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Form factor perturbation theory from finite volume

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Abstract

Using a regularization by putting the system in finite volume, we develop a novel approach to form factor perturbation theory for non-integrable models described as perturbations of integrable ones. This permits to go beyond first order in form factor perturbation theory and in principle works to any order. The procedure is carried out in detail for double sine-Gordon theory, where the vacuum energy density and breather mass correction is evaluated at second order. The results agree with those obtained from the truncated conformal space approach. The regularization procedure can also be used to compute other spectral sums involving disconnected pieces of form factors such as those that occur e.g. in finite temperature correlators.

1 Introduction

Form factor perturbation theory (FFPT) was developed in [1] in order to evaluate quantities in a non-integrable model obtained as a perturbation of an integrable one. Writing the Hamiltonian in the form

$$H_{\text{nonintegrable}} = H_{\text{integrable}} + \lambda \int dx \Psi(t,x)$$

where Ψ denotes the local perturbing field that breaks integrability, the first order corrections to the vacuum (bulk) energy density and particle masses are given as

$$\begin{aligned} \delta\mathcal{E}_{vac} &= \lambda \langle 0| \Psi |0\rangle_{\lambda=0} \\ \delta M^2_{ab} &= 2\lambda F^{\Psi}_{a\bar{b}}(i\pi, 0) \end{aligned}$$

where

$$F^{\Psi}_{i_1\dots i_n}(\vartheta_1,\dots,\vartheta_n) = \langle 0|\Psi(0,0)|A_{i_1}(\vartheta_1)\dots A_{i_n}(\vartheta_n)\rangle_{\lambda=0} \tag{1.1}$$

are the form factors of the perturbing operator calculated at the integrable point $\lambda = 0$ and $\bar{b}$ denotes the charge conjugate of particle species b. It is possible to evaluate first order corrections to the two-particle S matrix and also the widths of decays induced by the perturbation [2].

The evaluation of higher order corrections has not been developed; simple considerations along the lines of [1] lead to divergent expressions. However there is no place for mass renormalization by counter terms analogous to standard Feynman perturbation theory because the operator (1.1) defined by the form factors is already well-defined and physical. This is also

confirmed by the fact that when the non-integrable model is formulated using the truncated conformal space approach pioneered by Yurov and Zamolodchikov [3] the mass gaps turn out to be finite and well-defined (the vacuum energy can still have divergent contributions depending on the ultraviolet weight of Ψ, but the differences between energy levels are all finite).

This leads to the central idea of the paper: since the TCSA expression for the relative energy levels in a finite volume L is finite, and the ingredients necessary to evaluate finite volume perturbation theory can be determined from infinite volume form factors using the approach developed in [4, 5], one can write down finite and well-defined analytic expressions for the perturbation of finite volume energy levels (which are accurate up to so-called residual finite size effects that decay exponentially with the volume i.e. are non-analytic in $1/L$, i.e. valid to all orders in $1/L$). Then the quantity of interest (bulk energy density or mass correction) can be expressed directly in finite volume and the infinite volume limit is taken only at the end of the calculation. This is the same philosophy that was used to obtain the expression of finite temperature one-point functions in [5].

Eventually, since first order FFPT was used in [5] to derive the expressions of one-particle and two-particle diagonal matrix elements in finite volume, nothing new is to be gained from the application of finite volume techniques at first order. However, we get new results at second order: a consistent, generally valid way of calculating corrections to vacuum energy density and particle masses. It can also be extended to other quantities such as the S matrix, and to higher order FFPT corrections as well.

It is best to consider a concrete model to develop and test this approach. The model of choice is the double sine-Gordon model defined by the Hamiltonian

$$H_{\mathrm{DSG}} = \int dx \left(\frac{1}{2} \left(\partial_t \varphi\right)^2 + \frac{1}{2} \left(\partial_x \varphi\right)^2 - \mu : \cos \beta\varphi : + \lambda : \sin \frac{\beta}{2} \varphi : \right) \tag{1.2}$$

understood as a perturbation of the massless free boson (which also defines the normal ordering). It has attracted interest recently chiefly because it is a prototype of non-integrable field theory which can be understood by application of techniques developed in the context of integrable field theories [6, 9] and it also has several interesting applications [6, 7, 8] such as to the study of massive Schwinger model (two-dimensional quantum electrodynamics) and a generalized Ashkin-Teller model (a quantum spin system) which are discussed in [6].

The double sine-Gordon model (1.2) can be considered as a non-integrable perturbation of the integrable sine-Gordon field theory obtained by setting $\lambda = 0$ [6]. Form factor perturbation theory was applied to the double sine-Gordon model in [9]; for the particular version in eqn. (1.2) it was shown that the corrections to the breather masses vanish to first order in λ. However later semiclassical considerations [10] seemed to contradict these naive expectations, yielding mass corrections which were of first order in the coupling λ. In [12] it was shown that a precise numerical determination of the spectrum contradicts this conclusion and upholds the naive picture obtained from form factor perturbation theory: i.e. there are only second order corrections, and in fact all odd orders vanish since the entire spectrum turns out to be even under $\lambda \to -\lambda$. However, at that time the mass correction could not be calculated theoretically due to the lack of FFPT beyond first order. Therefore this model is an interesting testing ground for the present work, and it is also made ideal by the absence of first order corrections which makes comparison to numerical results easier. The numerical results which are compared with the theoretical predictions are obtained from TCSA which was first developed for the

sine-Gordon model in [11] and generalized to the double sine-Gordon model in [9]; to achieve a better precision we use an improved version developed for the work [12] and described therein.

2 Bulk energy correction

The general formula for second order corrections to energy levels can be written as

$$\delta E_i = \sum_{k \neq i} \frac{|\langle i\,|H_1|\,k\rangle|^2}{E_i^{(0)} - E_k^{(0)}}$$

where $E_i^{(0)}$ are the unperturbed energy eigenvalue corresponding to the eigenstate $|i\rangle$ and H_1 is the perturbation to the Hamiltonian. In our case

$$H_1 = \lambda \int_0^L dx\ :\sin\frac{\beta}{2}\varphi:$$

where L is the spatial volume of the system. With periodic boundary conditions the matrix elements of H_1 vanish unless the momenta of states $|i\rangle$ and $|k\rangle$ coincide; therefore when i is taken to be the vacuum, only states with zero total momentum contribute. In addition, the topological charge of $|k\rangle$ must also be zero, otherwise the amplitude vanishes. Furthermore, H_1 is odd under $C : \varphi \to -\varphi$ and so the C-parity of the contributing state must be odd as well (the nth breather B_n has C-parity $(-1)^n$). Only contributions of breather states are necessary to evaluate because our numerical data will come from a part of the attractive regime of sine-Gordon theory where solitons are heavy and contribute little to the summation over k, well below the available numerical precision.

