Existence methods in Nonlinear Partial Differential Equations

PhD Dissertation

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Contents

Summary

The present thesis considers three different problems from the theory of partial differential equations and handles them using the techniques of nonlinear analysis.

The first is a very general one: given an "energy" functional defined on a function space, what can we say about its minimizers, maximizers and saddle points? Since most physical principles are formulated as such a problem, the relevance of this question is evident. The classical field of calculus of variations was the first to take up this problem. The relatively recent field of critical point theory aims to qualitatively analyze such problems. Our contribution to this field is a generalization of a well-known set of results which concern critical points restricted to a fixed ball. We extend that theory into the so-called nonsmooth context, i.e. when the "energy" functional is not differentiable anymore, but only Lipschitz continuous. We obtain the existence of certain critical points called mountain-pass points, derive alternative theorems and finally, as an example "toy" application we prove a result on partial differential inclusions involving the p-Laplacian.

The second problem describes the equilibrium configuration of the contact between an elastic body and an obstacle. Such problems are intensely studied in the mathematical theory of elasticity. The strong form of the problem is a system of nonlinear partial differential equations with inequality boundary conditions. Through the use of the popular bipotential method, we reformulate the weak form of the problem and prove its equivalence to a primal-dual formulation. Then we prove solvability of the weak form using a recent existence result from the theory of hemivariational inequalities. Our novel approach allows the treatment of rather general boundary conditions.

The third problem is a nonlinear elliptic-type operator equation with nonlocal dependence defined on an unbounded domain. Such nonlocal – or functional – dependence means that the coefficient functions of the operator may depend on the whole solution instead of only local quantities. This has already been treated in the literature, but not on an unbounded domain, where compact embedding of Sobolev spaces does not hold in general. We prove solvability of the operator equation using the theory of pseudomonotone operators, and give a series of examples of concrete operators with nonlocal dependence.

Összefoglaló

Jelen értekezésben három különböző problémát tekintünk a parciális differenciálegyenletek elméletéből, amelyeket a nemlineáris analízis módszereivel kezelünk.

Az első roppant általános: adott egy függvénytéren értelmezett ”energiafunkcionál”, mit tudunk mondani a minimum-, maximum-, illetve nyeregpont-helyeiről? A kérdés relevanciája nyilvánvaló, minthogy a legtöbb fizikai alapelvet ilyen feladatként fogalmazzák meg. A klaszikus variációszámítás foglalkozott először ilyen ilyen kérdésekkel. A viszonylag új kritikus pont elmélet célja a feladat kvalitatív elemzése. A mi hozzájárulásunk ehhez a területhez egy olyan tételkör általánosítása, amely egy rögzített gömbön belüli kritikus pontokkal foglalkozik. Kiterjesztjük a meglévő elméletet a nem differenciálható kontextusba, azaz arra az esetre, amikor az energiafunkcionál csupán Lipschitz-folytonos. Mountain-pass típusú kritikus pontokok létezését bizonyítjuk, alternatívatételeket, és egy ”játék” alkalmazás gyanánt egy, a p-Laplace operátort tartalmazó differenciálinklúziókkal kapcsolatos eredményt bizonyítunk.

A második feladat egy rugalmas test és egy akadály közötti érintkezés egyensúlyi helyzetét írja le. Az ilyesfajta problémákat sokat tanulmányozták a rugalmasságtan matematikai elméletében. A feladat erős alakja egy nemlineáris parciális differenciálegyenlet-rendszer, egyenlőtlenségeket tartalmazó peremfeltételekkel. A népszerű bipotenciál-módszer segítségével átfogalmazzuk a feladat gyenge alakját és megmutatjuk ennek egyenértékűségét a primál-duál feladattal. Ezek után bebizonyítjuk a gyenge alak megoldhatóságát a hemivariációs egyenlőtlenségek elméletének egy új egziszenciatételével. Újszerű megközelítésünk igen általános peremfeltételek kezelését teszi lehetővé.

A harmadik feladat egy nemkorlátos tartományon definiált nemlineáris elliptikus-típusú operátoregyenlet, nemlokális függéssel. Az ilyen nemlokális, más szóval funkcionális függés azt jelenti, hogy az operátorban szereplő együtthatófüggvények függhetnek az egész megoldástól, nem csak lokális mennyiségektől. Ezt már kezelték az irodalomban, de nem nemkorlátos tartományokon, ahol a Sobolev-terek kompakt beágyazása nem érvényes általában. Az operátoregyenlet megoldhatóságát a pszeudomonoton operátorok elméletének használatával igazoljuk, és megadunk egy sor konkrét példát olyan operátorra, amely nemlokális függést tartalmaz.

Notations

$f'(u)$	The Fréchet derivative of f at u
$f'(u;v)$	The Gâteaux directional derivative of f at u in the direction v
$f^0(u;v)$	The Clarke generalized directional derivative of f at u in the direction v
$\partial f(u)$	The Clarke subdifferential of f at u
$\|\partial f\|(u)$	$= \inf\{\\|u^*\\| \,:\, u^* \in \partial f(u)\}$
f_a	$= \{u \,:\, f(u) \geq a\}$
f^b	$= \{u \,:\, f(u) \leq b\}$
f_a^b	$= f_a \cap f_b$
K	The set of critical points of some functional
K_c	The set of critical points of some functional at level $c \in \mathbb{R}$
$L^p(\Omega)$	Lebesgue space of exponent $1 < p < \infty$ on Ω
$W^{k,p}(\Omega)$	Sobolev space of order $k \in \mathbb{N}$ and exponent $1 < p < \infty$ on Ω
$W_0^{k,p}(\Omega)$	Homogeneous Sobolev space of order $k \in \mathbb{N}$ and exponent $1 < p < \infty$ on Ω
$W^{-k,q}(\Omega)$	Negative exponent Sobolev space of order $k \in \mathbb{N}$ and exponent $1 < q < \infty$ on Ω
$\mathcal{P}(A)$	Power set of a set A
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$,$\mathbb{R}_+$,$\mathbb{R}_-$	The set of all, nonnegative and nonpositive real numbers
E	Typically a real Banach space
E^*	Dual space of E^*
$\langle \cdot, \cdot \rangle$	Duality pairing
$\to$	Convergence in norm
$\rightharpoonup$	Weak convergence
$\overset{*}{\rightharpoonup}$	Weak* convergence
$B_X(u,r)$	Open ball of radius r and center u in the metric space X
$\mathrm{dist}(A,B)$	Distance between the sets A and B
$\mathrm{dist}(A)$	Distance between A and 0
$[A]_\theta$	$= A + B_E(0, \theta\,\mathrm{dist}(A))$, where $A \subset E$, $0 < \theta < 1$
$\overline{\mathrm{co}}(A)$	Closed convex hull of A

1 Introduction

"Man muss immer generalisieren." – C. G. J. Jacobi

Mathematical analysis clearly has its origins in classical physics. The fruitful interaction between the two fields initiated the development of many important methods in analysis. Physics continues to be an inspiring force for the development of mathematical theories to the present day. Therefore, it is worthwhile for mathematicians to directly study models originating from physics. More often than not, these models are difficult to formulate in a mathematically precise way and require sophisticated tools to analyze.

The common theme of this thesis is nonlinear analysis. The field of nonlinear analysis studies existence, uniqueness and multiplicity questions of nonlinear partial differential equations (and ordinary differential equations, as well) by employing tools from functional analysis (see e.g. [40, 82, 2] for a comprehensible account). In this thesis, we study three distinctively different problems and employ appropriate methods to prove solvability for each.

In **Chapter 2** we collect definitions and results mainly from functional analysis which are needed in the sequel.

In **Chapter 3**, we develop new tools for critical point theory and apply them to a nonlinear eigenvalue problem for the p-Laplacian. Critical point theory is a highly developed field of mathematical analysis with wide-ranging practical applications to ordinary differential equations, partial differential equations, differential geometry and optimization (see [50] or the monographies cited above). Roughly speaking, we might say that critical point theory is interested in the minimizers/maximizers and saddle points of functionals defined on a function space through the analysis of an appropriate concept of derivative of the functional. If the functional is smooth in the usual sense, then we are talking about "smooth" critical point theory.

The development of the nonsmooth variant of critical point theory was initiated in 1981 by Chang [17] who extended various minimax principles due to Ambrosetti and Rabinowitz [3] and Rabinowitz [77] to locally Lipschitz functions and then applied these theoretical results in the study partial differential equations with discontinuous nonlinearities. See Section 3.1 for more references on critical point theory.

Our starting point is Martin Schechter's bounded critical point theory for C^1-functionals (see [80] and [81] for a concise account). Schechter's most elementary results are concerned with critical points of the functional inside a ball, i.e. critical points of *bounded* norm. As expected, "boundary conditions" on the surface of the ball become relevant (see (3.14)). The theory is built up from a technical result called a "deformation lemma" (Theorem 3.3.1), from which various minimum and mountain pass theorems are derived (see Theorem 3.4.3, and [49] for an excellent introduction to the subject). The results – when combined with a suitable compactness assumption (Definition 3.6.1)– lead to the existence of minimizers (Theorem 3.6.3) or saddle points (Theorem 3.6.2) for the functional. These results are formulated in the generalized situation called "linking": there are two arbitrary sets in the function space which "cannot be pulled apart without intersecting" and the energy (i.e. the values of the functional) on one set is dominated by the energy on the other (Definition 3.4.2). The rationale behind this concept is that an appropriate choice of the linking sets enables one to exploit the symmetry properties of concrete functionals.

In summary, we generalize some of Schechter's results from the Hilbert space- to the Banach space-context, and more importantly, we allow locally Lipschitz functionals instead of only C^1. In other words, we extend those results to the nonsmooth setting. Since the differential of a nonsmooth functional is set-valued in general, certain conditions and relations become more complicated, as expected. For instance, the eigenvalue equation becomes a differential inclusion (Theorem 3.6.2). As a "toy" application, we study a nonlinear eigenvalue problem for the p-Laplacian (Section 3.7). The p-Laplacian is an intensely studied subject in nonlinear analysis, see [39] for a friendly introduction. Our results extend Schecter's original work for the usual Laplacian with a C^1 forcing term to the more general p-Laplacian with a locally Lipschitz forcing term (Theorem 3.7.3). The methods presented in Chapter 3 can be extended to a more complete theory – we barely demonstrated that the nonsmooth generalization of Schechter's theory is possible.

In **Chapter 4**, we formulate a very general problem in the mathematical theory of elasticity and through the use of the so-called bipotential method we can apply an existence result from the theory of hemivariational inequalities to prove the existence of a weak solution. The problem describes the equilibrium configuration of the contact between an elastic body and an obstacle. The first problem of this sort was Signorini's problem, for which existence and uniqueness of the weak solution was proved by Fichera [41]. The main difficulty of the problem lies in the nonstandard boundary conditions. The requirement that the body should not penetrate the obstacle leads to inequality constraints, called "ambiguous boundary conditions". The weak formulation of Signorini's problem is a so-called variational inequality, which are studied systematically in nonlinear analysis ever since. It is a fact of life that the strong form of a contact problem usually does not admit a solution.

Instead of the physically restrictive *linear* material law (Hooke's law) in Signorini's problem, researchers later handled more general nonlinear material laws. This relation between the stress tensor and the strain can be rather complicated in practice, which necessitates the use of convex analysis. Furthermore, more complicated boundary conditions can be considered, which may require nonsmooth analysis techniques to handle. These are called *nonmonotone* boundary conditions. For a short overview of the literature, see Section 4.1.

The key ingredient of our analysis is the *bipotential method* introduced by de Saxcé & Feng [35] (see Definition 4.3.2). The novelty of our approach is that it allows the treatment of nonmonotone boundary conditions through the use of bipotentials. We reformulate the weak form of the problem in terms of a bipotential and as a primal-dual variational formulation. We show that these two formulations are equivalent (Proposition 4.4.1). Then we prove the solvability of the weak problem (Theorem 4.4.2). This is done using a recent existence result for hemivariational inequalities by Costea and Varga [28] (see Theorem 2.5.1). Our treatment allows the incorporation of very general material laws and boundary conditions, and we give a series of examples in Section 4.2.

In **Chapter 5**, we prove the existence of a solution to a nonlinear, *nonlocal* elliptic PDE defined on an unbounded domain. By nonlocal, we mean that the coefficient functions of the operator may depend on the *whole* solution instead of only on local quantities, such as pointwise values. This is also called *functional dependence.* Such nonlocal dependence may occur in various models [67, 37, 62]. A nonlocal boundary value problem from plasma physics is treated in [8]. Further, see [90] for a linear elliptic functional differential equation, and [88, 86] for strongly nonlinear functional equations.

We invoke standard techniques from the theory of pseudomonotone operators (see Section 2.3 and [95, 89]). More precisely, we apply the well-known surjectivity result (Theorem 2.3.1) due to Browder to prove the solvability of our operator equation. The main difficulty in the treatment of the problem lies in the fact that the domain may be unbounded, for which the Rellich–Kondrachov theorem fails. This can be remedied by applying Browder's trick [12]. We also give a series of examples of operators with nonlocal dependence in Section 5.4.

In summary, we present three distinctively different methods to prove solvability of three different models. Naturally, the methods used here build on the earlier conceptual framework. But the novel ideas of Chapter 3 and 4 also extend the existing methods considerably, so that – hopefully – more general problems can be tackled through their use.

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2 Preliminaries

In chapter, we collect some notions and results from functional analysis which are needed in the sequel.

2.1 Basic Notions

Henceforth, elements of a vector space X are usually denoted as $u, v, w, \ldots$, and elements of its dual X^* are denoted as $u^*, v^*, w^*, \ldots$, where u^* is to be understood as *one symbol*, i.e. unrelated to u. This convention is common in nonlinear analysis.

Definition. Let X be a normed space and $f : X \to \mathbb{R}$ a functional.

(i) The functional f is said to be *locally Lipschitz* on $U \subset X$ open, if for every $u \in U$ there is a neighborhood $G_u \subset X$ of u and a Lipschitz constant $L_u > 0$ such that $|f(v) - f(w)| \leq L_u \|v - w\|$ for all $v, w \in G_u$.

(ii) The functional f is said to be *strongly (resp. weakly) lower semicontinuous*, if for every sequence $\{u_j\} \subset X$ and $u \in X$ such that $u_j \to u$ (resp. $u_j \rightharpoonup u$), there holds $f(u) \leq \liminf\limits_{j\to\infty} f(u_j)$.

(iii) The functional f is said to be *convex* on a convex set $C \subset X$, if for any $u, v \in C$ and any $0 \leq \lambda \leq 1$, there holds $f((1-\lambda)u + \lambda v) \leq (1-\lambda)f(u) + \lambda f(v)$.

(iv) The functional f is said to be *coercive* if for every sequence $\{u_j\} \subset X$ with $\|u_j\| \to \infty$, there holds $f(u_j) \to +\infty$.

Definition. Let X be a normed space, $U \subset X$ open. The functional $f : U \to \mathbb{R}$ is said to be *Fréchet differentiable at* $u \in U$ if there exists $u^* \in X^*$ such that

$$\lim_{h\to 0} \frac{f(u+h) - f(u) - \langle u^*, h\rangle}{\|h\|} = 0,$$

and the bounded linear functional u^* is called the *Fréchet derivative of f at u*, denoted by symbol $f'(u)$. The functional f is called *r-times continuously differentiable*, or C^r for short, if its iterated Fréchet derivatives f', f'', ..., $f^{(r)}$ exist everywhere and $f^{(r)}$ is continuous.

Definition. Let X be a normed space, $U \subset X$ open. The functional $f : U \to \mathbb{R}$ is said to be *Gâteaux differentiable at* $u \in U$ if

$$f'(u; v) = \lim_{t\to 0+} \frac{f(u+tv) - f(u)}{t}$$

exists for all $v \in X$ and the map $v \mapsto f'(u;v)$ is a bounded linear functional for every $u \in U$. The quantity $f'(u;v)$ is called the *directional derivative of f at u in the direction v.*

In other words, if f is Gâteaux differentiable at u, then there is a *unique* $u^* \in E^*$ such that $f'(u;v) = \langle u^*, v\rangle$ for all $v \in X$. In Clarke's generalization of the Gâteaux derivative where the definition of $f'(u;v)$ is generalized, there may be multiple such u^*'s.

Definition. Let X be a normed space, $U \subset X$ open. Let $f : U \to \mathbb{R}$ be a locally Lipschitz functional. For $u, v \in U$ quantity

$$f^0(u;v) = \limsup_{\substack{w\to u\\ t\to 0+}} \frac{f(w+tv)-f(w)}{t}$$

is called the *generalized directional derivative of f at u in the direction v.*

For a multivariate functional $f : X_1 \times \ldots \times X_N \to \mathbb{R}$, where the X_k's are normed spaces, the "generalized partial derivatives" $f_k^0(u_1,\ldots,u_N;v_k)$ are defined in the obvious way.

In the next remark we record some important correspondences between the above notions.

Remarks.

(i) Fréchet differentiability implies Gâteaux differentiability, but not conversely.

(ii) For a C^1-functional, the directional derivatives and the generalized directional derivatives coincide [51, Proposition 1.1.1].

(iii) For a *convex* and locally Lipschitz functional defined on convex set, the directional derivatives and the generalized directional derivatives coincide. [51, Proposition 1.1.1].

Clarke's generalized derivative may serve as a building block for a convenient calculus.

Proposition 2.1.1. Let X be a Banach space, $U \subset X$ open and $f : U \to \mathbb{R}$ a locally Lipschitz functional. Then the following properties hold true.

(i) For every fixed $u \in U$, the function $v \mapsto f^0(u;v)$ is positive homogeneous, subadditive and bounded by the Lipschitz constant $L_u > 0$ of f at u.

(ii) $f^0 : U \times X \to \mathbb{R}$ is upper semicontinuous, i.e. $-f^0$ is lower semicontinuous.

(iii) $f^0(u;-v) = -f^0(u;v)$ for all $u \in U$ and $v \in X$.

Proof. See [51, Proposition 1.1.3]. □

2.2 Functional analysis

Krein–Smulian theorem 2.2.1. Let X be a Banach space. If $K \subset X$ is weakly compact, then $\overline{\mathrm{co}}(K)$ is weakly compact.

Proof. See [21, V.13.4.]. □

Hahn–Banach separation theorem 2.2.1. Let X be a normed space, and let $A \subset X$ and $B \subset X$ be nonempty convex sets such that $A \cap B = \emptyset$. If A is closed and B is compact, then there exists a closed hyperplane that strongly separates A and B, i.e. there is a $u^* \in X^*$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\langle u^*, a\rangle \leq \beta < \alpha < \gamma \leq \langle u^*, b\rangle \quad \text{for all } a \in A,\ b \in B.$$

Proof. See [83, Theorem 5.10.6.]. □

Alaoglu's theorem 2.2.1. Let X be a normed space. Then $B_{X^*}(0,1)$ is weak*-compact.

Proof. See [21, V.3.1.]. □

Proposition 2.2.1. Let X be reflexive Banach space. Then the weak-, and the weak*-topologies on X^* coincide.

Proof. See [21, Theorem V.4.2.]. □

The following elementary result is sometimes useful in practice [23, p. 87].

Proposition 2.2.2. Let X be a normed space. Then for any $u^* \in X^*$ and $u \in X$,

$$|\langle u^*, u\rangle| = \|\varphi\| \operatorname{dist}(u, \ker u^*).$$

Proof. First we prove the "$\leq$" part. Let $u^* \in X^*$, $u \in X$ and $v \in \ker u^*$, then

$$|\langle u^*, u\rangle| = |\langle u^*, u - v\rangle| \leq \|u^*\|\|u - v\|.$$

Taking infimum with respect to $v \in \ker u^*$ yields

$$|\langle u^*, u\rangle| \leq \|u^*\| \inf\{\|u - v\| \ : \ v \in \ker u^*\} = \|u^*\| \operatorname{dist}(u, \ker u^*).$$

Finally, we prove the converse inequality. Let $v \in E \setminus \ker u^*$, then $u - v\frac{\langle u^*, u\rangle}{\langle u^*, v\rangle} \in \ker u^*$. Then

$$\operatorname{dist}(u, \ker u^*) \leq \left\| u - \left(u - \frac{\langle u^*, u\rangle}{\langle u^*, v\rangle} v \right) \right\| = \frac{|\langle u^*, u\rangle|}{|\langle u^*, v\rangle|}\|v\| \leq \frac{|\langle u^*, u\rangle|}{|\langle u^*, v\rangle|} \frac{|\langle u^*, v\rangle|}{\|u^*\|} = \frac{|\langle u^*, u\rangle|}{\|u^*\|}$$

□

The duality mapping is an important notion of Banach space geometry [18, Chapter 3].

Definition. A continuous and strictly increasing function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ with $\tau(0) = 0$ and $\tau(t) \to \infty$ as $t \to \infty$ is called a *gauge function*.

Definition. Let X be a normed space and τ a gauge function. The set-valued map $J_\tau : X \to \mathcal{P}(X^*)$ defined as

$$J_\tau(u) = \{u^* \in X^* \ : \ \langle u^*, u\rangle = \|u\|\|u^*\|, \ \|u^*\| = \tau(\|u\|)\}$$

is called the *duality map with gauge function* τ. If $\tau(t) = t$, then J_τ is called the *normalized duality map*.

When the gauge function τ is obvious from the context, we drop it for brevity,

Remarks.

(*i*) The duality map is always nonempty-valued due to the Hahn–Banach theorem.

(*ii*) The normalized duality map reduces to the Riesz–Fréchet isomorphism in the Hilbert space case.

Proposition. Let X be a Banach space. Then X^* is uniformly convex if and only if any duality mapping on X is single-valued and strongly uniformly continuous on $B_X(0, 1)$.

Proof. See [19, Theorem 2.16]. □

Let $\Omega \subset \mathbb{R}^n$ be a domain. As usual, Sobolev spaces of index $k \in \mathbb{N}$ and exponent $1 \leq p < \infty$ are defined as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \ : \ \partial^\alpha u \in L^p(\Omega), \ |\alpha| \leq k\},$$

where ∂^α denotes weak differentiation with respect to the multiindex α. The space $W^{k,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

We close this section with a simple result on the differentiation of norms.

Proposition 2.2.3. Let $\varphi : [0, \infty) \to X \setminus \{0\}$ be a C^1-curve and $\psi(t) = \int_0^t \tau(s)\, ds$. Then

$$\frac{d}{dt}\psi(\|\varphi(t)\|) = \langle J\varphi(t), \varphi'(t)\rangle .$$

Proof. Clearly, for all $t, s > 0$ the following relations hold

$$\langle J\varphi(t), \varphi(t)\rangle = \tau(\|\varphi(t)\|)\|\varphi(t)\|,$$

and

$$\langle J\varphi(t), \varphi(s)\rangle \leq \tau(\|\varphi(t)\|)\|\varphi(s)\|,$$

hence by subtraction we get

$$\langle J\varphi(t), \varphi(s) - \varphi(t)\rangle \le \tau(\|\varphi(t)\|)\left[\|\varphi(s)\| - \|\varphi(t)\|\right].$$

If $s > t$, then

$$\left\langle J\varphi(t), \frac{\varphi(s) - \varphi(t)}{s - t}\right\rangle \le \tau(\|\varphi(t)\|)\frac{\|\varphi(s)\| - \|\varphi(t)\|}{s - t},$$

and letting $s \downarrow t$ we get

$$\langle J\varphi(t), \varphi'(t)\rangle \le \tau(\|\varphi(t)\|)\frac{d}{dt}\|\varphi(t)\|.$$

For $s < t$ we get the converse inequality, hence

$$\langle J\varphi(t), \varphi'(t)\rangle = \tau(\|\varphi(t)\|)\frac{d}{dt}\|\varphi(t)\| = \frac{d}{dt}\psi(\|\varphi(t)\|).$$

□

2.3 Pseudomonotone operators

The concept of pseudomonotonicity was introduced by H. Brezis [10] in 1968.

Definition. Let V be a Banach space. A bounded operator $A\colon V \to V^*$ is said to be *pseudomonotone* if for any sequence $\{u_j\} \subset V$, such that

$$u_j \rightharpoonup u \quad (\text{in } V) \quad \text{and} \quad \limsup_{j\to\infty}\langle A(u_j), u_j - u\rangle \le 0,$$

then

(PM1) $\langle A(u_j), u_j - u\rangle \to 0$ as $j \to \infty$ and

(PM2) $A(u_j) \rightharpoonup A(u)$ in V^* as $j \to \infty$.

As usual, the symbol $\rightharpoonup$ denotes weak convergence.

The following abstract surjectivity result [89, Theorem 2.12] is widely used in the literature for proving the existence of a weak solution to a nonlinear elliptic partial differential equation.

Theorem 2.3.1. Let V be a reflexive separable Banach space and $A\colon V \to V^*$ a bounded, coercive and pseudomonotone operator. Then for arbitrary $F \in V^*$, there exists $u \in V$, such that $A(u) = F$ in V^*.

In this context, coercivity is defined as follows:

Definition 2.3.2. An operator $A\colon V \to V^*$ is called *coercive* if

$$\frac{\langle A(u), u\rangle}{\|u\|} \to +\infty \quad (\text{as } \|u\| \to \infty).$$

2.4 Clarke subdifferential

It was recognized in the early 1960's that the "subdifferential" of a convex function is a rather fruitful concept, and constitutes the basis of convex analysis [78]. Let $f : C \to \mathbb{R}$ be a convex functional defined on a convex subset $C \subset X$ of a normed space X. The *subdifferential of* f (in the sense of convex analysis) is a set-valued map $\partial f : C \to \mathcal{P}(X^*)$ defined as

$$\partial f(u) = \{u^* \in X^* \ : \ f(v) - f(u) \geq \langle u^*, v - u\rangle, \ v \in C\}.$$

Intuitively, $\partial f(u)$ is the collection of all "supporting hyperplanes" of the epigraph of f at the point $(u, f(u)) \in X \times \mathbb{R}$. The subdifferential $\partial f(u)$ is closed and convex. Furthermore, a global minimum of a (proper) convex function f is attained at u iff $0 \in \partial f(u)$. For further properies of the convex subdifferential and various other topics of convex analysis, see [7].

In the 1970s, R. T. Rockafellar's doctoral student, F. H. Clarke went on to generalize the convex subdifferential to nonconvex functions.

Definition. Let $f : U \to \mathbb{R}$ be a locally Lipschitz functional defined on an open subset $U \subset X$ of a Banach space X. The set-valed map $\partial f : U \to \mathcal{P}(X^*)$ defined as

$$\partial f(u) = \{u^* \in X^* \ : \ f^0(u; v) \geq \langle u^*, v\rangle, \ v \in X\}$$

is called the *Clarke subdifferential of* f.

The success of the Clarke subdifferential rests in its convenient functional-analytic properties.

Theorem 2.4.1. Let $f : U \to \mathbb{R}$ be a locally Lipschitz functional defined on an open subset $U \subset X$ of a Banach space X. Then the following properties hold true.

(*i*) $\partial(\lambda f)(u) = \lambda \partial f(u)$ for all $\lambda \in \mathbb{R}$ and $u \in U$.

(*ii*) If $g : U \to \mathbb{R}$ is locally Lipschitz, then $\partial(f + g)(u) \subset \partial f(u) + \partial g(u)$ for all $u \in U$. Further, equality holds if f and g are *regular* at $u \in U$, i.e. if the ordinary and the generalized directional derivatives coincide in all directions for both f and g.

(*iii*) For every $u \in U$, the set $\partial f(u) \subset X^*$ is nonempty, closed, weak*-compact and

$$\|u^*\|_{E^*} \leq L_u \quad \text{for all } u^* \in \partial f(u),$$

where $L_u > 0$ is the Lipschitz constant of f near u.

(*iv*) For each $u \in U$ and $v \in X$, there holds $f^0(u; v) = \max\{\langle u^*, v\rangle \ : \ u^* \in \partial f(u)\}$.

(*v*) The set-valed map $\partial f : U \to \mathcal{P}(X^*)$ is *weak*-closed* in the following sense: if $\{u_j\} \subset U$, $u_j \to u \in U$ and $\{u_j^*\} \subset X^*$ with $u_j^* \in \partial f(u_j)$ and $u_j^* \stackrel{*}{\rightharpoonup} u^* \in X^*$, then $u^* \in \partial f(u)$.

(*vi*) The set-valued map $\partial f : U \to \mathcal{P}(X^*)$ is *weak*-upper semicontinuous* in the following sense: for any $\varepsilon > 0$, $u \in U$ and $v \in X$ there exists $\delta > 0$ such that for all $w \in U$ with $u^* \in \partial f(w)$ and $\|w - u\| < \delta$, there holds $|\langle u^*, v\rangle - \langle v^*, v\rangle| < \varepsilon$ for some $v^* \in \partial f(u)$.

(*vii*) (Lebourg's mean value theorem) If $x, y \in X$ are distinct, then there is a $z \in (x, y)$ such that $f(y) - f(x) \in \langle \partial f(z), y - x\rangle$.

Proof. See [51, Proposition 1.1.4]. □

Remarks.

(*i*) If the locally Lipschitz functional f is Gâteaux differentiable at $u \in U$, then the Gâteaux derivative of f belongs to the Clarke subdifferential $f'(u) \in \partial f(u)$.

(*ii*) For a C^1-functional f, $\partial f(u) = \{f'(u)\}$.

(*iii*) The Clarke subdifferential is an extension of the convex subdifferential in the sense that for convex functions the two notions coincide.

(*iv*) If $u \in U$ is an extreme value of f, then $0 \in \partial f(u)$. This last relation is colloquially called a *"differential inclusion"* and may be regarded as a set-valued generalization of the classical Euler–Lagrange equations if f is an appropriate "energy functional" or "Lagrangian".

Furthermore, the Clarke subdifferential admits a rich set of calculus rules.

