ITP–Budapest Report No. 573

Finite size effects in boundary sine-Gordon theory

Z. Bajnok, L. Palla and G. Takács

November 15, 2018

Institute for Theoretical Physics
Roland Eötvös University,
H-1117 Budapest, Pázmány P. sétány 1/A, Hungary

Abstract

We examine the finite volume spectrum and boundary energy in boundary sine-Gordon theory, based on our recent results obtained by closing the boundary bootstrap. The spectrum and the reflection factors are checked against truncated conformal space, together with a (still unpublished) prediction by Al.B. Zamolodchikov for the boundary energy and the relation between the parameters of the scattering amplitudes and of the perturbed CFT Hamiltonian. In addition, a derivation of Zamolodchikov's formulae is given. We find an entirely consistent picture and strong evidence for the validity of the conjectured spectrum and scattering amplitudes, which together give a complete description of the boundary sine-Gordon theory on mass shell.

1 Introduction

Sine-Gordon field theory is one of the most important quantum field theoretic models with numerous applications ranging from particle theoretic problems to condensed matter systems, and one which has played a central role in our understanding of $1+1$ dimensional field theories. A crucial property of the model is integrability, which permits an exact analytic determination of many of its physical properties and characteristic quantities. Integrability can also be preserved in the presence of a boundary if suitable boundary conditions are imposed [1].

In this paper, continuing our work started in [2, 3], we investigate sine-Gordon field theory on the half-line and on a finite volume interval, with integrable boundary conditions. It was first pointed out by Ghoshal and Zamolodchikov [4] that the most general integrable boundary potential depends on two parameters. They also introduced the notion of 'boundary crossing unitarity', and combining it with the boundary version of the Yang

Baxter equations they were able to determine soliton reflection factors on the boundary; later Ghoshal completed this work by determining the breather reflection factors [5] using a boundary bootstrap equation first proposed by Fring and Köberle [6].

The results of Ghoshal and Zamolodchikov concerned only the reflection factors on the ground state boundary, although they already noticed that there are poles in the amplitudes which signal the existence of excited boundary states. The first (partial) results on the spectrum of these boundary states were obtained by Saleur and Skorik for Dirichlet boundary conditions [7]. However, they did not take into account the boundary analogue of the Coleman-Thun mechanism, the importance of which was first emphasized by Dorey et. al. [8]. Using this mechanism Mattsson and Dorey were able to close the bootstrap in the Dirichlet case and determine the complete spectrum and the reflection factors on the excited boundary states [9]. Recently we used their ideas to obtain the spectrum of excited boundary states and their reflection factors for the Neumann boundary condition [2] and then for the general two-parameter family of integrable boundary conditions [3]. For the Neumann case, extensive checks were performed using a boundary version of the so-called Truncated Conformal Space Approach (TCSA) [10, 11]; for the generic case, however, these checks were not carried out at that time.

Another interesting problem is that of the boundary energy. Namely, the boundary contributes a volume independent (constant) term to the free energy, in addition to the bulk energy density which gives a term proportional to the spatial volume. Just as in the case of the bulk energy density, the boundary energy in general QFT is not a universal quantity. However, in perturbed conformal field theories there is a preferred normalization[1] of the Hamiltonian which gives a unique definition for both the bulk and the boundary contributions. Therefore, this boundary energy is an interesting quantity to compute. For Dirichlet boundary condition it was obtained by Leclair et al. in [12]. A few years ago Al. B. Zamolodchikov presented a result for general integrable boundary conditions [13].

One crucial ingredient, that is needed e.g. for a TCSA check of the spectrum and reflection factors for the general integrable boundary conditions, is a relation between the ultraviolet (UV) parameters that appear in the perturbed CFT Hamiltonian and the infrared (IR) parameters in the reflection factors. This relation was also obtained by Al. B. Zamolodchikov [13]. Using his result, we perform an extensive check of the spectrum, boundary energy and reflection factors of boundary sine-Gordon theory. This provides strong evidence that all the results mentioned above form a consistent and complete description of the boundary sine-Gordon theory on mass shell (i.e. spectrum and scattering amplitudes).

The paper is organized as follows. In Section 2 we recall the results on the boundary bootstrap in boundary sine-Gordon theory. Section 3 describes Zamolodchikov's formulae on the UV-IR relation and the boundary energy. These formulae were presented at some seminars, but have not been published; however, we could get some notes taken by the

[1] In this preferred normalization, the perturbing bulk and boundary operators transform homogeneously under scale transformations.

audience.[2] In these notes we found several misprints; in order to determine the correct form of the formulae (which is of utmost importance in order to make the comparison to TCSA), we rederive here the boundary energy using the thermodynamic Bethe Ansatz (TBA) and then check the UV-IR relation using the exact vacuum expectation values of boundary fields conjectured by Fateev, Zamolodchikov and Zamolodchikov [14].[3] In Section 4 we describe the results coming from TCSA for generic (non Dirichlet) boundary conditions, while in Section 5 we present the results for Dirichlet boundary conditions, which are a singular limit of the generic case and so the TCSA must be set up differently. The paper ends with some brief conclusions and an outlook in Section 6.

2 Boundary bootstrap in sine-Gordon theory

Boundary sine-Gordon theory is defined by the action

$$\mathcal{A}_{sG} = \int_{-\infty}^{\infty} dt \left(\int_{-\infty}^{0} dx \left[\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{m_0^2}{\beta^2} \cos\beta\Phi \right] + M_0 \cos\frac{\beta}{2}\left(\Phi(0,\, t) - \phi_0\right) \right) \tag{2.1}$$

where $\Phi(x,\, t)$ is a real scalar field and M_0, ϕ_0 are the two parameters characterizing the boundary condition:

$$\partial_x \Phi(x,t)|_{x=0} = -M_0 \frac{\beta}{2} \sin\left(\frac{\beta}{2}(\Phi(0,t) - \phi_0) \right) . \tag{2.2}$$

Ghoshal and Zamolodchikov showed that the above model is integrable [4] and that the above boundary potential is the most general that permits the existence of higher spin conserved charges.

2.1 Bulk scattering properties

In the bulk sine-Gordon model the particle spectrum consists of the soliton s, the antisoliton $\bar{s}$, and the breathers B^n which appear as bound states in the $s\bar{s}$ scattering amplitude S_{+-}^{-+} . As a consequence of the integrable nature of the model any scattering amplitude factorizes into a product of two particle scattering amplitudes, from which the independent ones in the purely solitonic sector are [16]

$$\begin{aligned} a(u) = \;& S_{++}^{++}(u) = S_{--}^{--}(u) = & -\prod_{l=1}^{\infty}\left[\frac{\Gamma(2(l-1)\lambda - \frac{\lambda u}{\pi})\Gamma(2l\lambda + 1 - \frac{\lambda u}{\pi})}{\Gamma((2l-1)\lambda - \frac{\lambda u}{\pi})\Gamma((2l-1)\lambda + 1 - \frac{\lambda u}{\pi})} / (u \to -u) \right] \\ b(u) = \;& S_{+-}^{+-}(u) = S_{-+}^{-+}(u) = & \frac{\sin(\lambda u)}{\sin(\lambda(\pi - u))} a(u) \qquad ; \\ c(u) = \;& S_{+-}^{-+}(u) = S_{-+}^{+-}(u) = & \frac{\sin(\lambda \pi)}{\sin(\lambda(\pi - u))} a(u) \qquad ; \end{aligned} \tag{2.3}$$

[2] We thank P. Dorey and G. Watts for making the notes available to us.

[3] The derivation presented here is very similar to the way Al.B. Zamolodchikov arrived to the formulae (3.2-3.4) in Section 3.1, according to the hints he gave in his seminars.

where the parameter λ is determined by the sine-Gordon coupling constant

$$\lambda = \frac{8\pi}{\beta^2} - 1 \tag{2.4}$$

and $u = -i\theta$ denotes the purely imaginary rapidity. The other scattering amplitudes can be described in terms of the functions

$$\{y\} = \frac{\left(\frac{y+1}{2\lambda}\right)\left(\frac{y-1}{2\lambda}\right)}{\left(\frac{y+1}{2\lambda}-1\right)\left(\frac{y-1}{2\lambda}+1\right)} \quad , \quad (x) = \frac{\sin\left(\frac{u}{2}+\frac{x\pi}{2}\right)}{\sin\left(\frac{u}{2}-\frac{x\pi}{2}\right)} \ , \ \{y\}\{-y\} = 1 \quad , \ \{y+2\lambda\} = \{-y\}$$

as follows. For the scattering of the breathers B^n and B^m with $n \geq m$ and relative rapidity u the amplitude takes the form

$$S^{n\,m}(u) = S^{n\,m}_{n\,m}(u) = \{n+m-1\}\{n+m-3\}\dots\{n-m+3\}\{n-m+1\} \ ,$$

while for the scattering of the soliton (antisoliton) and B^n

$$S^n(u) = S^{+\,n}_{+\,n}(u) = S^{-\,n}_{-\,n}(u) = \{n-1+\lambda\}\{n-3+\lambda\}\dots\left\{\begin{array}{ll} \{1+\lambda\} & \text{if } n \text{ is even} \\ -\sqrt{\{\lambda\}} & \text{if } n \text{ is odd} \ . \end{array}\right.$$

2.2 Ground state reflection factors

The most general reflection factor - modulo CDD-type factors - of the soliton antisoliton multiplet $|s,\bar{s}\rangle$ on the ground state boundary, denoted by $|\,\rangle$, satisfying the boundary versions of the Yang Baxter, unitarity and crossing equations was found by Ghoshal and Zamolodchikov [4]:

$$\begin{aligned} R(\eta,\vartheta,u) &= \begin{pmatrix} P^+(\eta,\vartheta,u) & Q(\eta,\vartheta,u) \\ Q(\eta,\vartheta,u) & P^-(\eta,\vartheta,u) \end{pmatrix} \\ &= \begin{pmatrix} P_0^+(\eta,\vartheta,u) & Q_0(u) \\ Q_0(u) & P_0^-(\eta,\vartheta,u) \end{pmatrix} R_0(u)\frac{\sigma(\eta,u)}{\cos(\eta)}\frac{\sigma(i\vartheta,u)}{\cosh(\vartheta)} \ , \\ P_0^{\pm}(\eta,\vartheta,u) &= \cos(\lambda u)\cos(\eta)\cosh(\vartheta) \mp \sin(\lambda u)\sin(\eta)\sinh(\vartheta) \\ Q_0(u) &= -\sin(\lambda u)\cos(\lambda u) \end{aligned} \tag{2.5}$$

where η and ϑ are the two real parameters characterizing the solution,

$$R_0(u) = \prod_{l=1}^{\infty}\left[\frac{\Gamma(4l\lambda - \frac{2\lambda u}{\pi})\Gamma(4\lambda(l-1)+1-\frac{2\lambda u}{\pi})}{\Gamma((4l-3)\lambda - \frac{2\lambda u}{\pi})\Gamma((4l-1)\lambda+1-\frac{2\lambda u}{\pi})}/(u\to -u)\right]$$

is the boundary condition independent part and

$$\sigma(x,u) = \frac{\cos x}{\cos(x+\lambda u)}\prod_{l=1}^{\infty}\left[\frac{\Gamma(\frac{1}{2}+\frac{x}{\pi}+(2l-1)\lambda-\frac{\lambda u}{\pi})\Gamma(\frac{1}{2}-\frac{x}{\pi}+(2l-1)\lambda-\frac{\lambda u}{\pi})}{\Gamma(\frac{1}{2}-\frac{x}{\pi}+(2l-2)\lambda-\frac{\lambda u}{\pi})\Gamma(\frac{1}{2}+\frac{x}{\pi}+2l\lambda-\frac{\lambda u}{\pi})}/(u\to -u)\right]$$

describes the boundary condition dependence. Note that the topological charge may be changed by two in these reflections, thus the parity of the soliton number is conserved.