When the momentum of the state $|k\rangle$ is zero,

$$\langle 0\,|H_1|\,k\rangle = \lambda \left\langle 0 \right| : \sin\frac{\beta}{2}\varphi(t,x) : \left| k \right\rangle_L$$

is independent of x, therefore

$$\delta E_0 = -\lambda^2 L^2 \sum_{k\neq 0} \frac{\left|\left\langle 0 \right| : \exp i\tfrac{\beta}{2}\varphi(0,0) : \left| k \right\rangle_L\right|^2}{E_k^{(0)} - E_0^{(0)}} \tag{2.1}$$

where the subscript L designates finite volume matrix elements (the two exponential terms in the sine give equal contributions due to parity). For $\xi < 1/3$[1], the lowest lying states contributing to the sum are

$$\begin{aligned} & |B_1(0)\rangle\,, \quad |B_3(0)\rangle\,, \\ & |B_1(\theta_1)B_2(\theta_2)\rangle \quad \text{with} \quad m_1\sinh\theta_1 + m_2\sinh\theta_2 = 0 \\ \text{and} \quad & |B_1(\theta_1)B_1(\theta_2)B_1(\theta_3)\rangle \quad \text{with} \quad m_1\sinh\theta_1 + m_1\sinh\theta_2 + m_1\sinh\theta_3 = 0 \end{aligned}$$

where

$$m_k = 2M\sin\frac{\pi k\xi}{2}$$

[1]For the notations M and ξ cf. Appendix A.

are the breather masses, and the rapidities of the breathers are indicated in parentheses. In details

$$\begin{aligned}
\delta E_0(L) &= -\lambda^2 L^2 \frac{\left|\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_1(0)\right\rangle_L\right|^2}{m_1} - \lambda^2 L^2 \frac{\left|\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_3(0)\right\rangle_L\right|^2}{m_3} \\
&- \lambda^2 L^2 \sum_{\theta_1} \frac{\left|\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_1(\theta_1)B_2(\theta_2)\right\rangle_L\right|^2}{(m_1\cosh\theta_1 + m_2\cosh\theta_2)} \\
&- \lambda^2 L^2 \sum_{\theta_1,\theta_2} \frac{\left|\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_1(\theta_1)B_1(\theta_2)B_1(\theta_3)\right\rangle_L\right|^2}{(m_1\cosh\theta_1 + m_1\cosh\theta_2 + m_1\cosh\theta_3)} + O\left(\mathrm{e}^{-\mu L}\right) + \dots
\end{aligned}$$

where the presence of correction terms decaying exponentially with the volume is indicated, and the ellipsis denotes the terms corresponding to further multi-particle states. The summations run over all distinct solutions of the Bethe-Yang equations

$$Q_{k;a_1\dots a_n}(L|\theta_1,\dots,\theta_n) = m_{a_k} L\sinh\theta_k + \sum_{l\neq k} -i\log S_{a_k a_l}(\theta_k-\theta_l) = 2\pi I_k\ , \qquad I_k\in\mathbb{Z}$$

that have total momentum zero. Using the results of [4] one can write

$$\begin{aligned}
\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_1(0)\right\rangle_L &= \frac{F_1^{1/2}(0)}{\sqrt{\rho_1(L|0)}} + O\left(\mathrm{e}^{-\mu L}\right) \\
\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_3(0)\right\rangle_L &= \frac{F_3^{1/2}(0)}{\sqrt{\rho_3(L|0)}} + O\left(\mathrm{e}^{-\mu L}\right) \\
\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_1(\theta_1)B_2(\theta_2)\right\rangle_L &= \frac{F_{12}^{1/2}(\theta_1,\theta_2)}{\sqrt{\rho_{12}(L|\theta_1,\theta_2)}} + O\left(\mathrm{e}^{-\mu L}\right) \\
\left\langle 0\right|:\exp i\frac{\beta}{2}\varphi(0,0):\left|B_1(\theta_1)B_1(\theta_2)B_1(\theta_3)\right\rangle_L &= \frac{F_{111}^{1/2}(\theta_1,\theta_2)}{\sqrt{\rho_{111}(L|\theta_1,\theta_2,\theta_3)}} + O\left(\mathrm{e}^{-\mu L}\right)
\end{aligned}$$

The density factors ρ are obtained as

$$\rho_{i_1\dots i_n}(L|\theta_1,\dots,\theta_n) = \det\left\{\frac{\partial Q_{k;a_1\dots a_n}}{\partial\theta_l}\right\}_{k,l=1,\dots,n}$$

In particular, for the one-particle densities we obtain

$$\rho_k(L|\theta) = m_k L\cosh\theta \tag{2.2}$$

The next step is to take the limit $L\to\infty$: the exponential corrections can be dropped and the summations substituted with integrals

$$\begin{aligned}
\sum_{\theta_1} &\to \int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi}\tilde\rho_{12}(L|\theta_1) \\
\sum_{\theta_1,\theta_2} &\to \int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\tilde\rho_{111}(L|\theta_1,\theta_2)
\end{aligned}$$

where $\tilde{\rho}$ denotes the density of zero-momentum states. For the first integral, it can be obtained by inspecting the Bethe-Yang equations

$$\begin{aligned} m_1 L \sinh\theta_1 - i \log S_{12}(\theta_1 - \theta_2) &= 2\pi I_1 \\ m_2 L \sinh\theta_2 - i \log S_{12}(\theta_2 - \theta_1) &= 2\pi I_2 \end{aligned}$$

with S_{12} denoting the $B_1 B_2$ S-matrix (cf. eqn. (A.4)). The second equation is actually superfluous due to the zero-momentum constraint $m_1 \sinh\theta_1 + m_2 \sinh\theta_2 = 0$. Taking the derivative of the first equation gives