Definition 2.4.2. For a Banach space X and a nonempty, closed and convex subset $K \subset X$, the normal cone of K at u is defined by

$$N_K(u) = \left\{u^* \in X^* : \ \langle u^*, v - u\rangle_{E^* \times E} \le 0, \ \text{for all } v \in K\right\}.$$

It is well known that

$$N_K(u) = \partial I_K(u),$$

where I_K is the indicator function of K, that is,

$$I_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition. The *Fenchel conjugate* of a function $\varphi : X \to (-\infty, +\infty]$ is the function $\varphi^* : X^* \to (-\infty, +\infty]$ given by

$$\varphi^*(u^*) = \sup_{u \in X} \left\{\langle u^*, u\rangle_{X^* \times X} - \varphi(u)\right\}.$$

Proposition 2.4.3. Let $\varphi : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then

(i) φ^* is proper, convex and lower semicontinuous;

(ii) $\varphi(u) + \varphi^*(u^*) \geq \langle u^*, u\rangle_{X^*\times X}$, for all $u \in X, u^* \in X^*$;

(iii) $u^* \in \partial\varphi(u) \Leftrightarrow u \in \partial\varphi^*(u^*) \Leftrightarrow \varphi(u) + \varphi^*(u^*) = \langle u^*, u\rangle_{X^*\times X}$.

When dealing with concrete functionals, we often need to calculate the Clarke subdifferential of "integral functional" of the form

$$\Phi(u) = \int_\Omega \varphi(x, u)\, \mu(dx), \tag{2.1}$$

where $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space, E a Banach space and $\varphi : \Omega \times X \to \mathbb{R}$ satisfies the following assumptions.

Assumption 2.4.1.

(i) For each fixed $u \in X$, the map $x \mapsto \varphi(x, u)$ is $\mathcal{F}$-measurable.

(ii) For any bounded subset $B \subset X$, there is a $k_b \in L^1(\Omega)$ such that for a.a. $x \in \Omega$ and all $u, v \in B$

$$|\varphi(x, u) - \varphi(x, v)| \leq k_B(x)\|u - v\|.$$

Condition (ii) implies that the function $u \mapsto \varphi(x, u)$ is locally Lipschitz for all fixed $x \in \Omega$, so its partial Clarke subdifferential with respect to the second variable $\partial_2\varphi(x, u)$ makes sense. The following classical result of J.-P. Aubin and Clarke holds true [6][43, Theorem 1.3.9].

Aubin–Clarke theorem 2.4.1. Let $\Phi : X \to \mathbb{R}$ be given by (2.1), and suppose that Assumption 2.4.1 holds true. If Φ is finite at some point, then f is finite everywhere, Lipschitz continuous on every bounded subset of X and

$$\partial\Phi(u) \subset \int_\Omega \partial_2\varphi(x, u)\, \mu(dx), \tag{2.2}$$

by which we mean the following:[1] for all $u^* \in \partial\Phi(u)$ there exists $\xi^* : \Omega \to X^*$ weak*-measurable[2] such that $\xi^*(x) \in \partial_2\varphi(x, u)$ for a.a. $x \in \Omega$ and

$$\langle u^*, u\rangle = \int_\Omega \langle \xi^*(x), u\rangle\, \mu(dx).$$

Moreover, if $\varphi(x, \cdot)$ is a.e. regular at $u \in X$, then Φ is also regular at u and equality holds in (2.2).

[1]This is called *Gelfand-*, or *weak*-integral*.

[2]I.e. $\Omega \ni x \mapsto \langle v, \xi^*(x)\rangle \in \mathbb{R}$ is $\mathcal{F}$-measurable for all $v \in X$.

Remark. Finally, we briefly remark that there is a weaker subdifferential applicable to continuous funtionals [16] defined on normed spaces, This so-called Campa–Degiovanni subdifferential reduces to the Clarke subdifferential in the locally Lipschitz case. (Actually, the concept derives from the far weaker "weak slope" of [22], which is defined for continuous functionals on metric spaces.) However, the Campa–Degiovanni subdifferential may be unbounded which renders a number of techniques unapplicable, at least directly. Furthermore, the Campa–Degiovanni subdifferential *calculus* is less developed at the present.

2.5 Hemivariational inequalities

In this section we quote an existence result from Costea and Varga [28]. Let $X_1, \dots, X_N$ be reflexive Banach spaces for some $N \geq 1$ natural number. Let $Y_1, \dots, Y_N$ be Banach spaces such that there are compact operators $T_k : X_k \to Y_k$ for all $k = 1, \dots, N$. Consider the following *system of nonlinear hemivariational inequalities*, where we need to find $(u_1, \dots, u_N) \in K_1 \times \ldots \times K_N$ such that for all $(v_1, \dots, v_N) \in K_1 \times \ldots \times K_N$

$$\begin{cases} \psi_1(u_1, \dots, u_N; v_1) + J_1^0(\hat{u}_1, \dots, \hat{u}_N; \hat{v}_1 - \hat{u}_1) \geq \langle F_1(u_1, \dots, u_n), v_1 - u_1 \rangle_{X_1} \\ \vdots \\ \psi_N(u_1, \dots, u_N; v_N) + J_N^0(\hat{u}_1, \dots, \hat{u}_N; \hat{v}_N - \hat{u}_N) \geq \langle F_N(u_1, \dots, u_n), v_N - u_N \rangle_{X_N} \end{cases} \tag{2.3}$$

where for each $k = 1, \dots, N$, $\hat{u}_k = T_k(u_k)$ and

1. $K_k \subset X_k$ is a nonempty closed set;
2. $\psi_k : X_1 \times \cdots \times X_N \times X_k \to \mathbb{R}$ is such that
   a) $\psi_k(u_1, \dots, u_N; u_k) = 0$ for all $u_k \in X_k$,
   b) For fixed $v_k \in X_k$ the map $(u_1, \dots, u_N) \mapsto \psi_k(u_1, \dots, u_N; u_k)$ is weakly upper semicontinuous,
   c) For fixed $(u_1, \dots, u_N) \in X_1 \times \ldots \times X_N$ the map $v_k \mapsto \psi_k(u_1, \dots, u_N; u_k)$ is convex;
3. $F_k : X_1 \times \ldots \times X_n \to X_k^*$ is such that

$$\liminf_{n\to\infty} \langle F_k(u_1^n, \dots, u_N^n), v_k - u_k^n \rangle_{X_k} \geq \langle F_k(u_1, \dots, u_n), v_k - u_k \rangle_{X_k}$$

   for any $(u_1^n, \dots, u_N^n) \rightharpoonup (u_1, \dots, u_N)$ and $v_k \in X_k$.

Using Lin's fixed point theorem [59], the following existence result can be proved.

Theorem 2.5.1. [28, Theorem 3.1] Under the above assumptions the system of hemivariational inequalities (2.3) admits a solution.

2.6 Equiintegrability and tightness

In this section we collect a few elementary results from analysis needed in the sequel. See e.g. [79] for proofs.

Definition 2.6.1. A sequence $\{f_j\}$ of measurable functions $f_j\colon \Omega \to \mathbb{R}$ is said to be *equi-integrable* over Ω if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f_j| < \varepsilon \quad \text{for all } j \in \mathbb{N} \text{ and all } E \subset \Omega \text{ measurable with } |E| < \delta.$$

Definition 2.6.2. A sequence $\{f_j\}$ is said to be *tight* over Ω if for all $\varepsilon > 0$ there exists $E_0 \subset \Omega$ measurable with $|E_0| < \infty$ such that

$$\int_{\Omega\setminus E_0} |f_j| < \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

Clearly, a dominated sequence inherits equiintegrability (tightness). More precisely, if $|g_j| \leq |f_j|$ and $\{f_j\}$ is equiintegrable (tight), then $\{g_j\}$ is equiintegrable (tight). Similarly, equiintegrability (tightness) is inherited to a smaller domain $\Omega' \subset \Omega$. The following useful properties are easily established.

Proposition 2.6.3. The following statements hold.

1. If $\{f_j\} \subset L^1(\Omega)$, $f \in L^1(\Omega)$ and $f_j \to f$ in $L^1(\Omega)$, then $\{f_j\}$ is equiintegrable and tight.
2. If $\{f_j\}$ and $\{g_j\}$ are equiintegrable and tight, then $\{\alpha f_j + \beta g_j\}$ is equiintegrable and tight for all $\alpha, \beta \in \mathbb{R}$.
3. If $\{f_j\} \subset L^q(\Omega)$ is bounded and $\{g_j\} \subset L^{q'}(\Omega)$ (where $q' = q/(q-1)$ and $1 < q < \infty$) with $\{|g_j|^{q'}\}$ equiintegrable and tight, then $\{f_j g_j\}$ is equiintegrable and tight.

Theorem 2.6.4. Suppose that $|\Omega| < \infty$ and let $\{f_j\}$ be equiintegrable over Ω. If $f_j \to f$ a.e. on Ω, then $f \in L^1(\Omega)$ and

$$\int f_j \to \int f \quad \text{as } j \to \infty.$$

Theorem 2.6.5. Let $\{f_j\}$ be equiintegrable and tight over Ω. If $f_j \to f$ a.e. on Ω, then $f \in L^1(\Omega)$ and

$$\int f_j \to \int f \quad \text{as } j \to \infty.$$

Theorem 2.6.6. Suppose that $h_j \geq 0$ a.e. on Ω. Then

$$\int h_j \to 0 \quad \text{as } j \to \infty$$

if and only if $h_j \to 0$ a.e. on Ω and $\{h_j\}$ is equiintegrable and tight over Ω.

3 Linking-type results in nonsmooth critical point theory

This chapter is based on the paper [25].

3.1 Introduction

Nonsmooth critical point theory was initiated in 1981 by Chang [17] who extended various minimax principles due to Ambrosetti and Rabinowitz [3] and Rabinowitz [77] to locally Lipschitz functions and then applied these theoretical results in the study partial differential equations with discontinuous nonlinearities. Since then, this new field has undergone an explosive development as many important results for C^1-functionals have been adapted to non-differentiable functions which are either locally Lipschitz, a sum of a C^1-function and a convex lower semicontinuous function or, more generally, a sum of a locally Lipschitz function and a convex lower semincontinuous function, see e.g. Brezis and Nirenberg [11], Pucci and Serrin [76], Ghoussoub and Preiss [45] and their various nonsmooth generalizations in Livrea, Marano and Motreanu [61], Kurogenis and Papageorgiou [54], Szulkin [91], Marano and Motreanu [63], Arcoya and Carmona [4], just to name a few. More details and connections regarding critical point theory and its applications can be found in the books of Jabri [49] and Ghoussoub [44], for C^1-functions and Gasinski and Papageorgiou [43], Motreanu and Panagiotopoulos [70], Motreanu and Rădulescu [71], or more recently, Motreanu, Motreanu and Papageorgiou [69], for nonsmooth functions.

At the beginning of 1990's Schechter, see [80, 81], developed a critical point theory for C^1-functionals defined on a Hilbert space of the type $G : \overline{B}(0,R) \subset H \to \mathbb{R}$, by proving a deformation result which does not require the classical Palais-Smale compactness condition, but uses instead a boundary condition on a certain region of the sphere $\partial B(0,R)$ which prevents deformations from exiting the ball. The deformation lemma ensures the existence of a bounded Palais-Smale sequence, i.e. $\{u_n\} \subset \overline{B}(0,R)$ such that $\{G(u_n)\}$ is bounded and $G'(u_n) \to 0$ as $n \to \infty$. If the boundary condition is dropped, then one obtains an alternative: either a bounded Palais-Smale sequence exists, or there exists a sequence in $\partial B(0,R)$ tending to a negative eigenvalue.

The aim of this chapter is to extend Schechter's results to locally Lipschitz functions defined on closed ball of a real reflexive Banach space with strictly convex dual. In this new setting, Schechter's approach fails even for C^1-functions as the norm cannot be expressed in terms of an inner product. In order to overcome this difficulty we exploit the properties of the duality mapping. Another point of interest in our approach comes from the fact that problems involving the p-Laplacian (with $p \in (1, \infty)$) can now be tackled, in contrast to the case where only $p = 2$ is allowed.

3.2 Construction of the Pseudogradient Vector Field

The proof of the deformation lemma in Section 3.3 relies on the existence of a certain vector field. We begin with a simple compactness result.

Proposition 3.2.1. Let $E : X \to \mathbb{R}$ be a locally Lipschitz functional. If $u \in X$, $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ are such that $u_n \to u$ and $u_n^* \in \partial E(u_n)$, for all $n \in \mathbb{N}$, then there exist $u^* \in \partial E(u)$ and a subsequence $\{u_{n_k}^*\}$ of $\{u_n^*\}$ such that $u_{n_k}^* \rightharpoonup u^*$ in X^*.

Proof. The upper semicontinuity of ∂f together with Theorem 2.4.1 (iii) ensures that there exists $n_0 \in \mathbb{N}$ such that

$$\partial E(u_n) \subset B_{X^*}(0, 2L_u), \text{ for all } n \geq n_0,$$

with $L_u > 0$ the Lipschitz constant near u. Therefore $\{u_n^*\}$ is a bounded sequence in X^*. Since X is reflexive, X^* is also reflexive, hence $\{u_n^*\}$ possesses subsequence $\{u_{n_k}^*\}$ such that $u_{n_k}^* \rightharpoonup u^*$, for some $u^* \in X^*$. It follows at once that $u^* \in \partial E(u)$ since ∂f is weakly closed. $\square$

The following lemma ensures the existence of a locally Lipschitz vector field which plays the role of a pseudo-gradient field in the smooth case and will be used in the sequel.

Lemma 3.2.2. Let $E : X \to \mathbb{R}$ be a locally Lipschitz functional and let $F_0 \subset F \subset X$ be such that

(A) there exists $\gamma > 0$ such that $|\partial E|(u) \geq \gamma$, for all $u \in F$;

(B) there exists $\theta \in (0, 1)$ such that

$$0 \notin \mathcal{C}(u, \theta), \text{ for all } u \in F_0,$$

where

$$\mathcal{C}(u, \theta) = \overline{\text{co}}([\partial E]_\theta(u) \cup J(u)) \quad \text{where} \quad [\partial E]_\theta(u) := \partial E(u) + \overline{B}_{X^*}(0, \theta|\partial E|(u))$$

Then there exists a locally Lipschitz vector field $\Lambda : F \to X$ such that

(P1) $\|\Lambda(u)\| \le 1$, for all $u \in F$;
(P2) $\langle u^*, \Lambda(u)\rangle > \theta\gamma/2$, for all $u \in F$ and all $u^* \in \partial E(u)$;
(P3) $\langle J(u), \Lambda(u)\rangle > 0$, for all $u \in F_0$.

Proof. Let $u \in F_0$ be fixed. The Krein-Smulian theorem 2.2.1 implies that the convex set $\mathcal{C}(u,\theta)$ is weakly compact. Using the weak lower semicontinuity of the norm and assumption (B) we deduce that there exists $r_0 > 0$ such that

$$r_0 = \inf_{u^* \in \mathcal{C}(u,\theta)} \|u^*\|.$$

Since $B_{X^*}(0,r_0) \cap \mathcal{C}(u,\theta) = \emptyset$, the first geometric form of the Hahn-Banach separation theorem 2.2.1 implies that there exists $w_u \in \partial B_X(0,1)$ and $\alpha \in \mathbb{R}$ such that

$$\langle v^*, w_u\rangle \le \alpha \le \langle u^*, w_u\rangle \quad \text{for all} \quad v^* \in B_{X^*}(0,r_0) \quad \text{and all} \quad u^* \in \mathcal{C}(u,\theta).$$

Taking supremum with respect to v^*, we get

$$0 < r_0 \le \langle u^*, w_u\rangle, \quad \text{for all} \quad u^* \in \mathcal{C}(u,\theta). \tag{3.1}$$

In particular,

$$\langle J(u), w_u\rangle > 0. \tag{3.2}$$

We claim that

$$\langle u^*, w_u\rangle > \theta\gamma/2, \quad \text{for all} \quad u^* \in \partial E(u). \tag{3.3}$$

Recall that $\langle u^*, w_u\rangle = \operatorname{dist}(u^*, \ker w_u)$ (see Proposition 2.2.2). Therefore, it suffices to prove that $\operatorname{dist}(\partial E(u), \ker w_u) > \theta\gamma/2$. Let $v^* \in \ker w_u$ be fixed. Obviously $v^* \notin [\partial E]_\theta(u)$, otherwise v^* would belong to $\mathcal{C}(u,\theta)$ and (3.1) would be violated. By the definition of $[\partial E]_\theta(u)$, we have $\operatorname{dist}(v^*, \partial E(u)) \ge \theta|\partial E|(u)$. Since v^* was arbitrary it follows that

$$\operatorname{dist}(\partial E(u), \ker w_u) \ge \theta|\partial E|(u) > \theta\gamma/2.$$

We prove next that there exists $r_u > 0$ such that

$$\langle v^*, w_u\rangle > \theta\gamma/2, \text{ for all } v \in B_X(u,r_u) \cap F \text{ and all } v^* \in \partial E(v), \tag{3.4}$$

and

$$\langle J(v), w_u\rangle > 0, \text{ for all } v \in B_X(u,r_u). \tag{3.5}$$

Arguing by contradiction, assume that (3.4) does not hold, i.e. for each $r > 0$ there exist $v \in B_X(u,r) \cap F$ and $v^* \in \partial E(v)$ such that

$$\langle v^*, w_u\rangle \le \theta\gamma/2.$$

Taking $r = 1/n$ we obtain the existence of two sequences $\{v_n\} \subset X$ and $\{\zeta_n\} \subset X^*$ such that

$$v_n \to u, \ v_n^* \in \partial f(v_n) \text{ and } \langle v_n^*, w_u\rangle \leq \theta\gamma/2.$$

According to Proposition 3.2.1 there exists $v_0^* \in \partial E(u)$ such that, up to a subsequence,

$$v_n^* \rightharpoonup v_0^*, \text{ in } X^*.$$

Letting $n \to \infty$ we get $\langle v_0^*, w_u\rangle \leq \theta\gamma/2$ which contradicts (3.3). Relation (3.5) may be proved in a similar manner by using the fact that J is demicontinuous on reflexive Banach spaces, i.e. if $v_n \to u$ in X, then $J(v_n) \rightharpoonup J(u)$ in X^* (see e.g. [38, Proposition 3]).

If $u \in F \setminus F_0$, we can employ a similar argument as above with $\partial E(u)$ instead of $\mathcal{C}(u,\theta)$ to get the existence of an element $w_u \in \partial B_X(0,1)$ such that (3.3) holds.

Thus, the family $\{B_X(u, r_u)\}_{u\in F}$ is an open covering of F and it is paracompact, hence it possesses a locally finite refinement say $\{U_\alpha\}_{\alpha\in I}$. Standard arguments ensure the existence of a locally Lipschitz partition of unity, denoted $\{\rho_\alpha\}_{\alpha\in I}$, subordinated to the covering $\{U_\alpha\}_{\alpha\in I}$. The required locally Lipschitz vector field $\Lambda : F \to X$ can now be defined by

$$\Lambda(u) = \sum_{\alpha\in I} \rho_\alpha(u) w_u.$$

Simple computations show Λ satisfies the required conditions. □

The following proposition provides an equivalent form of condition (B) in the previous lemma, which will be useful the the following sections.

Proposition 3.2.3. Let $u \in X \setminus \{0\}$ and $\theta \in (0,1)$ be fixed. Then the following statements are equivalent

(*i*) $0 \notin \mathcal{C}(u,\theta)$;

(*ii*) $\mathbb{R}_- J(u) \cap [\partial E]_\theta(u) = \emptyset$.

Proof. $(i) \Rightarrow (ii)$ Arguing by contradiction, assume there exist $\alpha \in \mathbb{R}_-$ and $u^* \in [\partial E]_\theta(u)$ such that $u^* = \alpha J(u)$. Then, for $t = \frac{1}{1-\alpha} \in (0,1]$ we get

$$0 = \frac{1}{1-\alpha}(-\alpha J(u) + u^*) = (1-t)J(u) + tu^*,$$

which shows that $0 \in \mathcal{C}(u,\theta)$, contradicting (i).

$(ii) \Rightarrow (i)$ Assume by contradiction that $0 \in \mathcal{C}(u,\theta)$. Then there exist $t_n \in [0,1]$ and $u_n^* \in [\partial f]_\theta(u)$ such that

$$w_n^* := (1-t_n)J(u) + t_n u_n^* \to 0, \text{ as } n \to \infty.$$

Since $\{t_n\}$ is a bounded sequence in $\mathbb{R}$, it follows that it possesses a subsequence $\{t_{n_k}\}$ such that

$$t_{n_k} \to t \in [0,1].$$

Obviously the set $[\partial E]_\theta(u)$ is bounded, hence if $t = 0$, then $t_{n_k} u^*_{n_k} \to 0$. Thus $w^*_{n_k} \to J(u)$ and the uniqueness of the limit leads to $J(u) = 0$ which is a contradiction, as $u \neq 0$.

If $t \in (0,1]$, then

$$u^*_{n_k} = \frac{1}{t_{n_k}} w^*_{n_k} + \frac{t_{n_k} - 1}{t_{n_k}} J(u) \to \frac{t-1}{t} J(u) \in \mathbb{R}_- J(u).$$

Since $[\partial E]_\theta(u)$ is also closed, it follows that $\frac{t-1}{t} J(u) \in [\partial E]_\theta(u)$, but this contradicts (ii).

□

Remark. Assume $E : \overline{B}_R \to \mathbb{R}$ is a C^1-functional and fix $u \in S_R$ and $\theta \in (0,1)$. Then

$$[\partial E]_\theta(u) = E'(u) + \theta \|E'(u)\| \overline{B}_{X^*}(0,1) = \overline{B}_{X^*}\left(E'(u), \theta \|E'(u)\|\right),$$

and according to Proposition 3.2.3, $0 \notin \mathcal{C}(u,\theta)$ if and only if

$$\mathbb{R}_- J(u) \cap \overline{B}_{X^*}(E'(u), \theta \|E'(u)\|) = \emptyset. \tag{3.6}$$

If in addition X is a Hilbert space endowed with the inner product $(\cdot,\cdot)$, then $J =$identity and condition (3.6) becomes

$$\|\alpha u - E'(u)\| > \theta \|E'(u)\|, \text{ for all } \alpha \in \mathbb{R}_-,$$

which can be equivalently rewritten as

$$R^2\alpha^2 - 2\left(E'(u), u\right)\alpha + \left(1-\theta^2\right)\|E'(u)\|^2 > 0, \text{ for all } \alpha \in \mathbb{R}_-. \tag{3.7}$$

Taking $a = R^2$, $b = -2\left(E'(u), u\right)$, $c = \left(1-\theta^2\right)\|E'(u)\|^2$ and regarding the latter as a quadratic equation in α one can easily check that (3.7) holds if and only if one of the following cases occurs: either $\Delta = b^2 - 4ac < 0$, or $\Delta \geq 0$ and $\frac{-b-\sqrt{\Delta}}{2a} > 0$. Simple computations show that this reduces to

$$(E'(u), u) + \sqrt{1-\theta^2} R \|E'(u)\| > 0, \tag{3.8}$$

which coincides with Schechter's boundary condition

$$(E'(u), u) + \Theta R \|E'(u)\| \geq 0, \quad \Theta \in (0,1),$$

except when $E'(u) = 0$, that is, u is a critical point of E.

We conclude that our boundary condition $0 \notin \mathcal{C}(u,\theta)$ reduces to Schechter's original boundary condition if E is a C^1-functional and X is a Hilbert space.

3.3 A deformation lemma

We are now in position to prove the main technical tool of this chapter which is given by the following deformation theorem. In what follows, the set $Z \subset \overline{B}_R$ in the statement may be regarded as a "restriction" set that allows us to control the deformation. The reader may think of $Z = \overline{B}_R$ as the "unrestricted" case. Here and hereafter, if $E : \overline{B}_R \to \mathbb{R}$ is a functional and Z is a subset of $\overline{B}_R$, we adopt the following notations

$$E^a = \left\{u \in \overline{B}_R : \ E(u) \le a\right\},$$

and

$$Z_b = \left\{u \in \overline{B}_R : \ d(u, Z) \le b\right\}.$$

Theorem 3.3.1. Let $E : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz and $Z \subset \overline{B}_R$. Assume that there exist $c, \rho \in \mathbb{R}$, $\delta > 0$ and $\theta \in (0, 1)$ such that the following conditions hold:

(H1) $|\partial E|(u) \ge \dfrac{4\delta}{\rho\theta^2}$, on $\left\{u \in \overline{B}_R : \ |E(u) - c| \le 3\delta\right\} \cap Z_{3\rho}$;

(H2) $0 \notin \mathcal{C}(u, \theta)$, on $\{u \in S_R : \ |E(u) - c| \le 3\delta\} \cap Z_{3\rho}$.

Then there exists a continuous map $\sigma : [0, 1] \times \overline{B}_R \to \overline{B}_R$ such that:

(*i*) $\sigma(0, \cdot) = \mathrm{Id}$;

(*ii*) $\sigma(t, \cdot) : \overline{B}_R \to \overline{B}_R$ is a homeomorphism for all $t \in [0, 1]$;

(*iii*) $\sigma(t, u) = u$, for all $u \in \overline{\overline{B}_R \setminus \left\{u \in \overline{B}_R : \ d(u, Z) \le 2\rho, \ |E(u) - c| \le 2\delta\right\}}$;

(*iv*) The function $E(\sigma(\cdot, u))$ is nonincresing for all $u \in \overline{B}_R$. Moreover, $E(\sigma(t, u)) < E(u)$, whenever $\sigma(t, u) \ne u$;

(*v*) $\|\sigma(t_1, u) - \sigma(t_2, u)\| \le \rho\theta|t_1 - t_2|$ for all $t_1, t_2 \in [0, 1]$;

(*vi*) $\sigma\left(1, E^{c+\delta} \cap Z\right) \subseteq E^{c-\delta} \cap Z_\rho$.

Proof. Let us define the following subsets of $\overline{B}_R$ as follows

$$F = \left\{u \in \overline{B}_R : \ |\partial E|(u) \ge \frac{4\delta}{\rho\theta^2}\right\},$$

$$F_0 = \{u \in S_R : \ d(u, Z) \le 3\rho, \ |E(u) - c| \le 3\delta\},$$

$$F_1 = \left\{u \in \overline{B}_R : \ d(u, Z) \le 2\rho, \ |E(u) - c| \le 2\delta\right\},$$

and

$$F_2 = \left\{u \in \overline{B}_R : \ d(u, Z) \le \rho, \ |E(u) - c| \le \delta\right\},$$

and consider the locally Lipschitz function $\chi : \overline{B}_R \to \mathbb{R}$ defined as

$$\chi(u) = \frac{d(u, \overline{B}_R \setminus F_1)}{d(u, \overline{B}_R \setminus F_1) + d(u, F_2)}.$$

Obviously $\chi \equiv 0$ on $\overline{\overline{B}_R \setminus F_1}$, whereas $\chi \equiv 1$ on F_2 and $0 < \chi < 1$ in-between. Applying Lemma 3.2.2 with F and F_0 defined as above, we get the existence of a locally Lipschitz vector field $\Lambda : F \to X$ having the properties $(P1)$-$(P3)$. Using the cutoff function we define $V : \overline{B}_R \to X$ to be given by

$$V(u) = \begin{cases} -\chi(u)\Lambda(u), & \text{if } u \in F, \\ 0, & \text{otherwise.} \end{cases}$$

Then V can be extended to a locally Lipschitz and globally bounded map defined on the whole X by setting

$$V(u) = V\left(\frac{R}{\|u\|}u\right), \text{ whenever } \|u\| > R.$$

By an extended version of the Picard–Lindelöf existence theorem for Banach spaces (see e.g. [81, Lemma 2.11.1]) the initial value problem

$$\begin{cases} \dfrac{d}{dt}\eta(t,u) = V(\eta(t,u)), \\ \quad \eta(0,u) = u. \end{cases}$$

possesses a unique maximal solution $\eta : \mathbb{R} \times X \to X$. We define the required deformation via "time dilation",

$$\sigma(t,\cdot) = \eta(\rho\theta t,\cdot), \text{ for all } t \in \mathbb{R}.$$

The initial value ensures that $\sigma(0,\cdot) = \mathrm{Id}$, thus establishing (i). It follows from the aforementioned result that $\sigma(t,\cdot) : X \to X$ is a homeomorphism (with inverse $\sigma(t,\cdot)^{-1} = \sigma(-t,\cdot)$). For convenience, we denote by $\sigma_u : X \to X$, the orbit defined by $\sigma_u(t) = \sigma(t,u)$, for all $(t,u) \in \mathbb{R} \times X$.

We claim that, for each $u \in \overline{B}_R$, the orbit $\{\sigma_u(t)\}_{t \geq 0}$ lies entirely in $\overline{B}_R$. In order to check this, assume that $T_0 \geq 0$ is such that

$$u_1 := \sigma_u(T_0) \in S_R,$$

and

$$\|\sigma_u(t)\| \leq R, \text{ for all } t \in [0, T_0).$$

By Proposition 2.2.3 we have

$$\frac{d}{dt}\psi\left(\|\sigma_u(t)\|\right) = \rho\theta\left\langle J(\sigma_u(t)), V(\sigma_u(t))\right\rangle, \tag{3.9}$$

and

$$\langle J(\sigma_u(t)), V(\sigma_u(t))\rangle = \begin{cases} -\chi(\sigma_u(t))\langle J(\sigma_u(t)), \Lambda(\sigma_u(t))\rangle, & \text{if } \sigma_u(t) \in F, \\ 0, & \text{otherwise ,} \end{cases} \tag{3.10}$$

whenever $\sigma_u(t) \neq 0$.

If $u_1 \in F_0$, then $\langle J(u_1), \Lambda(u_1)\rangle > 0$, hence there exists a neighborhood U of u_1 such that

$$\langle J(v), \Lambda(v)\rangle > 0, \text{ for all } v \in U \cap F. \tag{3.11}$$

The continuity of $\sigma_u(\cdot)$ and relations (3.9)-(3.11) ensure that

$$\frac{d}{dt}\psi(\|\sigma_u(t)\|) \leq 0,$$

holds in a neighborhood $[T_0, T_0 + s)$ of T_0.

