Perturbative Quantum (In)equivalence of Dual σ Models in 2 dimensions

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Various examples of target space duality transformations are investigated up to two loop order in perturbation theory. Our results show that when using the tree level ('naive') transformation rules the dual theories are in general *inequivalent* at two loops to the original ones, (both for the Abelian and the non Abelian duality).

1. Introduction

Various duality transformations connecting two seemingly different sigma-models or string-backgrounds are playing an increasingly important role in string theories. It is assumed that models related by certain classical transformations are alternative descriptions of the same physical system (also at the quantum level). Here we shall consider several examples of the so called 'target space duality' (Abelian T-duality) [1], which is the generalization of the $R \to 1/R$ duality in toroidal compactification of string theory. T-duality is usually formulated in the σ-model description of the corresponding Conformal Field Theory (CFT) (for a recent review see [2]).

It has been recently shown that both the Abelian [3] and the non Abelian T-duality [4], [5], [6], [7] transformation rules can be recovered in an elegant way by performing a canonical transformation. This clearly shows that models related by T-duality are *classically* equivalent. By making some formal manipulations in the functional integral without going, however, into the thorny details of regularization, it is not difficult to argue that models which are related by duality transformations correspond to the the same Quantum Field Theory (QFT) [1], [6]. For Conformal Field Theories it has been been convincingly argued in Ref. [8] that the original and the dual models are nothing but two different functional integral representations of the same CFT. Nevertheless it has been already pointed out in Ref. [9] that the tree level transformations might be modified by higher order terms in α' and the first non-trivial correction (at the two-loop level) has been found for a special class of σ-model in Ref. [10]. Therefore we feel that the question of quantum equivalence between σ-models related by duality deserves further study.

In this contribution we shall consider T-duality transformations for σ-models, treated as 'ordinary' (i.e. not necessarily conformally invariant) two dimensional quantum field theories. More precisely we investigate the quantum equivalence of two dimensional (2d) σ-models related by either the Abelian [1], or the non-Abelian [6] version of T-duality in the framework of perturbation theory.

Throughout the paper we work in a field theoretic rather than string theoretic framework, that is we consider σ-models on a flat non-dynamical 2d space. Since the world-sheet is then flat we ignore the dilaton completely. Furthermore only purely bosonic σ-models shall be considered (in general also with torsion).

To investigate the quantum equivalence of dual σ-models we compute some 'physical' quantities

(up to two loops) in standard perturbation theory in both the original and in the dual models. The perturbative calculations are greatly simplified if the model admits a sufficient degree of symmetry. Therefore we have choosen models (with high enough symmetry) where the complete renormalization amounts to multiplicative renormalization of the coupling(s) and then it is not difficult to derive the corresponding β functions.

We have investigated various Abelian duals of a 'deformed' principal $SU(2)$ σ-model and the non Abelian dual of the principal $SU(2)$ σ-model. In all cases we found that up to the one loop order in perturbation theory the duals are indeed equivalent to the original models, though in some cases highly non trivial field redefinitions were needed to reach this conclusion. At the two loop level, however, we have found that the 'naive' (i.e. tree level) duality transformations break down and in most cases the dual models turned out to be non renormalizable (in the restricted, field theoretic sense). Then we could not even extract the β functions at the two loop level. At this point we should like to emphasize that this problem has nothing to do with the presence or absence of the dilaton field.

A natural way to try to overcome the problem of non renormalizability is that the duality transformation rules for the *renormalized* metric and antisymmetric tensor field are in general modified perturbatively beyond one loop order. This is not very surprising since in the σ-model framework the 'naive' duality relates *bare* quantities [21].

2. Abelian duality

Let us start with a brief summary of the Abelian T-duality [1], [8]. Consider the following gauged σ-model action:

$$
\begin{aligned}
S = &\frac{1}{4\pi\alpha'} \int d^2\xi \big[\sqrt{h} h^{\mu\nu} \big(g_{00} D_\mu D_\nu \\
&+ 2g_{0\alpha} D_\mu \partial_\nu x^\alpha + g_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta \big) \\
&+ i\epsilon^{\mu\nu} (2b_{0\alpha} D_\mu \partial_\nu x^\alpha + b_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta) \\
&+ 2i\epsilon^{\mu\nu} \tilde{\theta} \partial_\mu A_\nu \big]
\end{aligned}
\tag{1}
$$

where $D_\mu = \partial_\mu \theta + A_\mu$, g_{ij} is the target space metric, b_{ij} the torsion, and the target space indices

are decomposed as $i = (0, \alpha)$ corresponding to the coordinate decomposition $x^i = (\theta, x^\alpha)$. The target space metric and torsion are assumed to possess a Killing vector and are now written in the adopted coordinate system, i.e. they are independent of the coordinate θ. $h_{\mu\nu}$ is the world sheet metric and α' the inverse of the string tension. The $\tilde{\theta}$ variable is just a Lagrangian multiplier which on topologically trivial world sheet forces $A_\mu = \partial_\mu \epsilon$ leading to a standard (i.e. not gauged) σ-model, referred to as the original model. As already alluded to in the introduction the dilaton field is ignored in what follows and the world sheet metric, $h_{\mu\nu}$, is taken to be flat (e.g. that of a torus, to regulate the infrared divergences). Then the formal functional integration over $\tilde{\theta}$ can be made somewhat more precise, however, our problem is independent of zero modes and therefore we shall not enter into more details about them. On the orther hand since the action (1) is quadratic in the A_μ-fields by fixing the gauge $\theta = 0$ and integrating over them one finds the dual theory:

$$
\begin{aligned}
\tilde{S} = &\frac{1}{4\pi\alpha'} \int d^2\xi \big[\sqrt{h} h^{\mu\nu} \big(\tilde{g}_{00} \partial_\mu \tilde{\theta} \partial_\nu \tilde{\theta} \\
&+ 2\tilde{g}_{0\alpha} \partial_\mu \tilde{\theta} \partial_\nu x^\alpha + \tilde{g}_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta \big) \\
&+ i\epsilon^{\mu\nu} (2\tilde{b}_{0\alpha} \partial_\mu \tilde{\theta} \partial_\nu x^\alpha + \tilde{b}_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta) \big],
\end{aligned}
\tag{2}
$$

where:

$$
\begin{aligned}
\tilde{g}_{00} &= \frac{1}{g_{00}} \\
\tilde{g}_{0\alpha} &= \frac{b_{0\alpha}}{g_{00}}, \qquad \tilde{b}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}} \\
\tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}}{g_{00}} \\
\tilde{b}_{\alpha\beta} &= b_{\alpha\beta} - \frac{g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha}}{g_{00}}.
\end{aligned}
\tag{3}
$$

These formulae (Abelian T-duality) were first found by Buscher [1]. In this formal derivation there are, however, some hidden (potential) problems. First there is a field-dependent determinant multiplied by a quadratically divergent quantity ('$\delta^{(2)}(0)$') argued usually away by using dimensional regularization. In the exceptionally simple case when g_{00} is a constant, however, one would expect no problems whatsoever. As we

shall show on an example (the 'ψ-dual' model) while the equivalence of the dual model seems indeed true, (up to two loops in perturbation theory) there are nontrivial renormalization effects even in this case when using dimensional regularization. It is natural to assume that the use of dimensional regularization is part of the problem, as the two dimensional antisymmetric tensor, $\epsilon_{\mu\nu}$ is also present.

3. Counterterms and the renormalization of couplings

Our general strategy to carry out the renormalization of the 'original' and of the 'dual' models and to obtain the corresponding β functions is to simply use the one resp. two loop counterterms for the general σ-models (either with or without the torsion term) computed by several authors [12], [13], [14]. These counterterms were derived by the background field method in the dimensional regularization scheme. To carry out the coupling and wave function renormalization explicitly we recall the basic formalae needed. The general σ-model Lagrangian is written in the form

$$\mathcal{L} = \frac{1}{2\lambda}\big(g_{ij}(\xi) + b_{ij}(\xi)\big)\Xi^{ij} = \frac{1}{\lambda}\tilde{\mathcal{L}} \qquad (4)$$

where

$$\Xi^{ij} = (\partial_\mu\xi^i\partial^\mu\xi^j + \epsilon_{\mu\nu}\partial^\mu\xi^i\partial^\nu\xi^j). \qquad (5)$$

Expressing the loop expansion parameter, α', in terms of the coupling λ as $\alpha' = \lambda/(2\pi)$, the simple pole parts of the one ($i = 1$) and two ($i = 2$) loop counterterms, $\mathcal{L}_i$, apart from the $\mu^{-\epsilon}$ factor are given as:

$$\mu^\epsilon\mathcal{L}_1 = \frac{\alpha'}{2\epsilon\lambda}\hat{R}_{ij}\Xi^{ij} = \frac{1}{\pi\epsilon}\Sigma_1. \qquad (6)$$

The two loop counterterm then has the form:

$$\mu^\epsilon\mathcal{L}_2 = \frac{\alpha'^2}{16\epsilon\lambda}Y^{lmk}{}_j\hat{R}_{iklm}\Xi^{ij} = \frac{\lambda}{8\pi^2\epsilon}\Sigma_2, \qquad (7)$$

where

$$\begin{aligned}
Y_{lmkj} = &- 2\hat{R}_{lmkj} + 3\hat{R}_{[klm]j} \\
&+ 2(H^2_{kl}g_{mj} - H^2_{km}g_{lj}), \\
H^2_{ij} = &H_{ikl}H_j^{kl}, \\
2H_{ijk} = &\partial_i b_{jk} + \text{cyclic}.
\end{aligned} \qquad (8)$$

In Eqs. (6,7) $\hat{R}_{iklm}$ resp. $\hat{R}_{ij}$ denote the 'generalized' Riemann resp. Ricci tensors of the 'generalized' connection, G^i_{jk}, containing also the torsion term in addition to the Christoffel symbols, Γ^i_{jk} of the metric g_{ij};

$$G^i_{jk} = \Gamma^i_{jk} + H^i_{jk}. \qquad (9)$$

We shall also consider examples with an additional *parameter*, x, where x is not assumed to be small. Thus we do not expand anything in a parameter, the standard perturbative expansion is made only in the coupling λ. If the metric, g_{ij}, and the torsion potential, b_{ij}, also depend on a (bare) parameter, x i.e $g_{ij} = g_{ij}(\xi, x)$ and $b_{ij} = b_{ij}(\xi, x)$ then we convert the previous counterterms into coupling and parameter renormalization by assuming that in the one ($i = 1$) and two ($i = 2$) loop orders their bare and renormalized values are related as

$$\begin{aligned}
\lambda_0 &= \mu^\epsilon\lambda\Big(1 + \frac{\zeta_1(x)\lambda}{\pi\epsilon} + \frac{\zeta_2(x)\lambda^2}{8\pi^2\epsilon} + ...\Big) \\
&= \mu^\epsilon\lambda Z_\lambda(x, \lambda), \\
x_0 &= x + \frac{x_1(x)\lambda}{\pi\epsilon} + \frac{x_2(x)\lambda^2}{8\pi^2\epsilon} + ... \\
&= xZ_x(x, \lambda),
\end{aligned} \qquad (10)$$

where the dots stand for both the higher loop contributions and for the higher order pole terms. The unknown functions $\zeta_i(x)$ and $x_i(x)$ ($i = 1,2$) are determined from the following equations:

$$- \zeta_i(x)\tilde{\mathcal{L}} + \frac{\partial\tilde{\mathcal{L}}}{\partial x}x_i(x) + \frac{\delta\tilde{\mathcal{L}}}{\delta\xi^k}\xi^k_i(\xi, x) = \Sigma_i. \qquad (11)$$