$$\begin{aligned} \tilde{\rho}_{12}(L|\theta_1) &= m_1 L \cosh\theta_1 + \left(1 + \frac{m_2 \cosh\theta_2}{m_1 \cosh\theta_1}\right) \Phi_{12}(\theta_1 - \theta_2) \\ \Phi_{12}(\theta) &= -i \frac{\partial}{\partial\theta} \log S_{12}(\theta) \end{aligned}$$

using the zero-momentum constraint during the differentiation. On the other hand, the density factor ρ_{12} is

$$\rho_{12}(L|\theta_1,\theta_2) = m_1 L \cosh\theta_1 m_2 L \cosh\theta_2 + (m_1 L \cosh\theta_1 + m_2 L \cosh\theta_2)\Phi_{12}(\theta_1 - \theta_2) \qquad (2.3)$$

and therefore

$$\frac{\tilde{\rho}_{12}(L|\theta_1)}{\rho_{12}(L|\theta_1,\theta_2)} = \frac{1}{m_2 L \cosh\theta_2}$$

A similar calculation yields

$$\frac{\tilde{\rho}_{111}(L|\theta_1,\theta_2)}{\rho_{111}(L|\theta_1,\theta_2,\theta_3)} = \frac{1}{m_1 L \cosh\theta_3}$$

The end result is that the correction is proportional to the volume L, and therefore it represents a correction to the bulk energy density

$$\begin{aligned} \delta\mathcal{E} = \frac{\delta E_0(L)}{L} = -\lambda^2 \Bigg\{ & \frac{\left|F_1^{1/2}(0)\right|^2}{m_1^2} + \frac{\left|F_3^{1/2}(0)\right|^2}{m_3^2} \\ & + \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \left(\frac{\left|F_{12}^{1/2}(\theta_1,\theta_2)\right|^2}{(m_1\cosh\theta_1 + m_2\cosh\theta_2) m_2 \cosh\theta_2} \right)\Bigg|_{\theta_2 = -\operatorname{arsinh}(m_1 \sinh\theta_1/m_2)} \\ & + \frac{1}{3!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} \left(\frac{\left|F_{111}^{1/2}(\theta_1,\theta_2,\theta_3)\right|^2}{(m_1\cosh\theta_1 + m_1\cosh\theta_2 + m_1\cosh\theta_3) m_1 \cosh\theta_3} \right)\Bigg|_{\theta_3 = -\operatorname{arsinh}(\sinh\theta_1 + \sinh\theta_2)} \\ & + \ldots \Bigg\} + O\left(\lambda^4\right) \end{aligned} \qquad (2.4)$$

where the form factor functions are defined in Appendix A (the combinatorial factor in the last integral takes into account that states that only differ in the ordering of the rapidities are eventually identical).

$R=\sqrt{4\pi}/\beta$	1.5	1.9	2.2	2.5
b_2 (TCSA)	0.81	1.53	2.21	3.04
b_2 (FFPT)	0.82	1.53	2.20	3.01
$b_{s\bar{s}}$ (FFPT)	0.0025	4×10^{-5}	1×10^{-6}	1×10^{-8}

Table 2.1: Comparing vacuum energy density from FFPT to TCSA numerics. The parameter R is related to the compactification radius of the ultraviolet limiting $c=1$ free boson conformal field theory.

The bulk energy density corrections can now be evaluated explicitly in units of the soliton mass M. Introducing also the dimensionless coupling [12]

$$t=\lambda M^{-2+\beta^2/16\pi^2}$$

we can write

$$\frac{\delta\mathcal{E}}{M^2}=-b_2t^2+O(t^4)$$

The results of second order FFPT are summarized and compared to numerical values extracted from TCSA in Table 2.1. The accuracy of the data in the table corresponds to the estimated precision of the TCSA results; at this level, the contribution of the integral terms is negligible. The deviation between FFPT and TCSA comes from two sources. For lower values of R, TCSA was observed to converge slower, thereby limiting the accuracy of the numerical determination. Albeit there exists a renormalization group method for improving convergence [14, 15], implementing it comes with a cost (in terms of programming and running). It also does not seem to gain much compared to the simple-minded approach of evaluating bulk energy by the simpler method which was applied with success in many previous examples [9, 12]. Our method (also used in [9, 12]) is to find the scaling regime where the ground state level is most linear (the region where its second derivative in L is smallest) and evaluate the slope of the line there. Similarly, masses can be evaluated in the region where the gap between the appropriate excited state and the ground state becomes closest to constant (found by searching for the minimum of the first derivative) and taking the value of the gap there as the approximate mass.

For higher values of R, the spectrum of the theory becomes more and more dense as the sine-Gordon model is increasingly attractive (at the point $R=1.5$ there are three breather states in the spectrum, while at $R=2.5$ there are already eleven of them), therefore there are more multi-breather states to be included, and in addition there are also states containing solitons.

To demonstrate that solitons contribute very little, let us also compute the value of the first solitonic correction, which comes from the soliton-antisoliton two-particle state. It can be written in a form very similar to the B_1B_2 term:

$$-\lambda^2\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\frac{|F_{s\bar{s}}^{1/2}(\theta,-\theta)-F_{s\bar{s}}^{-1/2}(\theta,-\theta)|^2/4}{2M\cosh\theta\,M\cosh\theta}$$

where $F_{s\bar{s}}^{\pm1/2}$ is given in (A.5). The contribution of this integral to b_2 is denoted $b_{s\bar{s}}$ and is shown separately in table 2.1. The reason for the smallness of this integral is that it has a very limited effective support. The integrand is eventually symmetric in θ, and form factors

generally vanish on threshold ($\theta = 0$). On the other hand, the form factor combination in the numerator exhibits an exponential decay for large θ

$$|F_{s\bar{s}}^{1/2}(\theta, -\theta) - F_{s\bar{s}}^{-1/2}(\theta, -\theta)|^2 \sim \exp\left(-\frac{1-\xi}{\xi}\theta\right)$$

where $\xi < 1$ in the attractive regime. Together with the denominator this makes the integrand decay very fast with increasing θ. Similar behaviour happens in terms with larger number of particles, ensuring the convergence of all multi-particle integrals involved. Similar arguments hold also for the B_1B_2 term, but that is made larger by the appearance of smaller masses (m_1, m_2 instead of M) in the denominator.