As a consequence of the bootstrap equations [4] the breather reflection factors share the structure of the solitonic ones, [5]:

$$R^{(n)}(\eta,\vartheta,u)=R_0^{(n)}(u)S^{(n)}(\eta,u)S^{(n)}(i\vartheta,u)\ , \tag{2.6}$$

where

$$R_0^{(n)}(u)=\frac{\left(\frac{1}{2}\right)\left(\frac{n}{2\lambda}+1\right)}{\left(\frac{n}{2\lambda}+\frac{3}{2}\right)}\prod_{l=1}^{n-1}\frac{\left(\frac{l}{2\lambda}\right)\left(\frac{l}{2\lambda}+1\right)}{\left(\frac{l}{2\lambda}+\frac{3}{2}\right)^2}\quad ;\quad S^{(n)}(x,u)=\prod_{l=0}^{n-1}\frac{\left(\frac{x}{\lambda\pi}-\frac{1}{2}+\frac{n-2l-1}{2\lambda}\right)}{\left(\frac{x}{\lambda\pi}+\frac{1}{2}+\frac{n-2l-1}{2\lambda}\right)}\ . \tag{2.7}$$

In general $R_0^{(n)}$ describes the boundary independent properties and the other factors give the boundary dependent ones.

2.3 The spectrum of boundary bound states and the associated reflection factors

In the general case, the spectrum of boundary bound states was derived in [3]. It is a straightforward generalization of the spectrum in the Dirichlet limit previously obtained by Mattsson and Dorey [9]. The states can be labeled by a sequence of integers $|n_1,n_2,\dots,n_k\rangle$. Such a state exists whenever the

$$\frac{\pi}{2}\geq\nu_{n_1}>w_{n_2}>\nu_{n_3}>w_{n_4}>\ldots\geq 0$$

condition holds, where

$$\nu_n=\frac{\eta}{\lambda}-\frac{(2n+1)\pi}{2\lambda}\qquad\text{and}\qquad w_n=\pi-\frac{\eta}{\lambda}-\frac{(2n-1)\pi}{2\lambda}$$

denote the location of certain poles in $\sigma(\eta,u)$. The mass of such a state (i.e. its energy above the ground state) is

$$m_{|n_1,n_2,\dots,n_k\rangle}=M\sum_{i\ \mathrm{odd}}\cos(\nu_{n_i})+M\sum_{i\ \mathrm{even}}\cos(w_{n_i})\ , \tag{2.8}$$

where M is the soliton mass. The reflection factors of the various particles on these boundary states depend on whether k is even or odd. When k is even, the reflection factors take the form

$$Q_{|n_1,n_2,\dots,n_k\rangle}(\eta,\vartheta,u)=Q(\eta,\vartheta,u)\prod_{i\ \mathrm{odd}}a_{n_i}(\eta,u)\prod_{i\ \mathrm{even}}a_{n_i}(\bar{\eta},u)\ ,$$

and

$$P^{\pm}_{|n_1,n_2,\dots,n_k\rangle}(\eta,\vartheta,u)=P^{\pm}(\eta,\vartheta,u)\prod_{i\ \mathrm{odd}}a_{n_i}(\eta,u)\prod_{i\ \mathrm{even}}a_{n_i}(\bar{\eta},u)\ ,$$

for the solitonic processes, where

$$a_n(\eta,u)=\prod_{l=1}^{n}\left\{2\left(\frac{\eta}{\pi}-l\right)\right\}\quad ;\quad \bar{\eta}=\pi(\lambda+1)-\eta\ .$$

For the breather reflection factors the analogous formula is

$$R^{(n)}_{|n_1,n_2,\dots,n_k\rangle}(\eta,\vartheta,u)=R^{(n)}(\eta,\vartheta,u)\prod_{i\ \text{odd}}b^n_{n_i}(\eta,u)\prod_{i\ \text{even}}b^n_{n_i}(\bar{\eta},u) \tag{2.9}$$

where now

$$b^n_k(\eta,u)=\prod_{l=1}^{\min(n,k)}\left\{\frac{2\eta}{\pi}-\lambda+n-2l\right\}\left\{\frac{2\eta}{\pi}+\lambda-n-2(k+1-l)\right\}\ . \tag{2.10}$$

In the case when k is odd, the same formulae apply if in the $P^{\pm}$, Q and $R^{(n)}$ ground state reflection factors the $\eta\leftrightarrow\bar{\eta}$ and $s\leftrightarrow\bar{s}$ changes are made.

3 Boundary energy and UV-IR relation in sine-Gordon theory

3.1 Zamolodchikov's formulae

As mentioned in the introduction, recently Al. B. Zamolodchikov presented (but not yet published) [13] a formula for the relation between the UV and the IR parameters in the sine-Gordon model. To set the conventions for this relation, consider boundary sine-Gordon theory as a joint bulk and boundary perturbation of the $c=1$ free boson with Neumann boundary conditions (perturbed conformal field theory, pCFT):

$$\mathcal{A}_{pCFT}=\mathcal{A}^N_{c=1}+\mu\int_{-\infty}^{\infty}dt\int_{-\infty}^{0}dx\ :\cos\beta\Phi(x,\,t):+\tilde{\mu}\int_{-\infty}^{\infty}dt\ :\cos\frac{\beta}{2}\left(\Phi(0,\,t)-\phi_0\right): \tag{3.1}$$

where the colons denote the standard CFT normal ordering, which defines the normalization of the operators and of the coupling constants. The couplings μ and $\tilde{\mu}$ have nontrivial dimensions;

$$[\mu]=[\text{mass}]^{2-2h_\beta},\qquad [\tilde{\mu}]=[\text{mass}]^{1-h_\beta},\qquad h_\beta=\frac{\beta^2}{8\pi},$$

see the section on TCSA for more details. With these conventions the UV-IR relation [4] is

$$\begin{aligned}\cos\left(\frac{\beta^2\eta}{8\pi}\right)\cosh\left(\frac{\beta^2\vartheta}{8\pi}\right)&=\frac{\tilde{\mu}}{\tilde{\mu}_{\text{crit}}}\cos\left(\frac{\beta\phi_0}{2}\right)\ ,\\ \sin\left(\frac{\beta^2\eta}{8\pi}\right)\sinh\left(\frac{\beta^2\vartheta}{8\pi}\right)&=\frac{\tilde{\mu}}{\tilde{\mu}_{\text{crit}}}\sin\left(\frac{\beta\phi_0}{2}\right)\ ,\end{aligned} \tag{3.2}$$

[4] A similar relation was derived by Corrigan and Taormina [15] for sinh-Gordon theory, however, their normalization of the coupling constants is different from the one natural in the perturbed CFT framework.

where

$$\tilde{\mu}_{\rm crit} = \sqrt{\frac{2\mu}{\sin\left(\frac{\beta^2}{8}\right)}} \ . \tag{3.3}$$

Zamolodchikov also gave the boundary energy as

$$E(\eta,\vartheta) = -\frac{M}{2\cos\frac{\pi}{2\lambda}}\left(\cos\left(\frac{\eta}{\lambda}\right) + \cosh\left(\frac{\vartheta}{\lambda}\right) - \frac{1}{2}\cos\left(\frac{\pi}{2\lambda}\right) + \frac{1}{2}\sin\left(\frac{\pi}{2\lambda}\right) - \frac{1}{2}\right) \ . \tag{3.4}$$

3.2 Derivation of the boundary energy from TBA

In an integrable boundary theory with one scalar particle of mass m only, one can write down the TBA equation for the ground state energy on a strip with spatial volume L and integrable boundary conditions a and b at the two ends. The equation is of the form [12]:

$$\varepsilon\left(\theta\right) = 2l\cosh\theta + k_{ab}\left(\theta\right) - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\varphi\left(\theta-\theta'\right)\log\left(1+\mathrm{e}^{-\varepsilon(\theta')}\right) \tag{3.5}$$

where $l = mL$ is the dimensionless volume parameter. The kernel is expressed in terms of the two-body S-matrix $S(\theta)$ as

$$\varphi(\theta) = -i\frac{\partial}{\partial\theta}\log S\left(\theta\right) \ ,$$

while

$$k_{ab}(\theta) = -\log\left[R_a\left(\frac{i\pi}{2}-\theta\right)R_b\left(\frac{i\pi}{2}+\theta\right)\right] \ ,$$

where $R_a\left(\theta\right)$ and $R_b\left(\theta\right)$ are the reflection factors for the two ends. From the solution $\varepsilon\left(\theta\right)$ of the TBA equation the ground state energy can be calculated using the formula

$$E(L) = E_{\rm bulk}L + E_{\rm boundary} - \frac{\pi c(mL)}{24L} \ , \quad c(l) = \frac{6l}{\pi^2}\int_{-\infty}^{\infty} d\theta\, L(\theta)\cosh\theta \ , \tag{3.6}$$

where $L(\theta)$ is the usual short hand notation $L(\theta) = \log\left(1+\mathrm{e}^{-\varepsilon(\theta)}\right)$. It is well-known that no such TBA equation (or, for that matter, a finite system of TBA equations) can be written for sine-Gordon theory as a result of the nondiagonal bulk and boundary scattering of the solitons (except for special values of the parameters). Therefore, our approach is to calculate the boundary energy for sinh-Gordon theory and then analytically continue back to the sine-Gordon case. This is known to work e.g. for S-matrices, form factors and many other quantities, and so we simply assume it works for the boundary energy as well.

Consider therefore the boundary energy in boundary sinh-Gordon theory

$$\mathcal{A}_{shG} = \int_{-\infty}^{\infty} dt\left(\int_{\infty}^{0} dx\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m_0^2}{b^2}\cosh b\Phi\right] - M_0\cosh\frac{b}{2}\left(\Phi-\phi_0\right)\right) \ ,$$

which can be considered as the analytic continuation of the boundary sine-Gordon model (2.1) by substituting $b = i\beta$ (and changing the convention for the sign of M_0). Then from (2.4)

$$\lambda = -\frac{8\pi}{b^2} - 1$$

and, as a result, λ is negative for the sinh-Gordon case. Note that the analytic continuation is through the point $\lambda = \infty$ (complex infinity), therefore for the purposes of relating physical quantities between the two models the natural variable is λ^{-1}.

We now proceed to the calculation of the boundary energy. A similar calculation was performed by Dorey at al. [11] for the scaling Lee-Yang case. They presented the general idea with enough hints to reconstruct the method, but for the sake of completeness we write down the details for the interested reader. It is based on Zamolodchikov's method for obtaining the bulk energy from the TBA with periodic boundary conditions [17].

Suppose for simplicity that the boundary conditions a and b are identical and so $k = k_{aa}$ is even. Then in general the functions k and φ have the following asymptotic behaviour

$$\begin{aligned} k(\theta) &\sim k_0 + A\mathrm{e}^{-|\theta|} + \dots \\ \varphi(\theta) &\sim C\mathrm{e}^{-|\theta|} + \dots \end{aligned} \qquad (3.7)$$

for $|\theta| \to \infty$, where k_0, A and C are real constants.