Eqs. (11) express the finiteness of the generalized quantum effective action, $\Gamma(\xi)$, [12] [16] to the corresponding order in perturbation theory. In Eqs. (11) $\xi^k_i(\xi, x)$ may depend in an arbitrary way on the parameter, x, and on the fields, ξ^j, the only requirement being that $\xi^k_i(\xi, x)$ contain no derivatives of ξ^j. This freedom is related to the diffeomorphism invariance of the renormalized theory. Eqs. (11) admits a simple interpretation: the general counterterms of the σ-models may be absorbed by the renormalization of the coupling and the parameter(s) together with a (in general non-linear) redefinition of the fields ξ^j:

$$\xi^j_0 = \xi^j + \frac{\xi^j_1(\xi^k, x)\lambda}{\pi\epsilon} + \frac{\xi^j_2(\xi^k, x)\lambda^2}{8\pi^2\epsilon} + ... \qquad (12)$$

The functions ξ_1^j, ξ_2^j are also determined from Eqs. (11). In the special case when ξ_i^k depends linearly on ξ i.e. $\xi_i^k(\xi, x) = \xi^k y_i^k(x)$, Eqs. (12) simplify to an ordinary multiplicative wave function renormalization. We emphasize that it is not a priori guaranteed that Eqs. (11) may be solved at all for the functions $\zeta_i(x)$, $x_i(x)$ and $\xi_i^k(\xi, x)$. If this happens to be the case then the renormalization of the model is not possible within the restricted subspace characterized by the coupling(s) and the parameter(s) in the (infinite dimensional) space of metrics and torsions. This implies that the model is not renormalizable in the ordinary, field theoretical sense but only in the generalized sense [17], i.e. with an infinite number of couplings.

4. The deformed SU(2) principal σ-model and some of its duals

Consider the following one parameter deformation of the SU(2) principal σ-model Lagrangian:

$$\mathcal{L} = -\frac{1}{2\lambda}\Big(\sum_{a=1}^{3} J_\mu^a J^{\mu a} + g J_\mu^3 J^{\mu 3}\Big)\,, \qquad (13)$$

where $J_\mu = G^{-1}\partial_\mu G = J_\mu^a \tau^a$ where $\tau^a = \sigma^a/2$ and the σ^a are the standard Pauli matrices, with G being an element of SU(2) and g is the parameter of the deformation. From the Lagrangian (13) it is clear that the global $SU(2)_L \times SU(2)_R$ symmetry of the undeformed principal σ-model is broken to $SU(2)_L \times U(1)_R$ by the $J_\mu^3 J^{\mu 3}$ term. Setting $g = 0$ corresponds to the principal σ-model, while for $g = -1$ the $O(3)$ σ-model is obtained as can be seen from Eq. (15) below. In the following we shall make perturbation in the coupling λ while treating g as a parameter. Using the Euler angles (ϕ, θ, ψ) to parametrize the elements of SU(2) G is written as

$$G = e^{i\phi\tau^3} e^{i\theta\tau^1} e^{i\psi\tau^3}\,. \qquad (14)$$

Then the Lagrangian of the deformed model (13) becomes

$$\mathcal{L} = \frac{1}{2\lambda}\big\{(\partial_\mu\theta)^2 + (\partial_\mu\phi)^2\big(1 + g\cos^2\theta\big) + \\ (1+g)(\partial_\mu\psi)^2 + 2(1+g)\partial_\mu\phi\partial^\mu\psi\cos\theta\big\}\,. \qquad (15)$$

Clearly the deformed σ-model is a purely metric one.

Using the Killing vectors of the $SU(2)_L \times U(1)_R$ symmetry and exploiting the manifest target space covariance of the background field method one can prove that this model is renormalizable in the ordinary sense: there is no wave function renormalization for θ, ϕ and ψ, while the coupling constant and the parameter get renormalized as in Eq. (10):

$$\lambda_0 = \mu^\epsilon Z_\lambda(\lambda, g)\lambda, \qquad g_0 = Z_g(\lambda, g)g\,. \qquad (16)$$

Both in the one and in the two loop orders the residues of the single poles in $Z_\lambda(\lambda, g) = 1 + y_\lambda(\lambda, g)/\epsilon + ...$ and $Z_g(\lambda, g) = 1 + y_g(\lambda, g)/\epsilon + ...$ are determined from Eqs. (11). Though this yields five equations for the two unknown functions $y_\lambda(\lambda, g)$ and $y_g(\lambda, g)$, since the model is renormalizable these equations turn out to be compatible and their solution is given as:

$$y_\lambda = -\frac{\lambda}{4\pi}\Big(1 - g + \frac{\lambda}{16\pi}(1 - 2g + 5g^2)\Big)\,, \\ y_g = \frac{\lambda}{2\pi}(1 + g)\Big(1 + \frac{\lambda}{8\pi}(1 - g)\Big)\,. \qquad (17)$$

Let us next recall a useful relation between the β functions and the wavefunction renormalization in a theory with two couplings. Consider a theory with two couplings (or one coupling and one parameter) denoted by α and γ, whose bare and renormalized values are related as:

$$\alpha_0 = \mu^{a\epsilon} Z_\alpha(\alpha, \gamma)\alpha\,, \qquad \gamma_0 = \mu^{b\epsilon} Z_\gamma(\alpha, \gamma)\gamma\,. \qquad (18)$$