The issue of whether the summation over the states with increasing number of particles implied by (2.1) converges is more subtle since it is also necessary to take into account the various numerical prefactors (form factor normalization etc.) entering the individual contributions and is not considered here in detail. Just as in the above calculation, the contributions can be naturally ordered by the sum of the masses of the constituent particles in the intermediate state, and explicit numerical evaluations support the observation that they decrease very rapidly when going to more and more massive states.

3 Mass correction

Let us now turn to evaluating the mass correction for the first breather B_1. In finite volume, the B_1 one-particle state is just the next energy level $|1\rangle$ above the vacuum $|0\rangle$in the zero-momentum, zero topological charge sector. Therefore

$$\delta E_1 = -\lambda^2 L^2 \sum_{k\neq 1} \frac{\left|\left\langle 1 \right| : \exp i\tfrac{\beta}{2}\varphi(0,0) : \left| k \right\rangle_L\right|^2}{E_k^{(0)} - E_1^{(0)}} \tag{3.1}$$

and the correction to the mass gap is obtained by taking the difference to the vacuum level:

$$\delta m_1 = \lim_{L\to\infty} \delta E_1(L) - \delta E_0(L)$$

3.1 Bulk contributions: a puzzle and its solution

In particular, terms linear in the volume are expected to cancel, leaving us with a finite correction to the mass gap. However, right with the first term a serious problem appears. The first contribution to (3.1) is given by the vacuum state and can be written as

$$\lambda^2 L^2 \frac{\left|\left\langle B_1(0) \right| : \exp i\tfrac{\beta}{2}\varphi(0,0) : \left| 0 \right\rangle_L\right|^2}{m_1\rho_1(L|0)} + O\left(\mathrm{e}^{-\mu L}\right) \underset{L\to\infty}{\longrightarrow} \lambda^2 L \frac{\left|F_1^{1/2}(0)\right|^2}{m_1^2} \tag{3.2}$$

which has the wrong sign to cancel the corresponding contribution to the vacuum energy density i.e. the first term in eqn. (2.4).

The puzzle can be solved by observing that the contribution from B_1B_1 two-particle states (which are naively of order L^0 for large L) diverges as $L \to \infty$due to a disconnected piece. Such divergent pieces are finite for $L < \infty$, and have a dependence of L to the power of the number of particles involved in the disconnected part. In this case it leads to a piece

proportional to L, and we proceed to show that it gives the correct contributions to account for the mismatch noted above. The corresponding term can be written as

$$-\lambda^2 L^2 \sum_\theta \frac{\left|\left\langle B_1(0)\right|: \exp i\tfrac{\beta}{2}\varphi(0,0) :\left| B_1(\theta)B_1(-\theta)\right\rangle_L\right|^2}{2m_1\cosh\theta - m_1} + O\left(\mathrm{e}^{-\mu L}\right)$$

and using the results of [4] this can be written as

$$-\lambda^2 L^2 \sum_\theta \frac{\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2}{\rho_1(L|0)\rho_{11}(L|\theta,-\theta)(2m_1\cosh\theta - m_1)} + O\left(\mathrm{e}^{-\mu L}\right) \tag{3.3}$$

with ρ_1 as in (2.2) and

$$\begin{aligned}\rho_{11}(L|\theta_1,\theta_2) &= m_1^2 L^2\cosh\theta_1\cosh\theta_2 + m_1 L(\cosh\theta_1+\cosh\theta_2)\Phi_{11}(\theta_1-\theta_2)\\ &\Phi_{11}(\theta) = -i\frac{\partial}{\partial\theta}\log S_{11}(\theta)\end{aligned}$$

where

$$S_{11}(\theta) = \frac{\sinh\theta + i\sin\pi\xi}{\sinh\theta - i\sin\pi\xi} \tag{3.4}$$

is the B_1B_1 scattering amplitude. A simple calculation similar to that in the previous subsection gives the density of zero total momentum states as

$$\tilde\rho_{11}(L|\theta) = m_1 L\cosh\theta + 2\Phi_{11}(2\theta) = \frac{\rho_{11}(L|\theta,-\theta)}{m_1 L\cosh\theta}$$

Naive application of the infinite volume limit to (3.3) gives

$$-\lambda^2\int_0^\infty \frac{d\theta}{2\pi}\frac{\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2}{m_1^3\cosh\theta(2\cosh\theta-1)}$$

However, the integral is divergent due to kinematical poles of the form factor at $\theta=0$. The form factors in an integrable quantum field theory satisfy a number of axioms (for the details we refer to Smirnov's review [13]), among which there is the kinematical residue axiom of the form[2]

$$-i\operatorname*{Res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(\theta+i\pi,\theta',\theta_1,\dots,\theta_n)_{i\,j\,i_1\dots i_n} = \left(1-\delta_{i\,j}\prod_{k=1}^n S_{i\,i_k}(\theta-\theta_k)\right)F^{\mathcal{O}}_n(\theta_1,\dots,\theta_n)_{i_1\dots i_n} \tag{3.5}$$

which results in the following singularity

$$\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2 \sim \frac{16\left|F_1^{1/2}(0)\right|^2}{\theta^2} + O(\theta^0)$$

[2]This form of the axiom is valid for self-conjugate particles; for charged particles it involves the charge conjugation matrix.

using the fact that $S_{11}(0) = -1$ which expresses the Pauli exclusion principle satisfied by the B_1 particles. Therefore one must return to a more careful evaluation of the sum (3.3). The quantization of θ in a finite volume is given by

$$m_1 L \sinh\theta + \delta_{11}(2\theta) = 2\pi J \quad , \quad J \in \mathbb{N} + \frac{1}{2} \tag{3.6}$$

where the quantum number is shifted by $-1/2$ due to the following identification of the two-particle phase-shift:

$$S_{11}(\theta) = -\mathrm{e}^{i\delta_{11}(\theta)}$$

As a result we obtain that for fixed J

$$\theta = \frac{2\pi J}{m_1 L}$$

and the leading term in the sum (3.3) can be written as

$$-\lambda^2 L^2 \sum_J \frac{16\left|F_1^{1/2}(0)\right|^2}{m_1 L (m_1 L)^2 (2m_1 - m_1)} \left(\frac{m_1 L}{2\pi J}\right)^2$$

Using the identity

$$\sum_{J\in\mathbb{N}+1/2} \frac{1}{J^2} = \frac{\pi^2}{2}$$

we obtain

$$-\lambda^2 L \frac{2\left|F_1^{1/2}(0)\right|^2}{m_1^2}$$

which exactly compensates for the mismatch caused by the "wrong sign" in eqn. (3.2). The correction to the mass term comes from the subleading L^0 term in the sum (3.3), which requires a very careful evaluation that is carried out in Appendix B.