If $u_1 \notin F_0$, then V vanishes in a neighborhood of u_1 and by a similar reasoning we obtain

$$\frac{d}{dt}\psi(\|\sigma_u(t)\|) = 0, \text{ for all } t \in [T_0, T_0 + s).$$

Thus $\psi(\|\sigma_u(\cdot)\|)$ is nonincreasing in $[T_0, T_0+s)$, while $\psi(\cdot)$ is strictly increasing on $\mathbb{R}_+$, hence $\|\sigma_u(t)\| \leq R$ for all $t \in [T_0, T_0 + s)$. The argument can be repeated whenever $\{\sigma_u(t)\}_{t\geq 0}$ reaches S_R.

Henceforth we restrict σ to $[0,1] \times \overline{B}_R$, without changing the notation. It is clear from above that $\sigma(t, \cdot)$ is a homeomorphism for all $t \in [0,1]$ and $\chi \equiv 0$ on $\overline{\overline{B}_R \setminus F_1}$, therefore (ii) and (iii) hold.

In order to prove (iv), fix $u \in \overline{B}_R$ and define $h : [0,1] \to \mathbb{R}$ by $h(t) = E(\sigma_u(t))$. Then h is differentiable almost everywhere (see e.g. Chang [17, Proposition 9]) and for a.e. $s \in [0,1]$ we have

$$\begin{aligned} h'(s) &\leq \max\{\langle v^*, \sigma_u'(s)\rangle : \ v^* \in \partial E(\sigma_u(s))\} \\ &= \max\{\rho\theta\, \langle v^*, V(\sigma_u(s))\rangle : \ v^* \in \partial E(\sigma_u(s))\}. \end{aligned}$$

Since Λ satisfies property $(P2)$ and χ vanishes on $\overline{\overline{B}_R \setminus F_1}$, we get $h'(s) \leq 0$ if $\sigma_u(s) \in \overline{\overline{B}_R \setminus F_1}$ and

$$\begin{aligned} h'(s) &\leq -\rho\theta\chi(\sigma_u(s))\langle v^*, \Lambda(\sigma_u(s))\rangle \\ &\leq -\rho\theta\chi(\sigma_u(s))\frac{\theta}{2}\frac{4\delta}{\rho\theta^2} \\ &= -2\delta\chi(\sigma_u(s)), \end{aligned}$$

otherwise. This shows that $E(\sigma_u(\cdot))$ is nonincreasing.

If $\sigma_u(t) \neq u$, then $t > 0$ and $\sigma_u(t) \notin \overline{\overline{B}_R \setminus F_1}$. Therefore there exists $\varepsilon > 0$ such that $\sigma_u(s) \notin \overline{\overline{B}_R \setminus F_1}$ for all $s \in (t - \varepsilon, t + \varepsilon)$. Thus $\chi(\sigma_u(s)) > 0$ for all $s \in (t - \varepsilon, t)$ and

$$E(\sigma_u(t)) - E(u) = E(\sigma_u(t)) - E(\sigma_u(0)) = \int_0^t h'(s)ds < 0.$$

For a fixed $u \in \overline{B}_R$ and $0 \leq t_1 < t_2 \leq 1$ we have

$$\|\sigma_u(t_2) - \sigma_u(t_1)\| = \left\| \int_{t_1}^{t_2} \sigma_u'(s) ds \right\| \leq \rho\theta \int_{t_1}^{t_2} \|V(\sigma_u(s))\| ds \leq \rho\theta(t_2 - t_1),$$

which shows that (v) holds. Moreover, if $u \in Z$, then

$$\|\sigma_u(t) - u\| \leq \rho\theta t < \rho,$$

hence $\sigma_u(t) \in Z_\rho$, for all $t \in [0,1]$.

Finally, in order to complete the proof it suffices to show that for any $u \in Z \subset \overline{B}_R$ such that $E(u) \leq c + \delta$ we have $E(\sigma_u(1)) \leq c - \delta$. We distinguish two cases:

(i) $E(u) \leq c - \delta$. Then

$$E(\sigma_u(1)) \leq E(\sigma_u(0)) = E(u) \leq c - \delta.$$

(ii) $c - \delta < E(u) \leq c + \delta$. Then $u \in F_2$. Let $t_{max} \in [0,1]$ be the maximal time for which the $\sigma_u(\cdot)$ does not exit F_2, i.e.

$$\sigma_u(t) \in F_2 \text{ for } t \in [0, t_{max}].$$

If $t_{max} = 1$, then $\chi(\sigma_u(s)) = 1$ for all $s \in [0,1]$ and

$$E(\sigma_u(1)) - E(u) = \int_0^1 h'(s) ds \leq \int_0^1 -2\delta\chi(\sigma_u(s))\, ds = -2\delta,$$

which leads to

$$E(\sigma_u(1)) \leq E(u) - 2\delta \leq c + \delta - 2\delta = c - \delta.$$

If $t_{max} < 1$, then there exists $t_0 \in (t_{max}, 1]$ such that $\sigma_u(t_0) \notin F_2$. Since $\sigma_u(t_0) \in Z_\rho$, it follows that either $E(\sigma_u(t_0)) < c - \delta$, or $E(\sigma_u(t_0)) > c + \delta$. The latter cannot occur due to (i) and (iv).

$\square$

Before proceeding to more sophisticated results, we present a prototypical application of the deformation lemma above.

Theorem 3.3.2. Let $E : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz function such that

$$m_R := \inf_{B_R} E > -\infty. \tag{3.12}$$

Suppose that there exist $\theta \in (0,1)$ and $\varepsilon > 0$ such that

$$0 \notin \mathcal{C}(u,\theta), \text{ on } \{u \in S_R : \; |E(u) - m_R| \leq \varepsilon\}.$$

Then there exists a sequence $\{u_n\} \subset \overline{B}_R$ such that

$$E(u_n) \to m_R \text{ and } |\partial E|(u_n) \to 0.$$

Proof. Arguing by contradiction, assume that such a sequence does not exist. Then there exist γ, $\delta > 0$ such that

$$|\partial E|(u) \geq \gamma, \text{ on } \left\{u \in \overline{B}_R : \; |E(u) - m_R| \leq 3\delta\right\}.$$

Shrinking δ if necessary, we may assume that $3\delta \leq \varepsilon$. Applying Theorem 3.3.1 with $Z = \overline{B}_R$ and $c = m_R$ and $\rho = \frac{4\delta}{\gamma\theta^2}$ we get the existence of a continuous deformation $\sigma : [0,1] \times \overline{B}_R \to \overline{B}_R$ which satisfies

$$\sigma\left(1, E^{m_R+\delta}\right) \subseteq E^{m_R-\delta}. \tag{3.13}$$

Due to (3.12), the set in the left-hand side is nonempty, while the set in the right-hand side is empty, thus (3.13) yields a contradiction. □

3.4 A minimax theorem in the presence of linking

A number of different definitions are in use for "linking" see [81, p. 100], [49, p. 226] and [43, p. 136] for the relations between them. In this section we shall work with Schechter's definition of linking for the ball $\overline{B}_R$. To this end we introduce the *family of admissible deformations* to be the set $\mathcal{G} \subset C([0,1] \times \overline{B}_R, \overline{B}_R)$ whose elements $\Gamma \in \mathcal{G}$ satisfy:

(*G1*) For each $t \in [0,1)$, $\Gamma(t,\cdot) : \overline{B}_R \to \overline{B}_R$ is a homeomorphism;

(*G2*) $\Gamma(0,\cdot) = \mathrm{Id}$;

(*G3*) For each $\Gamma \in \mathcal{G}$, there exists $u_\Gamma \in \overline{B}_R$ such that $\Gamma(1,u) = u_\Gamma$ for all $u \in \overline{B}_R$ and $\Gamma(t,u) \to u_\Gamma$ uniformly as $t \to 1$.

If follows from property (G3) that every $\Gamma \in \mathcal{G}$ is a contraction of $\overline{B}_R$ to a point.

Definition 3.4.1. We say that $A \subset \overline{B}_R$ *links* $B \subset \overline{B}_R$ w.r.t. $\mathcal{G}$ if

(*L1*) $A \cap B = \emptyset$;

(*L2*) For every $\Gamma \in \mathcal{G}$ there exists $t \in (0,1]$ such that $\Gamma(t,A) \cap B \neq \emptyset$.

For various examples to linking, see [81].

The following linking-type theorem says that if A and B are linked, i.e. cannot be pulled apart without intersecting and the energy over A is dominated by the energy over B, then there is a *bounded* sequence whose energy is converging to a minimax level – given that a certain boundary condition holds on S_R. For later convenience we introduce the following notation for the above mentioned condition.

Definition 3.4.2. The sets $A, B \subset \overline{B}_R$ are *linked* (w.r.t. $E : \overline{B}_R \to \mathbb{R}$) if

$$(\mathrm{LC})_{A,B,E} \begin{cases} (i) & \overline{B}_R \supset A \text{ links } B \subset \overline{B}_R \text{ w.r.t } \mathcal{G}; \\ (ii) & \sup\limits_A E := a_0 \leq b_0 := \inf\limits_B E; \\ (iii) & c_R := \inf\limits_{\Gamma \in \mathcal{G}} \sup\limits_{\substack{t \in [0,1] \\ u \in A}} E(\Gamma(t,u)) < +\infty. \end{cases}$$

The following is a direct generalization of Schechter's result [81, Theorem 5.2.1] to the nonsmooth context.

Theorem 3.4.3. Let $E : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz functional such that $(\mathrm{LC})_{A,B,E}$ holds for some $A, B \subset \overline{B}_R$. Suppose that there exist $\theta \in (0,1)$ and $\varepsilon > 0$ such that

$$0 \notin \mathcal{C}(u,\theta), \text{ on } \{u \in S_R : \ |E(u) - c_R| \leq \varepsilon\}. \tag{3.14}$$

Then there exists a sequence $\{u_n\} \subset \overline{B}_R$ such that

$$E(u_n) \to c_R \text{ and } |\partial E|(u_n) \to 0.$$

Furthermore, if $c_R = b_0$, then $d(u_n, B) \to 0$ also holds.

Proof. Clearly, $b_0 \leq c_R$. We distinguish two cases.

Case 1. $b_0 < c_R$.

Assume by contradiction that a sequence satisfying the required properties does not exist. Then one can find $\gamma,\ \delta > 0$ such that

$$|\partial E|(u) \geq \gamma, \text{ on } \left\{u \in \overline{B}_R : \ |E(u) - c_R| \leq 3\delta\right\}.$$

Without loss of generality we may assume that $\delta < \min\{\varepsilon/3, c_R - b_0\}$. For $Z = \overline{B}_R$ and $c = c_R$ and $\rho = \frac{4\delta}{\gamma\theta^2}$, Theorem 3.3.1 ensures that there exists a continuous deformation $\sigma : [0,1] \times \overline{B}_R \to \overline{B}_R$ such that (i)–(vi) hold. We reach contradiction by constructing a deformation $\overline{\Gamma} \in \mathcal{G}$ for which the "sup" in the definition of c_R is actually lower than c_R. By the definition of c_R, there exists $\Gamma \in \mathcal{G}$ such that

$$\sup_{\substack{t\in[0,1]\\ u\in A}} E(\Gamma(t,u)) \leq c_R + \delta.$$

In other words

$$\Gamma(t,A) \subseteq E^{c_R+\delta}, \text{ for all } t \in [0,1]. \tag{3.15}$$

Now let $\overline{\Gamma} : [0,1] \times \overline{B}_R \to \overline{B}_R$ to be defined by

$$\overline{\Gamma}(t,u) = \begin{cases} \sigma\left(4t/3, u\right), & \text{if } t \in [0,3/4], \\ \sigma(1, \Gamma(4t-3,u)), & \text{if } t \in (3/4,1]. \end{cases} \tag{3.16}$$

We claim that $\overline{\Gamma} \in \mathcal{G}$. Obviously (G1) and (G2) follow directly from the deformation theorem. In order to check (G3), let $u_\Gamma \in \overline{B}_R$ be the element for which Γ satisfies (G3), then $u_{\overline{\Gamma}} = \sigma(1, u_\Gamma)$ is suitable for $\overline{\Gamma}$.

Furthermore, we claim that

$$\overline{\Gamma}(t,A) \subseteq E^{c_R-\delta}, \text{ for all } t \in [0,1].$$

Indeed, if $t \in [0,3/4]$, then

$$E\left(\overline{\Gamma}(t,u)\right) = E\left(\sigma(4t/3,u)\right) \leq E(u) \leq a_0 \leq b_0 < c_R - \delta,$$

for all $u \in A$. On the other hand, if $t \in (3/4,1]$ then

$$E\left(\overline{\Gamma}(t,u)\right) = E(\sigma(1,\Gamma(4t-3,u))) \leq c - \delta,$$

for all $u \in A$.

In conclusion we constructed $\overline{\Gamma} \in \mathcal{G}$ such that

$$E\left(\overline{\Gamma}(t,u)\right) \leq c_R - \delta, \text{ for all } u \in A \text{ and all } t \in [0,1],$$

which contradicts the definition of c_R.

Case 2. $b_0 = c_R$.

We point out the fact that it suffices to prove that for any $\gamma,\ \delta > 0$ there exists $u \in \overline{B}_R$ such that

$$|E(u) - c_R| \leq 3\delta,\ d(u,B) \leq \frac{16\delta}{\gamma\theta^2} \text{ and } |\partial E|(u) < \gamma, \tag{3.17}$$

as we can set $\delta = 1/n^2$ and $\gamma = 1/n$ to get the desired sequence.

Assume by contradiction that (3.17) does not hold, i.e. there exist $\gamma,\ \delta > 0$ such that

$$|\partial E|(u) \geq \gamma, \text{ on } \left\{ u \in \overline{B}_R : \ |E(u) - c_R| \leq 3\delta, \ d(u,B) \leq \frac{16\delta}{\gamma\theta^2} \right\}$$

and let $\sigma : [0,1] \times \overline{B}_R \to \overline{B}_R$ be the deformation given by Theorem 3.3.1 with $c = c_R$, $\rho = \frac{4\delta}{\gamma\theta^2}$ and $Z = \{u \in \overline{B}_R : \ d(u,B) \leq \rho\}$.

We claim that

$$\sigma\left(1, E^{c_R+\delta}\right) \cap B = \emptyset, \tag{3.18}$$

and

$$\sigma(t,A) \cap B = \emptyset, \text{ for all } t \in (0,1]. \tag{3.19}$$

If there exists $u \in E^{c_R+\delta}$ such that $\sigma(1,u) \in B$, then

$$\|\sigma(1,u) - u\| = \|\sigma(1,u) - \sigma(0,u)\| \leq \rho\theta < \rho,$$

hence $u \in Z$. Property (vi) implies that

$$E(\sigma(1,u)) \leq c_R - \delta = b_0 - \delta,$$

which violates the definition of b_0.

In order to show that (3.19) holds, assume by contradiction that there exists $(t,u) \in (0,1] \times A$ such that $\sigma(t,u) \in B$. If $\sigma(t,u) = u$, then $u \in A \cap B$, which contradicts the fact that A links B. If $\sigma(t,u) \neq u$, then

$$E(\sigma(t,u)) < E(u) \leq a_0 \leq b_0,$$

and this contradicts the definition of b_0.

Define $\overline{\Gamma} : [0,1] \times \overline{B}_R \to \overline{B}_R$ formally as in (3.16). Clearly, $\overline{\Gamma} \in \mathcal{G}$, but (3.15), (3.18) and (3.19) imply that $\overline{\Gamma}(t,A) \cap B = \emptyset$ for all $t \in (0,1]$ which contradicts the fact that A links B.

□

3.5 The minimax alternative

This section is devoted to the case when the boundary condition is dropped. Of course, one cannot expect to get the existence of a bounded Palais-Smale sequence in this case.

However, we are able to prove that the following alternative holds: either E possesses a Palais-Smale sequence in $\overline{B}_R$, or, there exist $\{u_n\} \subset S_R$ and $u_n^* \in \partial E(u_n)$ such that

$$u_n^* - \frac{\langle u_n^*, u_n\rangle}{R\tau(R)} Ju_n \to 0, \text{ as } n \to \infty.$$

Before stating the theorem, for each $u \in \overline{B}_R$ we define the projection $\pi_u : X^* \to \ker u$ as follows

$$\pi_u(u^*) = \begin{cases} u^* - \frac{\langle u^*, u\rangle}{\|u\|\tau(\|u\|)} Ju, & \text{if } u \neq 0, \\ u^*, & \text{if } u = 0. \end{cases}$$

Obviously,

$$\|\pi_u(u^*)\| = \left\| u^* - \frac{\langle u^*, u\rangle}{\|u\|\tau(\|u\|)} Ju \right\| \le \|u^*\| + \frac{|\langle u^*, u\rangle|}{\|u\|} \le 2\|u^*\|, \text{ for all } u^* \in X^*.$$

For $u \neq 0$ and $\alpha \in \mathbb{R}$ and $u^* \in X^*$ we have the following estimates

$$\begin{aligned} \|u^* - \alpha Ju\| &= \left\| \pi_u(u^*) + \left(\frac{\langle u^*, u\rangle}{\|u\|\tau(\|u\|)} - \alpha \right) Ju \right\| \\ &\le \|\pi_u(u^*)\| + \left| \frac{\langle u^*, u\rangle}{\|u\|\tau(\|u\|)} - \alpha \right| \tau(\|u\|), \end{aligned}$$

and

$$\|\pi_u(u^*)\| = \|\pi_u(u^* - \alpha Ju)\| \le 2\|u^* - \alpha Ju\|.$$

Taking the infimum as $\alpha \in \mathbb{R}$ we get

$$d(u^*, \mathbb{R}Ju) \le \|\pi_u(u^*)\| \le 2d(u^*, \mathbb{R}Ju), \text{ for all } u^* \in X^*. \tag{3.20}$$

Moreover, restricting the infimum to $\mathbb{R}_-$ or $\mathbb{R}_+^*$ we also have

$$\langle u^*, u\rangle \le 0 \Rightarrow d(u^*, \mathbb{R}_- Ju) \le \|\pi_u(u^*)\| \le 2d(u^*, \mathbb{R}_- Ju), \tag{3.21}$$

and

$$\langle u^*, u\rangle > 0 \Rightarrow d(u^*, \mathbb{R}_+^* Ju) \le \|\pi_u(u^*)\| \le 2d(u^*, \mathbb{R}_+^* Ju). \tag{3.22}$$

We will also make use of the following decomposition of $\partial E(u)$

$$\partial E^-(u) = \{u^* \in \partial E(u) : \ \langle u^*, u\rangle \le 0\},$$

and

$$\partial E^+(u) = \{u^* \in \partial E(u) : \ \langle u^*, u\rangle > 0\}.$$

Theorem 3.5.1. Let $E : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz functional and let $A, B \subset \overline{B}_R$ be such that $(\mathrm{LC})_{A,B,E}$ holds. Assume in addition that there exists $\Lambda_R > 0$ such that

$$|\langle u^*, u\rangle| \leq \Lambda_R, \text{ for all } u \in S_R \text{ and all } u^* \in \partial E(u). \tag{3.23}$$

Then the following alternative holds:

Either

(A1) there exists $\{u_n\} \subset \overline{B}_R$ such that

$$E(u_n) \to c_R \text{ and } |\partial E|(u_n) \to 0.$$

Furthermore, if $c_R = b_0$, then $d(u_n, B \cup S_R) \to 0$.

or,

(A2) there exists $\{u_n\} \subset S_R$ and $\{u_n^*\} \subset X^*$ with $u_n^* \in \partial E(u_n)$ such that

$$E(u_n) \to c_R, \ \|\pi_{u_n}(u_n^*)\| \to 0 \text{ and } \langle u_n^*, u_n\rangle \leq 0.$$

Proof. Assume option $(A2)$ does not hold. Then there exist $\gamma, \delta > 0$ such that

$$\|\pi_u(u^*)\| \geq \gamma, \tag{3.24}$$

whenever $u \in S_R$ and $u^* \in \partial E(u)$ satisfy

$$|E(u) - c_R| \leq \delta \text{ and } \langle u^*, u\rangle \leq 0. \tag{3.25}$$

Obviously if there exist $\theta \in (0, 1)$ and $\varepsilon > 0$ such that

$$0 \notin \mathcal{C}(u, \theta), \text{ on } \{u \in S_R : \ |E(u) - c_R| \leq \varepsilon\},$$

then $(A1)$ is obtained via Theorem 3.4.3.

If this is not the case, then for each $n \in \mathbb{N}$ there exists $u_n \in S_R$ such that

$$|E(u_n) - c_R| \leq \frac{1}{n} \text{ and } 0 \in \mathcal{C}\left(u_n, \frac{1}{n}\right).$$

Proposition 3.2.3 implies that $\mathbb{R}_- J u_n \cap [\partial E]_{\theta_n}(u_n) \neq \emptyset$, that is, there exist $u_n^* \in \partial E(u_n)$, $v_n^* \in \overline{B}_{X^*}(0, 1)$ and $w_n^* \in \mathbb{R}_- J u_n$ such that

$$u_n^* + \frac{1}{n}\lambda_E(u_n) v_n^* = w_n^*,$$

hence

$$\begin{aligned} d(u_n^*, \mathbb{R}_- J u_n) &\leq \|u_n^* - w_n^*\| \leq \frac{1}{n}\lambda_E(u_n) \leq \frac{1}{n}\|u_n^*\| \\ &\leq \frac{1}{n}\|\pi_{u_n}(u_n^*)\| + \frac{1}{n}\frac{|\langle u_n^*, u_n\rangle|}{R} \\ &\leq \frac{2}{n} d(u_n^*, \mathbb{R} J u_n) + \frac{\Lambda_R}{nR} \\ &\leq \frac{2}{n} d(u_n^*, \mathbb{R}_- J u_n) + \frac{\Lambda_R}{nR}, \end{aligned}$$

which leads to

$$d(u_n^*, \mathbb{R}_- J u_n) \to 0, \text{ as } n \to \infty. \tag{3.26}$$

Conditions (3.21), (3.24) and (3.25) ensure that there exists $n_0 \in \mathbb{N}$ such that

$$\langle u_n^*, u_n\rangle > 0, \text{ for all } n \geq n_0. \tag{3.27}$$

From (3.26) and (3.27) we deduce that

$$d(\partial E^+(u_n), \mathbb{R}_- J u_n) \to 0, \text{ as } n \to \infty. \tag{3.28}$$

On the other hand, taking the infimum as $u^* \in \partial E^+(u_n)$ in (3.22) and keeping in mind (3.20) we get

$$\begin{aligned} d(\partial E^+(u_n), \mathbb{R}_+^* J u_n) &\leq \inf_{u^* \in \partial E^+(u_n)} \|\pi_{u_n}(u^*)\| \leq 2 \inf_{u^* \in \partial E^+(u_n)} d(u^*, \mathbb{R} J u_n) \\ &\leq 2d(u_n^*, \mathbb{R} J u_n) \leq 2d(u_n^*, \mathbb{R}_- J u_n), \end{aligned}$$

hence

$$d(\partial E^+(u_n), \mathbb{R}_+^* J u_n) \to 0, \text{ as } n \to \infty. \tag{3.29}$$

Relations (3.28) and (3.29) ensure that for sufficiently large $n \in \mathbb{N}$ there exist $\alpha_n \in \mathbb{R}_-$, $\beta_n \in \mathbb{R}_+^*$ and $y_n^*, z_n^* \in \partial E^+(u_n)$ such that

$$\max\{\|y_n^* - \alpha_n J u_n\|, \|z_n^* - \beta_n J u_n\|\} \to 0, \text{ as } n \to \infty.$$

Define $t_n = \frac{\beta_n}{\beta_n - \alpha_n} \in (0,1]$ and $\bar{u}_n^* = t_n y_n^* + (1-t_n) z_n^*$. Since $\partial E^+(u_n)$ is convex it follows that $\bar{u}_n^* \in \partial E^+(u_n)$. Then

$$\begin{aligned} \|\bar{u}_n^*\| &= \|t_n y_n^* + (1-t_n) z_n^*\| \\ &= \|t_n(y_n^* - \alpha_n J u_n) + (1-t_n)(z_n^* - \beta_n J u_n)\| \\ &\leq t_n\|y_n^* - \alpha_n J u_n\| + (1-t_n)\|z_n^* - \beta_n J u_n\| \\ &\leq \max\{\|y_n^* - \alpha_n J u_n\|, \|z_n^* - \beta_n J u_n\|\}. \end{aligned}$$

We have proved thus that there exists $\{u_n\} \subseteq S_R$ and such that $|E(u_n) - c_R| \leq \frac{1}{n}$ and

$$|\partial E|(u_n) \leq \|\bar{u}_n\| \to 0, \text{ as } n \to \infty,$$

that is, $(A1)$ holds.

□

Corollary 3.5.2. Assume the hypotheses of Theorem 3.5.1 are fulfilled. Then there exists $\{u_n\} \subset \overline{B}_R$, $\{u_n^*\} \subset X^*$ with $u_n^* \in \partial E(u_n)$ and $\nu \in \mathbb{R}_-$ such that

$$E(u_n) \to c_R, \quad \|\pi_{u_n}(u_n^*)\| \to 0 \quad \text{and} \quad \langle u_n^*, u_n\rangle \to \nu.$$

Furthermore, if $c_R = b_0$, then $d(u_n, B \cup S_R) \to 0$.

Proof. Suppose that (A1) of the alternative theorem holds, i.e. $E(u_n) \to c_R$ and $|\partial E|(u_n) \to 0$ and let $u_n^* \in \partial E(u_n)$ be such that $\|u_n^*\| = \lambda_E(u_n)$. Then

$$\|\pi_{u_n}(u_n^*)\| \leq 2\|u_n^*\| \to 0, \text{ as } n \to \infty,$$

and

$$|\langle u_n^*, u_n\rangle| \leq \|u_n^*\|\|u_n\| \leq R\|u_n^*\| \to 0, \text{ as } n \to \infty,$$

hence we can choose $\nu = 0$ in this case.

On the other hand, if $(A2)$ holds, then condition (3.23) implies that the sequence $\nu_n := \langle u_n^*, u_n\rangle \leq 0$ is bounded in $\mathbb{R}$ hence possesses a convergent subsequence.

Finally, if $c_R = b_0$, then $(A1)$ implies $d(u_n, B \cup S_R) \to 0$, while $(A2)$ ensures that $d(u_n, S_R) = 0$, hence the proof is complete.

□

Remark. If $E : \overline{B}_R \to \mathbb{R}$ is a C^1-functional, then the conclusion of the previous corollary reads as follows: there exists $\{u_n\} \subset \overline{B}_R$ such that

$$E(u_n) \to c_R, \quad \left\| E'(u_n) - \frac{\langle E'(u_n), u_n\rangle}{\|u_n\|\tau(\|u_n\|)} Ju_n \right\| \to 0, \ \langle E'(u_n), u_n\rangle \to \nu \leq 0,$$

If, in addition, X is a Hilbert space, then this reduces to Schechter's conclusion (see e.g. [81, Corollary 5.3.2.]).

If E is bounded from below, then following similar steps as in the proofs of Theorems 3.3.2, 3.5.1 and Corollary 3.5.2 one can prove the following result.

Theorem 3.5.3. Let $E : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz satisfying (3.12) and (3.23). Then there exist $\{u_n\} \subset \overline{B}_R$, $\{u_n^*\} \subset X^*$ with $u_n^* \in \partial E(u_n)$ and $\nu \in \mathbb{R}_-$ such that

$$E(u_n) \to m_R, \ \|\pi_{u_n}(u_n^*)\| \to 0 \text{ and } \langle u_n^*, u_n\rangle \to \nu.$$

3.6 The Schechter-Palais-Smale compactness condition

In this section we revisit some of the results obtained in the previous sections under the additional assumption that a certain compactness condition holds. Recall that a locally Lipschitz functional $E : X \to \mathbb{R}$ satisfies the *Palais-Smale condition at level c*, $(PS)_c$ for short, if any sequence $\{u_n\} \subset X$ for which

$$E(u_n) \to c \text{ and } |\partial E|(u_n) \to 0, \tag{3.30}$$

possesses a (strongly) convergent subsequence.

Following Schechter, we introduce a compactness condition for locally Lipschitz functionals.

Definition 3.6.1. we say that a locally Lipschitz functional $E : X \to \mathbb{R}$ satisfies the *Schechter-Palais-Smale condition at level c* in $\overline{B}_R$, $(SPS)_c$ for short, if any sequence $\{u_n\} \subset \overline{B}_R$ satisfying:

(SPS1) $E(u_n) \to c$, as $n \to \infty$;

(SPS2) $u_n^* \in \partial E(u_n)$ and $\nu \leq 0$ such that $\|\pi_{u_n}(u_n^*)\| \to 0$ and $\langle u_n^*, u_n\rangle \to \nu \leq 0$,

possesses a (strongly) convergent subsequence.

Theorem 3.6.2. Let $E : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz functional such that the $(\mathrm{LC})_{A,B,E}$ holds for some $A, B \subset \overline{B}_R$. Assume in addition that (3.23) and $(SPS)_{c_R}$ hold. Then the following alternative holds:

(A1') Either there exists $u \in \overline{B}_R$ such that

$$E(u) = c_R \text{ and } 0 \in \partial E(u),$$

(A2') or there exist $u \in S_R$ and $\lambda < 0$ such that

$$E(u) = c_R \text{ and } \lambda Ju \in \partial E(u).$$

Furthermore, in case (A1'), if $c_R = b_0$, then $u \in \bar{B} \cup S_R$.

Proof. If case $(A1)$ of Theorem 3.5.1 holds, then there exists $\{u_n\} \subset \overline{B}_R$ such that

$$E(u_n) \to c_R, \text{ and } |\partial E|(u_n) \to 0.$$

Let $u_n^* \in \partial E(u_n)$ be such that $\|u_n^*\| = |\partial E|(u_n)$. Then

$$\|\pi_{u_n}(u_n^*)\| \leq 2\|u_n^*\| \to 0, \text{ as } n \to \infty,$$

and

$$|\langle u_n^*, u_n\rangle| \leq R\|u_n^*\| \to \nu = 0, \text{ as } n \to \infty.$$

The $(SPS)_{c_R}$ condition there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in \overline{B}_R$ such that $u_{n_k} \to u$ in X. Moreover, $u_{n_k}^* \in \partial E(u_{n_k})$ and $u_{n_k}^* \to 0$, thus Proposition 3.2.1 ensures that $0 \in \partial E(u)$.

