eigenvalues H, Ω and h, w respectively, generating the highest weight states from vacuum. We shall use the integral representation

$$W_n\Phi(z) = \oint_z \frac{d\xi}{2\pi i}(\xi - z)^{n+2}W(\xi)u(z)$$

and the freedom to deform the contour away from z to z_1 and z_3. For this we have to compute $W(\xi)\Phi_H(z_1)$ and $W(\xi)\Phi_h(z_3)$.

The singular terms in the operator product have the form:

$$W(\xi)\Phi_H(z) = \frac{\Omega\Phi_H(z)}{(\xi - z)^3} + \frac{A_H(z)}{(\xi - z)^2} + \frac{B_H(z)}{(\xi - z)} \qquad (B.2)$$

where $A_H(z) = W_{-1}\Phi_H(z)$ and $B_H(z) = W_{-2}\Phi_H(z)$ denote the W descendandts of the primary field $\Phi_H(z)$. It is important to realise that the irreducible W representation generated from $\Phi_H(z)$ may contain several Virasoro primary fields among the W descendants. If the representation built on $\Phi_H(z)$ is not degenerate on the first grade then the two fields $L_{-1}\Phi_H(z)$ $W_{-1}\Phi_H(z)$ are not related to each other. Therefore defining $\Xi_{H+1}(z)$ as

$$A_H(z) = \frac{3\Omega}{2H}L_{-1}\Phi_H(z) + \Xi_{H+1}(z) \qquad (B.3)$$

we see using eq.(18) that Ξ_{H+1} is a Virasoro primary field

$$L_0\Xi_{H+1}(z) = (H+1)\Xi_{H+1}(z), \qquad L_n\Xi_{H+1}(z) = 0 \quad n > 0.$$

In the same way we have

$$B_H = AL_{-2}\Phi H(z) + BL_{-1}^2\Phi_H(z) + DL_{-1}\Xi_{H+1}(z) + \Psi_{H+2}(z) \qquad (B.4)$$

where A, B and D are constants and

$$L_0\Psi_{H+2}(z) = (H+2)\Psi_{H+2}, \qquad L_n\Psi_{H+2}(z) = 0 \quad n > 0.$$

(Because u is in an irreducibile representation degenerate on the first and second grade its associated fields are null, i.e. $\Xi_{\Delta+1}(z) = 0$ and $\Psi_{\Delta+2} = 0$ respectively.) Since $\Xi_{H+1}(z)$ and $\Psi_{H+2}(z)$ are Virasoro primary fields, conformal symmetry restricts the z dependence of the 3-point functions they enter.

Deforming the contour in eq.(B.1) to z_1 and z_3 and substituting (B.2), (B.3) and (B.4) into (B.1) and taking the $z_1 \to \infty$ limit we get:

$$\langle H\Omega|W_{-3}u(z)|hw\rangle = w\frac{G(H,h,..)}{z^{y+3}} + \frac{1}{z^2}\langle h|u(z)|A_h\rangle + \frac{1}{z}\langle H|u(z)|B_h\rangle \qquad (B.5)$$

where

$$\langle H|u(z)|A_h\rangle = \lim_{z_1\to\infty}\lim_{z_3\to 0} z_1^{2H}\langle \Phi_H(z_1)u(z)A_h(z_3)\rangle$$

We shall determine the unknown $\langle H|u(z)|A_h\rangle$ and $\langle H|u(z)|B_h\rangle$ functions by computing $\langle H|W_{-1}u(z)|h\rangle$ and $\langle H|W_{-2}u(z)|h\rangle$. Repeating the same steps that lead from (B.1) to (B.5) we have

$$\langle H|W_{-2}u(z)|h\rangle = -\langle H|u(z)|B_h\rangle \qquad (B.6)$$

and

$$\langle H|W_{-1}u(z)|h\rangle = z\langle H|u(z)|B_h\rangle - \langle H|u(z)|A_h\rangle. \qquad (B.7)$$

Since the W representation built on $u(z,\bar{z})$ is characterized by the null vectors (21.1), (21.2) in (B.6) and (B.7) we can write

$$W_{-2}u(z) = \omega\beta^{-1-1}L_{-1}^2u(z) + \omega\beta^{-2}L_{-2}u(z) \qquad (B.8)$$

and

$$W_{-1}u(z) = \frac{3\omega}{2\Delta}L_{-1}u(z). \qquad (B.9)$$

Substituting (B.6) and (B.7) into (B.5) using (B.8) and (B.9) we get:

$$\langle H|W_{-3}u(z)|h\rangle = \frac{G(H,h,..)}{z^{y+3}}\big\{w + \omega(\frac{3}{2\Delta}y - 2\beta^{-2}(y+h) - 2\beta^{-1-1}y(y+1)\big\}$$

Appendix C

In this appendix we derive the $x \to x^{-1}$ transformation rule for the generalized hypergeometric functions ${}_3F_2$:

$$ {}_3F_2\left(\begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 \\ & \beta_1 & \beta_2\end{matrix}\Big| z\right) = \sum_{n=0}^{\infty} \frac{\alpha_1^{(n)}\alpha_2^{(n)}\alpha_3^{(n)}}{\beta_1^{(n)}\beta_2^{(n)}} \frac{z^n}{n!} \tag{C.1} $$

To derive the transformation rule we use an integral representation wich is a straightforward generalization of the corresponding one for the hypergeometric functions:

$$ \begin{aligned} &\frac{\prod_{i=1}^{3}\Gamma(\alpha_i)}{\prod_{j=1}^{2}\Gamma(\beta_j)}\, {}_3F_2\left(\begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 \\ & \beta_1 & \beta_2\end{matrix}\Big| z\right) = \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^{3}\Gamma(\alpha_i + s)\Gamma(-s)}{\prod_{j=1}^{2}\Gamma(\beta_j + s)} (-z)^s ds \end{aligned} \tag{C.2} $$

In (C.2) $|\arg(-z)| < \pi$ and the contour of integration is chosen in such a way that the poles of $\Gamma(\alpha_i + s)$, $\Gamma(\beta_j + s)$ lie to its left while the poles of $\Gamma(-s)$ lie to its right. We also assume that none of α_i is a negative integer. Deforming the contour to encircle the poles of $\Gamma(-s)$ we indeed recover (C.1). However deforming it to encircle the poles of $\Gamma(\alpha_i + s)$ we get

$$ \begin{aligned} &{}_3F_2\left(\begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 \\ & \beta_1 & \beta_2\end{matrix}\Big| z\right) = \\ &= \sum_{i=1}^{3} A_i (-z)^{-\alpha_i}\, {}_3F_2\left(\begin{matrix}\alpha_i & 1+\alpha_i-\beta_1 & 1+\alpha_i-\beta_2 \\ & 1+\alpha_i-\alpha_{i+1} & 1+\alpha_i-\alpha_{i+2}\end{matrix}\Big| z^{-1}\right) \end{aligned} $$

where the $i+1$, $i+2$ indeces are understood only mod 3 and

$$ A_i = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\prod_{j\neq i}\Gamma(\alpha_j - \alpha_i)}{\prod_{j=1}^{2}\Gamma(\beta_j - \alpha_i)\prod_{j\neq i}\Gamma(\alpha_j)} \; . $$

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