Moving to the next correction (B_3 term) to the bulk energy density in (2.4) and keeping in mind the above example, it is easy to see that its counterpart arises from the $B_1 B_3$ contribution to (3.1). Here we encounter a different mechanism for the generation of the bulk term. The appropriate sum to evaluate is

$$-\lambda^2 L^2 \sum_{\theta_1} \frac{\left|\left\langle B_1(0)\right| : \exp i\tfrac{\beta}{2}\varphi(0,0) : \left|B_1(\theta_1)B_3(\theta_2)\right\rangle_L\right|^2}{(m_1\cosh\theta_1 + m_3\cosh\theta_2) - m_1} \tag{3.7}$$
$$\text{with } m_1\sinh\theta_1 + m_3\sinh\theta_2 = 0$$

It turns out that due to $S_{13}(0) = +1$ the form factor

$$F_{113}^{1/2}(i\pi, \theta_1, \theta_2)$$

is regular as $\theta_1 \to 0$ (and therefore also $\theta_2 \sim m_1\theta_1/m_3 \to 0$) and so the above discrete sum converts directly to an integral of the form

$$-\lambda^2 \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \left(\frac{\left|F_{113}^{1/2}(i\pi,\theta_1,\theta_2)\right|^2}{(m_1\cosh\theta_1 + m_3\cosh\theta_2 - m_1)m_3\cosh\theta_2} \right)\Bigg|_{\theta_2 = -\mathrm{arsinh}(m_1\sinh\theta_1/m_3)} \tag{3.8}$$

However, the Bethe-Yang equations

$$\begin{aligned} m_1 L \sinh\theta_1 - i \log S_{13}(\theta_1 - \theta_2) &= 2\pi I_1 \\ m_3 L \sinh\theta_2 - i \log S_{13}(\theta_2 - \theta_1) &= 2\pi I_2 \end{aligned}$$

have the solution $\theta_1 = \theta_2 = 0$ for $I_1 = I_2 = 0$, which is allowed due to $S_{13}(0) = +1$. Using the results of the paper [5], the finite volume matrix element can be written as

$$\left\langle B_1(0) \right| : \exp i\frac{\beta}{2}\varphi(0,0) : \left| B_1(0)B_3(0) \right\rangle_L = \frac{1}{\sqrt{\rho_{13}(L|0,0)\rho_1(L|0)}} \left(F_{113}^{1/2}(i\pi,0,0) + m_1 L F_3^{1/2}(0) \right)$$

with ρ_{13} obtained from (2.3) by replacing the index 2 with 3. The $\theta_1 = 0$ term of (3.7) therefore takes the form

$$-\lambda^2 \left(\frac{\left| F_3^{1/2}(0) \right|^2}{m_3^2} L + 2\Re\mathrm{e} \frac{F_{113}^{1/2}(i\pi,0,0) F_3^{1/2}(0)}{m_1 m_3} - \frac{(m_1+m_3)\left| F_3^{1/2}(0) \right|^2 \Phi_{13}(0)}{m_1 m_3^2} + O(L^{-1}) \right)$$

$$\Phi_{13}(\theta) = -i\frac{\partial}{\partial\theta} \log S_{13}(\theta) \tag{3.9}$$

The first term is just the correct bulk contribution, the next two terms must be added to the mass correction and the L^{-1} corrections can be discarded.

The other bulk terms in (2.4), written as integrals, are expected to follow from terms of (3.1) with $B_1B_1B_2$ and $B_1B_1B_1B_1$ as intermediate state. However, their evaluation is rather tedious and will not be pursued here. The corresponding bulk parts in (2.4) are small and therefore it is plausible that their contributions to the mass shift are small as well, comparable to numerical accuracy of TCSA and the errors made by neglecting other states. This assumption will be justified by the later comparison to TCSA.

3.2 Evaluating the mass correction

Using the formulae in the previous subsection and the end result (B.2) of Appendix B, the correction to the first breather mass can be written as follows (the particle composition of the

$R=\sqrt{4\pi}/\beta$	1.6	1.9	2.2	2.5
a_2 (TCSA)	3.8 ± 0.3	4.7 ± 0.2	6.1 ± 0.1	7.6 ± 0.1
a_2 (FFPT)	3.66	4.91	6.23	7.82

Table 3.1: Comparing the mass correction coefficient a_2 from FFPT to TCSA numerics. The parameter R is related to the compactification radius of the ultraviolet limiting $c = 1$ free boson conformal field theory. The values of R are chosen to lie in a range to ensure a sufficient precision for the TCSA determination, for which an estimate of the numerical uncertainty is shown. FFPT values from (3.10) are reported with two decimal places accuracy.

contributing intermediate state is indicated below each term):