If $c_R = b_0$, then $d(u_{n_k}, B \cup S_R) \to 0$, hence $u \in \bar{B} \cup S_R$.

On the other hand, if case $(A2)$ of Theorem 3.5.1 holds, then there exist $\{u_n\} \in S_R$, $u_n^* \in \partial E(u_n)$ and $\nu \leq 0$ such that

$$E(u_n) \to c_R, \ \|\pi_{u_n}(u_n^*)\| \to 0 \text{ and } \langle u_n^*, u_n\rangle \to \nu.$$

The $(SPS)_{c_R}$ condition and Proposition 3.2.1 show that there exist $u \in S_R$, $u^* \in \partial E(u)$ and two subsequences $\{u_{n_k}\}$, $\{u_{n_k}^*\}$ of $\{u_n\}$ and $\{u_n^*\}$, respectively, such that

$$u_{n_k} \to u \text{ and } u_{n_k}^* \rightharpoonup u^*.$$

But J is demicontinuous, hence

$$\pi_{u_{n_k}}(u_{n_k}^*) = u_{n_k}^* - \frac{\langle u_{n_k}^*, u_{n_k}\rangle}{R\tau(R)} J u_{n_k} \rightharpoonup u^* - \frac{\nu}{R\tau(R)} J u,$$

which together with $\pi_{u_{n_k}}(u_{n_k}^*) \to 0$ gives

$$u^* = \frac{\nu}{R\tau(R)} J u \in \partial E(u).$$

If $\nu = 0$, then option $(A1')$ holds, while $\nu < 0$ implies that option $(A2')$ holds for $\lambda = \frac{\nu}{R\tau(R)}$. □

The next result follows directly from Theorem 3.5.3 and the (SPS)-condition.

Theorem 3.6.3. Assume the hypotheses of Theorem 3.5.3 are fulfilled and assume $(SPS)_{m_R}$ also holds. Then there exist $u \in \overline{B}_R$ and $\lambda \leq 0$ such that

$$E(u) = m_R \text{ and } \lambda J u \in \partial E(u).$$

Furthermore, $\lambda \neq 0 \ \Rightarrow \ u \in S_R$.

Assuming the hypotheses of Theorems 3.6.2 and 3.6.3 are simultaneously satisfied, one can obtain multiplicity results of the following type.

Theorem 3.6.4. Let $E : \overline{B}_R \to \mathbb{R}$ be a locally Lipschitz functional such that (3.12) and (3.23) hold. Suppose there exist two subsets A, B of $\overline{B}_R$ such that $(\mathrm{LC})_{A,B,E}$ holds and condition $(SPS)_c$ is satisfied for $c \in \{c_R, m_R\}$. Then there exist $u_1, u_2 \in \overline{B}_R$ and $\lambda_1, \lambda_2 \leq 0$ such that $u_1 \neq u_2$ and

$$\lambda_k J u_k \in \partial E(u_k), \quad k = 1, 2. \tag{3.31}$$

Furthermore, if $\lambda_k < 0$, then $u_k \in S_R$. Also, if there exist $v_0, v_1 \in A \cap B_R$ distinct such that $E(v_1) \leq E(v_0)$ and $v_0 \notin \overline{B}$, then u_1 and u_2 can be chosen in such a way that $v_0 \notin \{u_1, u_2\}$.

Proof. It follows from Theorems 3.6.2 and 3.6.3 that there exist $u_1, u_2 \in \overline{B}_R$ and $\lambda_1, \lambda_2 \leq 0$ such that

$$E(u_1) = m_R \leq c_R = E(u_2), \text{ and } \lambda_k J u_k \in \partial E(u_k), \ k = 1, 2.$$

The fact that $\lambda_k < 0 \Rightarrow u_k \in S_R$, follows directly from Theorems 3.6.2 and 3.6.3, respectively. In order to complete the proof we consider the following cases:

(*i*) $m_R \leq b_0 < c_R$. Then

$$E(u_1) = m_R \leq E(v_1) \leq E(v_0) \leq a_0 \leq b_0 < c_R = E(u_2),$$

hence $u_1 \neq u_2$ and $v_0 \neq u_2$. If $u_1 = v_0$, then $E(v_1) = m_R$, that is v_1 is a global minimum point of E on B_R. As any extremum point of a locally Lipschitz functional is in fact a critical point, we conclude that $0 \in \partial E(v_1)$, which shows that v_1, u_2 satisfy the conclusion of the theorem.

(*ii*) $m_R < b_0 = c_R$. Then

$$E(u_1) = m_R < b_0 = c_R = E(u_2),$$

hence $u_1 \neq u_2$. Moreover, $u_2 \in \overline{B} \cup S_R$ which shows that $v_0 \neq u_2$. Again, if $u_1 = v_0$, then we can replace u_1 with v_1.

(*iii*) $m_R = b_0 = c_R$. Then each point of A is a solution of (3.31). Note that A must have at least two points in order to link B. It is readily seen that we only need to discuss the case $A = \{v_0, v_1\} \subset B_R$ and $v_1 \in \overline{B}$. Let $\rho \in (0, \|v_1 - v_0\|)$ be such that $S_\rho(v_0) \subset B_R$. Then A links $S_\rho(v_0)$ (see [81, Example 1, p. 31]) and

$$m_R \leq \inf_{S_\rho(v_0)} E \leq \inf_{\Gamma \in \mathcal{G}} \sup_{\substack{t \in [0,1] \\ u \in A}} E(\Gamma(t, u)) = m_R.$$

Theorem 3.6.2 ensures that (3.31) possesses a solution $u_* \in S_\rho(v_0) \cup S_R$, hence $u_* \neq v_0$. □

3.7 Application: Differential inclusions

In this section we use the theoretical results obtained in the previous sections to study differential inclusions involving the p-Laplace operator. More exactly we prove that either the problem

$$(P)\begin{cases} -\Delta_p u \in \partial_2 f(x,u(x)), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

possesses at least two nontrivial weak solutions, or the the corresponding eigenvalue problem

$$(P_\lambda)\begin{cases} -\Delta_p u \in \lambda\partial_2 f(x,u(x)), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has a rich family of eigenfunctions corresponding to eigenvalues located in the interval $(0,1)$.

Here, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, is the p-Laplacian, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with $C^{1,\alpha}$ boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function with respect to the second variable and $\partial_2 f(x,t)$ denotes the Clarke subdifferential of the map $t \mapsto f(x,t)$. The p-Laplacian is used in many applications including

- Fluid flows in porous media [55, Example 14.1.2.][39, 93, 53],
- Lane–Emden and Emden–Fowler equation of stellar physics [55, 14.2-14.3],
- Growth and diffusion of sandpiles [5] and
- Image denoising [52].

As usual, we consider the Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \ \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = 1, \dots, N \right\}$$

endowed with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$, with $\|\cdot\|_p$ being the usual norm on $L^p(\Omega)$. Since we work with Dirichlet boundary condition, the natural space to seek weak solution of problem (P) is the Sobolev space

$$W_0^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,p}} = \left\{ u \in W^{1,p}(\Omega) : \ u = 0 \text{ on } \partial\Omega \right\},$$

with the value of u on $\partial\Omega$ understood in the sense of traces.

Definition 3.7.1. A function $u \in W_0^{1,p}(\Omega)$ is a *weak solution* of problem (P) if there exists $\xi \in W^{-1,p'}(\Omega)$ such that $\xi(x) \in \partial_2 f(x,u(x))$ for a.e. $x \in \Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega \xi(x)v(x) \, dx, \text{ for all } v \in W_0^{1,p}(\Omega),$$

Definition 3.7.2. A real number $\lambda \neq 1$ is said to be an *eigenvalue* of (P_λ) if there exist $u_\lambda \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\xi_\lambda \in W^{-1,p'}(\Omega)$ such that $\xi_\lambda(x) \in \partial_2 f(x, u_\lambda(x))$ for a.e. $x \in \Omega$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega \xi_\lambda(x) v(x) \, dx, \text{ for all } v \in W_0^{1,p}(\Omega).$$

The function u_λ satisfying the above relation is called an *eigenfunction* corresponding to λ.

Following a well-known idea of Lions [60], we may regard $-\Delta_p$ as an operator acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ by

$$\langle -\Delta_p u, v \rangle := \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \text{ for all } u, v \in W_0^{1,p}(\Omega).$$

Henceforth we consider $W_0^{1,p}(\Omega)$ to be endowed with the norm $|u|_{1,p} = \|\nabla u\|_p$, which is equivalent to $\|u\|_{1,p}$ due to the Poincaré inequality. Then the duality mapping corresponding to the normalization function $\tau(t) = t^{p-1}$, $J : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ satisfies

$$J(u) = -\Delta_p u. \tag{3.32}$$

It is also known that $-\Delta_p$ is a potential operator in the sense that

$$\psi'(u) = -\Delta_p u,$$

with $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ being the C^1-functional defined as follows

$$\psi(u) = \frac{1}{p} |u|_{1,p}^p = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx.$$

Finally, we note that $X = W_0^{1,p}(\Omega)$ is separable and uniformly convex (see [1, Theorem 3.6] or [38, Theorem 1.6]), therefore the theory developed in the preceding sections is applicable. Here and hereafter, we denote by p^* the critical Sobolev exponent, that is,

$$p^* = \begin{cases} \frac{Np}{N-p}, & \text{if } p < N, \\ \infty, & \text{otherwise.} \end{cases}$$

Assumption 3.7.1. The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

(f1) For all $t \in \mathbb{R}$ the map $x \mapsto f(x,t)$ is measurable and $f(x,0) = 0$;

(f2) For almost all $x \in \Omega$, the map $t \mapsto f(x,t)$ is locally Lipschitz;

(f3) There exists $C > 0$ and $q \in (p, p^*)$ such that

$$|\xi| \leq C|t|^{q-1},$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial_2 f(x,t)$.

Assumption 3.7.2. There exists $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$|u_0|_{1,p}^p \leq p \int_\Omega f(x, u_0(x))\, dx.$$

Theorem 3.7.3. Suppose that Assumptions 3.7.1–3.7.2 hold. Then the following alternative holds:

Either

(a) Problem (P) possesses at least two nontrivial weak solutions;

or,

(b) For each $R \in (|u_0|_{1,p}, \infty)$ problem (P_λ) possesses an eigenvalue $\lambda \in (0,1)$ with the corresponding eigenfunction satisfying $|u_\lambda|_{1,p} = R$.

Proof. Assumption 1 ensures that we can apply the Aubin-Clarke theorem (see e.g. [20, Theorem 2.7.5]) to conclude that the function $F : L^q(\Omega) \to \mathbb{R}$ defined by

$$F(w) = \int_\Omega f(x, w(x))\, dx,$$

is Lipschitz continuous on bounded domains and

$$\partial F(w) \subseteq \int_\Omega \partial_2 f(x, w(x))\, dx, \text{ for all } w \in L^q(\Omega),$$

in the sense that for each $\zeta \in \partial F(w)$, there exists $\xi \in L^{q'}(\Omega)$ such that $\xi(x) \in \partial_2 f(x, w(x))$ for a.e. $x \in \Omega$ and

$$\langle \zeta, w \rangle = \int_\Omega \xi(x) w(x)\, dx.$$

Define now the energy functional $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ as follows

$$E(u) = \frac{1}{p}|u|_{1,p}^p - F(u).$$

It follows from the Rellich-Kondrachov theorem (see e.g. [1, Theorem 6.3]) that the inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, hence E is well defined. Moreover, using Clarke's calculus with subgradients (see [20, Proposition 2.3.1 & Proposition 2.3.3]) we have

$$\partial E(u) \subset -\Delta_p u - \partial F(u).$$

In conclusion, if $\mu \leq 0$ and $u \in W_0^{1,p}(\Omega)$ are such that

$$\mu J u \in \partial E(u),$$

then there exists $\xi \in L^{q'}(\Omega) \subset W^{-1,p'}(\Omega)$ such that $\xi(x) \in \partial_2 f(x, u(x))$ for almost all $x \in \Omega$ and

$$\mu \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - \int_\Omega \xi(x) v(x) \, dx.$$

Moreover, if $\mu = 0$, then u is a weak solution of (P), while $\mu < 0$ implies that $\lambda = \frac{1}{1-\mu} \in (0,1)$ is an eigenvalue of (P_λ), provided that $u \neq 0$.

Fix $R \in (|u_0|_{1,p}, \infty)$. We prove next that $E|_{\overline{B}_R}$ satisfies the hypotheses of Theorem 3.6.4.

Claim 1. The functional E maps bounded sets into bounded sets.

Fix $u \in W_0^{1,p}(\Omega)$ and $M > 0$ such that $|u|_{1,p} \leq M$. According to Lebourg's mean value theorem (see [58]) there exist $t \in (0,1)$ and $\bar{\xi}(x) \in \partial_2 f(x, tu(x))$ such that

$$f(x, u(x)) = f(x, u(x)) - f(x, 0) = \bar{\xi}(x) u(x), \text{ for a.e. } x \in \Omega.$$

Therefore,

$$\begin{aligned} |F(u)| &\leq \int_\Omega |f(x, u(x))| \, dx \leq \int_\Omega |\bar{\xi}(x)||u(x)| \, dx \\ &\leq \int_\Omega C|t|^{q-1} |u(x)|^{q-1} |u(x)| \, dx \\ &\leq C\|u\|_q^q. \end{aligned}$$

Then

$$|E(u)| \leq \frac{1}{p} M^p + C C_q^q M^q,$$

with $C_q > 0$ being the constant given by the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

Claim 2. There exists $\rho \in (0, |u_0|_{1,p})$ such that $E(u) \geq 0$ for all $u \in S_\rho$.

By Assumption 3.7.2 and Claim 1 we have

$$\frac{1}{p} |u_0|_{1,p}^p \leq F(u_0) \leq C C_q^q |u_0|_{1,p}^q.$$

Pick $\rho = \frac{1}{2} \left(\frac{1}{pc_0} \right)^{\frac{1}{q-p}}$, with $c_0 = C C_q^q$. Then $\rho < \left(\frac{1}{pc_0} \right)^{\frac{1}{q-p}} \leq |u_0|_{1,p}$ and for all $u \in W_0^{1,p}(\Omega)$ satisfying $|u|_{1,p} = \rho$ we have

$$\begin{aligned} E(u) &= \frac{1}{p} |u|_{1,p}^p - F(u) \geq \frac{1}{p} |u|_{1,p}^p - c_0 |u|_{1,p}^q \\ &= \left(\frac{1}{p} \right)^{\frac{q}{q-p}} \left(\frac{1}{c_0} \right)^{\frac{p}{q-p}} \left(\frac{1}{2^p} - \frac{1}{2^q} \right) \geq 0. \end{aligned}$$

Claim 3. The functional E satisfies $(SPS)_c$ in $\overline{B}_R$ for all $c \in \mathbb{R}$.

Let $c \in \mathbb{R}$, $\{u_n\} \subset \overline{B}_R$ be such that $E(u_n) \to c$ and assume there exists $\{\zeta_n\} \subset W^{-1,p'}(\Omega)$ satisfying

$$\zeta_n \in \partial E(u_n), \quad \|\pi_{u_n}(\zeta_n)\| \to 0, \ \langle \zeta_n, u_n \rangle \to \nu \leq 0. \tag{3.33}$$

Since $\{u_n\}$ is bounded and $W_0^{1,p}(\Omega)$ is reflexive, it follows that there exist $u \in W_0^{1,p}(\Omega)$ and a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, such that

$$u_n \rightharpoonup u, \ \text{in } W_0^{1,p}(\Omega).$$

We may assume that $|u_n|_{1,p} \to r$. If $r = 0$, then $u_n \to 0$ in $W_0^{1,p}(\Omega)$. Assume now that $r > 0$. Then the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ implies

$$u_n \to u, \ \text{in } L^q(\Omega).$$

Since $\partial E(u_n) \subset -\Delta_p u_n - \partial F(u_n)$, it follows that there exists $\eta_n \in \partial F(u_n)$ such that

$$\zeta_n = -\Delta_p u_n - \eta_n.$$

Since $u_n \to u$ in $L^q(\Omega)$, it follows from Proposition 3.2.1 that there exists $\eta \in \partial F(u)$ such that

$$\eta_n \rightharpoonup \eta, \ \text{in } L^{q'}(\Omega).$$

But $L^{q'}(\Omega)$ is compactly embedded into $W^{-1,p'}(\Omega)$ which means

$$\eta_n \to \eta, \ \text{in } W^{-1,p'}(\Omega).$$

It follows that

$$-\zeta_n - \Delta_p u_n \to \eta, \ \text{in } W^{-1,p'}(\Omega). \tag{3.34}$$

On the other hand, the second relation of (3.33) implies

$$\zeta_n + \frac{\langle \zeta_n, u_n \rangle}{|u_n|_{1,p}^p} \Delta_p u_n \to 0 \text{ in } W^{-1,p'}. \tag{3.35}$$

Adding (3.34) and (3.35) we get

$$\left(1 - \frac{\langle \zeta_n, u_n \rangle}{|u_n|_{1,p}^p}\right)(-\Delta_p u_n) \to \eta, \ \text{in } W^{-1,p'}(\Omega).$$

Consequently,

$$\lim_{n\to\infty}\left(1-\frac{\langle\zeta_n,u_n\rangle}{|u_n|_{1,p}^p}\right)\langle-\Delta_p u_n,u_n-u\rangle=0.$$

But, $\lim_{n\to\infty}(1-\langle\zeta_n,u_n\rangle/|u_n|_{1,p}^p)=1-\nu/r^p\geq 1$, which combined with the above relation gives

$$\lim_{n\to\infty}\langle-\Delta_p u_n,u_n-u\rangle=0.$$

It follows that $u_n\to u$ in $W_0^{1,p}(\Omega)$ due to the fact that $-\Delta_p$ satisfies the $(S)_+$ condition (see e.g. [38, Proposition 2]), that is, if $u_n\rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle-\Delta_p u_n,u_n-u\rangle\leq 0$, then $u_n\to u$.

Claim 4. There exists $\Lambda_R>0$ such that $|\langle\zeta,u\rangle|\leq\Lambda_R$, for all $u\in S_R$ and all $\zeta\in\partial E(u)$.

Fix $u\in S_R$ and $\zeta\in\partial E(u)$. Then there exists $\xi\in W^{-1,p'}(\Omega)$ satisfying $\xi(x)\in\partial_2 f(x,u(x))$ such that

$$\begin{aligned}|\langle\zeta,u\rangle| &= \left|\langle-\Delta_p u,u\rangle-\int_\Omega\xi(x)u(x)\,dx\right|\\ &\leq |\langle-\Delta_p u,u\rangle|+\int_\Omega|\xi(x)|\,|u(x)|\,dx\\ &\leq R^p+C\|u\|_q^q\leq R^p+CC_q^qR^q:=\Lambda_R.\end{aligned}$$

Applying Theorem 3.6.4 with $A=\{0,u_0\}$, $B=S_\rho$ (with $\rho>0$ given by Claim 2), $v_0=0$, $v_1=u_0$ we get the desired conclusion.

$\square$

4 Weak solvability for a contact problem with nonmonotone boundary conditions

This chapter is based on the paper [24].

4.1 Introduction

This chapter focuses on the weak solvability of a general mathematical model which describes the contact between a body and an obstacle. The process is assumed to be static and we work under the small deformations hypothesis. The behavior of the materials is described by a possibly multivalued constitutive law written as a subdifferential inclusion, while the contact between the body and the foundation is described by two inclusions, corresponding to the normal and the tangential directions, each inclusion involving the sum of a Clarke subdifferential and the normal cone of a nonempty, closed and convex set.

Inspired and motivated by some recent papers in the literature we consider a variational formulation in terms of bipotentials for our model. This leads to a system of two inequalities: a hemivariational inequality related to the equilibrium law and a variational inequality related to the functional extension of the constitutive law. The unknown of the system is a pair $(\boldsymbol{u}, \boldsymbol{\sigma})$ consisting of the displacement field and the Cauchy stress field. A key role in our approach is played by the separable bipotential that can be defined as the sum of the constitutive map and its Fenchel conjugate. Bipotentials were introduced in 1991 by de Saxcé & Feng [35] and within a very short period of time this theory has undergone a remarkable development both in pure and applied mathematics as bipotentials were successfully applied in addressing various problems arising in mechanics (non-associated Drücker-Prager models in plasticity [15, 32], cam-clay models in soil mechanics [31, 96], cyclic plasticity [9, 30] and viscoplasticity of metals with kinematical hardening rule [48], Coulomb's friction law [13, 56, 65], displacement-traction models for elastic materials [66], contact models with Signorini's boundary condition [64]). For more details and connections regarding the theory of bipotentials see also [14, 33, 34]. The bipotential approach has the advantage that it allows to approximate simultaneously the displacement field and the Cauchy stress tensor and facilitated the implementation of new and efficient numerical algorithms (see e.g. [42, 36]). However, in all the works we are aware of, the bipotential method has been used only for problems with monotone boundary conditions, mostly expressed as inclusions involving the subdifferential of a proper, convex and lower semicontinuous function. Thus, the variational formulation for these problems leads to a coupled system of variational inequalities. Due to the nonmonotone boundary conditions two major differences arise:

- The set of admissible stress tensors is defined with respect to a given displacement field and depends explicitly on this displacement field, in contrast to the case of monotone boundary conditions when the set of admissible stress tensors is the same for all displacement fields;
- The variational formulation leads to a system of inequalities consisting of a hemivariational inequality and a variational inequality.

Consequently, several difficulties occur in determining the existence of weak solutions since the classical methods fail to be applied directly.

Here and hereafter, m is a positive integer, indices i and j run from 1 to m and the summation convention of the repeated indices is adopted. For a bounded open set $\Omega \subset \mathbb{R}^m$ with sufficiently smooth boundary Γ (e.g. Lipschitz continuous) we denote by $\boldsymbol{\nu}$ the outward unit vector to Γ and we introduce the following function spaces which will play a key role in our approach

$$\begin{aligned}
H &= L^2(\Omega;\mathbb{R}^m)\\
\mathcal{H} &= \left\{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^m :\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\} = L^2(\Omega;\mathcal{S}^m)\\
H_1 &= \{\mathbf{u} \in H :\ \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H}\} = H^1(\Omega;\mathbb{R}^m)\\
\mathcal{H}_1 &= \{\boldsymbol{\tau} \in \mathcal{H} :\ \mathrm{Div}\boldsymbol{\tau} \in H\},
\end{aligned}$$

where $\boldsymbol{\varepsilon}$ and Div are the deformation operator and the divergence operator, respectively and are defined in the following way

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\big(\partial_j u_i + \partial_i u_j\big), \qquad \mathrm{Div}\boldsymbol{\tau} = \sum_{j=1}^m \partial_j \tau_{ij}.$$

These Hilbert spaces are endowed with the following inner products

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega \mathbf{u}\cdot\mathbf{v}, \quad (\boldsymbol{\tau},\boldsymbol{\sigma})_{\mathcal{H}} = \int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma}, \quad (\mathbf{u},\mathbf{v})_{H_1} = (\mathbf{u},\mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}},$$

$$(\boldsymbol{\tau},\boldsymbol{\sigma})_{\mathcal{H}_1} = (\boldsymbol{\tau},\boldsymbol{\sigma})_{\mathcal{H}} + (\mathrm{Div}\boldsymbol{\tau},\mathrm{Div}\boldsymbol{\sigma})_H.$$

We recall that the trace operator $\gamma : H^1(\Omega;\mathbb{R}^m) \to H^{1/2}(\Gamma;\mathbb{R}^m) \subset L^2(\Gamma;\mathbb{R}^m)$ is compact [1]. For the sake of brevity, we will omit to write γ to indicate the Sobolev trace on the boundary, writing $\mathbf{v}$ instead of $\gamma\mathbf{v}$. For a given $\mathbf{v} \in H^{1/2}(\Gamma;\mathbb{R}^m)$ we denote by v_ν and v_τ the normal and the tangential components of $\mathbf{v}$ on the boundary, i.e.

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu} \quad \text{and} \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu},$$

respectively. Similarly, for a tensor field $\boldsymbol{\sigma}$, we define σ_ν and σ_τ to be the *normal* and the *tangential components* of the *Cauchy vector field* $\boldsymbol{\sigma\nu}$, that is

$$\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu},$$

respectively. Recall that the following Green formula holds

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}\boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\mathbf{v}, \text{ for all } \boldsymbol{v} \in H_1. \tag{4.1}$$

Let $\mathcal{S}^m$ denote the subspace of symmetric matrices in $\mathbb{R}^{m\times m}$ endowed with the Frobenius inner product.

4.2 The mechanical model and its strong formulation

Let us consider a body $\mathcal{B}$ which occupies the domain $\Omega \subset \mathbb{R}^m$ $(m = 2, 3)$ with a sufficiently smooth boundary Γ (e.g. Lipschitz continuous) and a unit outward normal $\boldsymbol{\nu}$. The body is acted upon by forces of density $\mathbf{f}_0$ and it is mechanically constrained on the boundary. In order to describe these constraints we assume Γ is partitioned into three Lebesgue measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that Γ_1 has positive Lebesgue measure.

- The body is *clamped* on Γ_1, i.e. the displacement field vanishes here.
- On Γ_2 *surface traction* of force density $\mathbf{f}_2$ act.
- On Γ_3 the body may come in contact with an obstacle which will be referred to as the *foundation.*

The process is assumed to be static and the behavior of the material is modeled by a (possibly multivalued) constitutive law expressed as a subdifferential inclusion. The contact between the body and the foundation is modeled with respect to the normal and the tangent direction respectively, to each corresponding an inclusion involving the sum between the Clarke subdifferential of a locally Lipschitz function and the normal cone of a nonempty, closed and convex set.

It is well-known that the subdifferential of a convex function is a monotone set-valued operator, while the Clarke subdifferential is a set-valued operator which is not necessarily monotone in general (See Section 2.4). This is why we say that the constitutive law is monotone and the boundary conditions are nonmonotone.

The mathematical model which describes the above process is the following.

(P) Find a *displacement* $\mathbf{u} : \Omega \to \mathbb{R}^m$ and a *stress tensor* $\boldsymbol{\sigma} : \Omega \to \mathcal{S}^m$ such that

$$-\mathrm{Div}\boldsymbol{\sigma} = \mathbf{f}_0, \quad \text{in } \Omega \tag{4.2}$$

$$\boldsymbol{\sigma} \in \partial\varphi(\boldsymbol{\varepsilon}(\mathbf{u})), \quad \text{in } \Omega \tag{4.3}$$

$$\mathbf{u} = 0, \quad \text{on } \Gamma_1 \tag{4.4}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2, \quad \text{on } \Gamma_2 \tag{4.5}$$

$$-\sigma_\nu \in \partial_2 j_\nu(\cdot, u_\nu) + N_{C_1}(u_\nu), \quad \text{on } \Gamma_3 \tag{4.6}$$

$$-\boldsymbol{\sigma}_\tau \in h(\cdot, \mathbf{u}_\tau)\partial_2 j_\tau(\cdot, \mathbf{u}_\tau) + N_{C_2}(\mathbf{u}_\tau), \quad \text{on } \Gamma_3 \tag{4.7}$$

where

- $\varphi : \mathcal{S}^m \to \mathbb{R}$ is convex and lower semicontinous,
- $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ and $j_\tau : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ are locally Lipschitz with respect to the second variable and $h : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is a prescribed function.

Here, $C_1 \subset \mathbb{R}$ and $C_2 \subset \mathbb{R}^m$ are nonempty closed and convex subsets and N_{C_k} denotes the normal cone of C_k $(k = 1, 2)$. See Definition 2.4.2.

Relation (4.2) represents the *equilibrium equation* (i.e. Newton's second law), (4.3) is the *constitutive law*, (4.4)-(4.5) are the *displacement and traction boundary conditions* and (4.6)-(4.7) describe the contact between body and the foundation.

Relations between the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$ of the type (4.3) describe the *constitutive laws* of the deformation theory of plasticity, of Hencky plasticity with convex yield function, of locking materials with convex locking functions, etc. For concrete examples and their physical interpretation one can consult Sections 3.3.1 and 3.3.2 in Panagiotopoulos [74] (see also Section 3.1 in [75]). A particular case of interest regarding (4.3) is when the constitutive map φ is Gâteaux differentiable, thus the subdifferential inclusion reducing to

$$\boldsymbol{\sigma} = \varphi'(\boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{4.8}$$

which corresponds to nonlinear elastic materials.

Some classical constitutive laws which can be written in the form (4.8) are presented below:

(i) Assume that φ is defined by

$$\varphi(\boldsymbol{\mu}) = \frac{1}{2}\mathcal{E}\boldsymbol{\mu} : \boldsymbol{\mu},$$

where $\mathcal{E} = (\mathcal{E}_{ijkl})$, $1 \leq i, j, k, l \leq m$ is a fourth order tensor which satisfies the symmetry property

$$\mathcal{E}\boldsymbol{\mu} : \boldsymbol{\tau} = \boldsymbol{\mu} : \mathcal{E}\boldsymbol{\tau}, \text{ for all } \boldsymbol{\mu}, \boldsymbol{\tau} \in \mathcal{S}^m,$$

and the ellipticity property

$$\mathcal{E}\boldsymbol{\mu} : \boldsymbol{\mu} \geq c|\boldsymbol{\mu}|^2, \text{ for all } \boldsymbol{\mu} \in \mathcal{S}^m.$$

In this case (4.8) reduces to the classical *Hooke's law*, that is, $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})$, and corresponds to linearly elastic materials.