Defining their β functions in the standard way: $\beta_\alpha = \mu\frac{d\alpha}{d\mu}$, $\beta_\gamma = \mu\frac{d\gamma}{d\mu}$, then these are determined by the residues of the simple poles in the renormalization constants Z_α and Z_γ, $y_\alpha(\alpha, \gamma)$ and $y_\gamma(\alpha, \gamma)$ as follows:

$$\beta_\alpha = \alpha\Big(a\alpha\frac{\partial y_\alpha}{\partial\alpha} + b\gamma\frac{\partial y_\alpha}{\partial\gamma}\Big)\,, \\ \beta_\gamma = \gamma\Big(b\gamma\frac{\partial y_\gamma}{\partial\gamma} + a\alpha\frac{\partial y_\gamma}{\partial\alpha}\Big)\,. \qquad (19)$$

From Eqs. (17,19) one obtains the β functions of the deformed σ-model:

$$\beta_\lambda = -\frac{\lambda^2}{4\pi}\Big(1 - g + \frac{\lambda}{8\pi}(1 - 2g + 5g^2)\Big)\,, \\ \beta_g = \frac{\lambda}{2\pi}g(1 + g)\Big(1 + \frac{\lambda}{4\pi}(1 - g)\Big)\,. \qquad (20)$$

It is easy to see, that the $g = 0$ resp. the $g = -1$ lines are fixed lines under the renormalization group, and β_λ reduces to the β function of the principal σ-model, resp. of the $O(3)$ σ-model on them. In the ($\lambda \geq 0$, $g < 0$) quarter of the (λ,g) plane the renorm trajectories run into $\lambda = 0$, $g = -1$; while for $g > 0$ they run to infinity. This implies that the $g = 0$ fixed line corresponding to the principal σ-model is 'unstable' under the deformation.

The Lagrangian of the deformed σ-model given by Eq. (15) exhibits two obvious Abelian isometries that can be used to construct two different (Abelian) duals: namely the translations in the ϕ and ψ fields; we call the models obtained this way the 'ϕ dual' and the 'ψ dual' of the deformed σ model (15). From Eq. (14) it is clear that these translations correspond to multiplying the $SU(2)$ element, G, by a constant, diagonal $SU(2)$ matrix from the left (respectively from the right). Since in the dual variables there is always a translational symmetry and the duality transformation amounts to a canonical transformation (classically) the 'ψ dual' is expected to show the full remaining $SU(2) \times U(1)$ symmetry of the original model (15), while for the ϕ dual only a $U(1) \times U(1)$ symmetry is expected.

4.1. The 'ψ dual' model

Let us start first with the 'ψ dual' of the deformed σ model. Its Lagrangian is easily found to be using Buscher's formulae (3) and Eq. (15):

$$
\mathcal{L} = \frac{1}{2\tilde{\lambda}}\Big((\partial_\mu\theta)^2 + (\partial_\mu\phi)^2\sin^2\theta + (\partial_\mu h)^2 \\
+ 2a\cos\theta\,\epsilon^{\mu\nu}\partial_\mu h\partial_\nu\phi\Big),
\tag{21}
$$

where h denotes the (appropriately scaled) variable dual to ψ and ($\tilde{\lambda},\tilde{g}$) stand for the couplings of the dual model. The couplings of the original (15) and of the dual model (21) are related (at the classical level) as

$$
\tilde{\lambda} = \lambda \quad a = \sqrt{1+\tilde{g}} \quad \tilde{g} = g.
\tag{22}
$$

Note the appearance of a non trivial torsion potential in Eq. (21) generated by the off diagonal $g_{\psi\phi}$ terms of the original purely metric model Eq. (15). For $a = 0$ Eq. (21) reduces to the Lagrangian of the $O(3)$ σ-model (apart from a decoupled free field), and it is easy to show that for all values of a it shows the expected $SU(2)\times U(1)$ symmetry, indeed. For $a = 1$ the Lagrangian of the 'ψ dual' becomes similar but not identical to that of the so called 'pseudo dual' of the $SU(2)$ principal model [18], [19]. The difference between the 'pseudo dual' model of Ref. [18] and the 'ψ dual' models is that the metric is flat for the first model while it is not for the 'ψ dual' one.

The three Killing vectors generating an $SU(2)$ of the (global) symmetry algebra of (21) act on a two-sphere, i.e. they are not linearly independent. For this reason the Killing equations expressing the symmetry of the counterterms are not restrictive enough, so from this $SU(2)$ symmetry alone one cannot conclude that the ψ dual model is renormalizable in the ordinary sense. Taking into account the additional restrictions following from the discrete symmetry $h, \theta, \phi \to -h, -\theta, -\phi$ of Eq. (21), makes it possible to show that the complete renormalization of the model amounts to just an ordinary multiplicative renormalization of the couplings and of the h field:

$$
\tilde{\lambda}_0 = \mu^\epsilon Z_{\tilde{\lambda}}(\tilde{\lambda}, a)\tilde{\lambda}, \quad a_0 = Z_a(\tilde{\lambda}, a)a, \\
h_0 = hZ_h(\tilde{\lambda}, a).
\tag{23}
$$

The one and two loop counterterms Σ_1, Σ_2 are relatively simple expressions:

$$
\Sigma_1 = \frac{1}{4}\Big((1 - \frac{a^2}{2})\Omega - \frac{a^2}{2}(\partial_\mu h)^2\Big), \\
\Sigma_2 = \frac{1}{8}\Big((\frac{3a^4}{2} + 4(1 - a^2))\Omega + \frac{3a^4}{2}(\partial_\mu h)^2\Big),
\tag{24}
$$

where $\Omega = (\partial_\mu\theta)^2 + (\partial_\mu\phi)^2\sin^2\theta$. A straightforward computation based on Eq. (11) yields for the simple pole parts of $Z_j(\tilde{\lambda}, a) = 1 + y_j(\tilde{\lambda}, a)/\epsilon + ...$, $j = \tilde{\lambda}, a, h$:

$$
y_{\tilde{\lambda}} = -\frac{\tilde{\lambda}}{2\pi}(1 - \frac{a^2}{2}) - \frac{\tilde{\lambda}^2}{8\pi^2}\Big(\frac{3a^4}{8} + 1 - a^2\Big), \\
y_a = -\frac{\tilde{\lambda}}{4\pi}(1 - a^2) - \frac{\tilde{\lambda}^2}{16\pi^2}\Big(\frac{3a^4}{4} + 1 - a^2\Big), \tag{25} \\
y_h = -\frac{\tilde{\lambda}}{4\pi} - \frac{\tilde{\lambda}^2}{16\pi^2}(1 - a^2).
$$