$$
\begin{aligned}
\delta m_1 &= \underbrace{\delta m_1^{(11)}}_{B_1B_1} + \underbrace{\delta m_1^{(2)}}_{B_2} + \underbrace{\delta m_1^{(13)}}_{B_1B_3} + \underbrace{\delta m_1^{(22)}}_{B_2B_2} + \underbrace{\delta m_1^{(4)}}_{B_4} + \dots \qquad (3.10)\\
\delta m_1^{(11)} &= -\lambda^2 \int_0^\infty \frac{d\theta}{2\pi} \left(\frac{\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2}{m_1^3 \cosh\theta(2\cosh\theta-1)} - \frac{16\left|F_1^{1/2}(0)\right|^2}{m_1^3\sinh^2\theta\cosh\theta} \right)\\
&- \lambda^2 \times 16\left|F_1^{1/2}(0)\right|^2 \left(\frac{\Phi_{11}(0)}{4m_1^3} - \frac{1}{4m_1^3} \right)\\
\delta m_1^{(2)} &= -\lambda^2 \frac{\left|F_{12}^{1/2}(i\pi,0)\right|^2}{m_1 m_2 (m_2-m_1)}\\
\delta m_1^{(4)} &= -\lambda^2 \frac{\left|F_{14}^{1/2}(i\pi,0)\right|^2}{m_1 m_4 (m_4-m_1)}\\
\delta m_1^{(13)} &= -\lambda^2 \left(2\Re\mathrm{e}\frac{F_{113}^{1/2}(i\pi,0,0)F_3^{1/2}(0)}{m_1 m_3} - \frac{(m_1+m_3)\left|F_3^{1/2}(0)\right|^2 \Phi_{13}(0)}{m_1 m_3^2} \right)\\
&- \lambda^2 \int_{-\infty}^\infty \frac{d\theta_1}{2\pi} \left(\frac{\left|F_{113}^{1/2}(i\pi,\theta_1,\theta_2)\right|^2}{(m_1\cosh\theta_1 + m_3\cosh\theta_2 - m_1)m_3\cosh\theta_2} \right)\Bigg|_{\theta_2=-\mathrm{arsinh}(m_1\sinh\theta_1/m_3)}\\
\delta m_1^{(22)} &= -\lambda^2 \frac{1}{2!} \int_{-\infty}^\infty \frac{d\theta}{2\pi} \left(\frac{\left|F_{122}^{1/2}(i\pi,\theta,-\theta)\right|^2}{(2m_2\cosh\theta - m_1)m_2\cosh\theta} \right)
\end{aligned}
$$

(the evaluation of the terms $\delta m_1^{(2)}$, $\delta m_1^{(4)}$ and $\delta m_1^{(22)}$ proceeds by the already discussed methods; there are no disconnected pieces in any of them). The mass correction can be parametrized with the dimensionless coefficient a_2 defined by

$$\frac{\delta m_1}{M} = -a_2 t^2 + O(t^4)$$

which is compared to TCSA data in Table 3.1.

4 Conclusions

In this paper it was shown how to use finite volume techniques to go beyond first order in form factor perturbation theory. The second-order corrections to vacuum energy and first breather mass was evaluated in double sine-Gordon theory. In principle, the method works for higher corrections, and for other quantities, such as the S matrix[3] as well.

The results of second order FFPT are in good agreement with numerical data from TCSA. In addition, the regularization techniques developed to evaluate disconnected contributions can also be used for the form factor expansion of finite temperature two-point correlators. Eventually, during the typing of this manuscript there appeared an independent work by Essler and Konik [16], which uses similar finite volume techniques for correlators, and also introduces another, novel infinite volume regularization procedure.

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A Breather form factors in sine-Gordon theory

To obtain matrix elements containing the first breather, one can analytically continue the form factors of sinh-Gordon theory obtained in [17] to imaginary values of the couplings. For the theory obtained by setting $\lambda = 0$ in (1.2), the result reads

$$\begin{aligned} F^a_{\underbrace{11\ldots1}_{n}}(\theta_1,\ldots,\theta_n) &= \left\langle 0 \left| \mathrm{e}^{ia\beta\varphi(0)} \right| B_1(\theta_1)\ldots B_1(\theta_n) \right\rangle \\ &= \mathcal{G}_a(\beta)\,[a]_\xi\,(i\bar\lambda(\xi))^n \prod_{i<j} \frac{f_\xi(\theta_j-\theta_i)}{\mathrm{e}^{\theta_i}+\mathrm{e}^{\theta_j}}\, Q^{(n)}_a\left(\mathrm{e}^{\theta_1},\ldots,\mathrm{e}^{\theta_n}\right) \qquad \text{(A.1)} \end{aligned}$$

where $\xi = \beta^2/(8\pi-\beta^2)$,

$$\begin{aligned} Q^{(n)}_a(x_1,\ldots,x_n) &= \det\left[a+i-j\right]_\xi \sigma^{(n)}_{2i-j}(x_1,\ldots,x_n)_{i,j=1,\ldots,n-1} \text{ if } n\geq 2 \\ Q^{(1)}_a = Q^{(2)}_a &= 1 \quad , \qquad [a]_\xi = \frac{\sin\pi\xi a}{\sin\pi\xi} \\ \bar\lambda(\xi) &= 2\cos\frac{\pi\xi}{2}\sqrt{2\sin\frac{\pi\xi}{2}}\exp\left(-\int_0^{\pi\xi}\frac{dt}{2\pi}\frac{t}{\sin t}\right) \end{aligned}$$

and

$$\begin{aligned} f_\xi(\theta) &= v(i\pi+\theta,-1)v(i\pi+\theta,-\xi)v(i\pi+\theta,1+\xi)v(-i\pi-\theta,-1)v(-i\pi-\theta,-\xi)v(-i\pi-\theta,1+\xi) \\ v(\theta,\zeta) &= \prod_{k=1}^{N}\left(\frac{\theta+i\pi(2k+\zeta)}{\theta+i\pi(2k-\zeta)}\right)^k \\ &\quad \times\exp\left\{\int_0^\infty\frac{dt}{t}\left(-\frac{\zeta}{4\sinh\frac{t}{2}}-\frac{i\zeta\theta}{2\pi\cosh\frac{t}{2}}+\left(N+1-N\mathrm{e}^{-2t}\right)\mathrm{e}^{-2Nt+\frac{it\theta}{\pi}}\frac{\sinh\zeta t}{2\sinh^2 t}\right)\right\} \end{aligned}$$

[3] The evaluation of S matrix corrections can be carried out by calculating the shifts of two-particle levels, which depend on the phase shift in finite volume.