(ii) Assume that φ is defined by

$$\varphi(\boldsymbol{\mu}) = \frac{1}{2}\mathcal{E}\boldsymbol{\mu} : \boldsymbol{\mu} + \beta \left|\boldsymbol{\mu} - P_{\mathcal{K}}\boldsymbol{\mu}\right|^2,$$

where $\mathcal{E}$ is the elasticity tensor and satisfies the same properties as in the previous example, $\beta > 0$ is a constant coefficient of the material, $P : \mathcal{S}^m \to \mathcal{K}$ is the projection operator and $\mathcal{K}$ is the nonempty, closed and convex *von Mises set*

$$\mathcal{K} = \left\{ \boldsymbol{\mu} \in \mathcal{S}^m : \ \frac{1}{2}\boldsymbol{\mu}^D : \boldsymbol{\mu}^D \leq a^2, \ a > 0 \right\}.$$

Here the notation $\boldsymbol{\mu}^D$ stands for the *deviator* of the tensor $\boldsymbol{\mu}$, that is, $\boldsymbol{\mu}^D = \boldsymbol{\mu} - \frac{1}{m}\operatorname{tr}(\boldsymbol{\mu})\mathbf{I}$, with $\mathbf{I}$ being the identity tensor.

In this case (4.8) becomes

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + 2\beta(\mathbf{I} - P_{\mathcal{K}})\boldsymbol{\varepsilon}(\mathbf{u}),$$

which is known in the literature as *piecewise linear constitutive law* (see e.g. Han & Sofonea [47]).

(iii) Assume φ is defined by

$$\varphi(\boldsymbol{\mu}) = \frac{k_0}{2}\operatorname{tr}(\boldsymbol{\mu})\mathbf{I} : \boldsymbol{\mu} + \frac{1}{2}\psi\left(\left|\boldsymbol{\mu}^D\right|^2\right),$$

where $k_0 > 0$ is a constant and $\psi : [0, \infty) \to [0, \infty)$ is a continuously differentiable constitutive function.

In this case (4.8) becomes

$$\boldsymbol{\sigma} = k_0 \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + \psi' \left(\left| \boldsymbol{\varepsilon}^D(\mathbf{u}) \right|^2 \right) \boldsymbol{\varepsilon}^D(u),$$

and this describes the behavior of the so-called "Hencky materials" (see e.g. Zeidler [94]).

Boundary conditions of the type (4.6) and (4.7) can model a large class of contact problems arising in mechanics and engineering. For the case $h \equiv 1$ many examples of "nonmonotone" laws of the type

$$-\sigma_\nu \in \partial j_\nu(u_\nu) \text{ and } -\boldsymbol{\sigma}_\tau \in \partial j_\tau(\mathbf{u}_\tau),$$

can be found in [75] Section 2.4, [73] Section 1.4 or [46] Section 2.8.

The case when the function h actually depends on the second variable allows the study of contact problems with *slip-dependent friction law* (see e.g. [26, 68] for antiplane models and [27] for general 3D models). This friction law reads as follows

$$-|\boldsymbol{\sigma}_\tau| \leq \mu(x, |u_\tau|), \ -\boldsymbol{\sigma}_\tau = \mu(x, |\mathbf{u}_\tau|)\frac{\mathbf{u}_\tau}{|\mathbf{u}_\tau|} \text{ if } \mathbf{u}_\tau \neq 0, \tag{4.9}$$

where $\mu : \Gamma_3 \times [0, +\infty) \to [0, +\infty)$ is the *sliding threshold* and it is assumed to satisfy

$$0 \leq \mu(x, t) \leq \mu_0, \text{ for a.e. } x \in \Gamma_3 \text{ and all } t \geq 0,$$

for some positive constant μ_0. It is easy to see that (4.7) can be cast in the form (4.9) simply by choosing

$$h(x, \mathbf{u}_\tau) = \mu(x, |\mathbf{u}_\tau|) \text{ and } j_\tau(x, \mathbf{u}_\tau) = |\mathbf{u}_\tau|.$$

We point out the fact that the above example cannot be written in the form $-\boldsymbol{\sigma}_\tau \in \partial j_\tau(\mathbf{u}_\tau)$ as, in general, for two locally Lipschitz functions h, g there does not exists j such that $\partial j(u) = h(u)\partial g(u)$. We would also like to point out that many boundary conditions of classical elasticity are particular cases of (4.6) and (4.7), in most of these cases the functions j_ν and j_τ being convex, hence leading to monotone boundary conditions. We list below some examples:

(a) *The Winkler boundary condition*

$$-\sigma_\nu = k_0 u_\nu, \ k_0 > 0.$$

This law is used in engineering as it describes the interaction between a deformable body and the soil and can be expressed in the form (4.6) by setting

$$C_1 = \mathbb{R} \text{ and } j_\nu(x, u_\nu) = \frac{k_0}{2} u_\nu^2,$$

More generally, if we want to describe the case when the body may lose contact with the foundation, we can consider the following law

$$\begin{cases} u_\nu < 0 \Rightarrow \sigma_\nu = 0, \\ u_\nu \geq 0 \Rightarrow -\sigma_\nu = k_0 u_\nu, \end{cases}$$

The first relation corresponds to the case when there is no contact, while the second models the contact case. Obviously the above law can be expressed in the form (4.6) by choosing

$$C_1 = \mathbb{R} \text{ and } j_\nu(x, u_\nu) = \begin{cases} 0, & \text{if } u_\nu < 0, \\ \frac{k_0}{2} u_\nu^2, & \text{if } u_\nu \geq 0, \end{cases}$$

In [72] the following nonmonotone boundary conditions were imposed to model the contact between a body and a Winkler-type foundation which may sustain limited values of efforts

$$\begin{cases} u_\nu < 0 \Rightarrow \sigma_\nu = 0, \\ u_\nu \in [0, a) \Rightarrow -\sigma_\nu = k_0 u_\nu, \\ u_\nu = a \Rightarrow -\sigma_\nu \in [0, k_0 a], \\ u_\nu > a \Rightarrow \sigma_\nu = 0. \end{cases}$$

This means that the rupture of the foundation is assumed to occur at those points in which the limit effort is attained. The first condition holds in the noncontact zone, the second describes the zone where the contact occurs and it is idealized by the Winkler law. The maximal value of reactions that can be maintained by the foundation is given by $k_0 a$ and it is accomplished when $u_\nu = a$, with k_0 being the Winkler coefficient. The fourth relation holds in the zone where the foundation has been destroyed. The above Winkler-type law can be written as an inclusion of the type (4.6) by setting

$$C_1 = \mathbb{R} \text{ and } j_\nu(x, u_\nu) = \begin{cases} 0, & \text{if } u_\nu < 0, \\ \frac{k_0}{2} u_\nu^2, & \text{if } 0 \leq u_\nu < a, \\ \frac{k_0}{2} a^2, & \text{if } u_\nu \geq a. \end{cases}$$

Since all of the above example only describe what happens in the normal direction, in order to complete the model we must combine these with boundary conditions concerning $\boldsymbol{\sigma}_\tau$, $\mathbf{u}_\tau$, or both. The simplest cases are $\mathbf{u}_\tau = 0$ (which corresponds to $C_2 = \{0\}$) and $\boldsymbol{\sigma}_\tau = S_\tau$, where $S_\tau = S_\tau(x)$ is given (which corresponds to $j_\tau(x, u_\tau) = -S_\tau \cdot \mathbf{u}_\tau$).

(b) *The Signorini boundary conditions*, which hold if the foundation is rigid and are as follows

$$\begin{cases} u_\nu < 0 \Rightarrow \sigma_\nu = 0, \\ u_\nu = 0 \Rightarrow \sigma_\nu \leq 0, \end{cases}$$

or equivalently,

$$u_\nu \leq 0,\ \ \sigma_\nu \leq 0 \text{ and } \sigma_\nu u_\nu = 0.$$

This can be written equivalently in form (4.6) by setting

$$C_1 = (-\infty, 0] \text{ and } j_\nu \equiv 0.$$

(c) In [65] the following *static version of Coulomb's law of dry friction with prescribed normal stress* was considered

$$\begin{cases} -\sigma_\nu(x) = F(x) \\ |\boldsymbol{\sigma}_\tau| \leq k(x)|\sigma_\nu|, \\ \boldsymbol{\sigma}_\tau = -k(x)|\sigma_\nu|\frac{\mathbf{u}_\tau}{|\mathbf{u}_\tau|},\ \text{if } \mathbf{u}_\tau(x) \neq 0. \end{cases}$$

We can write the above law in the form of (4.6) and (4.7) simply by setting

$$C_1 = \mathbb{R},\ C_2 = \mathbb{R}^m,\ j_\nu(x, u_\nu) = F(x)u_\nu,\ h(x, \mathbf{u}_\tau) = k(x)|F(x)| \text{ and } j_\tau(x, \mathbf{u}_\tau) = |\mathbf{u}_\tau|.$$

The strong formulation of problem **(P)** consists of finding $\mathbf{u} : \Omega \to \mathbb{R}^m$ and $\boldsymbol{\sigma} : \Omega \to \mathcal{S}^m$, regular enough, such that (4.2)-(4.7) are satisfied. However, it is a fact that for most contact problems the strong formulation does not admit a solution.

4.3 Variational formulation

It is useful to reformulate problem **(P)** in a weaker sense, i.e. we shall derive a variational formulation. The assumptions on the functions $f_0,\ f_2,\ \varphi,\ h,\ j_\nu$ and j_τ required to prove our main result are listed below.

($\boldsymbol{H_C}$) The constraint sets C_1 and C_2 are convex cones, i.e.

$$0 \in C_k \quad \text{and} \quad \lambda C_k \subset C_k \text{ for all } \lambda > 0, \quad k = 1, 2.$$

($\boldsymbol{H_f}$) The *force density* and the *traction* satisfy $\mathbf{f}_0 \in H$ and $\mathbf{f}_2 \in L^2(\Gamma_2; \mathbb{R}^m)$.

$(\boldsymbol{H_\varphi})$ The *constitutive function* $\varphi : \mathcal{S}^m \to \mathbb{R}$ and its Fenchel conjugate $\varphi^* : \mathcal{S}^m \to (-\infty, +\infty]$ satisfy

(i) φ is convex and lower semicontinuous;

(ii) there exists $\alpha_1 > 0$ such that $\varphi(\boldsymbol{\tau}) \geq \alpha_1 |\boldsymbol{\tau}|^2$, for all $\boldsymbol{\tau} \in \mathcal{S}^m$;

(iii) there exists $\alpha_2 > 0$ such that $\varphi^*(\boldsymbol{\mu}) \geq \alpha_2 |\boldsymbol{\mu}|^2$, for all $\boldsymbol{\mu} \in \mathcal{S}^m$;

(iv) $\varphi(\boldsymbol{\varepsilon}(\boldsymbol{v})) \in L^1(\Omega)$, for all $\boldsymbol{v} \in V$ and $\varphi^*(\boldsymbol{\tau}) \in L^1(\Omega)$, for all $\boldsymbol{\tau} \in \mathcal{H}$.

$(\boldsymbol{H_h})$ The function $h : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is such that

(i) $\Gamma_3 \ni x \mapsto h(x, \boldsymbol{\zeta})$ is measurable for each $\boldsymbol{\zeta} \in \mathbb{R}^m$;

(ii) $\mathbb{R}^m \ni \boldsymbol{\zeta} \mapsto h(x, \boldsymbol{\zeta})$ is continuous for a.e. $x \in \Gamma_3$;

(iii) there exists $h_0 > 0$ such that $0 \leq h(x, \boldsymbol{\zeta}) \leq h_0$ for a.e. $x \in \Gamma_3$ and all $\boldsymbol{\zeta} \in \mathbb{R}^m$.

$(\boldsymbol{H_{j_\nu}})$ The function $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $\Gamma_3 \ni x \mapsto j_\nu(x, t)$ is measurable for each $t \in \mathbb{R}$;

(ii) there exists $p \in L^2(\Gamma_3)$ such that for a.e. $x \in \Gamma_3$ and all $t_1, t_2 \in \mathbb{R}$

$$|j_\nu(x, t_1) - j_\nu(x, t_2)| \leq p(x)|t_1 - t_2|;$$

(iii) $j_\nu(x, 0) \in L^1(\Gamma_3)$.

$(\boldsymbol{H_{j_\tau}})$ The function $j_\tau : \Gamma_3 \times \mathbb{R}^m \to \mathbb{R}$ is such that

(i) $\Gamma_3 \ni x \mapsto j_\tau(x, \boldsymbol{\zeta})$ is measurable for each $\boldsymbol{\zeta} \in \mathbb{R}^m$;

(ii) there exist $q \in L^2(\Gamma_3)$ such that for a.e. $x \in \Gamma_3$ and all $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathbb{R}^m$

$$|j_\tau(x, \boldsymbol{\zeta}_1) - j_\tau(x, \boldsymbol{\zeta}_2)| \leq q(x)|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|;$$

(iii) $j_\tau(x, 0) \in L^1(\Gamma_3; \mathbb{R}^m)$.

We consider the following function space

$$V = \{\mathbf{v} \in H_1 : \ \mathbf{v} = 0 \text{ a.e. on } \Gamma_1\} \tag{4.10}$$

which is a closed subspace of H_1, hence a Hilbert space. Since the Lebesgue measure of Γ_1 is positive, it follows from Korn's inequality that the following inner product

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \tag{4.11}$$

generates a norm on V which is equivalent with the norm inherited from H_1.

Now we provide a variational formulation for problem **(P)**. To this end, let $\mathbf{u}$ be a strong solution, $\mathbf{v} \in V$ a test function and multiply the first line of **(P)** by $\mathbf{v} - \mathbf{u}$. Using Green's formula (4.1) we have

$$\begin{aligned}
(\mathbf{f}_0, \mathbf{v} - \mathbf{u})_H &= -\left(\mathrm{Div}\boldsymbol{\sigma}, \mathbf{v} - \mathbf{u}\right)_H \\
&= -\int_\Gamma (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot (\mathbf{v} - \mathbf{u}) + (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_{\mathcal{H}} \\
&= -\int_{\Gamma_2} \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) - \int_{\Gamma_3} [\sigma_\nu (v_\nu - u_\nu) + \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau)] + (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_{\mathcal{H}}
\end{aligned}$$

for all $\mathbf{v} \in V$. Since $V \ni \mathbf{v} \mapsto (\mathbf{f}_0, \mathbf{v})_H + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v}$ is linear and continuous, we can apply Riesz's representation theorem to conclude that there exists a unique element $\mathbf{f} \in V$ such that

$$(\mathbf{f}, \mathbf{v})_V = (\mathbf{f}_0, \mathbf{v})_H + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v}. \tag{4.12}$$

Definition 4.3.1. The following nonempty, closed and convex subset of V

$$\Lambda = \{\mathbf{v} \in V : \ v_\nu(x) \in C_1 \text{ and } \mathbf{v}_\tau(x) \in C_2 \text{ for a.e. } x \in \Gamma_3\},$$

is called the *set of admissible displacement fields.*

Since C_1, C_2 are convex cones, it follows that Λ is also a convex cone. Moreover, taking into account the definition of the Clarke subdifferential, we deduce that for all $\mathbf{v} \in \Lambda$ the following inequalities hold

$$-\int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu) \leq \int_{\Gamma_3} j_\nu^0(x, u_\nu; v_\nu - u_\nu)\, dx \tag{4.13}$$

and

$$-\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau) \leq \int_{\Gamma_3} h(x, \mathbf{u}_\tau) j_\tau^0(x, \mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau)\, dx. \tag{4.14}$$

Here, and hereafter, the generalized derivatives of the functions j_ν and j_τ are taken with respect to the second variable, i.e. of the functions $\mathbb{R} \ni t \mapsto j_\nu(x, t)$ and $\mathbb{R}^m \ni \boldsymbol{\zeta} \mapsto j_\tau(x, \boldsymbol{\zeta})$ respectively, but for simplicity we omit to mention that in fact these are partial generalized derivatives. On the other hand, taking Proposition 2.4.3 into account we can rewrite (4.3) as

$$\boldsymbol{\varepsilon}(\mathbf{u}) \in \partial \varphi^*(\boldsymbol{\sigma}), \text{ a.e. in } \Omega,$$

and which after integration over Ω leads to

$$-(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\mu} - \boldsymbol{\sigma})_{\mathcal{H}} + \int_\Omega (\varphi^*(\boldsymbol{\mu}) - \varphi^*(\boldsymbol{\sigma})) \geq 0, \text{ for all } \boldsymbol{\mu} \in \mathcal{H}. \tag{4.15}$$

Let us define the operator $L : V \to \mathcal{H}$ by $L\mathbf{v} = \boldsymbol{\varepsilon}(\mathbf{v})$ and denote by $L^* : \mathcal{H} \to V$ its adjoint, that is,

$$(L^*\boldsymbol{\mu}, \mathbf{v})_V = (\boldsymbol{\mu}, L\mathbf{v})_{\mathcal{H}}, \text{ for all } \mathbf{v} \in V \text{ and all } \boldsymbol{\mu} \in \mathcal{H}.$$

Using (4.12)-(4.15) we arrive at the following system of inequalities

$$(\tilde{\boldsymbol{P}}) \begin{cases} \text{Find } \mathbf{u} \in \Lambda \text{ and } \boldsymbol{\sigma} \in \mathcal{H} \text{ such that} \\ (L^*\boldsymbol{\sigma}, \mathbf{v} - \mathbf{u})_V \\ \quad + \displaystyle\int_{\Gamma_3} \left[j_\nu^0(x, u_\nu; v_\nu - u_\nu) + h(x, u_\tau) j_\tau^0(x, \mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau) \right] dx \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_V, \ \forall \mathbf{v} \in \Lambda, \\ -(L\mathbf{u}, \boldsymbol{\mu} - \boldsymbol{\sigma})_{\mathcal{H}} + \displaystyle\int_\Omega (\varphi^*(\boldsymbol{\mu}) - \varphi^*(\boldsymbol{\sigma})) \geq 0, \ \forall \boldsymbol{\mu} \in \mathcal{H}. \end{cases}$$

Here, the first inequality is related to the equilibrium relation, while the second represents the functional extension of the constitutive law (4.3). It is well-known (see e.g. [46], Theorem 1.3.21) that the second relation implies $L\mathbf{u} \in \partial\varphi^*(\boldsymbol{\sigma})$ almost everywhere in Ω.

Definition 4.3.2. A *bipotential* is a function $B : E \times E^* \to (-\infty, +\infty]$ satisfying the following conditions

(i) for any $x \in E$, if $D(B(x, \cdot)) \neq \emptyset$, then $B(x, \cdot)$ is proper and lower semicontinuous; for any $\xi \in E^*$, if $D(B(\cdot, \xi)) \neq \emptyset$, then $B(\cdot, \xi)$ is proper, convex and lower semicontinuous;

(ii) $B(x, \xi) \geq \langle \xi, x \rangle_{E^* \times E}$, for all $x \in E$, $\xi \in E^*$;

(iii) $\xi \in \partial B(\cdot, \xi)(x) \Leftrightarrow x \in \partial B(x, \cdot)(\xi) \Leftrightarrow B(x, \xi) = \langle \xi, x \rangle_{E^* \times E}$.

Proposition 2.4.3 allows us to construct the separable *bipotential* $a : \mathcal{S}^m \times \mathcal{S}^m \to (-\infty, +\infty]$, which connects the constitutive law, the function φ and its Fenchel conjugate φ^*, as follows

$$a(\boldsymbol{\tau}, \boldsymbol{\mu}) = \varphi(\boldsymbol{\tau}) + \varphi^*(\boldsymbol{\mu}), \text{ for all } \boldsymbol{\tau}, \boldsymbol{\mu} \in \mathcal{S}^m.$$

Using the bipotential a let us define $A : V \times \mathcal{H} \to \mathbb{R}$ by

$$A(\mathbf{v}, \boldsymbol{\mu}) = \int_\Omega a(L\mathbf{v}, \boldsymbol{\mu})\, dx, \text{ for all } \mathbf{v} \in V, \boldsymbol{\mu} \in \mathcal{H}.$$

and note that, due to $(\boldsymbol{H_\varphi})$, A is well defined and

$$A(\mathbf{v}, \boldsymbol{\mu}) \geq \alpha_1 \|\mathbf{v}\|_V^2 + \alpha_2 \|\boldsymbol{\mu}\|_{\mathcal{H}}^2, \text{ for all } \mathbf{v} \in V, \boldsymbol{\mu} \in \mathcal{H}.$$

Moreover, Proposition 2.4.3 ensures that

$$A(\mathbf{u}, \boldsymbol{\sigma}) = (L^*\boldsymbol{\sigma}, \mathbf{u})_V \text{ and } A(\mathbf{v}, \boldsymbol{\mu}) \geq (L^*\boldsymbol{\mu}, \mathbf{v})_V, \text{ for all } \mathbf{v} \in V, \ \boldsymbol{\mu} \in \mathcal{H}. \tag{4.16}$$

Combining the first inequality of $(\tilde{\boldsymbol{P}})$ and (4.16) we get

$$A(\mathbf{v},\boldsymbol{\sigma})-A(\mathbf{u},\boldsymbol{\sigma})+\int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;v_\nu-u_\nu)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;\mathbf{v}_\tau-\mathbf{u}_\tau)\right]\,dx\geq(\mathbf{f},\mathbf{v}-\mathbf{u})_V, \tag{4.17}$$

for all $\mathbf{v}\in\Lambda$.

Definition 4.3.3. Let us define now the *set of admissible stress tensors with respect to the displacement* $\mathbf{u}$, to be the following subset of $\mathcal{H}$

$$\Theta_{\mathbf{u}}=\Big\{\boldsymbol{\mu}\in\mathcal{H}:$$
$$(L^*\boldsymbol{\mu},\mathbf{v})_V+\int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;v_\nu)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;\mathbf{v}_\tau)\right]\,dx\geq(\mathbf{f},\mathbf{v})_V,\ \forall\mathbf{v}\in\Lambda\Big\}.$$

Now let $\mathbf{w}\in\Lambda$ be fixed. Choosing $\mathbf{v}=\mathbf{u}+\mathbf{w}\in\Lambda$ in ($(\tilde{\boldsymbol{P}})$ shows that $\boldsymbol{\sigma}\in\Theta_{\mathbf{u}}$, hence $\Theta_{\mathbf{u}}\neq\emptyset$. It is easy to check that $\Theta_{\mathbf{u}}$ is an unbounded, closed and convex subset of $\mathcal{H}$. Taking into account (4.16) we have

$$A(\mathbf{u},\boldsymbol{\mu})+\int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;u_\nu)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;\mathbf{u}_\tau)\right]\,dx\geq(\mathbf{f},\mathbf{u})_V,\ \text{for all }\boldsymbol{\mu}\in\Theta_{\mathbf{u}},$$

while for $\mathbf{v}=0\in\Lambda$ in $(\tilde{\boldsymbol{P}})$ we have

$$-A(\mathbf{u},\boldsymbol{\sigma})+\int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;-u_\nu)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;-\mathbf{u}_\tau)\right]\,dx\geq-(\mathbf{f},\mathbf{u})_V.$$

Adding the above relations, for all $\boldsymbol{\mu}\in\Theta_{\mathbf{u}}$ we have

$$\begin{aligned}&A(\mathbf{u},\boldsymbol{\mu})-A(\mathbf{u},\boldsymbol{\sigma})\\&+\int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;u_\nu)+j_\nu^0(x,u_\nu;-u_\nu)+h(x,\mathbf{u}_\tau)\left(j_\tau^0(x,\mathbf{u}_\tau;\mathbf{u}_\tau)+j_\tau^0(x,\mathbf{u}_\tau;-\mathbf{u}_\tau)\right)\right]\,dx\geq0.\end{aligned} \tag{4.18}$$

On the other hand, Proposition 2.1.1 and $(\boldsymbol{H_h})$ ensure that

$$\int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;u_\nu)+j_\nu^0(x,u_\nu;-u_\nu)+h(x,\mathbf{u}_\tau)\left(j_\tau^0(x,\mathbf{u}_\tau;\mathbf{u}_\tau)+j_\tau^0(x,\mathbf{u}_\tau;-\mathbf{u}_\tau)\right)\right]\,dx\geq0, \tag{4.19}$$

as

$$\begin{aligned}0&=j_\nu^0(x,u_\nu;0)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;0)\\&=j_\nu^0(x,u_\nu;u_\nu-u_\nu)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;\mathbf{u}_\tau-\mathbf{u}_\tau)\\&\leq\left(j_\nu^0(x,u_\nu;u_\nu)+j_\nu^0(x,u_\nu;-u_\nu)\right)+h(x,\mathbf{u}_\tau)\left(j_\tau^0(x,\mathbf{u}_\tau;\mathbf{u}_\tau)+j_\tau^0(x,\mathbf{u}_\tau;-\mathbf{u}_\tau)\right).\end{aligned}$$

Putting together (4.17)-(4.19) we derive the variational formulation in terms of bipotentials of problem **(P)** which reads as follows:

$$
\left(\mathcal{P}_{var}^{b}\right)\begin{cases}\text{Find } \mathbf{u}\in\Lambda \text{ and } \boldsymbol{\sigma}\in\Theta_{\mathbf{u}} \text{ such that}\\ A(\mathbf{v},\boldsymbol{\sigma})-A(\mathbf{u},\boldsymbol{\sigma})+\\ \qquad\displaystyle\int_{\Gamma_3}\left[j_\nu^0(x,u_\nu;v_\nu-u_\nu)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;\mathbf{v}_\tau-\mathbf{u}_\tau)\right]\,dx\geq(\mathbf{f},\mathbf{v}-\mathbf{u})_V,\quad\forall\mathbf{v}\in\Lambda,\\ A(\mathbf{u},\boldsymbol{\mu})-A(\mathbf{u},\boldsymbol{\sigma})\geq 0,\forall\boldsymbol{\mu}\in\Theta_{\mathbf{u}}.\end{cases}
$$

Each solution $(\mathbf{u},\boldsymbol{\sigma})\in\Lambda\times\Theta_{\mathbf{u}}$ of problem $\left(\mathcal{P}_{var}^{b}\right)$ is called a *weak solution* for problem **(P)**.

4.4 Existence of weak solutions

In this section we prove an existence result concerning the solutions of problem $\left(\mathcal{P}_{var}^{b}\right)$ by using a recent result due to Costea and Varga [28], see Section 2.5. First, we highlight the connection between the variational formulation in terms of bipotentials and other variational formulations such as the primal and dual variational formulations. As we have seen in the previous section, multiplying the first line of problem $(\mathbf{P})$ by $\mathbf{v}-\mathbf{u}$, integrating over Ω and then taking the functional extension of the constitutive law, we get a coupled system of inequalities, namely problem $(\tilde{\boldsymbol{P}})$. The primal variational formulation consists in rewriting $(\tilde{\boldsymbol{P}})$ as an inequality which depends only on the displacement field $\mathbf{u}$, while the dual variational formulation consists in rewriting $(\tilde{\boldsymbol{P}})$ in terms of the stress tensor $\boldsymbol{\sigma}$. The primal variational formulation can be derived by reasoning in the following way.

The second line of $(\tilde{\boldsymbol{P}})$ implies that $L\mathbf{u}\in\partial\varphi^*(\boldsymbol{\sigma})$ and this can be written equivalently as $\boldsymbol{\sigma}\in\partial\varphi(L\mathbf{u})$, hence

$$
\boldsymbol{\sigma}:(\boldsymbol{\mu}-L\mathbf{u})\leq\varphi(\boldsymbol{\mu})-\varphi(L\mathbf{u}),\ \text{for all } \boldsymbol{\mu}\in\mathcal{S}^m.
$$

For each $\mathbf{v}\in\Lambda$, taking $\boldsymbol{\mu}=L\mathbf{v}$ in the previous inequality and integrating over Ω yields

$$
(L^*\boldsymbol{\sigma},\mathbf{v}-\mathbf{u})_V\leq\int_\Omega[\varphi(L\mathbf{v})-\varphi(L\mathbf{u})],\ \text{for all } \mathbf{v}\in\Lambda.
$$

Now, combining the above relation and the first line of $(\tilde{\boldsymbol{P}})$ we get the following problem

$$
(\mathcal{P}_{var}^{p})\begin{cases}\text{Find } \mathbf{u}\in\Lambda \text{ such that}\\ F(\mathbf{v})-F(\mathbf{u})\\ \displaystyle +\int_{\Gamma_3}\left[j_\nu^0(x,\mathbf{u}_\nu;\mathbf{v}_\nu-\mathbf{u}_\nu)+h(x,\mathbf{u}_\tau)j_\tau^0(x,\mathbf{u}_\tau;\mathbf{v}_\tau-\mathbf{u}_\tau)\right]\,dx\geq(\mathbf{f},\mathbf{v}-\mathbf{u})_V,\ \forall\mathbf{v}\in\Lambda,\end{cases}
$$

where $F : V \to \mathbb{R}$ is the convex and lower semicontinous function defined by

$$F(\mathbf{v}) = \int_\Omega \varphi(L\mathbf{v}).$$

Problem $(\mathcal{P}^p_{var})$ is called the *primal variational formulation* of problem $(\boldsymbol{P})$.

Conversely, in order to transform $(\tilde{\boldsymbol{P}})$ into a problem formulated in terms of the stress tensor we reason in the following way. First, let us define $G : \mathcal{H} \to \mathbb{R}$ by

$$G(\boldsymbol{\mu}) = \int_\Omega \varphi^*(\boldsymbol{\mu}),$$

and for a fixed $\mathbf{w} \in \Lambda$ let $\Theta_\mathbf{w}$ be the following subset of $\mathcal{H}$

$$\Theta_\mathbf{w} = \Big\{\boldsymbol{\mu} \in \mathcal{H} : (L^*\boldsymbol{\mu}, \mathbf{v})_V \\ + \int_{\Gamma_3} \left[j^0_\nu(x, \mathbf{w}_\nu; \mathbf{v}_\nu) + h(x, \mathbf{w}_\tau) j^0_\tau(x, \mathbf{w}_\tau; \mathbf{v}_\tau)\right] \, dx \geq (\mathbf{f}, \mathbf{v})_V, \ \forall \mathbf{v} \in \Lambda\Big\}.$$

Let us consider the following inclusion

$$\left(\mathcal{P}^d_w\right) \begin{cases} \text{Find } \boldsymbol{\sigma} \in \mathcal{H} \text{ such that} \\ 0 \in \partial G(\boldsymbol{\sigma}) + \partial I_{\Theta_\mathbf{w}}(\boldsymbol{\sigma}) \end{cases}$$

which we call the *dual variational formulation with respect to* $\mathbf{w}$.