The 'kink' functions, defined as

$$\varepsilon_\pm(\theta) = \lim_{l \to 0} \varepsilon\left(\theta \pm \log\frac{1}{l}\right) ,$$

satisfy the 'kink' equation

$$\varepsilon_\pm(\theta) = \mathrm{e}^{\pm\theta} + k_0 - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\varphi(\theta - \theta') \log\left(1 + \mathrm{e}^{-\varepsilon_\pm(\theta')}\right)$$

and are related as

$$\varepsilon_-(\theta) = \varepsilon_+(-\theta) .$$

Let us also introduce the following definitions

$$L_\pm(\theta) = \log\left(1 + \mathrm{e}^{-\varepsilon_\pm(\theta)}\right)$$

and define the asymptotic values

$$\varepsilon_0 = \varepsilon_+(-\infty) \quad , \quad L_0 = L_+(-\infty) = \log\left(1 + \mathrm{e}^{-\varepsilon_0}\right)$$

which satisfy the standard 'plateau' equation

$$\varepsilon_0 = k_0 - NL_0 \quad , \quad N = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\varphi(\theta) . \qquad (3.8)$$

Our aim is to expand $c(l)$ around $l=0$. To calculate the first few terms, it is convenient to define the functions δ and $\tilde{L}$ in the following way:

$$\begin{aligned} \varepsilon(\theta) &= \varepsilon_+\left(\theta-\log\frac{1}{l}\right)+\varepsilon_+\left(-\theta-\log\frac{1}{l}\right)+\delta\left(\theta\right)-\varepsilon_0\,, \\ L(\theta) &= L_+\left(\theta-\log\frac{1}{l}\right)+L_+\left(-\theta-\log\frac{1}{l}\right)+\tilde{L}\left(\theta\right)-L_0\,. \end{aligned} \tag{3.9}$$

They satisfy

$$\begin{aligned} \delta\left(\theta\right) &= k\left(\theta\right)-k_0-\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\varphi\left(\theta-\theta'\right)\tilde{L}\left(\theta'\right)\,, \\ & \delta\left(\theta\right)\,,\;\tilde{L}\left(\theta\right)\;\rightarrow\;0 \text{ as } l\;\rightarrow\;0. \end{aligned} \tag{3.10}$$

We can then rewrite

$$c(l)=\frac{6}{\pi^2}\int_{-\infty}^{\infty}d\theta\,\mathrm{e}^{\theta}L_+(\theta)+\frac{6l^2}{\pi^2}\int_{-\infty}^{\infty}d\theta\,\mathrm{e}^{-\theta}\frac{\partial L_+}{\partial\theta}+\frac{6l}{\pi^2}\int_{-\infty}^{\infty}d\theta\,\tilde{L}\left(\theta\right)\cosh\theta$$

The first term gives the UV central charge and can be calculated using the standard dilogarithm sum rules. The second term is the (anti) bulk energy density, that can be calculated self-consistently by examining the $\theta\rightarrow-\infty$ asymptotics of the integrand [17]:

$$\frac{\partial L_+}{\partial\theta}=-\frac{1}{1+\mathrm{e}^{\varepsilon_+(\theta)}}\frac{\partial\varepsilon_+}{\partial\theta}\sim-\frac{1}{1+\mathrm{e}^{\varepsilon_0}}\frac{\partial\varepsilon_+}{\partial\theta}\quad\text{for}\quad\theta\rightarrow-\infty$$

The terms proportional to e^{θ} must cancel for the integral to converge on its lower bound. Using the kink equation and the asymptotics of φ, to leading order

$$\frac{\partial\varepsilon_+}{\partial\theta}=\mathrm{e}^{\theta}\left(1-\frac{C}{2\pi}\int_{-\infty}^{\infty}d\theta'\,\mathrm{e}^{-\theta'}\frac{\partial L_+}{\partial\theta'}\right)$$

from which

$$\int_{-\infty}^{\infty}d\theta\,\mathrm{e}^{-\theta}\frac{\partial L_+}{\partial\theta}=\frac{2\pi}{C}\,.$$

In the perturbed conformal field theory formalism, the ground state energy can be expanded as

$$E(L)=\frac{\pi}{L}\sum_{n=0}^{\infty}\mathcal{C}_n\left(mL\right)^{n(1-\Delta)}$$

so the terms linear in L must cancel from (3.6). Therefore we obtain the bulk energy density as

$$E_{\text{bulk}}=\frac{1}{2C}m^2\,.$$

The third term can be rewritten using that $\tilde{L}\left(\theta\right)=\tilde{L}\left(-\theta\right)$:

$$\int_{-\infty}^{\infty}d\theta\,\tilde{L}\left(\theta\right)\cosh\theta=\int_{-\infty}^{\infty}d\theta\,\tilde{L}\left(\theta\right)\,\mathrm{e}^{-\theta}$$

After a partially integration, it can be seen that once again, the integral is convergent if terms proportional to e^{θ} cancel in $\frac{\partial \tilde{L}}{\partial \theta}$. Using equations (3.9) this is equivalent to cancellation of all terms proportional e^{θ} in δ, at least to leading order in l. From (3.10) we obtain

$$\delta\left(\theta\right) = \mathrm{e}^{\theta}\left(A - \frac{C}{2\pi}\int_{-\infty}^{\infty} d\theta' \tilde{L}\left(\theta'\right)\,\mathrm{e}^{-\theta'}\right)$$

from which we obtain (to leading order)

$$\int_{-\infty}^{\infty} d\theta\,\tilde{L}\left(\theta\right)\,\mathrm{e}^{-\theta} = \frac{2\pi A}{C}\,.$$

None of the subleading terms contains any contribution which are independent of the volume and therefore in (3.6) E_{boundary} must cancel against this particular term, leading to

$$E_{\mathrm{boundary}} = \frac{A}{2C}m\ .$$

3.3 The sinh-Gordon case

In sinh-Gordon theory the two particle S-matrix can be written as (remember, that in our convention λ is negative in its physical range):

$$S\left(\theta\right) = \frac{\sinh\theta + i\sin\frac{\pi}{\lambda}}{\sinh\theta - i\sin\frac{\pi}{\lambda}}\ . \tag{3.11}$$

As a result, the TBA kernel is

$$\varphi(\theta) = -\frac{2\cosh\theta\sin\frac{\pi}{\lambda}}{\sinh^2\theta + \sin^2\frac{\pi}{\lambda}} \ \sim\ -4\sin\frac{\pi}{\lambda}\,\mathrm{e}^{-|\theta|} + O\left(\mathrm{e}^{-2|\theta|}\right)\ ,$$

and so we get

$$C = -4\sin\frac{\pi}{\lambda}\ .$$

The integral N takes the value

$$N = \int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\varphi\left(\theta\right) = \begin{cases} 1 & \text{for} \quad \Re\mathrm{e}\,\lambda < 0 \\ -1 & \text{for} \quad \Re\mathrm{e}\,\lambda > 0 \end{cases}$$

which means that the plateau equation (3.8) has the solution

$$\mathrm{e}^{-\varepsilon_0} = \frac{\mathrm{e}^{-k_0}}{1 + \mathrm{e}^{-k_0}\,\mathrm{sign}\,\Re\mathrm{e}\,\lambda}\ .$$

Note that there is no real solution for $\lambda < 0$, $k_0 \leq 0$. This peculiarity of the sinh-Gordon TBA equation was already noted by Al.B. Zamolodchikov in the case of periodic boundary condition [18]. We simply assume that we are working for parameter values for which such a solution exists, so the considerations of the previous subsection apply. This is always the case for $\Re e\,\lambda > 0$, which is however not a physical range of the parameter λ in sinh-Gordon theory. Therefore we treat the sinh-Gordon TBA in this range as a mathematical problem only, without a corresponding physical field theory (except for the case $\lambda = 3/2$, see later). We further assume that all physical quantities that we wish to calculate are meromorphic functions of λ^{-1} and so they have a unique analytic continuation to the values of λ^{-1} that we are interested in[5]. Note that this argument is not a proper derivation; however, for the time being this is the only way we can arrive at the desired result, and we show that the results fit with the bootstrap spectrum, TCSA data and known results from previous literature.

E.g. for the bulk energy density we obtain

$$E^{\mathrm{shG}}_{\mathrm{bulk}} = -\frac{m^2}{8\sin\frac{\pi}{\lambda}}\ .$$

This is meromorphic in λ^{-1} and so we trust that it is the true bulk energy constant of the sinh-Gordon theory in the regime $\lambda < 0$. Furthermore, it is equal to the known result [19]. Now we can try and continue this result to the sine-Gordon regime $\lambda > 0$. Under this continuation the sinh-Gordon particle is identified with the first breather of sine-Gordon theory and so we have

$$m = 2M\sin\frac{\pi}{2\lambda}\,,$$

where M is the soliton mass. We then obtain

$$E^{\mathrm{sG}}_{\mathrm{bulk}} = -\frac{M^2}{4}\tan\frac{\pi}{2\lambda}\,,$$

which is the correct bulk energy density of sine-Gordon theory [24]. Hence the above method of continuation works for the bulk energy constant.

Now let us calculate the boundary energy. From eqns. (2.6,2.7), the reflection factor of the first breather in sine-Gordon theory can be written as

$$R^{(1)}(\theta) = \frac{\left(\frac{1}{2}\right)_\theta\left(\frac{1}{2\lambda}+1\right)_\theta}{\left(\frac{1}{2\lambda}+\frac{3}{2}\right)_\theta}\frac{\left(\frac{\eta}{\pi\lambda}-\frac{1}{2}\right)_\theta\left(\frac{i\vartheta}{\pi\lambda}-\frac{1}{2}\right)_\theta}{\left(\frac{\eta}{\pi\lambda}+\frac{1}{2}\right)_\theta\left(\frac{i\vartheta}{\pi\lambda}+\frac{1}{2}\right)_\theta}\quad,\quad (x)_\theta\equiv(x)=\frac{\sinh\left(\frac{\theta}{2}+i\frac{\pi x}{2}\right)}{\sinh\left(\frac{\theta}{2}-i\frac{\pi x}{2}\right)}\,, \tag{3.12}$$

where η and ϑ parametrize the boundary conditions. The sinh-Gordon reflection factor can be obtained by continuing the reflection factor to negative values of λ^{-1} (for sinh-Gordon theory, η is real and ϑ is purely imaginary, while for sine-Gordon theory both parameters are real). Putting the same boundary condition on the two boundaries of the strip (with the same values of ϑ and η) we obtain

$$E^{\mathrm{shG}}_{\mathrm{boundary}} = 2E^{\mathrm{shG}}(\eta,\,\vartheta)$$

[5]It is clear that the relevant variable to consider is λ^{-1} because the continuation in the coupling goes through the value $\beta = 0$ which corresponds to $\lambda = \infty$

where $E(\eta,\,\vartheta)$ is the energy of a single boundary. The term $k\,(\theta)$ in the TBA equation (3.5) is

$$k\,(\theta) = -\log\left[K\,(\theta)\,K\,(-\theta)\right] \quad , \quad K\,(\theta) = R^{(1)}\left(i\frac{\pi}{2}-\theta\right) \ .$$

Using the identity

$$(x)_{i\frac{\pi}{2}+\theta}(x)_{i\frac{\pi}{2}-\theta} = \frac{\cosh\theta+\sin\pi x}{\cosh\theta-\sin\pi x}\,,$$

we get

$$-\log\left[(x)_{i\frac{\pi}{2}+\theta}(x)_{i\frac{\pi}{2}-\theta}\right] \sim -4\sin\pi x\,\mathrm{e}^{-|\theta|}+O\left(\mathrm{e}^{-2|\theta|}\right) \ .$$

Note that k_0 and A in (3.7) can be calculated additively from the asymptotics of the contribution of a single (x) block above. As a result, $k_0 = 0$ and so the plateau eqn. (3.8) has no solution in the sinh-Gordon regime $\lambda < 0$, thus the analytic continuation described above cannot be avoided. Putting the ingredients together, the boundary energy in sinh-Gordon theory takes the form

$$E^{\mathrm{shG}}(\eta,\vartheta) = -\frac{m}{2\sin\frac{\pi}{\lambda}}\left(\cos\left(\frac{\eta}{\lambda}\right)+\cosh\left(\frac{\vartheta}{\lambda}\right)-\frac{1}{2}\cos\left(\frac{\pi}{2\lambda}\right)+\frac{1}{2}\sin\left(\frac{\pi}{2\lambda}\right)-\frac{1}{2}\right) \ . \quad (3.13)$$

It is now easy to recover Zamolodchikov's formula (3.4) for the boundary energy in sine-Gordon theory.

As an immediate check on this calculation, we wish to note that for $\lambda = \frac{3}{2}$ the S-matrix (3.11) is identical to that of the scaling Lee-Yang model, and the reflection factors of the scaling Lee-Yang model corresponding to integrable boundary conditions are reproduced by specifying some complex values for η and ϑ. It can be easily checked that the formula (3.13) reproduces correctly the results of Dorey et al. [11].

3.4 Special cases

Since we obtained the boundary energy of sine-Gordon/sinh-Gordon theory under some non trivial assumptions we check the results in some known cases.