gives the minimal B_1B_1 form factor[4], while $\sigma_k^{(n)}$ denotes the elementary symmetric polynomial of n variables and order k defined by

$$\prod_{i=1}^{n}(x+x_i)=\sum_{k=0}^{n}x^{n-k}\sigma_k^{(n)}(x_1,\ldots,x_n)$$

Furthermore

$$\begin{aligned}\mathcal{G}_a(\beta)=\langle e^{ia\beta\varphi}\rangle &= \left[\frac{M\sqrt{\pi}\Gamma\left(\frac{4\pi}{8\pi-\beta^2}\right)}{2\Gamma\left(\frac{\beta^2/2}{8\pi-\beta^2}\right)}\right]^{\frac{a^2\beta^2}{4\pi}} \\ &\times \exp\left\{\int_0^\infty\frac{dt}{t}\left[\frac{\sinh^2\left(\frac{a}{4\pi}t\right)}{2\sinh\left(\frac{\beta^2}{8\pi}t\right)\cosh\left(\left(1-\frac{\beta^2}{8\pi}\right)t\right)\sinh t}-\frac{a^2\beta^2}{4\pi}e^{-2t}\right]\right\}\end{aligned}$$

is the exact vacuum expectation value of the exponential field [18], with M denoting the soliton mass related to the coupling μ via [19]

$$\mu=\frac{2\Gamma(\Delta)}{\pi\Gamma(1-\Delta)}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2-2\Delta}\right)M}{2\Gamma\left(\frac{\Delta}{2-2\Delta}\right)}\right)^{2-2\Delta}\qquad,\qquad\Delta=\frac{\beta^2}{8\pi}\tag{A.2}$$

Formula (A.1) also coincides with the result given in [20].

Form factors containing higher breathers can be obtained using that B_n is a bound state of B_1 and B_{n-1}; therefore sequentially fusing n adjacent first breathers gives B_n. Following the lines of reasoning of Appendix A of the paper [21] one obtains

$$\begin{aligned}&F^a_{k_1\ldots k_r n l_1\ldots l_s}(\theta_1,\ldots,\theta_r,\theta,\theta'_1,\ldots,\theta'_s)= \\ &\left\langle 0\left|\mathrm{e}^{ia\beta\Phi(0)}\right|B_{k_1}(\theta_1)\ldots B_{k_r}(\theta_r)B_n(\theta)B_{l_1}(\theta'_1)\ldots B_{l_s}(\theta'_s)\right\rangle=\gamma_{11}^2\gamma_{12}^3\ldots\gamma_{1n-1}^n \\ &\times F^a_{k_1\ldots k_r\underbrace{11\ldots1}_{n}l_1\ldots l_s}\left(\theta_1,\ldots,\theta_r,\theta+\frac{1-n}{2}i\pi\xi,\theta+\frac{3-n}{2}i\pi\xi,\ldots,\theta+\frac{n-1}{2}i\pi\xi,\theta'_1,\ldots,\theta'_s\right)\end{aligned}\tag{A.3}$$

where

$$\gamma_{1k}^{k+1}=\sqrt{\frac{2\tan\frac{k\pi\xi}{2}\tan\frac{(k+1)\pi\xi}{2}}{\tan\frac{\pi\xi}{2}}}$$

is the $B_1B_k\to B_{k+1}$ coupling, defined as the residue of the appropriate scattering amplitude:

$$\begin{aligned}i\left(\gamma_{1k}^{k+1}\right)^2 &= \operatorname*{Res}_{\theta=\frac{i\pi(k+1)\xi}{2}}S_{1k}(\theta) \\ S_{1k}(\theta) &= \frac{\sinh\theta+i\sin\frac{\pi(k+1)\xi}{2}}{\sinh\theta-i\sin\frac{\pi(k+1)\xi}{2}}\frac{\sinh\theta+i\sin\frac{\pi(k-1)\xi}{2}}{\sinh\theta-i\sin\frac{\pi(k-1)\xi}{2}}\end{aligned}\tag{A.4}$$

[4]The formula for the function v is in fact independent of N; choosing N large extends the width of the strip where the integral converges and also speeds up convergence.

Using the results of [20] we also quote here the simplest solitonic form factor, which is needed in the main text

$$F_{s\bar{s}}^{\pm 1/2}(\theta_1,\theta_2) = \mathcal{G}_a(\beta)\frac{G(\theta_2-\theta_1)}{G(-i\pi)}\frac{2ie^{\mp\frac{\theta+i\pi}{2\xi}}}{\xi\sinh\left(\frac{\theta+i\pi}{\xi}\right)} \tag{A.5}$$

where

$$\begin{aligned} G(\theta) &= i\mathcal{C}_1\sinh\frac{\theta}{2}\exp\left(\int_0^\infty\frac{dt}{t}\frac{\sinh^2 t(1-\frac{i\theta}{\pi})\sinh t(\xi-1)}{\sinh 2t\,\cosh t\,\sinh t\xi}\right) \\ \mathcal{C}_1 &= \exp\left(-\int_0^\infty\frac{dt}{t}\frac{\sinh^2\frac{t}{2}\sinh t(\xi-1)}{\sinh 2t\,\cosh t\,\sinh t\xi}\right) \end{aligned}$$

B Evaluation of disconnected contributions

To obtain $\delta m_1^{(11)}$ we need to evaluate the $O(L^0)$ of the sum (3.3), which is

$$-\lambda^2L^2\sum_\theta\frac{\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2}{\rho_1(L|0)\rho_{11}(L|\theta,-\theta)(2m_1\cosh\theta-m_1)}+O\left(e^{-\mu L}\right)$$

Using

$$\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2\sim\frac{16\left|F_1^{1/2}(0)\right|^2}{\theta^2}+O(\theta^0)$$

we can subtract the singular piece to write

$$-\lambda^2L^2\sum_\theta\left(\frac{\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2}{\rho_1(L|0)\rho_{11}(L|\theta,-\theta)(2m_1\cosh\theta-m_1)}-\frac{16\left|F_1^{1/2}(0)\right|^2}{\sinh^2\theta\rho_1(L|0)\rho_{11}(L|\theta,-\theta)m_1}\right)$$

which can be readily converted for $L\to\infty$ to the following integral:

$$-\lambda^2\int_0^\infty\frac{d\theta}{2\pi}\left(\frac{\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2}{m_1^3\cosh\theta(2\cosh\theta-1)}-\frac{16\left|F_1^{1/2}(0)\right|^2}{m_1^3\sinh^2\theta\cosh\theta}\right)$$

Therefore what we need is the $O(L^0)$ part of the subtracted term

$$\lambda^2L^2\sum_\theta\frac{16\left|F_1^{1/2}(0)\right|^2}{\sinh^2\theta\rho_1(L|0)\rho_{11}(L|\theta,-\theta)m_1} \tag{B.1}$$

where

$$\rho_1(L|0)=m_1L\,,\quad \rho_{11}(L|\theta,-\theta)=m_1^2L^2\cosh^2\theta+2m_1L\cosh\theta\Phi_{11}(2\theta)$$