Now, looking at the first line of $(\tilde{\boldsymbol{P}})$ and keeping in mind the above notations, we deduce that $\Theta_\mathbf{u} \neq \emptyset$ as $\boldsymbol{\sigma} \in \Theta_\mathbf{u}$. Moreover, for each $\boldsymbol{\mu} \in \Theta_\mathbf{u}$ we have

$$-(L^*(\boldsymbol{\mu} - \boldsymbol{\sigma}), u)_V \leq \int_{\Gamma_3} [j^0_\nu(x, u_\nu; u_\nu) + j^0_\nu(x, u_\nu; -u_\nu) \\ + h(x, \mathbf{u}_\tau) \left(j^0_\tau(x, \mathbf{u}_\tau; \mathbf{u}_\tau) + j^0_\tau(x, \mathbf{u}_\tau; -\mathbf{u}_\tau)\right)] \, dx,$$

which combined with the second line of $(\tilde{\boldsymbol{P}})$ leads to

$$G(\boldsymbol{\mu}) - G(\boldsymbol{\sigma}) \geq -\int_{\Gamma_3} [j^0_\nu(x, u_\nu; u_\nu) + j^0_\nu(x, u_\nu; -u_\nu) \\ + h(x, \mathbf{u}_\tau) \left(j^0_\tau(x, \mathbf{u}_\tau; \mathbf{u}_\tau) + j^0_\tau(x, \mathbf{u}_\tau; -\mathbf{u}_\tau)\right)] \, dx, \tag{4.20}$$

for all $\boldsymbol{\mu} \in \Theta_u$. A simple computation shows that any solution of $\left(\mathcal{P}^d_u\right)$ will also solve (4.20).

A particular case of interest regarding problem $\left(\mathcal{P}^d_w\right)$ is when the set $\Theta_\mathbf{w}$ *does not actually depend on* $\mathbf{w}$. In this case problem $\left(\mathcal{P}^d_w\right)$ will be simply denoted $\left(\mathcal{P}^d\right)$ and will be called *the dual variational formulation of problem* $(\boldsymbol{P})$. For example, this case is encountered when the functions j_ν and j_τ are convex and positive homogeneous, as it is the case of examples (a)-(c) presented in Section 3.

In the above particular case, problem $(\tilde{\boldsymbol{P}})$ reduces to the following system of variational inequalities

$$(\tilde{\boldsymbol{P}}')\begin{cases}\text{Find } \mathbf{u}\in\Lambda \text{ and } \boldsymbol{\sigma}\in\mathcal{H} \text{ such that}\\ (L^*\boldsymbol{\sigma},\mathbf{v}-\mathbf{u})_V+H(\mathbf{v})-H(\mathbf{u})\geq(\mathbf{f},\mathbf{v}-\mathbf{u})_V, & \text{for all } \mathbf{v}\in\Lambda\\ -(L\mathbf{u},\boldsymbol{\mu}-\boldsymbol{\sigma})_{\mathcal{H}}+G(\boldsymbol{\mu})-G(\boldsymbol{\sigma})\geq 0, & \text{for all } \boldsymbol{\mu}\in\mathcal{H},\end{cases}$$

where $H=j\circ T$, $j:L^2(\Gamma_3;\mathbb{R}^m)\to\mathbb{R}$ is defined by

$$j(\boldsymbol{y})=\int_{\Gamma_3}\left[j_\nu(x,y_\nu)+j_\tau(x,\mathbf{y}_\tau)\right]dx,$$

and $T:V\to L^2(\Gamma_3;\mathbb{R}^m)$ is given by $T\boldsymbol{v}=[(\gamma\circ i)(\boldsymbol{v})]|_{\Gamma_3}$, with $i:V\to H_1$ being the embedding operator and $\gamma:H_1\to H^{1/2}(\Gamma;\mathbb{R}^m)$ being the trace operator. On the other hand, for each $\mathbf{w}\in\Lambda$,

$$\Theta_{\mathbf{w}}:=\Theta=\left\{\boldsymbol{\mu}\in\mathcal{H}:\ (L^*\boldsymbol{\mu},\mathbf{v})_V+H(\mathbf{v})\geq(\mathbf{f},\mathbf{v})_V,\quad\text{for all } \mathbf{v}\in\Lambda\right\},$$

and thus by taking $\mathbf{v}=2\mathbf{u}$ and $\mathbf{v}=0$ in the first line of $(\tilde{\boldsymbol{P}}')$ we get

$$(L^*\boldsymbol{\sigma},\mathbf{v})_V+H(\mathbf{u})=(\mathbf{f},\mathbf{u})_V,$$

hence

$$-(L\mathbf{u},\boldsymbol{\mu}-\boldsymbol{\sigma})_{\mathcal{H}}\leq 0,\ \text{for all } \boldsymbol{\mu}\in\Theta.$$

Combining this and the second line of $(\tilde{\boldsymbol{P}}')$ we get

$$G(\boldsymbol{\mu})-G(\boldsymbol{\sigma})\geq 0,\ \text{for all } \boldsymbol{\mu}\in\Theta,$$

which can be formulated equivalently as

$$\left(\mathcal{P}^d\right)\begin{cases}\text{Find } \boldsymbol{\sigma}\in\mathcal{H} \text{ such that}\\ 0\in\partial G(\boldsymbol{\sigma})+\partial I_\Theta(\boldsymbol{\sigma}).\end{cases}$$

The following proposition points out the connection between the variational formulations presented above.

Proposition 4.4.1. A pair $(\mathbf{u},\boldsymbol{\sigma})\in V\times\mathcal{H}$ is a solution for $\left(\mathcal{P}^b_{var}\right)$ if and only if $\mathbf{u}$ solves $(\mathcal{P}^p_{var})$ and $\boldsymbol{\sigma}$ solves $\left(\mathcal{P}^d_{\mathbf{u}}\right)$.

Proof. " $\Rightarrow$ " Let $(\mathbf{u},\boldsymbol{\sigma})\in V\times\mathcal{H}$ be a solution for $\left(\mathcal{P}^b_{var}\right)$. Then $\mathbf{u}\in\Lambda$, $\boldsymbol{\sigma}\in\Theta_{\mathbf{u}}$ and

$$\begin{cases}A(\mathbf{v},\boldsymbol{\sigma})-A(\mathbf{u},\boldsymbol{\sigma})+\displaystyle\int_{\Gamma_3}\left[j^0_\nu(x,u_\nu;v_\nu-u_\nu)+h(x,\mathbf{u}_\tau)j^0_\tau(x,\mathbf{u}_\tau;\mathbf{v}_\tau-\mathbf{u}_\tau)\right]\,dx\\ \qquad\geq(\mathbf{f},\mathbf{v}-\mathbf{u})_V,\\ A(\mathbf{u},\boldsymbol{\mu})-A(\mathbf{u},\boldsymbol{\sigma})\geq 0,\end{cases}$$

for all $(\mathbf{v}, \boldsymbol{\mu}) \in \Lambda \times \Theta_{\mathbf{u}}$.

Taking into account the way A, F and G were defined we get

$$A(\mathbf{v}, \boldsymbol{\sigma}) - A(\mathbf{u}, \boldsymbol{\sigma}) = F(\mathbf{v}) - F(\mathbf{u}), \text{ for all } \mathbf{v} \in V, \tag{4.21}$$

and

$$A(\mathbf{u}, \boldsymbol{\mu}) - A(\mathbf{u}, \boldsymbol{\sigma}) = G(\boldsymbol{\mu}) - G(\boldsymbol{\sigma}), \text{ for all } \boldsymbol{\mu} \in \mathcal{H}, \tag{4.22}$$

which shows that $\mathbf{u}$ is a solution for $(\mathcal{P}^p_{var})$ and

$$[G(\boldsymbol{\mu}) + I_{\Theta_{\mathbf{u}}}(\boldsymbol{\mu})] - [G(\boldsymbol{\sigma}) + I_{\Theta_{\mathbf{u}}}(\boldsymbol{\sigma})] \geq 0, \text{ for all } \boldsymbol{\mu} \in \mathcal{H}.$$

The last inequality can be written equivalently as

$$0 \in \partial(G + I_{\Theta_{\mathbf{u}}})(\boldsymbol{\sigma}).$$

On the other hand, applying Proposition 1.3.10 in [46] we deduce that

$$\partial(G + I_{\Theta_{\mathbf{u}}})(\boldsymbol{\sigma}) = \partial G(\boldsymbol{\sigma}) + \partial I_{\Theta_{\mathbf{u}}}(\boldsymbol{\sigma}),$$

hence $\boldsymbol{\sigma}$ solves $(\mathcal{P}^d_u)$.

" $\Leftarrow$ " Assume now that $\mathbf{u} \in V$ is a solution of $(\mathcal{P}^p_{var})$ and $\boldsymbol{\sigma} \in \mathcal{H}$ solves $\left(\mathcal{P}^d_{var}\right)$. The fact that $\boldsymbol{\sigma}$ solves $(\mathcal{P}^d_u)$ implies that $D(\partial I_{\Theta_{\mathbf{u}}}) \neq \emptyset$ and

$$\boldsymbol{\sigma} \in D(\partial I_{\Theta_{\mathbf{u}}}).$$

On the other hand, it is well known that

$$D(\partial I_{\Theta_{\mathbf{u}}}) \subseteq D(I_{\Theta_{\mathbf{u}}}) = \Theta_{\mathbf{u}},$$

hence $\boldsymbol{\sigma} \in \Theta_{\mathbf{u}}$. Moreover,

$$\begin{cases} F(\mathbf{v}) - F(\mathbf{u}) + \int_{\Gamma_3} [j^0_\nu(x, u_\nu; v_\nu - u_\nu) + h(x, \mathbf{u}_\tau) j^0_\tau(x, \mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau)] \, dx \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_V, \\ G(\boldsymbol{\mu}) - G(\boldsymbol{\sigma}) \geq 0, \end{cases}$$

for all $(\mathbf{v}, \boldsymbol{\mu}) \in \Lambda \times \Theta_{\mathbf{u}}$, which combined with (4.21) and (4.22) shows that $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution for problem $(\mathcal{P}^b_{var})$.

□

The main result of this chapter is given by the following theorem.

Theorem 4.4.2. Assume $(\boldsymbol{H_C})$, $(\boldsymbol{H_f})$, $(\boldsymbol{H_h})$, $(\boldsymbol{H_{j_\nu}})$, $(\boldsymbol{H_{j_\tau}})$ and $(\boldsymbol{H_\varphi})$ hold. Then problem $(\mathcal{P}^b_{var})$ has at least one solution.

Before proving the main result we need the following Aubin-Clarke type result concerning the Clarke subdifferential of integral functions. Let us consider the function $j : L^2(\Gamma_3;\mathbb{R}^m)\times L^2(\Gamma_3;\mathbb{R}^m) \to \mathbb{R}$ defined by

$$j(\mathbf{y},\mathbf{z}) = \int_{\Gamma_3} j_\nu(x, z_\nu) + h(x,\mathbf{y}_\tau) j_\tau(x,\mathbf{z}_\tau)\, dx. \tag{4.23}$$

Lemma 4.4.3. Assume $(\boldsymbol{H_h})$, $(\boldsymbol{H_{j_\nu}})$ and $(\boldsymbol{H_{j_\tau}})$ are fulfilled. Then, for each $\mathbf{y} \in L^2(\Gamma_3;\mathbb{R}^m)$, the function $\mathbf{z} \mapsto j(\mathbf{y},\mathbf{z})$ is Lipschitz continuous and

$$j_2^0(\mathbf{y},\mathbf{z};\bar{\mathbf{z}}) \le \int_{\Gamma_3} [j_\nu^0(x, z_\nu;\bar z_\nu) + h(x,\mathbf{y}_\tau) j_\tau^0(x,\mathbf{z}_\tau;\bar{\mathbf{z}}_\tau)]\, dx. \tag{4.24}$$

Proof. Let $\mathbf{y},\mathbf{z}^1,\mathbf{z}^2 \in L^2(\Gamma_3;\mathbb{R}^m)$ be fixed. Then

$$\begin{aligned} |j(\mathbf{y},\mathbf{z}^1) - j(\mathbf{y},\mathbf{z}^2)| &= \left| \int_{\Gamma_3} [j_\nu(x,\mathbf{z}_\nu^1) - j_\nu(x,\mathbf{z}_\nu^2) + h(x,\mathbf{y}_\tau)\left(j_\tau(x,\mathbf{z}_\tau^1) - j_\tau(x,\mathbf{z}_\tau^2)\right)]\, dx\right| \\ &\le \int_{\Gamma_3} \left|j_\nu(x,z_\nu^1) - j_\nu(x,z_\nu^2)\right| dx + h_0 \int_{\Gamma_3} \left|j_\tau(x,\mathbf{z}_\tau^1) - j_\tau(x,\mathbf{z}_\tau^2)\right| dx. \end{aligned}$$

The equality

$$|\mathbf{z}|^2 = \mathbf{z}\cdot\mathbf{z} = z_\nu z_\nu + \mathbf{z}_\tau\cdot\mathbf{z}_\tau = z_\nu^2 + |\mathbf{z}_\tau|^2,$$

shows that if $\mathbf{z} \in L^2(\Gamma_3;\mathbb{R}^m)$, then $z_\nu \in L^2(\Gamma_3)$ and $\mathbf{z}_\tau \in L^2(\Gamma_3;\mathbb{R}^m)$ and

$$\|z_\nu\|_{L^2(\Gamma_3)}, \|\mathbf{z}_\tau\|_{L^2(\Gamma_3;\mathbb{R}^m)} \le \|\mathbf{z}\|_{L^2(\Gamma_3;\mathbb{R}^m)}.$$

Thus, from the hypotheses and Hölder's inequality we get

$$\begin{aligned} \left|j(\mathbf{y},\mathbf{z}^1) - j(\mathbf{y},\mathbf{z}^2)\right| &\le \|p\|_{L^2(\Gamma_3)}\|z_\nu^1 - z_\nu^2\|_{L^2(\Gamma_3)} + h_0\|q\|_{L^2(\Gamma_3)}\|\mathbf{z}_\tau^1 - \mathbf{z}_\tau^2\|_{L^2(\Gamma_3;\mathbb{R}^m)} \\ &\le \left(\|p\|_{L^2(\Gamma_3)} + h_0\|q\|_{L^2(\Gamma_3)}\right)\|\mathbf{z}^1 - \mathbf{z}^2\|_{L^2(\Gamma_3;\mathbb{R}^m)}, \end{aligned}$$

which shows that j is Lipschitz continuous.

In order to prove (4.24) we use Fatou's lemma and the fact that the convergence in $L^2(\Gamma_3;\mathbb{R}^m)$ implies, up to a subsequence a.e. convergence on Γ_3

$$j_2^0(\mathbf{y},\mathbf{z};\bar{\mathbf{z}}) = \limsup_{\substack{\mathbf{u}\to\mathbf{z}\\ \lambda\downarrow 0}} \frac{j(\mathbf{y},\mathbf{u}+\lambda\bar{\mathbf{z}}) - j(\mathbf{y},\mathbf{u})}{\lambda}$$

$$= \limsup_{\substack{\mathbf{u}\to\mathbf{z}\\ \lambda\downarrow 0}} \int_{\Gamma_3} \left[\frac{j_\nu(x,u_\nu+\lambda\bar z_\nu) - j_\nu(x,u_\nu)}{\lambda} + h(x,\mathbf{y}_\tau)\frac{j_\tau(x,\mathbf{u}_\tau+\lambda\bar{\mathbf{z}}_\tau) - j_\tau(x,\mathbf{u}_\tau)}{\lambda}\right] dx$$

$$\le \int_{\Gamma_3} \left[\limsup_{\substack{\mathbf{u}\to\mathbf{z}\\ \lambda\downarrow 0}} \frac{j_\nu(x,u_\nu+\lambda\bar z_\nu) - j_\nu(x,u_\nu)}{\lambda} + h(x,\mathbf{y}_\tau)\limsup_{\substack{\mathbf{u}\to\mathbf{z}\\ \lambda\downarrow 0}} \frac{j_\tau(x,\mathbf{u}_\tau+\lambda\bar{\mathbf{z}}_\tau) - j_\tau(x,\mathbf{u}_\tau)}{\lambda}\right] dx$$

$$\le \int_{\Gamma_3} [j_\nu^0(x,z_\nu;\bar z_\nu) + h(x,\mathbf{y}_\tau) j_\tau^0(x,\mathbf{z}_\tau;\bar{\mathbf{z}}_\tau)]\, dx.$$

$\square$

In order to prove Theorem 4.4.2 we consider the following system of nonlinear hemivariational inequalities according to Section 2.5,

$$(\mathcal{S}_{K_1,K_2})\begin{cases}\text{Find } (\mathbf{u},\boldsymbol{\sigma})\in K_1\times K_2 \text{ such that}\\ \psi_1(\mathbf{u},\boldsymbol{\sigma},\mathbf{v})+J_1^0(T\mathbf{u},S\boldsymbol{\sigma};T\mathbf{v}-T\mathbf{u})\geq(F_1(\mathbf{u},\boldsymbol{\sigma}),\mathbf{v}-\mathbf{u})_{X_1}, & \text{for all } \mathbf{v}\in K_1,\\ \psi_2(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\mu})+J_2^0(T\mathbf{u},S\boldsymbol{\sigma};S\boldsymbol{\mu}-S\boldsymbol{\sigma})\geq(F_2(\mathbf{u},\boldsymbol{\sigma}),\boldsymbol{\mu}-\boldsymbol{\sigma})_{X_2}, & \text{for all } \boldsymbol{\mu}\in K_2,\end{cases}$$

where

- $X_1=V$, $X_2=\mathcal{H}$, $K_i\subset X_i$ is closed and convex ($i=1,2$), $Y_1=L^2(\Gamma_3;\mathbb{R}^m)$, $Y_2=\{0\}$;
- $\psi_1: X_1\times X_2\times X_1\to\mathbb{R}$ is defined by $\psi_1(\mathbf{u},\boldsymbol{\sigma},\mathbf{v})=A(\mathbf{v},\boldsymbol{\sigma})-A(\mathbf{u},\boldsymbol{\sigma})$;
- $\psi_2: X_1\times X_2\times X_2\to\mathbb{R}$ is defined by $\psi_2(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\mu})=A(\mathbf{u},\boldsymbol{\mu})-A(\mathbf{u},\boldsymbol{\sigma})$;
- $T: X_1\to Y_1$ is defined by $T\mathbf{v}=[(\gamma\circ i)(\mathbf{v})]|_{\Gamma_3}$, with $i: V\to H_1$ the embedding operator and $\gamma: H_1\to H^{1/2}(\Gamma;\mathbb{R}^m)$ is the trace operator;
- $S: X_2\to Y_2$ is defined by $S\boldsymbol{\tau}=0$, for all $\boldsymbol{\tau}\in X_2$;
- $J: Y_1\times Y_2\to\mathbb{R}$ is defined by $J(\mathbf{y}^1,\mathbf{y}^2)=j(\mathbf{y}^0,\mathbf{y}^1)$, where $j: L^2(\Gamma_3;\mathbb{R}^m)\times L^2(\Gamma_3;\mathbb{R}^m)\to\mathbb{R}$ is as in (4.23) and $\mathbf{y}^0$ is a fixed element of $L^2(\Gamma_3;\mathbb{R}^m)$;
- $F_1: X_1\times X_2\to X_1$ is defined by $F_1(\mathbf{v},\boldsymbol{\mu})=\mathbf{f}$;
- $F_2: X_1\times X_2\to X_2$ is defined by $F_2(\mathbf{v},\boldsymbol{\mu})=0$.

Lemma 4.4.4. Assume $(\boldsymbol{H_h})$, $(\boldsymbol{H_{j_\nu}})$, $(\boldsymbol{H_{j_\tau}})$ and $(\boldsymbol{H_\varphi})$ are fulfilled. Then the following statements hold:

(i) $\psi_1(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{u})=0$ and $\psi_2(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\sigma})=0$, for all $(\mathbf{u},\boldsymbol{\sigma})\in X_1\times X_2$;

(ii) for each $v\in X_1$ and each $\mu\in X_2$ the maps $(\mathbf{u},\boldsymbol{\sigma})\mapsto\psi_1(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{v})$ and $(\boldsymbol{u},\boldsymbol{\sigma})\mapsto\psi_2(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mu})$ are weakly upper semicontinuous;

(iii) for each $(\boldsymbol{u},\boldsymbol{\sigma})\in X_1\times X_2$ the maps $\boldsymbol{v}\mapsto\psi_1(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{v})$ and $\boldsymbol{\mu}\mapsto\psi_2(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mu})$ are convex;

(iv) $\liminf_{k\to+\infty}(F_1(\boldsymbol{u}_k,\boldsymbol{\sigma}_k),\boldsymbol{v}-\boldsymbol{u}_k)_{X_1}\geq(F_1(\boldsymbol{u},\boldsymbol{\sigma}),\boldsymbol{v}-\boldsymbol{u})_{X_1}$ and $\liminf_{k\to+\infty}(F_2(\boldsymbol{u}_k,\boldsymbol{\sigma}_k),\boldsymbol{\mu}-\boldsymbol{\sigma}_k)_{X_2}\geq(F_2(\boldsymbol{u},\boldsymbol{\sigma}),\boldsymbol{\mu}-\boldsymbol{\sigma})_{X_2}$ whenever $(\boldsymbol{u}_k,\boldsymbol{\sigma}_k)\rightharpoonup(\boldsymbol{u},\boldsymbol{\sigma})$ as $k\to+\infty$;

(v) there exists $c:\mathbb{R}_+\to\mathbb{R}_+$ with the property $\lim_{t\to+\infty}c(t)=+\infty$ such that

$$\psi_1(\mathbf{u},\boldsymbol{\sigma},0)+\psi_2(\mathbf{u},\boldsymbol{\sigma},0)\leq -c\left(\sqrt{\|\boldsymbol{u}\|_{X_1}^2+\|\boldsymbol{\sigma}\|_{X_2}^2}\right)\sqrt{\|\boldsymbol{u}\|_{X_1}^2+\|\boldsymbol{\sigma}\|_{X_2}^2}$$

for all $(\boldsymbol{u},\boldsymbol{\sigma})\in X_1\times X_2$.

(vi) The function $J : Y_1 \times Y_2 \to \mathbb{R}$ is Lipschitz with respect to each variable. Moreover, for all $(\boldsymbol{y}^1, \boldsymbol{y}^2), (\boldsymbol{z}^1, \boldsymbol{z}^2) \in Y_1 \times Y_2$ we have

$$J_1^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^1) = j_2^0(\boldsymbol{y}^0, \boldsymbol{y}^1; \boldsymbol{z}^1)$$

and

$$J_2^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^2) = 0;$$

(vii) There exists $M > 0$ such that

$$J_1^0(\boldsymbol{y}^1, \boldsymbol{y}^2; -\boldsymbol{y}^1) \leq M\|\boldsymbol{y}^1\|_{Y_1}, \text{ for all } (\boldsymbol{y}^1, \boldsymbol{y}^2) \in Y_1 \times Y_2;$$

(viii) there exist $m_i > 0$, $i = 1, 2$, such that $\|F_i(\boldsymbol{u}, \boldsymbol{\sigma})\|_{X_i} \leq m_i$, for all $(\boldsymbol{u}, \boldsymbol{\sigma}) \in X_1 \times X_2$.

Proof. (i) Trivial.

(ii) Let $\boldsymbol{v} \in X_1$ be fixed and let $\{(\boldsymbol{u}_k, \boldsymbol{\sigma}_k)\}_k$ be a sequence such that $(\boldsymbol{u}_k, \boldsymbol{\sigma}_k)$ converges weakly in $X_1 \times X_2$ to $(\boldsymbol{u}, \boldsymbol{\sigma})$ as $k \to \infty$. Using the fact that L is linear, φ is convex and lower semicontinuous, hence weakly lower semicontinuous and using Fatou's lemma, we have

$$\begin{aligned}
\limsup_{k\to\infty} \psi_1(\boldsymbol{u}_k, \boldsymbol{\sigma}_k, \boldsymbol{v}) &= \limsup_{k\to\infty} [A(\boldsymbol{v}, \boldsymbol{\sigma}_k) - A(\boldsymbol{u}_k, \boldsymbol{\sigma}_k)] \\
&= \limsup_{k\to\infty} \int_\Omega [\varphi(L\mathbf{v}) - \varphi(L\mathbf{u}_k)] \\
&\leq \int_\Omega \varphi(L\mathbf{v}) - \int_\Omega \liminf_{k\to\infty} \varphi(L\mathbf{u}_k) \\
&\leq \int_\Omega [\varphi(L\boldsymbol{v}) - \varphi(L\boldsymbol{u}) + \varphi^*(\boldsymbol{\sigma}) - \varphi^*(\boldsymbol{\sigma})] \\
&= A(\boldsymbol{v}, \boldsymbol{\sigma}) - A(\boldsymbol{u}, \boldsymbol{\sigma}) \\
&= \psi_1(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{v}),
\end{aligned}$$

which shows that the map $(\boldsymbol{u}, \boldsymbol{\sigma}) \mapsto \psi_1(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{v})$ is weakly upper semicontinuous.

In a similar fashion we prove that for $\boldsymbol{\mu} \in X_2$ fixed, the map $(\boldsymbol{u}, \boldsymbol{\sigma}) \mapsto \psi_2(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mu})$ is weakly upper semicontinuous.

(iii) Follows from the convexity of φ and φ^*;

(iv) Let $\{(\boldsymbol{u}_k, \boldsymbol{\sigma}_k)\}$ be a sequence which converges weakly to $(\boldsymbol{u}, \boldsymbol{\sigma})$ in $X_1 \times X_2$ as $k \to +\infty$. Then $\boldsymbol{u}_k \to \boldsymbol{u}$ in X_1 as $k \to +\infty$ and

$$\liminf_{k\to\infty} \ (F_1(\boldsymbol{u}_k, \boldsymbol{\sigma}_k), \boldsymbol{v} - \boldsymbol{u}_k)_{X_1} = \liminf_{k\to\infty} (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_k)_{X_1} = (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{X_1},$$

and

$$\liminf_{k\to\infty} (F_2(\boldsymbol{u}_k,\boldsymbol{\sigma}_k),\boldsymbol{\mu}-\boldsymbol{\sigma}_k)_{X_2} = 0 = (F_2(\boldsymbol{u},\boldsymbol{\sigma}),\boldsymbol{\mu}-\boldsymbol{\sigma})_{X_2}.$$

(v) Let $(\boldsymbol{u},\boldsymbol{\sigma}) \in X_1 \times X_2$. Using $(\boldsymbol{H_\varphi})$ we get the following estimates

$$\begin{aligned}
\psi_1(\boldsymbol{u},\boldsymbol{\sigma},0)+\psi_2(\boldsymbol{u},\boldsymbol{\sigma},0) &= A(0,\boldsymbol{\sigma}) - A(\boldsymbol{u},\boldsymbol{\sigma}) + A(\boldsymbol{u},0) - A(\boldsymbol{u},\boldsymbol{\sigma}) \\
&= \int_\Omega [\varphi(0)+\varphi^*(0) - (\varphi(\boldsymbol{\varepsilon}(\boldsymbol{u}))+\varphi^*(\boldsymbol{\sigma}))] \\
&\le \tilde{c} - \min\{\alpha_1,\alpha_2\}\left(\|\boldsymbol{u}\|_{X_1}^2 + \|\boldsymbol{\sigma}\|_{X_2}^2\right).
\end{aligned}$$

Choosing $c(t) = b_0 t$, with $b_0 > 0$ a suitable constant, we get the desired inequality.

(vi) It follows directly from Lemma 4.4.3 and the definition of J.

(vii) From (vi) and Lemma 4.4.3 we deduce

$$\begin{aligned}
J_1^0(\boldsymbol{y}^1,\boldsymbol{y}^2;-\boldsymbol{y}^1) &= j_2^0(\boldsymbol{y}^0,\boldsymbol{y}^1;-\boldsymbol{y}^1) \\
&\le \int_{\Gamma_3} [j_\nu^0(x,y_\nu^1;-y_\nu^1) + h(x,\boldsymbol{y}_\tau^0) j_\tau^0(x,\boldsymbol{y}_\tau^1;-\boldsymbol{y}_\tau^1)]\,dx
\end{aligned}$$

On the other hand, assumptions $(\boldsymbol{H_{j_\nu}})$ and $(\boldsymbol{H_{j_\nu}})$ imply

$$j_\nu^0(x,t_1;t_2) \le p(x)|t_2|, \text{ for all } t_1,t_2 \in \mathbb{R},$$

and

$$j_\tau^0(x,\boldsymbol{\zeta}_1;\boldsymbol{\zeta}_2) \le q(x)|\boldsymbol{\zeta}_2|, \text{ for all } \boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2 \in \mathbb{R}^m.$$

Thus, invoking Hölder's inequality we get

$$J_1^0(\boldsymbol{y}^1,\boldsymbol{y}^2;-\boldsymbol{y}^1) \le \left(\|p\|_{L^2(\Gamma_3)} + h_0\|\boldsymbol{q}\|_{L^2(\Gamma_3;\mathbb{R}^m)}\right)\|\boldsymbol{y}^1\|_{L^2(\Gamma_3;\mathbb{R}^m)}.$$

(viii) Trivial.

□ □

Proof of Theorem 4.4.2 The proof will be carried out in three steps as follows.

Step 1. Let $K_1 \subset X_1$ and $K_2 \subset X_2$ be closed and convex sets. Then $(\mathcal{S}_{K_1,K_2})$ admits at least one solution.

This will be done by applying a slightly modified version of Corollary 3.2 in [28]. Lemma 4.4.4 ensures that all the conditions of the aforementioned corollary are satisfied except the regularity of J. We point out the fact that in our case this condition needs not to be imposed because the only reason it is imposed in the paper of Costea & Varga is to ensure the following inequality

$$J^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^1, \boldsymbol{z}^2) \leq J_1^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^1) + J_2^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^2),$$

which in our case is automatically fulfilled because J does not depend on the second variable and the following equalities take place

$$J^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^1, \boldsymbol{z}^2) = J_1^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^1)$$

and

$$J_2^0(\boldsymbol{y}^1, \boldsymbol{y}^2; \boldsymbol{z}^2) = 0,$$

and this completes the first step.