Using now Eq. (19) leads immediately to the β functions:

$$\beta_{\tilde{\lambda}} = -\frac{\tilde{\lambda}^2}{2\pi}(1 - \frac{a^2}{2}) - \frac{\tilde{\lambda}^3}{4\pi^2}\Big(\frac{3a^4}{8} + 1 - a^2\Big)\,,$$
$$\beta_a = -\frac{\tilde{\lambda}a}{4\pi}(1 - a^2) - \frac{\tilde{\lambda}^2 a}{8\pi^2}\Big(\frac{3a^4}{4} + 1 - a^2\Big)\,. \quad (26)$$

For convenience let us give here also $\beta_{\tilde{\lambda}}$ expressed in terms of $\tilde{g}$ together with $\beta_{\tilde{g}} = 2a\beta_a$:

$$\beta_{\tilde{\lambda}} = -\frac{\tilde{\lambda}^2}{4\pi}\Big(1 - \tilde{g} + \frac{\tilde{\lambda}}{8\pi}(3 - 2\tilde{g} + 3\tilde{g}^2)\Big)\,,$$
$$\beta_{\tilde{g}} = \frac{\tilde{\lambda}}{2\pi}(1 + \tilde{g})\big(\tilde{g} - \frac{\tilde{\lambda}}{8\pi}(3 + 2\tilde{g} + 3\tilde{g}^2)\big)\,. \quad (27)$$

These β functions show a number of interesting properties. First of all in the one loop order they are completely equivalent to those of Eq. (20) taking into account the relation (22) between the couplings (λ,g) and $(\tilde{\lambda},a)$ i.e. $g = a^2 - 1$, which holds true at the classical level. This equivalence is seemingly broken, however, at the two loop level, that is to that order the two sets of β functions, Eqs. (20) and (27) are different. Nevertheless this does not imply the inequivalence of the two models in perturbation theory since in the case of more than one coupling the two loop β functions are already scheme dependent quantities. Indeed the perturbative redefinition of the couplings (a change of scheme)

$$\tilde{\lambda} = \lambda + \frac{\lambda^2}{4\pi}F(g) \quad \tilde{g} = g + \frac{\lambda}{4\pi}H(g)\,, \quad (28)$$

can be easily seen to change the two loop coefficients of the β functions (27). In fact one can simply try to determine the arbitrary functions $F(g)$, $H(g)$, so that the β functions of the original and the dual model agree up to this order. A direct calculation to equate the corresponding β functions yields an explicitly solvable system of two first order differential equations for $F(g)$ and $H(g)$. The general solution contains of course two constants of integration which can only be determined from some other arguments (regularity in g and the constraint coming from self-dual point at $g = -1$). The following change of scheme (transforming Eqs. (20), (27) into each other up to second order in λ by construction) determined from

the two previous arguments are

$$\tilde{\lambda} = \lambda + \frac{\lambda^2}{4\pi}(1 + g)\,, \quad \tilde{g} = g + \frac{\lambda}{4\pi}(1 + g)^2\,. \quad (29)$$

To really establish the perturbative equivalence of (15) (21) up to two loops one has to make a more direct comparison between some 'physical' quantities in the two models. As one cannot directly relate operators under the T-duality transformation, we have chosen to compare the $< \theta(k_1)\theta(k_2)\phi(k_3)\phi(k_4) >$ four point functions computed both in the original (15) and in the dual model (21). Clearly the two four point functions must agree as θ and ϕ were not even touched upon. As already alluded to one has every right to expect that the T-duality transformation together with the use of dimensional regularization corresponds to a change of scheme. Then by comparing some 'physical' quantities in the two different looking theories corresponding to a change of scheme, one directly obtains the relation between them. It is of course clear that such four point functions are not really physical as they are e.g. coordinate dependent. For relating the couplings (λ,g) and $(\tilde{\lambda},\tilde{g})$ they are, however, perfectly well suited. Furthermore by computing $< \theta(k_1)\theta(k_2)\phi(k_3)\phi(k_4) >$ up to only one loop order, one can check the two loop relation between the couplings! The reason for this 'miracle' is simply that the tree level amplitudes in (15) and (21) are already proportional to λ. It is worth pointing out that the calculation of $< \theta(k_1)\theta(k_2)\phi(k_3)\phi(k_4) >$ provides a nontrivial cross check on the two loop σ-model counterterms (7). The computation of these four point functions is a straightforward though a somewhat tedious exercise in perturbation theory and we omit the details here. The final outcome of the calculation is precisely the previously deduced relation, Eq. (29), between the two sets of couplings. This is in agreement with our expectation that when g_{00} is constant the 'naive' T-duality transformation yields an equivalent model.