3.4.1 Dirichlet boundary conditions

Dirichlet boundary conditions correspond to the limit $\mu \to \infty$ in (3.1), which leads to

$$\Phi(0,\,t) = \phi_0 \mod \frac{2\pi}{\beta} \ .$$

The reflection factor of the first breather can be obtained as the $\vartheta \to \infty$ limit of (3.12):

$$R^{(1)}(\theta) = \frac{\left(\frac{1}{2}\right)_\theta\left(\frac{1}{2\lambda}+1\right)_\theta}{\left(\frac{1}{2\lambda}+\frac{3}{2}\right)_\theta}\frac{\left(\frac{\eta}{\pi\lambda}-\frac{1}{2}\right)_\theta}{\left(\frac{\eta}{\pi\lambda}+\frac{1}{2}\right)_\theta} \ .$$

The derivation of the previous subsection then gives the boundary energy

$$E_D(\eta) = -\frac{M}{2\cos\frac{\pi}{2\lambda}}\left(\cos\left(\frac{\eta}{\lambda}\right) - \frac{1}{2}\cos\left(\frac{\pi}{2\lambda}\right) + \frac{1}{2}\sin\left(\frac{\pi}{2\lambda}\right) - \frac{1}{2}\right), \tag{3.14}$$

which is exactly identical to the formula obtained by Leclair et al. in [12]. The parameter η is related to ϕ_0 in the following way

$$\eta = \pi\,(\lambda+1)\,\frac{\beta\phi_0}{2\pi}\,,$$

which was conjectured by Ghoshal and Zamolodchikov [4], and is a straightforward consequence of eqns. (3.2) as well.

Note that $E_D(\eta)$ cannot be obtained as the $\vartheta \to \infty$ limit of the general boundary energy eqn. (3.4). The reason is clear: the boundary potential is normalized in different ways in the two cases: classically to obtain finite energy in the Dirichlet limit one has to add M_0 to the general $-M_0\cos\left(\frac{\beta}{2}(\Phi-\phi_0)\right)$ boundary potential. Clearly in the quantum case, when the boundary vertex operator has a non trivial dimension, we can not simply subtract $\tilde{\mu}$ from $E(\eta,\vartheta)$. Since the quantity we subtract must have the dimension of mass and should depend on $\tilde{\mu}$, it must be proportional to $\tilde{\mu}^{1/(1-h_\beta)} = \tilde{\mu}^{\lambda/(\lambda+1)}$. The question is whether we can make this subtraction such that in the $\vartheta \to \infty$ limit the leading term cancels and the constant terms just reproduce $E_D(\eta)$. The UV-IR relations, eqn. (3.2-3.3) guarantee, that

$$\tilde{\mu} \to \frac{\mu_{\rm crit}}{2}\exp\left(\frac{\vartheta}{\lambda+1}\right)\left(1+\exp\left(-\frac{2\vartheta}{\lambda+1}\right)\cos(\frac{2\eta}{\lambda+1}) + \mathcal{O}\exp\left(-\frac{4\vartheta}{\lambda+1}\right)\right) \quad \text{as} \quad \vartheta \to \infty\,.$$

Thus, upon using the bulk mass gap relation (cf. eqn. (4.2)), $\tilde{\mu}^{\lambda/(\lambda+1)}$ becomes proportional to $Me^{\vartheta/\lambda}$ up to exponentially small terms for $\vartheta \to \infty$. Therefore, if we subtract this term with an appropriate coefficient then in the Dirichlet limit the surviving constant terms exactly reproduce (3.14).

3.4.2 The first excited state

It was noted in [9] (for Dirichlet boundary condition) and in [3] (for the general case) that continuing analytically

$$\eta \;\to\; \pi(\lambda+1) - \eta$$

the role of the boundary ground state $|\rangle$ and the boundary first excited state $|0\rangle$ are interchanged. Therefore we can calculate the energy difference between these two states from the formula for the boundary energy, eqn. (3.4). The result is

$$E(\pi(\lambda+1)-\eta,\vartheta) - E(\eta,\vartheta) = M\cos\left(\frac{\eta}{\lambda} - \frac{\pi}{2\lambda}\right)$$

which exactly equals the prediction of the bootstrap, i.e.

$$E_{|0\rangle} - E_{|\rangle} = M\cos\nu_0$$

that follows from eqn. (2.8).

3.5 UV-IR relations and vacuum expectation values (VEVs)

As it is well known in the bulk sine-Gordon theory there is a relation among the following three exactly calculable quantities: the ground state energy density, the dimensionless constant entering the mass gap relation connecting the UV and IR parameters, and the VEV of the exponential field $\langle e^{i\beta\Phi(x)}\rangle$ [20]. This relation is such that knowing any two of these quantities determines the third one. It generalizes to sine-Gordon theory with boundaries, where it connects the boundary energy, the UV-IR relations (3.2-3.3), and the VEV of the boundary field $\langle e^{i\frac{\beta}{2}\Phi(0)}\rangle$ in a similar way. As the VEV of the boundary operators has been determined by Fateev, Zamolodchikov and Zamolodchikov (FZZ), we can use it to show that the UV-IR relations, (3.2-3.3) and the boundary energy, (3.4), are indeed consistent with the VEVs given in [14]. For simplicity we consider only the special case when $\phi_0 = 0$, as this case already illustrates the point. (More precisely the condition $\phi_0 = 0$ can be satisfied in two different ways [3]: either by $\vartheta = 0$ or by $\eta = 0$, and we consider the former possibility).

Writing the functional integral representation of the partition function $Z_{ab} = \mathrm{Tr} e^{-RH_{ab}(L)}$ on a cylinder of length R and circumference L with boundary states a and b on the boundary circles and considering the $R \to \infty$ limit (when $Z_{ab} \sim e^{-RE_{ab}(L)}$) one readily derives that in this limit the ground state energy E_{aa} satisfies

$$\frac{\partial E_{aa}}{\partial \tilde{\mu}} = -\langle e^{i\frac{\beta}{2}\Phi(0)}\rangle \equiv -G(\beta,\tilde{\mu}). \tag{3.15}$$

(In writing this equation we assumed that $G(\beta,\tilde{\mu}) = G(-\beta,\tilde{\mu})$). Since for $\vartheta = 0$ the ground state energy depends on $\tilde{\mu}$ only through the η parameter appearing in the boundary energy, eqn. (3.15) actually determines the dependence of η on $\tilde{\mu}$. Furthermore, both sides of (3.15) can be integrated to obtain the following expression for the boundary energy

$$E(\eta) = -\int d\tilde{\mu} G(\beta,\tilde{\mu})\,. \tag{3.16}$$

What we show below is that using the FZZ expression for $G(\beta,\tilde{\mu})$ on the r.h.s. gives (3.4) for the boundary energy only if (3.2-3.3) hold.

The expression given in [14] for $G(\beta,\tilde{\mu})$ depends on $\tilde{\mu}$ through a parameter z, which, for $\phi_0 = 0$, we take to be pure imaginary $z = iZ$ (Z real):

$$\cos^2(\pi Z) = \frac{\tilde{\mu}^2}{2\mu}\sin\left(\frac{\beta^2}{8}\right)\,. \tag{3.17}$$

Then

$$G(\beta,\tilde{\mu}) = \left(\frac{\pi\mu\Gamma(\frac{\lambda}{\lambda+1})}{2\Gamma(\frac{1}{\lambda+1})}\right)^{\frac{1}{2\lambda}} g_0(\beta) g_S(\beta,Z)\,,$$

where g_0 and g_S are given by the integral representations[6]:

$$\log g_0(\beta) = \int_0^\infty \frac{dt}{t}\left[\frac{2\sinh(t/(\lambda+1)}{\sinh(t)\sinh(t\lambda/(\lambda+1))}\left(e^{t\lambda/(2\lambda+2)}\cosh\left(\frac{t}{2}\right)\cosh\left(\frac{t}{2\lambda+2}\right)-1\right)-\frac{e^{-t}}{\lambda+1}\right],$$

$$\log g_S(\beta,Z) = -\int_0^\infty \frac{dt}{t}\frac{2\sinh(t/(\lambda+1)\sinh^2(Zt)}{\sinh(t)\sinh(t\lambda/(\lambda+1))}.$$

The integrals appearing here can be computed analytically after some efforts. Finally, expressing μ in terms of the soliton mass M via (4.2), and converting the integral over $\tilde{\mu}$ to an integral over πZ by using (3.17), after some algebra one finds

$$-\int d\tilde{\mu}G(\beta,\tilde{\mu}) = -\frac{M}{2\cos\left(\frac{\pi}{2\lambda}\right)}\cos\left(\frac{Z\pi(\lambda+1)}{\lambda}\right)+f(\lambda)\ .$$

This agrees with the boundary energy, (3.4), if $Z\pi = \frac{\eta}{\lambda+1}$, i.e. when eqn. (3.17) becomes identical to (the $\vartheta = 0$ case of) (3.2-3.3).

4 TCSA: general integrable boundary condition

4.1 TCSA for the boundary sine-Gordon model

First we describe the Hamiltonian of boundary sine-Gordon model (BSG) living on the line segment $0 \leq x \leq L$ as that of a bulk and boundary perturbed free boson with suitable boundary conditions. This is the starting point of the TCSA analysis.

The basic idea of TCSA is to describe certain $2d$ models in finite volume as relevant perturbations of their ultraviolet limiting CFT-s [10]. If we consider boundary field theories, then the CFT-s in the ultraviolet are in fact boundary CFT-s. The use of TCSA to investigate boundary theories was advocated in [11, 21].

As the bulk SG model can be successfully described in TCSA as a perturbation of the $c = 1$ free boson [22], it is natural to expect that the various BSG models are appropriate perturbations of $c = 1$ theories with Neumann or Dirichlet boundary conditions. Therefore we take the strip $0 \leq x \leq L$ and consider the following perturbations of the free boson, as described in detail in [2]:

$$\begin{aligned} S &= \int_{-\infty}^{\infty}\int_0^L \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \mu\cos(\beta\Phi)\right)dxdt + \\ &+ \int_{-\infty}^{\infty}\left(\tilde{\mu}_0\cos\left(\frac{\beta}{2}(\Phi_B-\phi_0)\right)+\tilde{\mu}_L\cos\left(\frac{\beta}{2}(\Phi_B-\phi_L)\right)\right)dt\ . \end{aligned}$$

[6]Note that the integral for $\log g_S$ contains a factor of $1/2$ compared to the expression in [14] even after accounting for the difference between the parameters of this paper and of [14]. Without the inclusion of this factor it would be impossible to obtain the correct η dependence of the boundary energy as in eqn. (3.4). The fact that this factor should be present was later confirmed to us by Al.B. Zamolodchikov in a private discussion.

Here, for finite $\tilde{\mu}$'s, Neumann boundary conditions are imposed in the underlying $c = 1$ theory on the boundaries, while if any of the $\tilde{\mu}$ -s is infinite then the corresponding term is absent and the boundary condition in the underlying conformal theory on that boundary is Dirichlet. The Hamiltonian of the system can be rewritten in terms of the variables associated to the plane using the map $(x, it) = \xi \ \rightarrow \ z = e^{i\frac{\pi}{L}\xi}$:

$$\begin{aligned} H = \quad & H_{CFT} - \frac{\mu}{2}\left(\frac{\pi}{L}\right)^{2h_\beta - 1}\int_0^\pi \left(V_\beta(e^{i\theta}, e^{-i\theta}) + V_{-\beta}(e^{i\theta}, e^{-i\theta})\right) d\theta - \\ & \frac{\tilde{\mu}_0}{2}\left(\frac{\pi}{L}\right)^{h_\beta}\left(e^{-i\frac{\beta}{2}\phi_0}\Psi_{\frac{\beta}{2}}(1) + e^{i\frac{\beta}{2}\phi_0}\Psi_{-\frac{\beta}{2}}(1)\right) - \\ & \frac{\tilde{\mu}_L}{2}\left(\frac{\pi}{L}\right)^{h_\beta}\left(e^{-i\frac{\beta}{2}\phi_L}\Psi_{\frac{\beta}{2}}(-1) + e^{i\frac{\beta}{2}\phi_L}\Psi_{-\frac{\beta}{2}}(-1)\right) \ . \end{aligned} \tag{4.1}$$

Here $V_\beta(z, \bar{z}) = n(z, \bar{z}) : e^{i\beta\Phi(z,\bar{z})} :$ and $\Psi_{\frac{\beta}{2}}(y) =: e^{i\frac{\beta}{2}\Phi(y,y)} :$ are the bulk and boundary vertex operators and the normal ordering coefficient $n(z, \bar{z})$ depends on the boundary conditions chosen [2].