Step 2. Let $K_1^1, K_1^2 \subset X_1$ and $K_2^1, K_2^2 \subset X_2$ be closed and convex sets and let $(\boldsymbol{u}^1, \boldsymbol{\sigma}^1)$ and $(\boldsymbol{u}^2, \boldsymbol{\sigma}^2)$ be solutions for $(\mathcal{S}_{K_1^1, K_2^1})$ and $(\mathcal{S}_{K_1^2, K_2^2})$, respectively. Then $(\boldsymbol{u}^1, \boldsymbol{\sigma}^2)$ solves $(\mathcal{S}_{K_1^1, K_2^2})$ and $(\boldsymbol{u}^2, \boldsymbol{\sigma}^1)$ solves $(\mathcal{S}_{K_1^2, K_2^1})$.

The fact that $(\boldsymbol{u}^1, \boldsymbol{\sigma}^1)$ solves $(\mathcal{S}_{K_1^1, K_2^1})$ means

$$\begin{cases} \psi_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^1, \boldsymbol{v}) + J_1^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^1; T\boldsymbol{v} - T\boldsymbol{u}^1) \geq (F_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^1), \boldsymbol{v} - \boldsymbol{u}^1)_{X_1}, & \text{for all } \boldsymbol{v} \in K_1^1 \\ \psi_2(\boldsymbol{u}^1, \boldsymbol{\sigma}^1, \boldsymbol{\mu}) + J_2^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^1; S\boldsymbol{\mu} - S\boldsymbol{\sigma}^1) \geq (F_2(\boldsymbol{u}^1, \boldsymbol{\sigma}^1), \boldsymbol{\mu} - \boldsymbol{\sigma}^1)_{X_2}, & \text{for all } \boldsymbol{\mu} \in K_2^1 \end{cases} \tag{4.25}$$

while the fact that $(\boldsymbol{u}^2, \boldsymbol{\sigma}^2)$ solves $(\mathcal{S}_{K_1^2, K_2^2})$ shows

$$\begin{cases} \psi_1(\boldsymbol{u}^2, \boldsymbol{\sigma}^2, \boldsymbol{v}) + J_1^0(T\boldsymbol{u}^2, S\boldsymbol{\sigma}^2; T\boldsymbol{v} - T\boldsymbol{u}^2) \geq (F_1(\boldsymbol{u}^2, \boldsymbol{\sigma}^2), \boldsymbol{v} - \boldsymbol{u}^2)_{X_1}, & \text{for all } \boldsymbol{v} \in K_1^2 \\ \psi_2(\boldsymbol{u}^2, \boldsymbol{\sigma}^2, \boldsymbol{\mu}) + J_2^0(T\boldsymbol{u}^2, S\boldsymbol{\sigma}^2; S\boldsymbol{\mu} - S\boldsymbol{\sigma}^2) \geq (F_2(\boldsymbol{u}^2, \boldsymbol{\sigma}^2), \boldsymbol{\mu} - \boldsymbol{\sigma}^2)_{X_2}, & \text{for all } \boldsymbol{\mu} \in K_2^2 \end{cases} \tag{4.26}$$

Putting together the first line of (4.25) and the second line of (4.26) we get

$$\begin{cases} \psi_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^1, \boldsymbol{v}) + J_1^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^1; T\boldsymbol{v} - T\boldsymbol{u}^1) \geq (F_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^1), \boldsymbol{v} - \boldsymbol{u}^1)_{X_1}, & \text{for all } \boldsymbol{v} \in K_1^1 \\ \psi_2(\boldsymbol{u}^2, \boldsymbol{\sigma}^2, \boldsymbol{\mu}) + J_2^0(T\boldsymbol{u}^2, S\boldsymbol{\sigma}^2; S\boldsymbol{\mu} - S\boldsymbol{\sigma}^2) \geq (F_2(\boldsymbol{u}^2, \boldsymbol{\sigma}^2), \boldsymbol{\mu} - \boldsymbol{\sigma}^2)_{X_2}, & \text{for all } \boldsymbol{\mu} \in K_2^2 \end{cases} \tag{4.27}$$

On the other hand, keeping in mind the way $\psi_1, \psi_2, J, F_1, F_2$ were defined is it easy to check that for any $(\boldsymbol{v}, \boldsymbol{\mu}) \in K_1^1 \times K_2^2$ the following equalities hold

$$\psi_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^1, \boldsymbol{v}) = \psi_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^2, \boldsymbol{v}) \text{ and } \psi_2(\boldsymbol{u}^2, \boldsymbol{\sigma}^2, \boldsymbol{\mu}) = \psi_2(\boldsymbol{u}^1, \boldsymbol{\sigma}^2, \boldsymbol{\mu}),$$

$$J_1^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^1; T\boldsymbol{v} - T\boldsymbol{u}^1) = J_1^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^2; T\boldsymbol{v} - T\boldsymbol{u}^1)$$

$$J_2^0(T\boldsymbol{u}^2, S\boldsymbol{\sigma}^2; S\boldsymbol{\mu} - S\boldsymbol{\sigma}^2) = J_2^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^2; S\boldsymbol{\mu} - S\boldsymbol{\sigma}^1)$$

$$F_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^1) = F_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^2) \text{ and } F_2(\boldsymbol{u}^2, \boldsymbol{\sigma}^2) = F_2(\boldsymbol{u}^1, \boldsymbol{\sigma}^2).$$

Using these equalities and (4.27) we obtain

$$\begin{cases} \psi_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^2, \boldsymbol{v}) + J_1^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^2; T\boldsymbol{v} - T\boldsymbol{u}^1) \geq (F_1(\boldsymbol{u}^1, \boldsymbol{\sigma}^2), \boldsymbol{v} - \boldsymbol{u}^1)_{X_1}, & \text{for all } \boldsymbol{v} \in K_1^1 \\ \psi_2(\boldsymbol{u}^1, \boldsymbol{\sigma}^2, \boldsymbol{\mu}) + J_2^0(T\boldsymbol{u}^1, S\boldsymbol{\sigma}^2; S\boldsymbol{\mu} - S\boldsymbol{\sigma}^2) \geq (F_2(\boldsymbol{u}^1, \boldsymbol{\sigma}^2), \boldsymbol{\mu} - \boldsymbol{\sigma}^2)_{X_2}, & \text{for all } \boldsymbol{\mu} \in K_2^2 \end{cases}$$

hence $(\boldsymbol{u}^1, \boldsymbol{\sigma}^2)$ solves $(\mathcal{S}_{K_1^1, K_2^2})$. In a similar way we can prove that $(\boldsymbol{u}^2, \boldsymbol{\sigma}^1)$ solves $(\mathcal{S}_{K_1^2, K_2^1})$.

Step 3. There exists $\mathbf{u} \in \Lambda$ and $\boldsymbol{\sigma} \in \Theta_{\mathbf{u}}$ such that $(\boldsymbol{u}, \boldsymbol{\sigma})$ solves $(\mathcal{P}_{var}^b)$.

Let us choose $K_1^1 = \Lambda$ and $K_2^1 = X_2$. According to **Step 1** there exists a pair $(\boldsymbol{u}^1, \boldsymbol{\sigma}^1)$ which solves $(\mathcal{S}_{K_1^1, K_2^1})$. Next, we choose $K_1^2 = \Lambda$ and $K_2^2 = \Theta_{\boldsymbol{u}^1}$ and use again **Step 1** to deduce that there exists a pair $(\boldsymbol{u}^2, \boldsymbol{\sigma}^2)$ which solves $(\mathcal{S}_{K_1^2, K_2^2})$. Then, according to **Step 2**, the pair $(\boldsymbol{u}^1, \boldsymbol{\sigma}^2)$ will solve $(\mathcal{S}_{K_1^1, K_2^2})$. Invoking the way $\psi_1, \psi_2, J, F_1, F_2, K_1^1, K_2^2$ were defined, it is clear that the pair $(\boldsymbol{u}, \boldsymbol{\sigma}) = (\boldsymbol{u}^1, \boldsymbol{\sigma}^2) \in \Lambda \times \Theta_{\boldsymbol{u}}$ is a solution of the system

$$\begin{cases} A(\boldsymbol{v}, \boldsymbol{\sigma}) - A(\boldsymbol{u}, \boldsymbol{\sigma}) + j_2^0(\boldsymbol{y}^0, T\boldsymbol{u}; T\boldsymbol{v} - T\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V, & \text{for all } \boldsymbol{v} \in \Lambda, \\ A(\boldsymbol{u}, \boldsymbol{\mu}) - A(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0, & \text{for all } \boldsymbol{\mu} \in \Theta_{\boldsymbol{u}}, \end{cases}$$

for all $\boldsymbol{y}^0 \in L^2(\Gamma_3; \mathbb{R}^m)$, since $\boldsymbol{y}^0$ was arbitrary fixed. Choosing $\boldsymbol{y}^0 = T\boldsymbol{u}$ an taking into account (4.24) we conclude that $(\boldsymbol{u}, \boldsymbol{\sigma}) \in \Lambda \times \Theta_{\boldsymbol{u}}$ solves $(\mathcal{P}_{var}^b)$, hence the proof is complete.

□

We close this section with some comments and remarks concerning the particular case when the boundary conditions (4.6) and (4.7) reduce to the Signorini boundary condition combined with a frictionless condition, that is $\boldsymbol{\sigma}_\tau = 0$. In this case

$$C_1 = (-\infty, 0], \ C_2 = \mathbb{R}^m \text{ and } j_\nu, \boldsymbol{j}_\tau, h \equiv 0,$$

while

$$\Lambda = \{\boldsymbol{v} \in V : \ v_\nu \leq 0 \text{ on } \Gamma_3\},$$

and

$$\Theta = \{\boldsymbol{\mu} \in \mathcal{H} : \ (\boldsymbol{\mu}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \geq (\boldsymbol{f}, \boldsymbol{v})_V \text{ for all } \boldsymbol{v} \in \Lambda\}.$$

Problem $(\mathcal{P}_{var}^b)$ reduces to the following system of variational inequalities

Find $(\boldsymbol{u}, \boldsymbol{\sigma}) \in \Lambda \times \Theta$ such that for all $(\boldsymbol{v}, \boldsymbol{\mu}) \in \Lambda \times \Theta$

$$\begin{cases} A(\boldsymbol{v}, \boldsymbol{\sigma}) - A(\boldsymbol{u}, \boldsymbol{\sigma}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V, \\ A(\boldsymbol{u}, \boldsymbol{\mu}) - A(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0. \end{cases} \tag{4.28}$$

This case was studied recently by Matei [64] who used the Direct Method in the Calculus of Variations to prove that the functional $\mathcal{L} : \Lambda \times \Theta \to \mathbb{R}$

$$\mathcal{L}(\boldsymbol{v}, \boldsymbol{\mu}) = A(\boldsymbol{v}, \boldsymbol{\mu}) - (\boldsymbol{f}, \boldsymbol{v})_V,$$

admits a global minimizer and each minimizer $(\boldsymbol{u}, \boldsymbol{\sigma})$ of $\mathcal{L}$ is in fact a solution for (4.26). However, our proof is different, so even in this particular case our approach is new and supplements the result obtained by Matei in [64]. Furthermore, as far as we are aware, there were no papers in the literature in which the existence of the solutions for the variational approach via bipotentials is proved by using systems of hemivariational inequalities.

5 Existence result for a nonlocal elliptic problem

This chapter is based on the paper [29].

5.1 Introduction

In this chapter we generalize F. E. Browder's results concerning pseudomonotone elliptic partial differential operators defined on unbounded domains. Browder treated equations for quasilinear operators of divergence form

$$\sum_{|\alpha|\le k} \partial_\alpha a_\alpha(x, u(x), \dots, \partial^\beta u(x)) = f(x),$$

on an arbitrary *unbounded* domain Ω, where $|\beta| \le k$ for some $k \ge 1$. We show that under suitable assumptions, Browder's result holds true if the functions a_α are *functionals* of u.

We applying the theory of pseudomonotone operators, see Section 2.3, guaranteeing boundedness and coercivity is usually a trivial matter. The proof of pseudomonotonicity usually involves the Rellich–Kondrachov compactness theorem as a crucial step. On unbounded domains however, a compact embedding result seems to require more complicated conditions on the domain, see e.g. [1, Theorem 6.52]. F. E. Browder managed to avoid the use of such compactness results in [12]. To establish pseudomonotonicity, it turns out that the main task is to prove the a.e. convergence of the sequences $\{\partial^\alpha u_j\}_{j=1}^\infty$. Browder's idea is a natural one: let the unbounded domain Ω be exhausted by an increasing sequence $\{\Omega_i\}$ of bounded domains with smooth boundary – such that on each Ω_i the Rellich–Kondrachov theorem holds. Combining this with a diagonal argument, we extract a subsequence of the lower-order derivatives $\{\partial^\alpha u_j\}$ converging a.e. to $\partial^\alpha u$ ($|\alpha| \le k-1$). Proving a.e. convergence of the highest-order derivatives $\partial^\alpha u_j \to \partial^\alpha u$ ($|\alpha| = k$) is more involved.

The results of F. E. Browder on nonlinear elliptic equations on unbounded domains have been extended in [92], [57] and [84] to strongly nonlinear elliptic equations, i.e. equations containing a term which is arbitrarily quickly increasing with respect to the values of the unknown function u. Further, there are some results in [85] and [87] on elliptic problems where the lower order terms or the boundary condition contains nonlocal (e.g. integral type) dependence on u.

The aim of this chapter is to extend Browder's theorem to elliptic operators with nonlocal dependence in the main (highest order) terms, too: we shall modify the assumptions and the proof of the original theorem for $2k$-order divergence-type nonlinear functional elliptic equations. After formulating sufficient conditions for such a nonlocal operator to be bounded, coercive and pseudomonotone, we prove our main result. Finally, we give concrete examples that satisfy our assumptions.

5.2 Problem formulation and main result

Let $\Omega \subset \mathbb{R}^n$ be a possibly unbounded domain with sufficiently smooth boundary, and let $W_0^{k,p}(\Omega) \subset V \subset W^{k,p}(\Omega)$ be a closed linear subspace with $1 < p < \infty$ and $k \geq 1$. Let $A\colon V \to V^*$ be defined by

$$\langle A(u), v\rangle = \sum_{|\alpha|\leq k} \int_\Omega a_\alpha(x, u(x), \ldots, \partial^\beta u(x), \ldots; u)\partial^\alpha v(x)\, dx \tag{5.1}$$

for all $u, v \in V$, where $|\beta| \leq k$ is a multiindex. The function a_α may depend on the pointwise values of any of the partial derivatives of u. Furthermore, "$;u$" notation signifies that a_α may be a *functional* of u. In other words, a_α may depend on the *whole* solution u.

The arguments of the functions a_α are denoted as $a_\alpha(x, \eta; u)$, and we sometimes split η as $\eta = (\zeta, \xi)$ where $\zeta \in \mathbb{R}^{N_1}$ and $\xi \in \mathbb{R}^{N_2}$, so that $\eta \in \mathbb{R}^N$ with $N = N_1 + N_2$ and write $a_\alpha(x, \zeta, \xi; u)$, where the numbers N_1 and N_2 denote number of multiindexes β such that $|\beta| \leq k-1$ and $|\beta| = k$, respectively. Furthermore, the notation

$$\eta^{(\ell)} = \{\eta_\beta \ : \ |\beta| = \ell\}$$

is used, where $\ell = 0, 1, \ldots, k$. Note that

$$\zeta = \left\{\eta^{(\ell)} \ : \ \ell = 0, 1, \ldots, k-1\right\} \quad \text{and} \quad \xi = \left\{\eta^{(\ell)} \ : \ \ell = k\right\}.$$

We impose the following assumptions on the structure of A and Ω.

(A0) Suppose that there exist a sequence $\{\Omega_i\} \subset \mathbb{R}^n$ of bounded domains such that $\Omega_i \subset \Omega_{i+1}$ $(i = 1, 2, \ldots)$ and $\Omega = \bigcup_{i=1}^\infty \Omega_i$. Furthermore, assume that each $\partial\Omega_i$ is sufficiently smooth so that the Rellich–Kondrachov theorem holds: $W^{k,p}(\Omega_i) \subset\subset W^{k-1,p}(\Omega_i)$ $(i = 1, 2, \ldots)$.

(A1) Let a_α be Carathéodory functions for fixed $u \in V$ and all multiindex $|\alpha| \leq k$, i.e. let $a_\alpha(\,\cdot\,, \eta; u)$ be measurable for every fixed $\eta \in \mathbb{R}^N$, and let $a_\alpha(x, \,\cdot\,; u)$ be continuous for almost every fixed $x \in \Omega$.

(A2) Suppose that there exist a bounded functional $g_1\colon V \to \mathbb{R}_+$ and a compact map

$$k_1^\alpha\colon V \to L^{r'_\ell}(\Omega)$$

with $k_1^\alpha(u) \geq 0$, where $p' = p/(p-1)$, $r'_\ell = r_\ell/(r_\ell - 1)$ and

$$p \leq r_\ell < p_\ell^*, \qquad p_\ell^* = \begin{cases} \dfrac{np}{n-(k-\ell)p}, & \text{if } n > (k-\ell)p \\ > 0, & \text{otherwise.} \end{cases}$$

such that

$$|a_\alpha(x,\eta;u)| \leq g_1(u)\Big[|\eta^{(\ell)}|^{p-1} + |\eta^{(\ell)}|^{r_\ell - 1}\Big] + [k_1^\alpha(u)](x)$$

for each multiindex $\ell = |\alpha| \leq k$, almost all $x \in \Omega$, all $\eta \in \mathbb{R}^N$ and all $u \in V$. Note that for $|\alpha| = \ell = k$, we must have $r_k = p$. Here, we introduce the notation

$$[\mathcal{K}_1^{(\ell)}(u)](x) = \max_{|\alpha|=\ell}\,[k_1^\alpha(u)](x)$$

for all $\ell = 1, \ldots, k$.

(A3) Suppose that

$$\sum_{|\alpha|=k} \big(a_\alpha(x,\zeta,\xi;u) - a_\alpha(x,\zeta,\xi';u)\big)(\xi_\alpha - \xi'_\alpha) > 0$$

for almost all $x \in \Omega$, all $\zeta \in \mathbb{R}^{N_1}$, $\xi \neq \xi' \in \mathbb{R}^{N_2}$ and all $u \in V$.

(A4) Suppose that there exist a bounded and lower semicontinuous functional $g_2\colon V \to \mathbb{R}_+$ and a compact map $k_2\colon V \to L^1(\Omega)$ such that

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta;u)\eta_\alpha \geq g_2(u)|\xi|^p - [k_2(u)](x)$$

for almost all $x \in \Omega$, every $u \in V$, and all $\eta = (\zeta,\xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

Note that the preceding coercivity-like assumption requires the inequality to hold for *all* $u \in V$ and η – contrary to usual asymptotic version, which is prescribed only for *large* $\|u\|_V$ and $|\eta|$. The reason for this is that the proof of pseudomonotonicity employs a certain inequality which is needed for all u and η and is derived from this coercivity estimate. We now state a significant strengthening of **(A4)** that ensures coercivity in the sense of Definition 2.3.2.

(A4') Suppose that there exist a bounded functional $g_2\colon V \to \mathbb{R}_+$ and a compact map $k_2\colon V \to L^1(\Omega)$ such that

$$\sum_{|\alpha|\leq k} a_\alpha(x,\eta;u)\eta_\alpha \geq \begin{cases} g_2(u)|\xi|^p - [k_2(u)](x), & \text{for every } u \in V \\ g_2(u)\Big[|\xi|^p + \sum_{\ell=0}^{k-1}(|\eta^{(\ell)}|^p + |\eta^{(\ell)}|^{r_\ell})\Big] - [k_2(u)](x), & \text{for large } \|u\|_V \end{cases}$$

for almost all $x \in \Omega$ and all $\eta = (\zeta, \xi) \in \mathbb{R}^N$. Here, the functional g_2 satisfies the estimate

$$g_2(u) \geq c^* \|u\|_V^{-\sigma^*}$$

for all $u \in V$ with sufficiently large $\|u\|_V$, with some $c^* > 0$ and $0 \leq \sigma^* < p - 1$. Also, the map k_2 satisfies

$$\|k_2(u)\|_{L^1(\Omega)} \leq c^* \|u\|_V^{\sigma}$$

for all $u \in V$ with sufficiently large $\|u\|_V$ and some $0 \leq \sigma < p - \sigma^*$.

(A5) Whenever $u_j \rightharpoonup u$ in V and $\{\eta_j\} \subset \mathbb{R}^N$ with $\eta_j \to \eta$, then $a_\alpha(x, \eta_j; u_j) \to a_\alpha(x, \eta; u)$ for a.e. $x \in \Omega$ up to a subsequence.

We state the main result of this chapter as

Theorem 5.2.1. Suppose Assumptions ***(A0)*-*(A5)*** and ***(A4′)*** holds true. Then for any $F \in V^*$ there is a $u \in V$ such that $A(u) = F$ holds in V^*.

The proof is simply a verification of the assumptions of Theorem 2.3.1. It is straightforward to see that under these assumption the operator A is bounded using Hölder's inequality. In order to apply the abstract surjectivity result (Theorem 2.3.1), it remains to prove that A is pseudomonotone and coercive.

5.3 Proof of pseudomonotonicity and coercivity

Theorem 5.3.1. Assume **(A0)**, **(A1)**, **(A2)**, **(A3)** and **(A4)**. Then the operator $A\colon V \to V^*$ defined in (5.1) is pseudomonotone.

Proof. Let $\{u_j\} \subset V$ be a sequence that satisfies $u_j \rightharpoonup u$ in V and

$$\limsup_{j\to\infty} \langle A(u_j), u_j - u\rangle \leq 0. \tag{5.2}$$

Assumption **(A0)** implies that there exists a sequence $\{\Omega_i\} \subset \mathbb{R}^n$ of bounded domains such that $\Omega_i \subset \Omega_{i+1}$, $\Omega = \bigcup_{i=1}^\infty \Omega_i$ and the Rellich–Kondrachov theorem holds on each Ω_i: $W^{k,p}(\Omega_i) \subset\subset W^{k-1,p}(\Omega_i)$. For every $i \in \mathbb{N}$ there is a subsequence $\{u_j^{(i)}\}_{j=1}^\infty \subset \{u_j\}_{j=1}^\infty$ (indexed by the same j for simplicity) such that $\{u_j^{(i)}\}_{j=1}^\infty \supset \{u_j^{(i+1)}\}_{j=1}^\infty$ and $u_j^{(i)} \to u$ in $W^{k-1,p}(\Omega_i)$ as $j \to \infty$. The diagonal sequence $\{u_j\}_{j=1}^\infty = \{u_j^{(j)}\}_{j=1}^\infty$ satisfies $u_j \to u$ in $W^{k-1,p}(\Omega_i)$ for any $i \in \mathbb{N}$. Then

$$\partial^\gamma u_j \to \partial^\gamma u \quad \text{a.e. in } \Omega \text{ for all } |\gamma| \leq k-1 \tag{5.3}$$

up to a subsequence. Further, by **(A2)** and **(A4)** we may assume that the sequences $\{\mathcal{K}_1^{(\ell)}(u_j)\} \subset L^{r'_\ell}(\Omega)$ (for every $\ell = 1, \dots, k$) and $\{k_2(u_j)\} \subset L^1(\Omega)$ are convergent. Note however, that we *do not* have $u_j \to u$ in $W^{k-1,p}(\Omega)$.

The following notations are used throughout the proof:

$$\left.\begin{aligned} \zeta(x) &= \{\partial^\beta u(x) \;:\; |\beta| \le k-1\}, \\ \zeta_j(x) &= \{\partial^\beta u_j(x) \;:\; |\beta| \le k-1\}, \\ \xi(x) &= \{\partial^\beta u(x) \;:\; |\beta| = k\}, \\ \xi_j(x) &= \{\partial^\beta u_j(x) \;:\; |\beta| = k\} \\ \eta^{(\ell)}(x) &= \{\partial^\beta u(x) \;:\; |\beta| = \ell\}, \\ \eta_j^{(\ell)}(x) &= \{\partial^\beta u_j(x) \;:\; |\beta| = \ell\}, \\ \eta(x) &= \{\eta^{(\ell)}(x) \;:\; \ell = 1, \dots, k\}, \\ \eta_j(x) &= \{\eta_j^{(\ell)}(x) \;:\; \ell = 1, \dots, k\}. \end{aligned}\right\} \tag{5.4}$$

Using these, we may write

$$\langle A(u_j) - A(u), u_j - u\rangle = \int_\Omega p_j,$$

where

$$p_j(x) = \sum_{|\alpha|\le k} \big[a_\alpha(x, \zeta_j(x), \xi_j(x); u_j) - a_\alpha(x, \zeta(x), \xi(x); u)\big](\partial^\alpha u_j - \partial^\alpha u),$$

Also, (5.3) may be written as $\zeta_j \to \zeta$ a.e. or $\eta_j^{(\ell)} \to \eta^{(\ell)}$ a.e. for all $\ell = 0, 1, \dots, k-1$.

First we derive conclusion **(PM1)** of pseudomonotonicity. The following trivial lemma is well-known.

Lemma 5.3.2. Relation (5.2) implies

$$\limsup_{j\to\infty} \langle A(u_j) - A(u), u_j - u\rangle \le 0.$$

Proof. We have

$$\limsup_{j\to\infty} \langle A(u_j) - A(u), u_j - u\rangle \le \limsup_{j\to\infty} \langle A(u_j), u_j - u\rangle - \liminf_{j\to\infty} \langle A(u), u - u_j\rangle.$$

By (5.2), the first term is nonpositive. For the second term, note that the functional $v \mapsto \langle A(u), u - v\rangle$ is weakly lower semicontinuous, so $\liminf \langle A(u), u - u_j\rangle \ge 0$. □

The conclusion of Lemma 5.3.2 can be written briefly as

$$\limsup_{j\to\infty} \int_\Omega p_j \le 0. \tag{5.5}$$

Using the positive-negative decomposition $p_j(x) = p_j^+(x) - p_j^-(x)$, we have $0 \leq p_j^+(x) = p_j(x) + p_j^-(x)$ hence (5.5) immediately implies

$$\int_\Omega p_j^+ \to 0 \tag{5.6}$$

as $j \to \infty$. Hence, the convergence $\int_\Omega p_j^- \to 0$ $(j \to \infty)$ needs to be established, so that $\int_\Omega p_j \to 0$ $(j \to \infty)$ holds, which implies **(PM1)**. This will be done via Vitali's convergence theorem (see Theorem 2.6.6) applied to the sequence $\{p_j^-\}$.

Lemma 5.3.3. The sequence $\{p_j^-\}$ is equiintegrable and tight over Ω. Furthermore, there exist $C_1 > 0$ and an a.e. bounded function $\beta\colon \Omega \to \mathbb{R}_+$ such that for a.a. $x \in \Omega$,

$$p_j(x) \geq C_1 |\xi_j(x)|^p - \beta(x) \tag{5.7}$$

Proof. Expand $p_j(x)$ as

$$p_j(x) = \sum_{|\alpha|=k} a_\alpha(x, \zeta_j, \xi_j; u_j)\partial^\alpha u_j + \sum_{|\alpha|\leq k-1} a_\alpha(x, \zeta_j, \xi_j; u_j)\partial^\alpha u_j - w_j(x),$$

where

$$\begin{aligned} w_j(x) &= \sum_{|\alpha|\leq k} \Big[a_\alpha(x,\zeta,\xi;u)\big(\partial^\alpha u_j - \partial^\alpha u\big) + a_\alpha(x,\zeta_j,\xi_j;u_j)\partial^\alpha u\Big] \\ &=: \sum_{\ell=0}^{k} w_j^{(\ell)}(x). \end{aligned}$$

We prove that $\{w_j\}$ is equiintegrable and tight. Assumption **(A2)** implies that

$$\begin{aligned} |w_j^{(\ell)}(x)| &\leq C_2\Big(g_1(u)\Big[|\eta^{(\ell)}|^{p-1} + |\eta^{(\ell)}|^{r_\ell - 1}\Big] + [\mathcal{K}_1^{(\ell)}(u)](x)\Big)\big(|\eta_j^{(\ell)}| + |\eta^{(\ell)}|\big) \\ &\quad + C_2\Big(g_1(u_j)\Big[|\eta_j^{(\ell)}|^{p-1} + |\eta_j^{(\ell)}|^{r_\ell - 1}\Big] + [\mathcal{K}_1^{(\ell)}(u_j)](x)\Big)|\eta^{(\ell)}| \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\leq C_3\Big(|\eta^{(\ell)}|^{p-1}|\eta_j^{(\ell)}| + |\eta^{(\ell)}|^p + |\eta^{(\ell)}|^{r_\ell-1}|\eta_j^{(\ell)}| + |\eta^{(\ell)}|^{r_\ell} \\ &\quad + |\eta_j^{(\ell)}|^{p-1}|\eta^{(\ell)}| + |\eta_j^{(\ell)}|^{r_\ell-1}|\eta^{(\ell)}| \\ &\quad + [\mathcal{K}_1^{\ell}(u)](x)\big(|\eta_j^{(\ell)}| + |\eta^{(\ell)}|\big) + [\mathcal{K}_1^{(\ell)}(u_j)](x)|\eta^{(\ell)}|\Big) \end{aligned} \tag{5.9}$$

where $C_2, C_3 > 0$ are constants. We shall apply Proposition 2.6.3 to prove that the function dominating $w_j^{(\ell)}(x)$ is equiintegrable and tight. The weak convergence $u_j \rightharpoonup u$ in $V \subset W^{k,p}(\Omega)$ implies that the sequence $\{\eta_j^{(\ell)}\} \subset W^{k-\ell,p}(\Omega)$ is bounded, hence by the Sobolev embedding $W^{k-\ell,p}(\Omega) \subset L^q(\Omega)$ (where $p \leq q \leq p_\ell^*$) we have that $\{\eta_j^{(\ell)}\} \subset L^q(\Omega)$ is bounded. In particular, $\{|\eta_j^{(\ell)}|^{r_\ell}\}, \{|\eta_j^{(\ell)}|^p\} \subset L^1(\Omega)$ are bounded.