An alternative way to express the correspondence between the two models is to get rid of the scheme dependence of the β functions by eliminating one of the parameters (g or a) in β_λ in favour of a *renormalization group invariant pa-*

rameter (M resp. $\tilde{M}$). A straightforward computation yields, that the invariant parameter characterizing the trajectories under the renormalization group equations (20) has the form:

$$M = -\frac{\lambda^2 g}{(1+g)^2} - \frac{\lambda^3 g}{4\pi(1+g)} , \qquad (30)$$

while for Eqs. (27) it is given by

$$\tilde{M} = -\tilde{\lambda}^2 \frac{a^2 - 1}{a^4} + \frac{\tilde{\lambda}^3}{4\pi a^2} . \qquad (31)$$

(The signs have been chosen here to guarantee that M ($\tilde{M}$) > 0 in the most interesting domain $\lambda > 0, 0 > g > -1, (1 > a > 0)$). If $M \neq 0$ ($\tilde{M} \neq 0$), then, expressing perturbatively g (respectively a) from Eq. (30) (resp. Eq. (31)) yields

$$g(\lambda, M) = -1 + \frac{\lambda}{\sqrt{M}} , \quad a^2(\tilde{\lambda}, \tilde{M}) = \frac{\tilde{\lambda}}{\sqrt{\tilde{M}}} , \quad (32)$$

and using them to compute β_λ in Eqs. (20) (resp. $\beta_{\tilde{\lambda}}$ in Eqs. (27)) shows that the two expressions become identical provided $M = \tilde{M}$. If $M = 0$, then Eq. (30) immediately yields $g = 0$ (implying that this case is the $SU(2)$ principal model), while for $\tilde{M} = 0$ from Eq. (31) one obtains $a^2 = 1 + \frac{\tilde{\lambda}}{4\pi}$, and after eliminating it from Eqs. (26) $\beta_{\tilde{\lambda}}$ becomes identical with that of the β function of the principal model. Of course all these findings are compatible with the deformed σ-model being two loop quantum equivalent to its 'ψ dual'.

4.2. The 'ϕ dual' model

The Lagrangian of the ϕ dual of the deformed σ-model has the form:

$$\mathcal{L} = \frac{1}{2\lambda\Theta} \big[(\partial_\mu f)^2 + (1+g)\sin^2\theta(\partial_\mu\psi)^2 \\ + \Theta(\partial_\mu\theta)^2 + 2(1+g)\cos\theta\epsilon^{\mu\nu}\partial_\mu f\partial_\nu\psi \big], \qquad (33)$$

where $\Theta = 1 + g\cos^2\theta$ and f denotes the variable dual to ϕ. Setting $g = 0$ in Eq. (33) gives the same 'pseudo dual'– like Lagrangian as in the case of the ψ dual, while for $g = -1$ one obtains a model resembling the $O(3)$ σ-model. From the $SU(2){\times}U(1)$ symmetry of the original model (15) only a $U(1) \times U(1)$ remains manifest in Eq. (33) (the two translational symmetries in the variables f and ψ). Since this $U(1) \times U(1)$ symmetry is

not restrictive enough (the only constraint on the counterterms coming from it is that they can only depend on the θ field), to prove renormalizability of (33) in the restricted sense. Computing, nevertheless, Σ_1 and Σ_2 reveals that up to this order the structure of the Lagrangian Eq. (33) is preserved: both in the metric and in the torsion potential only the non-vanishing elements receive corrections, while the vanishing elements do not. The explicit form of Σ_1 is:

$$\Sigma_1 = -\frac{1}{8\Theta^3} \big[-8g_+ z\sin^2\theta\epsilon^{\mu\nu}\partial_\mu f\partial_\nu\psi \\ - (1 + 3g - 6gg_+ z^2 + g^2 g_- z^4)\Theta(\partial_\mu\theta)^2 \\ - g_+^2\sin^2\theta(1 - 4gz^2 - g^2 z^4)(\partial_\mu\psi)^2 \\ + (g_-(1 - g^2 z^4) + 4gg_+ z^2)(\partial_\mu f)^2 \big] , \qquad (34)$$

where $z \equiv \cos\theta$, $g_+ \equiv 1 + g$ and $g_- \equiv 1 - g$. Let us now try to convert this one loop counterterm into a coupling and parameter renormalization as in Eqs. (10), accompanied by a non-linear redefiniton of θ together with some multiplicative renormalization of the ψ and f fields:

$$\lambda_0 = \mu^\epsilon \lambda \Big(1 + \frac{\zeta_1(g)\lambda}{\pi\epsilon} \Big), \ g_0 = g + \frac{g_1(g)\lambda}{\pi\epsilon} ,$$

$$f_0 = f \Big(1 + \frac{y_f^{(1)}(g)\lambda}{\pi\epsilon} \Big), \qquad (35)$$

$$\theta_0 = \theta + \frac{T_1(\theta, g)\lambda}{\pi\epsilon} , \ \psi_0 = \psi \Big(1 + \frac{y_p^{(1)}(g)\lambda}{\pi\epsilon} \Big) .$$

Eq. (11) yields four equations (corresponding to the four non vanishing elements of the metric and the torsion potential) for the five unknown functions in Eqs. (35) with only one depending on two variables (g, θ). Therefore it is by no means obvious that this problem has a solution at all. This is especially so, since equating the coefficients of $(\partial_\mu\theta)^2$ on the two sides of Eq. (11) yields a differential equation for $T_1(\theta, g)$ from which we have found

$$T_1(\theta, g) = -\frac{g\cos\theta\sin\theta}{2\Theta} , \quad \zeta_1(g) = -\frac{g_-}{4} . \qquad (36)$$

Thus we have three functions of one variable (g) at our disposal to satisfy the three remaining equations in two variables (θ and g). Nevertheless, after some effort one finds that choosing

$$g_1(g) = \frac{gg_+}{2} , \ y_f^{(1)}(g) = -\frac{g_-}{4} , \ y_p^{(1)}(g) = 0 , \quad (37)$$

guarantees that all equations are satisfied. Therefore in the one loop order the 'ϕ dual' model is also renormalizable in the restricted sense. Furthermore extracting the residues of the simple poles of Z_λ and Z_g from Eqs. (35, 37) shows that up to this order the β functions of (33) are just identical to that of the deformed σ-model in complete agreement with their one loop equivalence.