Now the computation of the matrix elements of the bulk and boundary vertex operators $V_{\pm\beta}$ and $\Psi_{\pm\beta/2}$ (with conformal dimension $h_\beta = \frac{\beta^2}{8\pi}$) between the vectors of the appropriate conform Hilbert spaces is straightforward and the integrals can also be calculated explicitly. The TCSA method consists of truncating the Hilbert space at a certain conformal energy level E_{cut} (which is nothing but the eigenvalue of the zeroth Virasoro generator) and diagonalizing the Hamiltonian numerically.

It is important to realize that one has to write separate TCSA programs for checking the Dirichlet limit and the general two parameter case. In the Dirichlet case there are no relevant operators on the boundary, thus both $\tilde{\mu}_0$ and $\tilde{\mu}_L$ must be set to zero, and we can have $\tilde{\mu}$-s different from zero only if we perturb a CFT with Neumann boundary condition. Therefore we investigate the general two parameter boundary sine-Gordon theory by describing it as an appropriately perturbed $c = 1$ CFT with Neumann boundary conditions at both ends. The Hilbert spaces of the $c = 1$ CFT-s with Dirichlet or Neumann boundary conditions at the two ends are rather different: while in the former case it basically consists of the vacuum module only, in the latter it is the direct sum of modules built on the highest weight vectors carrying the allowed values of the field momentum.

Let us investigate the general two parameter BSG first. Then the simplest choice (i.e. the one resulting in the least complex spectrum which is enough to compare to the predictions) is to switch on the boundary perturbation only at one end of the strip. The TCSA Hamiltonian for BSG with Neumann boundary condition at one end and perturbed Neumann condition, (2.2), at the other, is obtained from (4.1) by setting $\tilde{\mu}_L = 0$, $\tilde{\mu}_0 \equiv \tilde{\mu} \neq 0$. The spectrum of vertex operators in this case is $V_{\frac{n}{r}}(z, \bar{z})$ and $\Psi_{\frac{m}{r}}(y)$, where r is the compactification radius of the free boson of the $c = 1$ theory in the UV, and n, m are integers. These fields are primary under the chiral algebra $\widehat{U(1)}$ (i.e. $U(1)$ affine Lie algebra). However the compactification radius must be chosen so that both $V_{\pm\beta}$ and $\Psi_{\pm\frac{\beta}{2}}$ be in the spectrum:[7] $r = 2\sqrt{4\pi}/\beta = 2r_0$. Then $V_{\pm\beta}$ are represented as $V_{\pm\frac{2}{r}}$ while $\Psi_{\pm\frac{\beta}{2}}$

[7]The $\sqrt{4\pi}$ has its origin in the different normalizations of the SG scalar field Φ and the $c = 1$ CFT one.

as $\Psi_{\pm\frac{1}{r}}$. In other words we have to consider the boundary perturbation of the 2-folded sine-Gordon model in the sense of [23].

We choose our units in terms of the soliton mass M. The bulk coupling μ is related to M by

$$\mu = \kappa(\beta) M^{2-2h_\beta}, \qquad h_\beta = \frac{\beta^2}{8\pi}, \tag{4.2}$$

where $\kappa(\beta)$ is a dimensionless constant. In the bulk SG, from TBA considerations, the exact form of $\kappa(\beta)$ was obtained in [24], and we use the same form also here in BSG. Once we expressed μ and used the UV-IR relation (3.2,3.3) to rewrite $\tilde{\mu} \exp\left(\pm i \frac{\beta}{2}\phi_0\right)$ in terms of the IR parameters, the Hamiltonian can be made dimensionless $h = H/M$, depending only on the dimensionless volume $l = ML$, the coupling constant β and η, ϑ. We compare the predictions on the spectrum, ground state energy etc. of the general two parameter BSG model to the truncated spectrum of this Hamiltonian.

4.2 Finite size corrections from scattering theory

Here we briefly recall the method to calculate the finite size corrections for large volumes ($l \gg 1$) from the knowledge of the bulk S-matrices and boundary reflection factors. To simplify the presentation, let us consider a single scalar particle of mass m with reflection factors $R_a(\theta)$ and $R_b(\theta)$ on the boundaries at $x = 0$ and $x = L$ respectively. Then the energy as a function of the volume can be obtained by solving the Bethe-Yang equation

$$mL \sinh\theta - i \log R_a(\theta) - i \log R_b(\theta) = 2\pi I \tag{4.3}$$

for θ, where I is an integer (half integer) quantum number (corresponding to quantization of momentum in finite volume). From the solution $\theta(L)$ of (4.3) the energy with respect to the state with no particles is obtained as

$$E(L) - E_0^{ab}(L) = m \cosh\theta(L). \tag{4.4}$$

Eqn. (4.3-4.4) can also be used to give the $(E(L), L)$ 'Bethe-Yang line' in a parametric form. When $I = 0$, eqn. (4.3) may have solutions corresponding to purely imaginary θ, which may (in turn) correspond to boundary excited states obtained from the particle binding to one of the walls, cf. [2] for details.

4.3 Results

In the TCSA for the general two parameter case the number of states with conformal energies below E_{cut} depends very sensitively on the coupling constant β (compactification radius r), since the Hilbert space of the conformal free boson with Neumann boundary conditions is the direct sum of modules corresponding to the various momenta, which are integer multiple of $1/r$. Therefore it is not a surprise that in the range $r_0 \geq 3/2$, where TCSA is expected to converge, there are so many states even for moderate E_{cut}-s, that the time needed for diagonalizing H practically makes it impossible to proceed.

We overcome this difficulty partly by considering first only models on a "special line" in the parameter space described by $\phi_0 = 0$ or $\vartheta = 0$. As pointed out in [3], the models on this line admit the $\Phi \mapsto -\Phi$ 'charge conjugation' symmetry as a result of the equality $P^+ = P^- \equiv P$. As a consequence in these models there are two sectors, namely the even and the odd ones. It is straightforward to implement the projection onto the even and odd sectors in the conformal Hilbert spaces used in TCSA. This projection has two beneficial effects: on the one hand it effectively halves the number of states below E_{cut}[8], thus it drastically reduces the time needed to obtain the complete TCSA spectrum, and on the other the separate spectra of the even and odd sectors are less complex and therefore easier to study than the combined one. Furthermore, the spectrum of boundary states in the most general case depends only on η [3], and so our considerations can be restricted to $\vartheta = 0$ without any loss of generality in this respect.

4.3.1 Boundary energy

First we investigate the ground state energy of these models to check the predictions of the BSG model. Since at one end of the strip we imposed ordinary Neumann boundary condition and switched on the boundary perturbation only at the other end, the ground state energy (in units of the soliton mass) for large enough L-s should depend on the dimensionless volume $l = ML$ as

$$\frac{E}{M}(l) = -\frac{l}{4}\tan(\frac{\pi}{2\lambda}) + \frac{E(\eta_N,0)}{M} + \frac{E(\eta,0)}{M} + O\left(\mathrm{e}^{-l}\right)\,, \tag{4.5}$$

where $E(\eta,\vartheta)$ is the boundary energy, eqn. (3.4), and $\eta_N = \frac{\pi}{2}(1+\lambda)$ is the η parameter of the Neumann boundary [4]. This prediction is compared to the TCSA data on Fig.s(4.1-4.2), where the dashed lines are given by eqn. (4.5). The agreement between the predictions and the data is so good that in the interval $5 \leq l \leq 15$ the bulk energy constant and the sum of boundary energies can be measured with a reasonable accuracy.

In our earlier paper [2] when numerically investigating the ground state energy of the BSG model with Neumann boundary condition at both ends we made a conjecture that

$$E(\eta_N,0) = -E_D(0)$$

holds. (E_D is the boundary energy of the BSG model with Dirichlet boundary condition, eqn. (3.14)). Clearly the exact expressions eqn. (3.4) and eqn. (3.14) do not satisfy this, but the violation of this relation is practically undetectable (using numerical methods) in the λ range investigated in [2].

We also checked the ϑ dependence of the boundary energy $E(\eta,\vartheta)$, eqn. (3.4): the rapid growth in the number of states, caused by the absence of the two sectors, can be compensated by going to a sufficiently attractive value of λ ($\lambda = 17$) where TCSA is known to converge faster. In this case the choice $E_{\text{cut}} = 13$ resulted in 4147 conformal states and

[8]In our numerical studies of these models E_{cut} varied between 15 and 18 and this resulted in 3×10^3 - 5×10^3 conformal states per *sectors*.

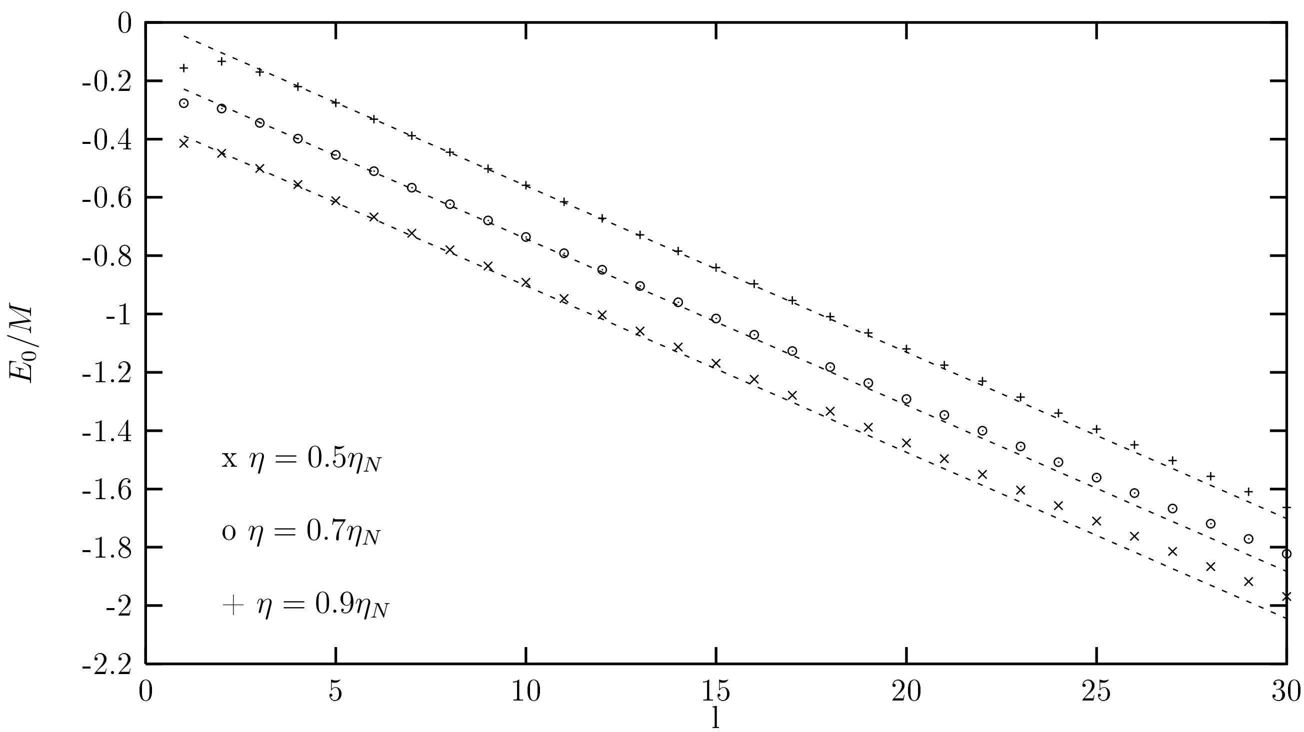


Figure 4.1: Ground state energy versus l in three BSG models with $r_0 = \sqrt{4\pi}/\beta = 2$ and $\vartheta = 0$.