The second and fourth terms in (5.9) are equiintegrable and tight by part (1) of Proposition 2.6.3. Further, the first term is equiintegrable and tight by part (3) of Proposition 2.6.3 applied to the constant sequence $|\eta^{(\ell)}|^{p-1} \in L^{p'}(\Omega)$ (with $|\eta^{(\ell)}|^p \in L^1(\Omega)$ being equiintegrable and tight by part (1) of the said Proposition) and to the bounded sequence $\{|\eta_j^{(\ell)}|\} \subset L^p(\Omega)$. The third term is similar. The fifth term is also equiintegrable and tight by part (3) of Proposition 2.6.3 applied to the bounded $\{|\eta_j^{(\ell)}|^{p-1}\} \subset L^{p'}(\Omega)$ and the constant $|\eta^{(\ell)}| \in L^p(\Omega)$ sequences. The sixth term is handled in a similar way. Finally, $\{\mathcal{K}_1^{(\ell)}(u_j)^{r'_\ell}\} \subset L^1(\Omega)$ is convergent by construction. Therefore the last two terms are equiintegrable and tight, too.

Moreover, assumption **(A4)** implies that

$$p_j(x) \geq g_2(u_j)|\xi_j|^p - k_2(u_j)(x) - |w_j(x)| \geq -k_2(u_j)(x) - |w_j(x)|. \tag{5.10}$$

It follows that

$$0 \leq p_j^-(x) \leq [k_2(u_j)](x) + |w_j(x)|,$$

hence $\{p_j^-\}$ is equiintegrable and tight, where we have used the fact that $\{k_2(u_j)\}$ is equiintegrable and tight, since it is convergent in $L^1(\Omega)$.

Finally, we turn to the proof of inequality (5.7). Young's inequality applied to the products on the right side of (5.8) implies that

$$\begin{aligned}|w_j^{(\ell)}(x)| \leq K_3(\varepsilon)\Big(|\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta^{(\ell)}|^{r_\ell} + [\mathcal{K}_1^{(\ell)}(u)](x)^{r'_\ell}\Big) + C_3\varepsilon\Big(|\eta_j^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{r_\ell}\Big)\\ + C_4\varepsilon\Big(|\eta_j^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + [\mathcal{K}_1^{(\ell)}(u_j)](x)^{r'_\ell}\Big) + K_4(\varepsilon)|\eta^{(\ell)}|^{r_\ell}.\end{aligned}$$

By summing over $j = 0, 1, \dots, k$, and noting that $r_k = p$, we get

$$\begin{aligned}|w_j(x)| &\leq C_5\varepsilon|\xi_j|^p + K_5(\varepsilon)\Big(2|\xi|^p + [\mathcal{K}_1^{(k)}(u)](x)^{p'} + [\mathcal{K}_1^{(k)}(u_j)](x)^{p'}\Big) + \sum_{\ell=0}^{k-1}|w_j^{(\ell)}(x)|\\
&\leq C_5\varepsilon|\xi_j|^p + K(\varepsilon)\Bigg(2|\xi|^p + [\mathcal{K}_1^{(k)}(u)](x)^{p'} + [\mathcal{K}_1^{(k)}(u_j)](x)^{p'}\\
&\qquad + \sum_{\ell=0}^{k-1}\Big[|\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + |\eta_j^{(\ell)}|^{(p-1)r'_\ell}\\
&\qquad + [\mathcal{K}_1^{(\ell)}(u)](x)^{r'_\ell} + [\mathcal{K}_1^{(\ell)}(u_j)](x)^{r'_\ell}\Big]\Bigg)\\
&=: C_5\varepsilon|\xi_j|^p\\
&\quad + K(\varepsilon)\Bigg(2|\xi|^p + \sum_{\ell=0}^{k-1}\Big[|\eta^{(\ell)}|^{r_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{(p-1)r'_\ell}\Big] + [\mathcal{K}_3(u, u_j)](x)\Bigg),\end{aligned}$$

where $\{\mathcal{K}_3(u, u_j)\} \subset L^1(\Omega)$ is convergent, hence it is convergent a.e. up to a subsequence, thus it is a.e. bounded. Therefore, using the a.e. convergence $\eta^{(\ell)} \to \eta$ ($\ell = 0, \dots, k-1$) we have that the function

$$\beta_1(x) = 2|\xi|^p + \sum_{\ell=0}^{k-1} \left[|\eta^{(\ell)}|^{r_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{(p-1)r'_\ell} \right] + [\mathcal{K}_3(u, u_j)](x)$$

is bounded a.e.

The first inequality of (5.10) combined with the preceding estimate and assumption **(A4)** leads to

$$\begin{aligned} p_j(x) &\geq g_2(u_j)|\xi_j|^p - [k_2(u_j)](x) - |w_j(x)| \\ &\geq g_2(u_j)|\xi_j|^p - C_5\varepsilon|\xi_j|^p - [k_2(u_j)](x) - K(\varepsilon)\beta_1(x) \\ &\geq |\xi_j|^p(A - C_5\varepsilon) - \beta(x) \end{aligned}$$

where $g_2(u_j) \geq A > 0$ (due to the weak lower semicontinuity of $g_2 \colon V \to \mathbb{R}_+$ and the weak convergence $u_j \rightharpoonup u$) and $\beta(x) = K(\varepsilon)\beta_1(x) + [k_2(u_j)](x)$ is still bounded a.e., because $\{k_2(u_j)\} \subset L^1(\Omega)$ is bounded and therefore convergent a.e. up to a subsequence. The desired inequality follows by choosing $\varepsilon = A/(2C_5)$. □

Claim. The convergence $p_j^- \to 0$ a.e. holds.

Proof. Split $p_j(x)$ as

$$\begin{aligned} p_j(x) = &\sum_{|\alpha|=k} \left[a_\alpha(x, \zeta_j, \xi_j; u_j) - a_\alpha(x, \zeta_j, \xi; u_j)\right](\partial^\alpha u_j - \partial^\alpha u) \\ &+ \sum_{|\alpha|=k} \left[a_\alpha(x, \zeta_j, \xi; u_j) - a_\alpha(x, \zeta, \xi; u)\right](\partial^\alpha u_j - \partial^\alpha u) \\ &+ \sum_{|\alpha|\leq k-1} \left[a_\alpha(x, \zeta_j, \xi_j; u_j) - a_\alpha(x, \zeta, \xi; u)\right](\partial^\alpha u_j - \partial^\alpha u) \\ =: &\; q_j(x) + r_j(x) + s_j(x) \end{aligned} \tag{5.11}$$

Let χ_j be the characteristic function of the level set $\{x \in \Omega \;:\; p_j^-(x) > 0\}$ and write

$$-p_j^- = \chi_j q_j + \chi_j r_j + \chi_j s_j.$$

First, note that $\chi_j q_j \geq 0$ a.e. due to the monotonicity assumption **(A3)**, so it is enough to prove $\chi_j r_j \to 0$ a.e. and $\chi_j s_j \to 0$ a.e. Lemma 5.3.3 ensures that there exists $\beta \colon \Omega \to \mathbb{R}$ a.e. bounded such that

$$|\xi_j(x)|^p \leq \beta(x),$$

for all $x \in \Omega$ such that $p_j(x) < 0$. Therefore $\{\chi_j(x)\xi_j(x)\}$ is bounded for a.e. $x \in \Omega$. By **(A2)**, **(A5)** and $\zeta_j \to \zeta$ a.e. (from (5.3)), we find that $\chi_j r_j \to 0$ a.e. and $\chi_j s_j \to 0$ a.e. for a subsequence, from which $p_j^- \to 0$ a.e. follows. □

In summary, we have that $\{p_j^-\}$ is equiintegrable and tight, and $p_j^- \to 0$ a.e. A corollary of the Vitali convergence theorem (Theorem 2.6.6 below) yields that these conditions are actually *necessary and sufficient* to ensure the convergence

$$\int_\Omega p_j^- \to 0,$$

as $j \to \infty$. Recalling (5.6), we have in summary

$$\int_\Omega p_j \to 0 \tag{5.12}$$

as $j \to \infty$. Then conclusion **(PM1)** of pseudomonotonicity is established:

$$\begin{aligned}\langle A(u_j), u_j - u\rangle &= \langle A(u_j) - A(u), u_j - u\rangle + \langle A(u), u_j - u\rangle \\ &= \int_\Omega p_j + \langle A(u), u_j - u\rangle \to 0.\end{aligned}$$

Turning to the proof of **(PM2)**, first note that (5.12) implies that $p_j \to 0$ a.e. up to a subsequence.

Claim. The convergence $\xi_j \to \xi$ a.e. holds.

Proof. It follows from estimate (5.7) that $\{\xi_j\}$ is bounded a.e. Fix an $x_0 \in \Omega$ such that $\{\xi_j(x_0)\}$ is bounded and $p_j(x_0) \to 0$. Assume for contradiction that $\xi_j(x_0) \to \xi'$ for a subsequence and some ξ' such that $\xi' \neq \xi(x_0)$. Since we have $\zeta_j \to \zeta$ a.e., by using decomposition (5.11) and **(A1)**, it follows that $r_j \to 0$ and $s_j \to 0$ a.e. But then the continuity assumption **(A5)** implies

$$p_j(x_0) \to 0 = \sum_{|\alpha|=k} \big[a_\alpha(x_0, \zeta, \xi'; u) - a_\alpha(x_0, \zeta, \xi; u)\big](\xi'_\alpha - \partial^\alpha u(x_0))$$

Thus **(A3)** yields $\xi'_\alpha = \partial^\alpha u(x_0)$, which is a contradiction. □

Finally, we prove $A(u_j) \rightharpoonup A(u)$ in V^*. By the Vitali convergence theorem

$$\begin{aligned}\langle A(u_j), v\rangle &= \sum_{|\alpha|\le k} \int_\Omega a_\alpha(x, \eta_j; u_j)\partial^\alpha v(x)\,dx \\ &\to \sum_{|\alpha|\le k} \int_\Omega a_\alpha(x, \eta; u)\partial^\alpha v(x)\,dx,\end{aligned}$$

because the integrand is equiintegrable and tight by Proposition 2.6.3 (3) and the a.e. convergence $a_\alpha(x, \eta_j; u_j) \to a_\alpha(x, \eta; u)$ follows from **(A5)**. □

Proposition 5.3.4. If **(A4')** holds then $A\colon V \to V^*$ is coercive.

Proof. We have for $u \in V$ with sufficiently large $\|u\|_V$,

$$\begin{aligned}\langle A(u), u\rangle &\geq g_2(u)\int_\Omega |\xi|^p + \sum_{\ell=0}^{k-1}(|\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^p)\,dx - \int_\Omega [k_2(u)](x)\,dx \\ &\geq C\|u\|_V^{-\sigma^*}\|u\|_V^p - c^*\|u\|_V^\sigma \\ &\geq C'\|u\|_V^{p-\sigma^*}\end{aligned}$$

for some $C, C' > 0$. Therefore $\langle A(u), u\rangle/\|u\|_V \to +\infty$ if $\|u\|_V \to \infty$, because $p - \sigma^* > 1$. $\square$

5.4 Examples

Here we formulate examples satisfying **(A1)**–**(A5)** and **(A4')**. For all $|\alpha| = \ell$, with $\ell = 0, 1, \ldots, k$ consider

$$a_\alpha(x, \eta; u) = \Psi_\ell(H_\ell(u))\Big[a_\ell(x)\chi_\ell(G_\ell(u))\big(|\eta^{(\ell)}|^{r_\ell - 2} + |\eta^{(\ell)}|^{p-2}\big)\eta_\alpha + b_\alpha(x)M_\alpha(u)\Big],$$

where $p \leq r_\ell \leq p_\ell^*$ and $m \leq a_\ell(x) \leq M$ for some constants $m, M > 0$. (We remind the reader that $\eta^{(k)} = \xi$ and $p_k^* = p$, so that the highest order a_α reads

$$a_\alpha(x, \eta; u) = \Psi_k(H_k(u))\big[a_k(x)\chi_k(G_k(u))|\xi|^{p-2}\xi_\alpha + b_\alpha(x)M_\alpha(u)\big],$$

where $|\alpha| = k$, which is reminiscent of the p-Laplacian.) We propose the following two possibilities for the choice of Ψ_ℓ and H_ℓ.

1. Let $H_\ell\colon W^{k-1,p}(\Omega') \to L^\infty(\Omega)$ be a bounded linear map (with $\Omega' \subset \Omega$ a bounded domain) and let $\Psi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $\Psi_\ell(\nu) \geq C_\Psi/(1 + |\nu|)^{-\sigma^*}$ for some $C_\Psi > 0$ and large $|\nu|$.
2. Let $H_\ell\colon V \to \mathbb{R}$ be a bounded linear functional and let $\Psi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $\Psi_\ell(\nu) \geq C_\Psi/(1 + |\nu|^{\sigma^*})$ for some $C_\Psi > 0$.

Again, we may choose χ_ℓ and G_ℓ as follows.

1. Let $G_\ell\colon W^{k-1,p}(\Omega') \to L^{p'}(\Omega)$ be a bounded linear map and let $\chi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $m \leq \chi_\ell(\nu) \leq M$ for some constants $m, M > 0$.
2. Let $G_\ell\colon V \to \mathbb{R}$ be a bounded linear functional and let $\chi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $m \leq \chi_\ell(\nu) \leq M$ for some constants $m, M > 0$.

Finally, for fixed any $|\alpha| = \ell$, let $2 \leq p_1 \leq p$, $m = 1, \ldots, k$ and let

$$M_\alpha\colon V \to W^{m,p_1}(\Omega) \text{ (or } \mathbb{R})$$

be a bounded map such that

$$\|M_\alpha(u)\|_{W^{m,p_1}(\Omega)} \leq \text{const}\|u\|_V^{\gamma_\alpha}, \tag{5.13}$$

$$\|M_\alpha(u) - M_\alpha(v)\|_{W^{m,p_1}(\Omega)} \leq \text{const}\|u-v\|_V^{\gamma_\alpha}, \tag{5.14}$$

where $0 < \gamma_\alpha < \frac{p}{r'_\ell}$; also, let $\lambda_\alpha = q_\alpha/r'_\ell$ and $b_\alpha \in L^{r'_\ell \lambda'_\alpha}(\Omega)$ where

$$\begin{cases} p_1 < q_\alpha < \frac{np_1}{n-mp_1} & \text{if} \quad m < \frac{n}{p_1} \\ q_\alpha > 0 & \text{otherwise} \end{cases}$$

(or, if $M_\alpha \colon V \to \mathbb{R}$, then

$$|M_\alpha(u)| \leq \text{const}\|u\|_V^{\gamma_\alpha}, \tag{5.15}$$

$$|M_\alpha(u) - M_\alpha(v)| \leq \text{const}\|u-v\|_V^{\gamma_\alpha}, \tag{5.16}$$

with $\gamma_\alpha = \sigma/r'_\ell$, $b_\alpha \in L^{r'_\ell}(\Omega)$).

Under these hypotheses, **(A1)** and **(A3)** are satisfied. Note that the continuous embeddings $W^{m,p_1}(\Omega) \subset L^{q_\alpha}(\Omega)$ hold, so

$$\|M_\alpha(u)\|_{L^{q_\alpha}(\Omega)} \leq \text{const}\|M_\alpha(u)\|_{W^{m,p_1}(\Omega)} \leq \text{const}\|u\|_V^{\gamma_\alpha}.$$

Therefore, by Hölder's inequality and (5.13)

$$\begin{aligned} \int_\Omega |b_\alpha(x)|^{r'_\ell}|M_\alpha(u)|^{r'_\ell}\,dx &\leq \|b_\alpha\|^{r'_\ell}_{L^{r'_\ell\lambda'_\alpha}(\Omega)}\Big[\int_\Omega |M_\alpha(u)|^{r'_\ell\lambda_\alpha}\Big]^{1/\lambda_\alpha} \\ &= \|b_\alpha\|^{r'_\ell}_{L^{r'_\ell\lambda'_\alpha}(\Omega)}\|M_\alpha(u)\|^{q_\alpha/\lambda_\alpha}_{L^{q_\alpha}(\Omega)} \\ &\leq \text{const}\|b_\alpha\|^{r'_\ell}_{L^{r'_\ell\lambda'_\alpha}(\Omega)}\|u\|_V^{q_\alpha\gamma_\alpha/\lambda_\alpha} \\ &\leq c^*\|u\|_V^\sigma, \end{aligned} \tag{5.17}$$

where $\sigma = q_\alpha\gamma_\alpha/\lambda_\alpha = r'_\ell\gamma_\alpha < p$ for the case $M_\alpha \colon V \to W^{m,p_1}(\Omega)$. The case $M_\alpha \colon V \to \mathbb{R}$ is treated similarly.

Claim. Assumption **(A2)** holds.

Proof. The growth condition reads

$$\begin{aligned} |a_\alpha(x,\eta,\xi;u)| \leq{}& \Psi_\ell(H_\ell(u))|a_\ell(x)|\chi_\ell(G_\ell(u))\big(|\eta^{(\ell)}|^{r_\ell-1} + |\eta^{(\ell)}|^{p-1}\big) \\ &+ \Psi_\ell(H_\ell(u))|b_\alpha(x)M_\alpha(u)|. \end{aligned}$$

Then $g_1(u) = \Psi_\ell(H_\ell(u))M^2$ is a bounded functional by assumption. Letting

$$[k_1^\alpha(u)](x) = \Psi_\ell(H_\ell(u))|b_\alpha(x)M_\alpha(u)|,$$

we find by (5.17) that $k_1^\alpha\colon V \to L^{r'_\ell}(\Omega)$ is bounded.

Proving the compactness of k_1^α requires more effort (except when $M_\alpha\colon V \to \mathbb{R}$). To this end, suppose that $\{u_j\} \subset V$ is a bounded sequence. Let $\{\Omega_i\}$ be the sequence guaranteed to exist by assumption **(A0)**. Then $\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega\setminus\Omega_i)} \to 0$. Using the compact embedding $W^{m,p_1}(\Omega_i) \subset\subset L^{q_\alpha}(\Omega_i)$ we can choose subsequences of $\{u_j\}$ as follows. Let $\{u_{1j}\} \subset \{u_j\}$ be a subsequence such that

$$\|M_\alpha(u_{1j}) - M_\alpha(u_{1m})\|_{L^{q_\alpha}(\Omega_1)} < 1 \quad \text{for } j,m = 1,2,3,\dots$$

Let $\{u_{2j}\} \subset \{u_{1j}\}$ be a subsequence such that

$$\|M_\alpha(u_{2j}) - M_\alpha(u_{2m})\|_{L^{q_\alpha}(\Omega_2)} < \frac{1}{2} \quad \text{for } j,m = 2,3,\dots$$

Continuing this way, for fixed i let $\{u_{ij}\} \subset \{u_{i-1,j}\}$ be a subsequence such that

$$\|M_\alpha(u_{ij}) - M_\alpha(u_{im})\|_{L^{q_\alpha}(\Omega_i)} < \frac{1}{i} \quad \text{for } j,m = i,i+1,\dots$$

It follows that the diagonal sequence $\{u_{jj}\}$ satisfies

$$\|M_\alpha(u_{jj}) - M_\alpha(u_{mm})\|_{L^{q_\alpha}(\Omega_i)} < \frac{1}{i} \quad \text{for } j,m = i,i+1,\dots$$

Using Hölder's inequality, we find for $j,m \geq i$

$$\begin{aligned}
&\int_\Omega |b_\alpha(x)|^{r'_\ell} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{r'_\ell}\,dx \\
&= \left(\int_{\Omega\setminus\Omega_i} + \int_{\Omega_i}\right) |b_\alpha(x)|^{r'_\ell} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{r'_\ell}\,dx \\
&\leq \text{const}\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega\setminus\Omega_i)} \left[\int_{\Omega\setminus\Omega_i} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{q_\alpha}\,dx\right]^{1/\lambda_\alpha} \\
&\quad + \text{const}\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega_i)} \left[\int_{\Omega_i} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{q_\alpha}\,dx\right]^{1/\lambda_\alpha}.
\end{aligned}$$

Here, $\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega\setminus\Omega_i)} \to 0$ and $\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega_i)}$ is bounded. By assumption (5.14), the first integral is bounded and for the second integral we have

$$\int_{\Omega_i} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{q_\alpha}\,dx \leq \frac{1}{i^{q_\alpha}} \to 0$$

if $j,m \geq i$ and $i \to \infty$. □

We now show that **(A4')** holds. It is enough to estimate the terms of

$$\begin{aligned}
&\sum_{|\alpha|=\ell} a_\alpha(x,\eta;u)\eta_\alpha \\
&\quad = \Psi_\ell(H_\ell(u)) a_\ell(x) \chi_\ell(G_\ell(u)) (|\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^p) + \sum_{|\alpha|=\ell} \Psi_\ell(H_\ell(u)) b_\alpha(x) M_\alpha(u) \eta_\alpha
\end{aligned}$$

for all $\ell = 0, 1, \ldots, k$. The first term may be estimated from below by

$$C\Psi_\ell(H_\ell(u))\Big(|\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^p\Big)$$

for some constant $C > 0$. Here, the quantity $\Psi_\ell(H_\ell(u))$ satisfies

$$\Psi_\ell(H_\ell(u)) \geq \frac{C_\Psi}{|H_\ell(u)|^{\sigma^*} + 1} \geq \frac{C_\Psi}{\|H_\ell(u)\|^{\sigma^*}_{L^\infty(\Omega)} + 1} \geq \frac{C'_\Psi}{\|u\|^{\sigma^*}_{W^{k-1,p}(\Omega')} + 1} \geq \frac{C'_\Psi}{\|u\|^{\sigma^*}_V + 1}.$$

The terms of the sum may be bounded from above by Young's inequality,

$$\begin{aligned}\Psi_\ell(H_\ell(u))|b_\alpha(x)M_\alpha(u)\eta_\alpha| &\leq \varepsilon\Psi_\ell(H_\ell(u))|\eta_\alpha|^{r_\ell} + C^*(\varepsilon)|b_\alpha(x)|^{r'_\ell}|M_\alpha(u)|^{r'_\ell}\\ &\leq \varepsilon\Psi_\ell(H_\ell(u))|\eta^{(\ell)}|^{r_\ell} + C^*(\varepsilon)|b_\alpha(x)|^{r'_\ell}|M_\alpha(u)|^{r'_\ell}.\end{aligned}$$

Choosing a sufficiently small $\varepsilon > 0$, it turns out that it is enough to estimate the $L^1(\Omega)$-norm of the expression

$$[k_2^\alpha(u)](x) = |b_\alpha(x)|^{r'_\ell}|M_\alpha(u)|^{r'_\ell},$$

which, using (5.17), satisfies

$$\|k_2^\alpha\|_{L^1(\Omega)} \leq c^*\|u\|^\sigma_V.$$

The proof of compactness of k_2^α is analogous to that of k_1^α. The required k_2 in Assumption **(A4')** is given by the pointwise maximum of k_2^α over all $|\alpha| \leq k$.

To finish the argument, note that assumption **(A5)** is satisfied since the functions Φ_ℓ, χ_ℓ and Ψ_α are continuous and the operators H_ℓ, G_ℓ and M_α are continuous in the respective Sobolev and Lebesgue spaces. Thus if $u_j \rightharpoonup u$ in V, then for a subsequence $H_\ell(u_j)$, $G_\ell(u_j)$, $M_\alpha(u_j)$ are convergent a.e. in Ω.

Example 5.4.1. For a more concrete example to M_α, consider the following. In the case $M_\alpha\colon V \to W^{m,p_1}(\Omega)$, let $M_\alpha(u) = \widetilde{H}_\alpha(u)$ where $\widetilde{H}_\alpha\colon V \to W^{m,p_1}(\Omega)$ is a continuous linear operator. For a more concrete example, consider

$$[\widetilde{H}_\alpha(u)](x) = \sum_{|\alpha|\leq k}\int_\Omega G_\alpha(x,y)\partial^\alpha u(y)\,dy,$$

where the functions $G_\alpha\colon \Omega \times \Omega \to \mathbb{R}$ satisfy

$$x \mapsto \left[\int_\Omega |\partial^\beta G_\alpha(x,y)|^{p'}\,dy\right]^{1/p'} \in L^{p_1}(\Omega) \quad \text{for } |\beta| \leq m.$$

In the case $M_\alpha\colon V \to \mathbb{R}$, let $M_\alpha(u) = \Phi_\alpha(\widetilde{H}_\alpha(u))$, where $\widetilde{H}_\alpha\colon V \to \mathbb{R}_+$ is a bounded linear functional and $\Phi_\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $|\Phi_\alpha(\nu_1) - \Phi_\alpha(\nu_2)| \leq C_\Phi|\nu_1 - \nu_2|^{\sigma/r'_\ell}$. Note that $\Phi_\alpha(\nu) \leq C_\Phi|\nu|^{\sigma/r'_\ell}$ follows automatically.

The operators $H_\ell\colon W^{k-1,p}(\Omega') \to L^\infty(\Omega)$ and $G_\ell\colon W^{k-1,p}(\Omega') \to L^{p'}(\Omega)$ can be defined by the formula

$$(Bu)(x) = \sum_{|\alpha|\le k-1} \int_{\Omega'} G_\alpha(x,y)\partial^\alpha u(u)\,dy,$$

where the measurable functions $G_\alpha\colon \Omega\times\Omega' \to \mathbb{R}$ satisfy

$$x \mapsto \left[\int_{\Omega'} |G_\alpha(x,y)|^{p'}\,dy\right]^{1/p'} \in L^\infty(\Omega) \text{ and } L^{p'}(\Omega),$$

respectively.

Example 5.4.2. Now suppose that $\Omega' \subset \Omega$ is a bounded domain with sufficiently smooth boundary. Let $V = H_0^1(\Omega)$, $V_1 = H_0^1(\Omega') \subset W^{1,2}(\Omega')$, $m = 1$, $p_1 = 2$ and let $B_\alpha\colon V_1 \to V_1^*$ be an elliptic operator given by

$$\langle B_\alpha(v), w\rangle = \int_\Omega \left[\sum_{j,k=1}^n a_{jk}^\alpha(x)\partial_j v\partial_k w + c_\alpha(x)vw\right] dx,$$

where $v, w \in V_1$ and $a_{jk}^\alpha \in L^\infty(\Omega)$ form a uniformly elliptic coefficient matrix and $c_\alpha(x) \ge c_0 > 0$. The strong form of this operator is "$-\mathrm{div}\mathbf{A}^\alpha Dv + c_\alpha v$", where $\mathbf{A}^\alpha = (a_{jk}^\alpha)$. Then we may take $M_\alpha(u) = \widetilde{H}_\alpha(u) = v$, where $v \in V_1$ is a unique solution to $B_\alpha(v) = u\big|_{\Omega'} \in V_1^*$. Then $\widetilde{H}_\alpha = B_\alpha^{-1}\colon V \to V_1 \subset W^{1,2}(\Omega)$ is a continuous linear operator. (A function $v \in H_0^1(\Omega')$ belongs to $W^{1,2}(\Omega)$ if it is extended by 0 in $\Omega \setminus \Omega'$.)

Example 5.4.3. More generally, let $V_1 \subset W^{m,p_1}(\Omega)$ be a closed subspace (which may depend on α) and let $N_\alpha\colon V_1 \to V_1^*$ be a bounded, strictly monotone and coercive operator that satisfies

$$\langle N_\alpha(v_1) - N_\alpha(v_2), v_1 - v_2\rangle \ge c_2\|v_1 - v_2\|_{V_1}^{p_1},$$

and

$$\langle N_\alpha(v), v\rangle \ge c_3\|v\|_{V_1}^{p_1}.$$

Then for every $w \in V_1^*$ there exists a unique element $v \in V_1$ such that $N_\alpha(v) = w$ and the mapping $N_\alpha^{-1}\colon V_1^* \to V_1$ is Hölder continuous:

$$\|N_\alpha^{-1}(w_1) - N_\alpha^{-1}(w_2)\|_{V_1}^{1/(p_1-1)} \le \mathrm{const}\|w_1 - w_2\|_{V_1^*}.$$

Now let

$$M_\alpha(u) := N_\alpha^{-1}(h_\alpha u),$$

for all $u \in V$, where $h_\alpha \in L^{p_1' r}(\Omega)$ is some fixed function that makes $h_\alpha u \in L^{p_1'}(\Omega) \subset V_1^*$ if $p > 2$, and we may take $h_\alpha \equiv 1$ if $p = 2$. We have that $M_\alpha(u) \in V_1$ and $M_\alpha\colon V \to W^{m,p_1}(\Omega)$

is bounded map:

$$\begin{aligned}
\|M_\alpha(u)\|_{W^{m,p_1}(\Omega)} &= \|N_\alpha^{-1}(h_\alpha u)\|_{W^{m,p_1}(\Omega)} \leq \|h_\alpha u\|_{V_1^*}^{1/(p_1-1)} \\
&\leq \mathrm{const}\|h_\alpha u\|_{L^{p_1'}(\Omega)}^{1/(p_1-1)} = \mathrm{const}\left[\int_\Omega |h_\alpha u|^{p_1'}\right]^{1/p_1} \\
&\leq \mathrm{const}\left[\left[\int_\Omega |h_\alpha|^{p_1' r}\right]^{1/r}\left[\int_\Omega |u|^p\right]^{p_1'/p}\right]^{1/p_1} \\
&\leq \mathrm{const}\|h_\alpha\|_{L^{p_1' r}(\Omega)}^{1/(p_1-1)}\|u\|_V^{p_1'-1},
\end{aligned}$$

where $r = p/(p - p_1')$. The exponent $\gamma_\alpha' = p_1' - 1$ satisfies $\gamma_\alpha < p/r_\ell'$ if $2 \leq p_1 < p$.

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