At the two loop order it still remains true that in (33) only the non vanishing metric and torsion potential terms receive corrections. As the explicit form of the two loop counterm, Σ_2, is rather complicated we do not display it here. Σ_2 is again a rational function of $\cos\theta$ similarly to Σ_1. There is a dramatic change, however, as compared to the previous (one loop) case when we try to determine from Σ_2 the renormalization of the couplings together with the same type of field renormalizations as in Eqs. (35). Integrating the differential equation for $T_2(\theta, g)$ yields namely

$$
2T_2(\theta, g) = \left[\zeta_2 + \frac{3g_-^2 + 4g}{8}\right]\theta + \frac{(gz)^3 \sin\theta}{\Theta^2}
$$
$$
+ C - \frac{g}{2g_+}\left[\frac{z\sin\theta}{\Theta} - \frac{1}{\sqrt{g_+}}\mathrm{arctg}(\sqrt{g_+}\,\cot g\theta)\right], \tag{38}
$$

where C is the constant of integration. The problem is now that no choice of $\zeta_2(g)$ and C in Eq. (38) could make $T_2(\theta, g)$ a purely rational expression of $\cos\theta$ and $\sin\theta$. Recalling that Σ_2 is a rational expression in $\cos\theta$ and looking at the remaining three equations for $g_2(g)$, $y_f^{(2)}(g)$ and $y_p^{(2)}(g)$ it is not difficult to see that there is no way to satisfy Eqs. (11) in the two loop order. The implication of this result is that in the two loop order in perturbation theory, the 'ϕ dual' model is not renormalizable in the restricted (field theoretical) sense. This clearly shows that application of the *classical* T-duality transformations (3) in the standard perturbative σ-model renormalization framework may lead to inequivalent dual models. As we have already mentioned the quantum inequivalence of the dual model *defined by the classical* Buscher formulae (3) is expected to be related to the a presence of a nonconstant g_{00}, hence to renormalization effects.

5. The non Abelian dual of the principal σ-model

In this section we investigate the problem of perturbative quantum equivalence of the principal σ-model with its non Abelian dual [4], [5], [6], [7]. The non Abelian dual of the principal σ-model can be deduced in a way similar to that of the Abelian T-duality transformation, by making some formal manipulations in the partition function. One can start for example with the 2d Freedman-Townsend model

$$
\mathcal{L}_{\mathrm{FT}} = B^a \epsilon^{\mu\nu} F_{\mu\nu}^a + A_\mu^a A^{a\mu}, \tag{39}
$$

where $a = 1, 2, \ldots$ denotes the Lie algebra (semi simple) indices, $F_{\mu\nu}^a$ is the standard field-strength tensor of the non Abelian vector fields, A_μ^a, and B^a are auxiliary fields (Lagrange multipliers enforcing the vanishing of the field tensor). By integrating over the auxiliary fields one obtains the principal σ-model while integration over A_μ^a yields the corresponding non Abelian dual. An alternative derivation of the non Abelian duality generalizing the gauging procedure of [8] for the Abelian duality was given in [22].

In what follows we shall only consider the simplest case, namely the Lie algebra being $SU(2)$. Let us present first a 'universal' Lagrangian contaiing both the principal σ-model and its non Abelian dual:

$$
\mathcal{L} = \frac{1}{2e^2}\{\partial_\mu r \partial^\mu r + A(r)\partial_\mu n^a \partial^\mu n^a
$$
$$
+ B(r)\epsilon^{abc}\epsilon_{\mu\nu}n^a \partial_\mu n^b \partial_\nu n^c\}, \tag{40}
$$

where an element of $SU(2)$ has been parametrized by a unit vector, n^a, and by a radial variable r. The principal σ-model corresponds to the choice $A(r) = \sin^2 r$ and $B(r) \equiv 0$ while for the non Abelian dual $A(r) = r^2/r_+$ and $B(r) = 2r^3/r_+$, where $r_+ = 1 + 4r^2$. To make contact between the present formulation of the principal model and its previously given Lagrangian (13) (for $g = 0$) we note that (40) corresponds to parametrizing G in (13) as $G = \cos r + i \sin r n^a \sigma^a$ (and writing $\lambda = 2e^2$).

After the results of the previous section it is natural to guess that troubles are likely to arise when checking the quantum equivalence between

the principal σ-model and its 'naive' (i.e. classical) non Abelian dual. The one loop agreement of the corresponding β functions has been established long ago in Ref. [5] and in Ref. [6] it has been even argued that the β functions are completely equivalent. It will be demonstrated below, however, that in analogy to the 'ϕ dual' studied in the previous section, the 'naive' non Abelian dual is simply not renormalizable with a single coupling at the two loop level, while the β functions in the two models agree up to one loop order, indeed.

From Eq. (40) it is clear that both the original and the dual model have a manifest $SU(2)$ symmetry. Eq. (13) for $g = 0$ clearly shows the full $SU(2) \times SU(2)$ symmetry of the principal σ-model. It is well known of course that the principal σ-model is renormalizable in the restricted sense. This fact is due to its high degree of symmetry, in particular the corresponding (3d) metric in (40) possesses three linearly independent Killing vectors. This restricts the counterterms so strongly that the standard multiplicative renormalization is sufficient to remove the divergences. For the non Abelian dual model where the Killing vectors of the single $SU(2)$ symmetry act on a two sphere (that only two of them is linearly independent) this symmetry is not restrictive enough to establish its renormalizability.