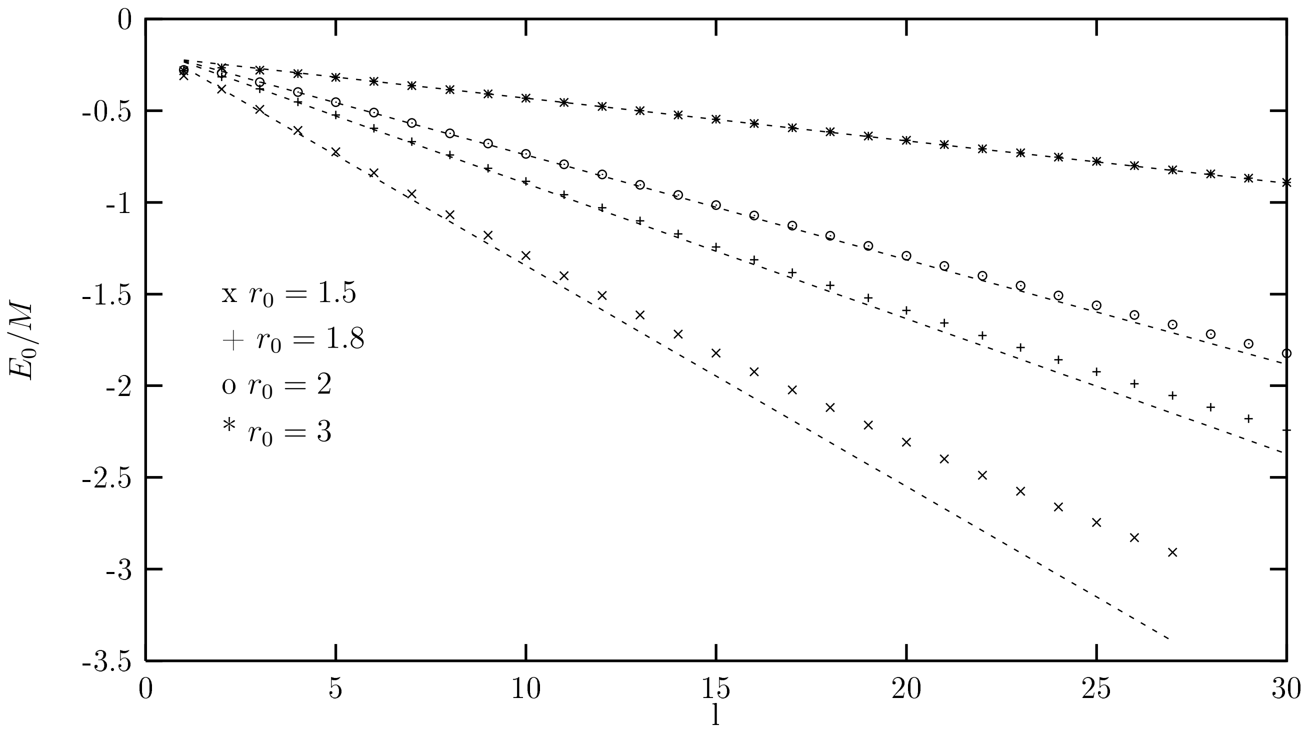


Figure 4.2: Ground state energy versus l in four BSG models with $\eta = 0.7\eta_N$ and $\vartheta = 0$.

ϑ	$E(\eta_N,0)+E(\eta,\vartheta)$ (predicted)	$E(\eta_N,0)+E(\eta,\vartheta)$ (TCSA)
5	-0.22259	-0.226959
10	-0.29012	-0.29986

Table 4.1: Boundary energies (in units of soliton mass) of two BSG models with $\lambda = 17$ and $\eta = 0.7\eta_N$ as measured from TCSA

we could measure the sum of the two boundary energies fitting the volume dependence of the ground state energy by a straight line in the range $6 \leq l \leq 17$, the results are collected in table 4.1.

To sum up, we showed that the prediction eqn. (3.4) for the boundary energy of the general two parameter boundary sine-Gordon model is in perfect agreement with the TCSA data. This agreement indirectly confirms also the UV-IR relations, eqn. (3.2-3.3), since they were built into the TCSA program. The case of Dirichlet boundary conditions is investigated in the next section.

4.3.2 Reflection factors and the spectrum of excited states

We compare the reflection factors and the spectrum of excited states to the TCSA data in the case of models with $\phi_0 = 0$ (which is realized here as $\vartheta = 0$). The bulk breathers naturally belong to one of the sectors, as the **C** parity of the n-th breather is $(-1)^n$. However, since solitons and anti solitons can reflect into themselves as well as into their charge conjugate partners, solitonic one particle states (i.e. states, whose energy and momentum are related by $E = \sqrt{P^2 + M^2}$ where M is the soliton mass) are present in both sectors.

To associate the various boundary bound states to the two sectors we have to determine the C parity of the poles ν_n and w_m in the soliton/antisoliton reflection factors. As in the even/odd sectors the reflection factors are given by $P \pm Q$ (where $P \equiv P^+ = P^-$ for $\vartheta = 0$), the possible cancellation between the zeroes of $P_0 \pm Q_0$ and the poles of $\sigma(\eta, u)$ have to be investigated. The outcome is that the poles at ν_{2k} and w_{2k} $(k = 0, 1, 2, ..)$ appear in $P + Q$ (i.e. the corresponding bound states are in the even sector), while the poles ν_{2k+1}, w_{2k+1} appear in $P - Q$ (i.e. the corresponding bound states are in the odd sector).

We analyzed the appearance of boundary bound states in the TCSA spectra of a number of BSG models. The results are illustrated on the example of a model when $\lambda = 7$ and $\eta = 0.9\ \eta_N$. For these values of the parameters the sequence of ν_n-s and w_m-s in the physical strip is

$$\nu_0 > w_1 > \nu_1 > w_2 > \nu_2 > w_3 > \nu_3\,. \tag{4.6}$$

Therefore in the even sector we expect the following low lying bound states (i.e. ones with not more then three labels[9]):

$$|0\rangle,\quad |2\rangle,\quad |0,2\rangle,\quad |1,3\rangle,\quad |0,1,1\rangle,\quad |0,1,3\rangle,\quad |1,2,3\rangle,\quad |2,3,3\rangle, \tag{4.7}$$

[9] States having more labels are heavier thus they correspond to higher TCSA lines.

while in the odd sector

$$|1\rangle, \quad |3\rangle, \quad |1,2\rangle, \quad |2,3\rangle, \quad |0,1,2\rangle, \quad |1,2,2\rangle. \tag{4.8}$$

Since at one end of the strip the unperturbed Neumann boundary condition is imposed, the corresponding bound states are also expected to appear in the TCSA spectrum. As described in [2]-[3] for $\eta = \eta_N$ the ν_n-s and the w_m-s coincide and the bound states can be labeled by an increasing sequence of positive integers $|n_1, ..., n_k\rangle_N$ with $n_k \leq \lambda/2$. Therefore in the even sector there should be TCSA lines corresponding to the

$$|2\rangle_N, \quad |1,3\rangle_N, \quad |1,2,3\rangle_N, \tag{4.9}$$

'Neumann bound states', while in the odd one to

$$|1\rangle_N, \quad |3\rangle_N, \quad |1,2\rangle_N, \quad |2,3\rangle_N. \tag{4.10}$$

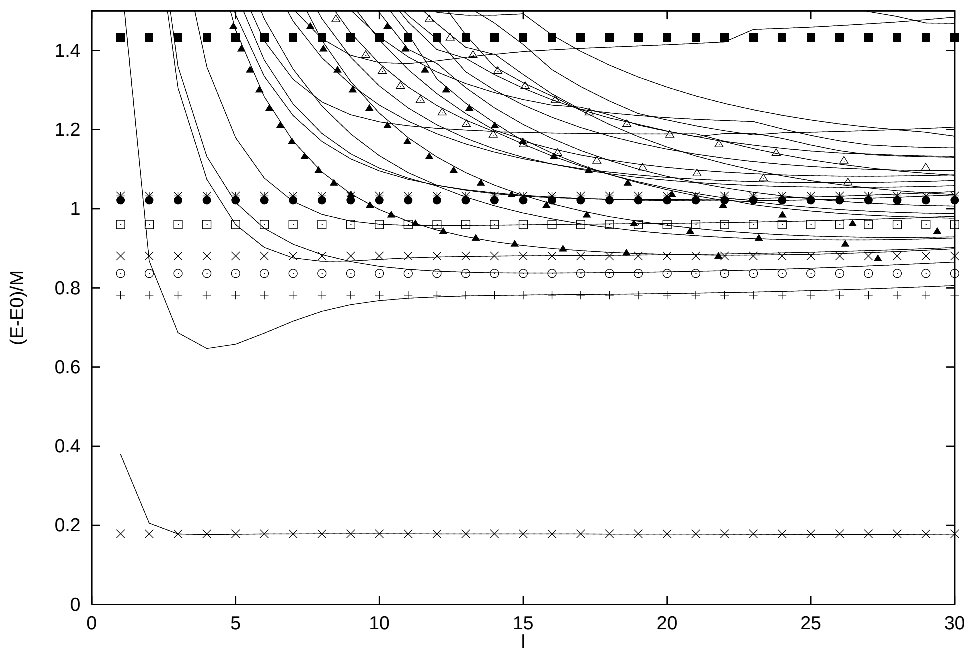
Finally there should be TCSA lines describing the situation when both boundaries are in excited states with no particle(s) moving between them, thus e.g. one expects a line in the even (odd) sector that corresponds to $|0\rangle \otimes |2\rangle_N$ ($|0\rangle \otimes |1\rangle_N$).

We compare the predictions about these bound states to the TCSA data on Fig.(4.3) where the dimensionless energy levels above the ground state are plotted against l. On both plots the continuous lines are the interpolated TCSA data and the various symbols mark the data corresponding to the various boundary bound states and Bethe-Yang lines[10].

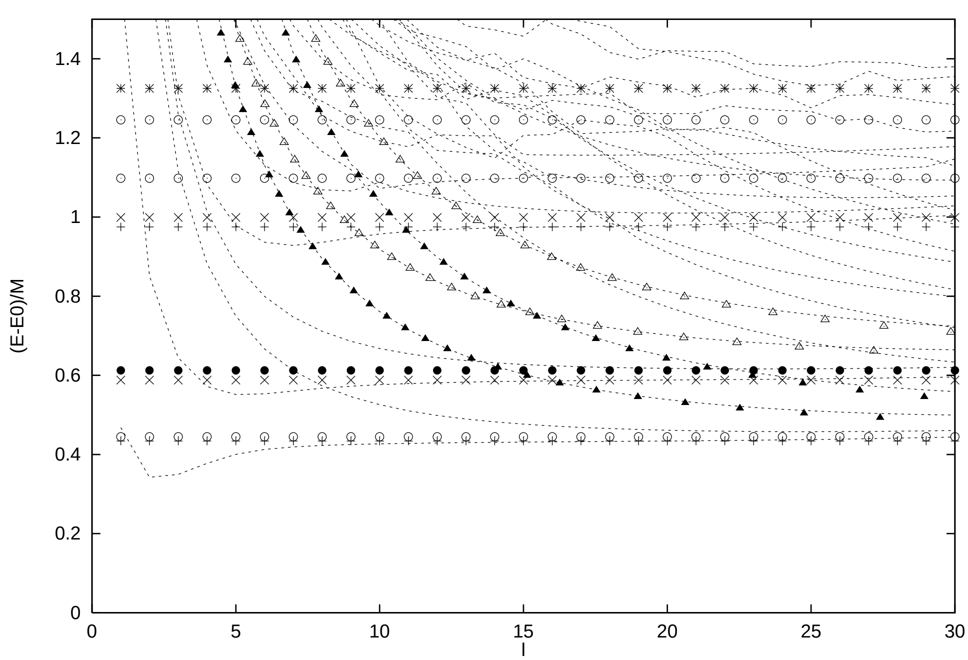
The two plots on Fig.(4.3) show in a convincing way that the low lying boundary states indeed appear as predicted by the bootstrap solution. (We show only those really low lying ones whose identification is beyond any doubt; the higher lying ones may be lost among the multitude of other TCSA lines). The reader's attention is called to two relevant points: first there is no TCSA line that would correspond to a $|1,1\rangle$ bound state. The absence of this state is explained in the bootstrap solution [3] by a Coleman-Thun diagram, that exists only if $w_1 > \nu_1$. Second, both in the even and in the odd sectors, there is evidence for the existence of the lowest bound states with three labels. These states are predicted in the bootstrap solution by the absence of any Coleman-Thun diagrams when $\nu_{n_1} > w_{n_2} > \nu_{n_3}$ holds. These two findings together give an indirect argument for the correctness of the boundary Coleman-Thun mechanism. This is most welcome, as the theoretical foundations of the boundary version of this mechanism are less solid than that of the bulk one.