According to eqn. (3.6) the rapidity is quantized as

$$m_1L\sinh\theta+\delta_{11}(2\theta)=2\pi J$$

where J is a positive half-integer. Using the arguments of Subsection 3.1

$$\lambda^2 L^2 \sum_J \frac{16\left|F_1^{1/2}(0)\right|^2}{m_1^2 L (m_1 L)^2}\left(\frac{m_1 L}{2\pi J}\right)^2 = \lambda^2 L^2 \sum_\theta \frac{16\left|F_1^{1/2}(0)\right|^2}{m_1^2 L (m_1 L)^2}\left(\frac{1}{\sinh\theta + \frac{\delta_{11}(2\theta)}{m_1 L}}\right)^2 = \lambda^2 L \frac{2\left|F_1^{1/2}(0)\right|^2}{m_1^2}$$

is just the $O(L)$ part of the sum (B.1), which can be explicitly subtracted without affecting the $O(L^0)$ part. Thus the $O(L^0)$ term of (B.1) can be obtained as the $L\to\infty$ limit of

$$\begin{aligned} & 16\frac{\lambda^2\left|F_1^{1/2}(0)\right|^2}{m_1^3}\sum_\theta\left(\frac{1}{\sinh^2\theta\cosh\theta\,(m_1 L\cosh\theta + 2\Phi_{11}(2\theta))} - \frac{1}{m_1 L\sinh^2\theta}\right) \\ + \quad & 16\frac{\lambda^2\left|F_1^{1/2}(0)\right|^2}{m_1^4 L}\sum_\theta\left(\frac{1}{\sinh^2\theta} - \frac{1}{\left(\sinh\theta + \frac{\delta_{11}(2\theta)}{m_1 L}\right)^2}\right) \end{aligned}$$

where an intermediate subtraction was inserted to simplify the evaluation. The second sum can be written

$$\sum_\theta\left(\frac{1}{\sinh^2\theta} - \frac{1}{\left(\sinh\theta + \frac{\delta_{11}(2\theta)}{m_1 L}\right)^2}\right) = \sum_J \frac{m_1^2 L^2}{4\pi^2 J^2}\left(\frac{2\delta_{11}(2\theta)}{m_1 L\sinh\theta} + \left(\frac{\delta_{11}(2\theta)}{m_1 L\sinh\theta}\right)^2\right)$$

Now the singular part is in the J^{-2} prefactor, and the remainder is finite for any fixed J when $L\to\infty$. Since in the infinite volume limit $\theta\to 0$ for any fixed J, and the summation over J is uniformly convergent, permitting to exchange the limit with the sum, we can write

$$\sum_\theta\left(\frac{1}{\sinh^2\theta} - \frac{1}{\left(\sinh\theta + \frac{\delta_{11}(2\theta)}{m_1 L}\right)^2}\right) = \sum_J \frac{m_1 L}{4\pi^2 J^2} 4\Phi_{11}(0) + O(L^0) = \frac{m_1 L\Phi_{11}(0)}{2} + O(L^0)$$

For the first sum we obtain

$$\begin{aligned} & \sum_\theta\left(\frac{1}{\sinh^2\theta\cosh\theta\,(m_1 L\cosh\theta + 2\Phi_{11}(2\theta))} - \frac{1}{m_1 L\sinh^2\theta}\right) = \\ & -\sum_\theta \frac{1}{\sinh^2\theta}\left(\frac{1 - 1/\cosh^2\theta}{m_1 L} + 2\frac{\Phi_{11}(2\theta)}{m_1^2 L^2\cosh^3\theta} + O(L^{-3})\right) = \\ & -\frac{1}{4} - \frac{\Phi_{11}(0)}{4} + O(L^{-1}) \end{aligned}$$

where in the first term we used

$$\sum_\theta \frac{1 - 1/\cosh^2\theta}{m_1 L\sinh^2\theta} = \int_0^\infty \frac{d\theta}{2\pi}(m_1 L\cosh\theta + \Phi_{11}(2\theta))\frac{1}{m_1 L\cosh^2\theta} = \frac{1}{4} + O(L^{-1})$$

(the integrand is non-singular at $\theta = 0$, hence it can be evaluated by a density integral), while for the second term:

$$\sum_\theta \frac{1}{\sinh^2\theta}\frac{2\Phi_{11}(2\theta)}{m_1^2 L^2\cosh^3\theta} = \sum_\theta \frac{2\Phi_{11}(0)}{m_1^2 L^2\sinh^2\theta} + O(L^{-1}) = \sum_J \frac{2\Phi_{11}(0)}{4\pi^2 J^2} + O(L^{-1}) = \frac{\Phi_{11}(0)}{4} + O(L^{-1})$$

where again, all parts non-singular at $\theta = 0$ were evaluated at the origin. Collecting all the pieces we obtain that the subtracted part (B.1) equals

$$\lambda^2 L^2 \sum_\theta \frac{16\left|F_1^{1/2}(0)\right|^2}{\sinh^2\theta\rho_1(L|0)\rho_{11}(L|\theta,-\theta)m_1} = \lambda^2 L\,\frac{2\left|F_1^{1/2}(0)\right|^2}{m_1^2} + \lambda^2\times 16\left|F_1^{1/2}(0)\right|^2\left(\frac{\Phi_{11}(0)}{4m_1^3}-\frac{1}{4m_1^3}\right)+O(L^{-1})$$

and therefore

$$\begin{aligned}\delta m_1^{(11)} &= -\lambda^2\int_0^\infty\frac{d\theta}{2\pi}\left(\frac{\left|F_{111}^{1/2}(i\pi,\theta,-\theta)\right|^2}{m_1^3\cosh\theta(2\cosh\theta-1)}-\frac{16\left|F_1^{1/2}(0)\right|^2}{m_1^3\sinh^2\theta\cosh\theta}\right) \\ &\quad -\lambda^2\times 16\left|F_1^{1/2}(0)\right|^2\left(\frac{\Phi_{11}(0)}{4m_1^3}-\frac{1}{4m_1^3}\right)\end{aligned} \tag{B.2}$$

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