The one and two loop counterterms for both cases can be written in a unified way:

$$\Sigma_i = N_i \Big\{ Z_i(r) \partial_\mu r \partial^\mu r + A_i(r) \partial_\mu n^a \partial^\mu n^a \\ + B_i(r) \epsilon^{abc} \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c \Big\}, \quad (41)$$

where $i = 1, 2$ and $N_1 = 1/4$ and $N_2 = 1/8$. This form of the counterterms agrees with the known property of dimensional regularization of preserving the global symmetries of the target manifold. To respect the manifest $SU(2)$ symmetry of (41) when discussing the nonlinear field renormalizations, the only thing one can allow for is a redefinition of the radial variable, r. Therefore when the models are renormalizable the counterterms should just give the renormalization of the (sin-

gle) coupling and the r field:

$$e_0^2 = \mu^\epsilon e^2 \Big(1 + \frac{\zeta_1 e^2}{\pi\epsilon} + \frac{\zeta_2 e^4}{8\pi^2\epsilon} \Big) = \mu^\epsilon e^2 Z_{e^2}, \\ r_0 = r + \frac{r_1(r)e^2}{\pi\epsilon} + \frac{r_2(r)e^4}{8\pi^2\epsilon}. \quad (42)$$

In the case of the principal σ-model the actual expressions for A_i, B_i and Z_i are given as:

$$A_i(r) = \delta_i A(r), \qquad B_i(r) \equiv 0, \\ \delta_1 = Z_1 = 1, \qquad \delta_2 = Z_2 = 2. \quad (43)$$

Using now (43) in Eqs. (11) yields that $r_1(r) \equiv 0$, $r_2(r) \equiv 0$ and $Z_{e^2} = 1 - e^2/\pi\epsilon - e^4/4\pi^2\epsilon$. Then it is easy reproduce the known result for the two loop β function of the principal σ-model:

$$\beta_{e^2} = -\frac{e^4}{\pi} \left(1 + \frac{e^2}{2\pi} \right). \quad (44)$$

In the case of the non Abelian dual the A_i, B_i and Z_i functions are somewhat more complicated:

$$A_1(r) = \frac{r^2(r_+^2 - 2r_-)}{r_+^3}, \ Z_1(r) = -\frac{r_+^2 + 4r_-}{r_+^3}, \\ B_1(r) = \frac{8r^3}{r_+^3}, \ A_2(r) = \frac{2r^2(3r_+^4 - 80r_+ + 32)}{r_+^5}, \\ B_2(r) = -\frac{64r^3(r_+^2 + 4r_+ - 2)}{r_+^5}, \\ Z_2(r) = \frac{6(r_+^3 + 8(r_+^2 + r_- - 2))}{r_+^3}, \quad (45)$$

where we have introduced $r_- = 4r^2 - 1$. Eqs. (11) lead to three equations for the two unknown functions r_i, ζ_i (coming from equating the coefficients of the three different tensor structures arising on the two sides):

$$-\zeta_i f(r) + r_i(r) \frac{\partial f}{\partial r}(r) - f_i(r) = 0, \\ -\zeta_i + 2\frac{dr_i}{dr}(r) - Z_i(r) = 0, \quad (46)$$

where $f = A, B$. The first two equations in (46) can be solved algebraically for ζ_i and $r_i(r)$; defining $b_i = A_i(r)/A(r)$ and $c_i = B_i(r)/rA(r)$ the solutions are simply

$$\zeta_i = \frac{c_i - (3 + 4r^2)b_i}{r_+}, \quad r_i(r) = r(\frac{c_i}{2} - b_i). \quad (47)$$

Clearly, to be self-consistent, ζ_i must be independent of r and $r_i(r)$ must still satisfy the third equation in (46). From the given expressions for A_i, B_i and Z_i in Eqs (45) one finds that at one loop $\zeta_1 = -1$ and $r_1(r) = -rr_-/r_+$, which indeed solves the third equation in (46). Then the one loop β function of the principal σ-model can be immediately reproduced indicating the equivalence of the two theories to this order. In the two loop order, however, we find that ζ_2 is not r independent, thus the non Abelian dual is not renormalizable! This shows that the 'naive' version of the non Abelian duality transformation as it stands in Eq. (40) yields an inequivalent model at the two loop level in perturbation theory. This last conclusion has also been reached in Ref. [20], we disagree, however, with some of the other claims of that paper.

Let us remark that in all of the examples investigated so far we have also checked that our conclusions on the two loop (non)renormalizability of the models in question is independent of the well known ambiguity (or freedom) in the counterterms (corresponding to target space diffeomorphism invariance) [12], [13], [14], [15], [16]. In all of the cases we have found that the appropriate covariant derivatives of $H^2 = H_{ijk}H^{ijk}$ either vanish identically or give no contribution to the consistency conditions like the one in Eq. (47).

6. Conclusions

We have shown on a number of examples that the 'naive' (tree level) T-duality transformations in 2d σ-models cannot be exact symmetries of the quantum theory. The 'naive' Abelian duality transformations Eqs. (3) are correct to one loop in perturbation theory, they break down in general, however, at the two loop order. This two loop problem is expected to be connected to regularization issues in the functional integral when deriving the duality transformation formulae Eqs. (3) [10]. In the very simple case of the Abelian duality transformations (3) with g_{00} *constant* when the derivation amounts to just a standard gaussian integration, no problems are expected with the quantum equivalence of the dual theory ('ψ-dual' model). We have found that in

this case the dual model is indeed equivalent to two loops to the original one, however, there is a nontrivial change of scheme involved when insisting on dimensional regularization. That the 'ψ-dual' model corresponds to changing the regularization scheme in the original theory has also been checked by computing a suitable four point function.

Our conclusion from the above (somewhat discouraging) results concerning T-duality at the quantum level is certainly *not* that there is no T-duality symmetry. In the simplest conceivable case of two free fields, when simililar problems arise at the two loop level, we have found a suitable modification of the duality formulae to ensure quantum equivalence to two loops [21]. Encouraged by this positive result we do expect that a suitable modification of the 'naive' duality transformations is possible in general order by order in perturbation theory making (Abelian) T-duality a true *quantum* symmetry.

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