On the plots on Fig.(4.3) we also show in case of the lightest breathers B^1, B^2 the excellent agreement between the TCSA data and the energy levels as predicted by the Bethe-Yang equations (4.3,4.4), using either the ground state reflection factors (2.6, 2.7) or the ones on the $|0\rangle$ excited boundary (2.9, 2.10). (In the latter case one has to take into account that now k is odd, $b_0^n(\eta, u) = 1$, and the energy above the ground state also contains the energy of $|0\rangle$).

[10] Some of the higher TCSA lines appear to have been broken, the apparent turning points are in fact level crossings with the other line not shown. This happens because our numerical routine, instead of giving the eigenvalues of the Hamiltonian in increasing order at each value of l, fixes their order at a particular small l and follows them – keeping their order – according to some criteria as l is changing to higher values.



The even sector: x denote the energy of $|0\rangle$ and $|2\rangle$, + that of $|2\rangle_N$, $\circ$ of $|0,2\rangle$,
$\bullet$, the empty/full squares stand for $|1\rangle_N \otimes |1\rangle$, $|2\rangle_N \otimes |0\rangle$ and $|1\rangle_N \otimes |3\rangle$, $*$ for $|0,1,1\rangle$,
the full/empty triangles are B^2 lines on ground state/$|0\rangle$ boundary.



The odd sector: x stand for the energy of $|1\rangle$, $|3\rangle$, + for $|1\rangle_N$, $|3\rangle_N$, $\bullet$ for $|1\rangle_N \otimes |0\rangle$,
$\circ$ stand for $|0,1\rangle$, $|0,3\rangle$ and $|1,2\rangle$, $*$ for $|0,1,2\rangle$,
the full/empty triangles are B^1 lines on ground state/$|0\rangle$ boundary.

Figure 4.3: TCSA data, boundary bound states and breather Bethe Yang lines in the BSG model with $\lambda = 7$ and $\eta = 0.9\ \eta_N$.

λ	$\frac{\beta\phi_0}{2\pi}$	E_{bulk} (exact)	E_{bulk} (TCSA)	E_{boundary} (exact)	E_{boundary} (TCSA)
31	0	−0.01267857	−0.01267(2)	−0.0259997	−0.026(17)
31	0.2	−0.01267857	−0.0126(14)	0.1773231	0.17(30)
31	0.495	−0.01267857	−0.012(25)	1.009779	0.97(75)
17	0	−0.02316291	−0.0231(22)	−0.0484739	−0.049(06)
17	0.485	−0.02316291	−0.022(67)	0.998483	0.97(78)
17	0.5	−0.02316291	−0.022(69)	1.048474	1.02(84)
7	0.25	−0.05706087	−0.056(25)	0.259213	0.24(88)
7	0.48	−0.05706087	−0.055(62)	1.054646	1.03(23)
41/8	0.25	−0.07911730	−0.077(36)	0.2464426	0.23(14)
41/8	0.36	−0.07911730	−0.076(92)	0.6381842	0.61(72)
41/8	0.44	−0.07911730	−0.076(56)	0.9513045	0.92(52)
7/2	0	−0.1203937	−0.118(22)	−0.2957454	−0.30(11)
7/2	0.3	−0.1203937	−0.114(69)	0.4241742	0.39(37)
7/2	0.42	−0.1203937	−0.11(34)	0.9532802	0.91(12)
7/2	0.5	−0.1203937	−0.11(18)	1.295745	1.23(60)

Table 5.1: Boundary energy for Dirichlet boundary conditions: comparison to the TCSA data. The values for the boundary energy are for two identical boundary conditions at both ends of the strip. Energies are given in units of the soliton mass.

5 TCSA: Dirichlet boundary conditions

For Dirichlet boundary conditions, the formula (4.1) has to be changed: the terms containing boundary perturbations must be omitted, since there are no relevant boundary operators on a Dirichlet boundary. Furthermore, one must quantize the $c = 1$ free boson with Dirichlet boundary condition, which preserves boundary conformal invariance as well as the Neumann one. The Hilbert space is also changed, because there is a single vertex operator (the identity) living on the boundary, therefore it is essentially the same as the vacuum module of the chiral algebra (which in this case is the $\widehat{U(1)}$ affine Lie algebra). In all numerical computations the truncation level was $E_{\text{cut}} = 22$, which corresponds to 4508 vectors.

5.1 Boundary energy

Here we summarize the agreement between the formula (3.14), first derived in [12] and TCSA with Dirichlet boundary conditions. At both ends of the strip, identical boundary conditions are imposed. In this case, it is easier to vary the field value ϕ_0: the interaction needs to be calculated for each given value of the sine-Gordon coupling parameter λ only once. The agreement between the predicted values of the bulk and boundary energy and the TCSA vacuum energy levels is illustrated on Figure 5.1, while numerical results are summarized in table 5.1.

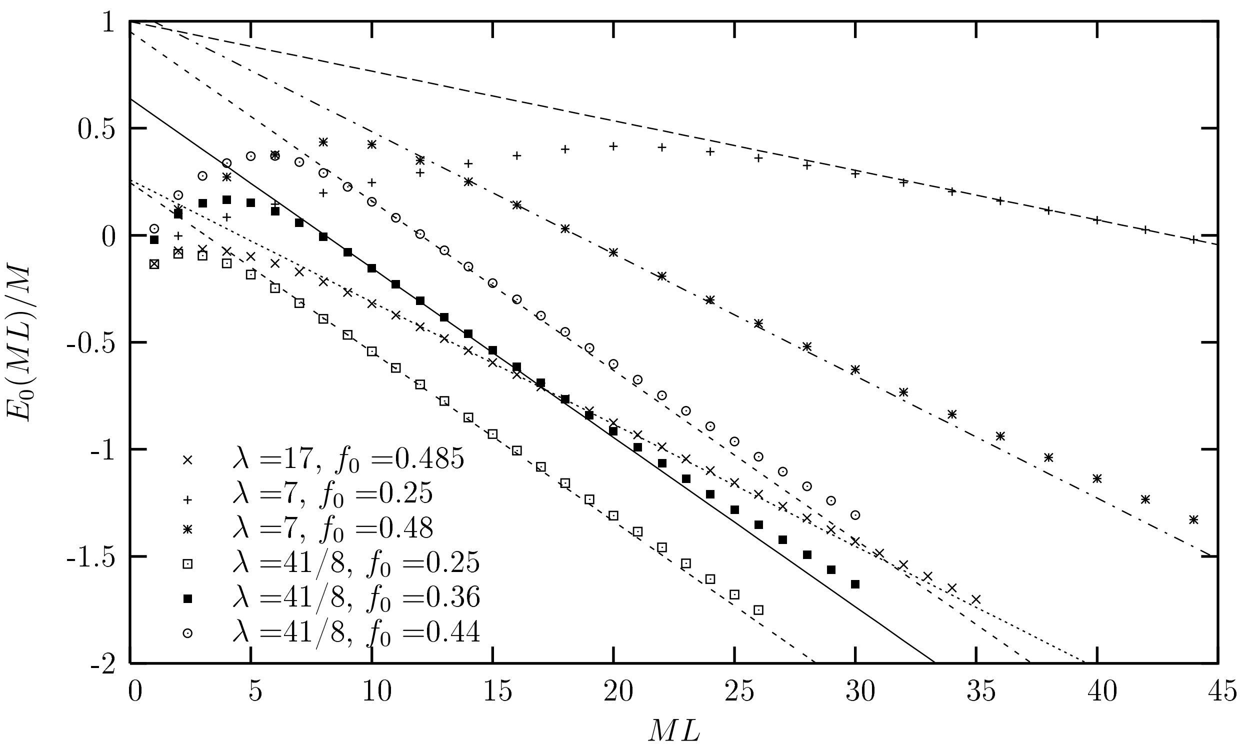


Figure 5.1: Comparing the predicted bulk and boundary energies to the TCSA data for Dirichlet boundary conditions. The dots are the TCSA data for various values of λ and $f_0 = \frac{\beta\phi_0}{2\pi}$, while the lines are their predicted asymptotic behaviour for large volume.

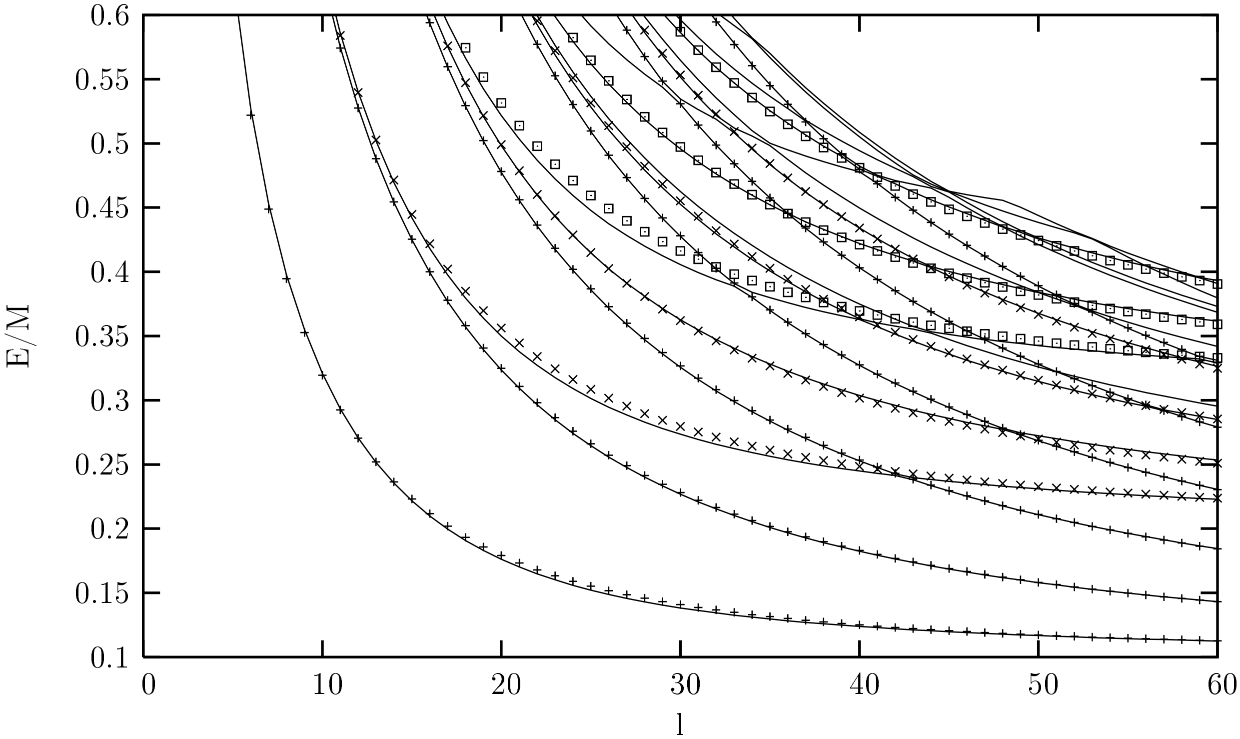


Figure 5.2: Checking the reflection factors of B_1, B_2 and B_3 for $\lambda = 31$ and $\frac{\beta\phi_0}{2\pi} = 0.2$. The dots show the one-particle energies predicted from the Bethe-Yang equations, while the continuous lines are the TCSA results. All energies are relative to the ground state and are in units of the soliton mass.

5.2 Reflection factors

Using the Bethe-Yang equations (4.3,4.4), we checked that the predictions for the energy levels from the ground state reflection factors are in excellent agreement with the TCSA data. Figure 5.2 is just an illustrative example; for all the other values of λ and ϕ_0 in Table 5.1 we had similar results. The deviations are partly due to truncation effects, but partly signal the fact that the Bethe-Yang equation only gives an approximate description of the finite size corrections.

On Figure 5.3, we illustrate how to obtain excited boundary states by analytic continuation of one-particle lines.

5.3 Spectrum of boundary excited states

We also performed an analysis of boundary excited states for Dirichlet boundary conditions. As there are two identical boundaries, the states come in doublets with symmetric/antisymmetric wave functions if the two boundaries are in a different state, and are singlets if the two boundaries are in the same state. There is also a selection rule due to

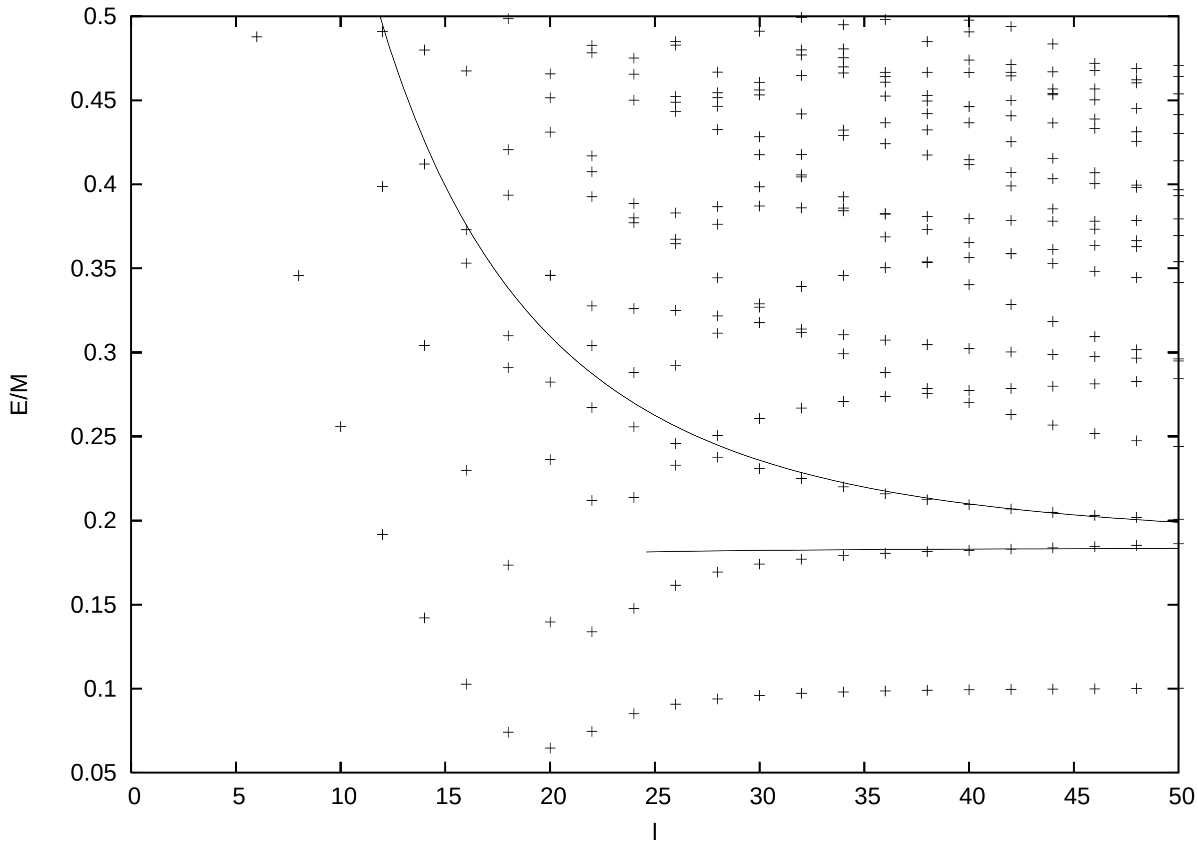


Figure 5.3: Boundary excited states at $\lambda = 17$ and $\frac{\beta\phi_0}{2\pi} = 0.485$. The upper line is the $I = 0$ one-particle B_1 line, including its continuation to imaginary rapidities, while the lower line is another portion of the imaginary rapidity continuation coming from another solution of the Bethe-Yang equations. The two lines together fit very well to the energy level doublet corresponding to the combination of a boundary in its ground state $|\rangle$ and the other in the excited state $|0,1\rangle$, at least for sufficiently large values of the volume parameter l.

λ	$\frac{\beta\phi_0}{2\pi}$	$E_1 - E_0$ (predicted)	$E_1 - E_0$ (TCSA)
31	0.495	0.032428	0.0323(62)
17	0.485	0.099750	0.0997(68)
7	0.48	0.14349	0.143(82)
7	0.45	0.35711	0.357(94)
41/8	0.44	0.44675	0.447(62)
41/8	0.36	1.0035	1.00(64)
17/8	0.4	0.89148	0.89(73)

Table 5.2: Energy of the first boundary excited state as measured from TCSA

a parity introduced by Mattsson and Dorey; namely, whenever the excited state of the left boundary has an even/odd number of indices, the right boundary also has even/odd number of indices, respectively.

For the cases when $\phi_0 = \frac{\pi}{\beta}$, the first excited state is expected to be degenerate with the ground state and this is indeed what we found within numerical precision. For the other cases, the energies of the first excited state are summarized in table 5.2. This state corresponds to both boundaries being in the same excited state, so it must be a singlet and its energy with respect to the ground state (in infinite volume) is predicted to equal

$$E_1 - E_0 = 2M \cos \frac{1}{\lambda}\left(\eta - \frac{\pi}{2}\right) = 2M \cos \pi \left(\frac{\lambda + 1}{\lambda}\frac{\beta\phi_0}{2\pi} - \frac{1}{2\lambda}\right)$$

We can measure this energy difference using the TCSA data. The results are illustrated in table 5.2.

For higher excited states one can introduce the notion of level. For a state labeled as $|n_1, \ldots, n_k\rangle$ it can be defined as the sum of the integers labels $\sum n_i$. It turns out that the energies are more or less hierarchically ordered and increase with the level. We considered excited states up to and including level 2 (the first excited state is at level 0) and found excellent agreement with the predicted spectrum apart from cases when the TCSA spectrum was too dense to come up with a meaningful identification of the TCSA data points with individual states. We also fitted them with analytic continuation of breather lines where this was possible, which also agreed very well with the TCSA data (see e.g. figure 5.3).

6 Conclusions

In this paper we described an extensive verification of some results on boundary sine-Gordon theory, comparing numerical TCSA calculations to predictions concerning the spectrum, scattering amplitudes, boundary energy and the identification of Lagrangian and bootstrap parameters of the theory. We found an excellent agreement and confirmed the general picture that was formed of boundary sine-Gordon theory in the previous literature.

The main open problems are the calculation of off-shell quantities (e.g. correlation functions) and exact finite size spectra. While correlation functions in general present a very hard problem even in theories without boundaries, in integrable theories significant progress was made using form factors. One-point functions of bulk operators have already been computed using form factor expansions in some simple boundary quantum field theories [25] and one could hope to extend these results further. In addition, the vacuum expectation values of boundary operators in sine-Gordon theory are also known exactly [14]. It would be interesting to make further progress in this direction.

Concerning finite size spectra, there is already a version of the so-called nonlinear integral equation for the vacuum (Casimir) energy with Dirichlet boundary conditions [12], but it is not yet clear how to extend it to describe excited states and more general boundary conditions, which also seems to be a fascinating problem.

Acknowledgments

We would like to thank P. Dorey, G. Watts and especially Al.B. Zamolodchikov for very useful discussions. G.T. was supported by a Magyary postdoctoral fellowship from the Hungarian Ministry of Education. This research was supported in part by the Hungarian Ministry of Education under FKFP 0178/1999, 0043/2001 and by the Hungarian National Science Fund (OTKA) T029802/99.

References

[1] E.K. Sklyanin, *Funct. Anal. Appl.* **21** (1987) 164.
E.K. Sklyanin, *J. Phys.* **A21** (1988) 2375-2389.

[2] Z. Bajnok, L. Palla and G. Takács, *Nucl. Phys.* **B614** (2001) 405-448, hep-th/0106069.

[3] Z. Bajnok, L. Palla, G. Takács and G.Z. Tóth: *The spectrum of boundary states in sine-Gordon model with integrable boundary conditions,* preprint ITP-BUDAPEST-571, hep-th/0106070.

[4] S. Ghoshal and A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A9** (1994) 3841-3886 (Erratum-ibid. **A9** (1994) 4353), hep-th/9306002.

[5] S. Ghoshal, *Int. J. Mod. Phys.* **A9** (1994) 4801-4810, hep-th/9310188.

[6] A. Fring and R. Köberle, *Nucl. Phys.* **B421** (1994) 159, hep-th/9304141.

[7] S. Skorik and H. Saleur, *J. Phys.* **A28** (1995) 6605-6622, hep-th/9502011.

[8] P. Dorey, R. Tateo and Gerard Watts, *Phys. Lett.* **B448** (1999) 249-256, hep-th/9810098.

[9] P. Mattsson and P. Dorey, *J. Phys.* **A33** (2000) 9065-9094, hep-th/0008071.

[10] V.P. Yurov and A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A5** (1990) 3221-3246.

[11] P. Dorey, A. Pocklington, R. Tateo and G. Watts, *Nucl. Phys.* **B525** (1998) 641-663, hep-th/9712197.

[12] A. LeClair, G. Mussardo, H. Saleur and S. Skorik, *Nucl. Phys.* **B453** (1995) 581-618, hep-th/9503227.

[13] Al.B. Zamolodchikov, unpublished.

[14] V. Fateev, A.B. Zamolodchikov and Al.B. Zamolodchikov: *Boundary Liouville field theory. 1. Boundary state and boundary two point function,* preprint RUNHETC-2000-01, hep-th/0001012.

[15] E. Corrigan and A. Taormina *J. Phys.* **A33** (2000) 8739. (`hep-th/0008237`), E. Corrigan `hep-th/0010094`.

[16] A.B. Zamolodchikov and Al.B. Zamolodchikov *Ann. Phys.* **120** (1979) 253.

[17] A.B. Zamolodchikov, *Nucl. Phys.* **B342** (1990) 695-720.

[18] Al.B. Zamolodchikov: *On the thermodynamic Bethe Ansatz equation in sinh-Gordon model*, preprint LPM-00-15, hep-th/0005181.

[19] C. Destri and H. de Vega, *Nucl. Phys.* **B358** (1991) 251.

[20] S. Lukyanov, A.B. Zamolodchikov, *Nucl. Phys.* **B493** (1997) 571, hep-th/9611238.

[21] K. Graham , I. Runkel and G. Watts: *Minimal model boundary flows and c=1 CFT*, preprint KCL-MTH-01-01, PAR-LPTHE-01-01, hep-th/0101187.
K. Graham, I. Runkel, G. Watts: *Renormalization group flows of boundary theories,* preprint KCL-MTH-00-54, hep-th/0010082. Talk presented at 4th Annual European TMR Conference on Integrability Nonperturbative Effects and Symmetry in Quantum Field Theory, Paris, France, 7-13 Sep 2000.

[22] G. Feverati, F. Ravanini and G. Takács, *Phys. Lett.* **B430** (1998) 264-273, hep-th/9803104.
G. Feverati, F. Ravanini and G. Takács, *Nucl. Phys.* **B540** (1999) 543-586, hep-th/9805117.

[23] Z. Bajnok, L. Palla, G. Takács and F. Wágner, *Nucl. Phys.* **B587** (2000) 585-618, hep-th/0004181.

[24] Al. B. Zamolodchikov, *Int. J. Mod. Phys.* **A10** (1995) 1125-1150.

[25] R. Konik, A. LeClair and G. Mussardo, *Int. J. Mod. Phys.* **A11** (1998) 2765-2782, hep-th/9508099.
F. Lesage and H. Saleur, *J. Phys.* **A30** (1997) L457-463, cond-mat/9608112.
P. Dorey, M. Pillin, R. Tateo and G. Watts, *Nucl. Phys.* **B594** (2001) 625-659, hep-th/0007077.