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Restricting bid withdrawal: a new efficient and incentive compatible dynamic auction for heterogeneous commodities

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Abstract

This paper studies the problem of how to restrict bid withdrawal in dynamic auctions for heterogeneous commodities. We formulate a withdrawal restriction rule as a function from submitted bids to irrevocable bids. In the unit-demand auction due to Demange et al. (1986), we identify the least restrictive rule under which the auction reaches an equilibrium allocation on any termination. The key idea of the rule is to restrict withdrawing bids on items that have been affected most favorably by the price change. The resulting auction also supports sincere bidding as an ex post Nash equilibrium. In the latter part of the paper, we generalize the restriction rule to the multi-demand setting due to Ausubel (2006) and show that the above desirable properties are preserved. Along the way, we also show that Ausubel's (2006) original auction is not well-defined with respect to outcomes off the equilibrium path and propose a way to overcome this problem.

JEL classification: C72; D44; D47
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1 Introduction

Auctions have contributed to allocating various heterogeneous commodities in the real world, such as spectrum licenses (Ausubel and Milgrom 2002, Milgrom 2004, Ausubel and

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Baranov 2014, Milgrom and Segal 2020), airport time slots (Rassenti et al. 1982) and bus routes (Cantillon and Pesendorfer 2006), among others. In carrying out auctions, one of the following two formats is typically chosen: one-shot (static) auctions or dynamic auctions. Ausubel (2006) notes that dynamic auctions "offer advantages of simplicity, transparency, and privacy preservation". In addition, the auction format helps bidders increase their knowledge about the valuations of items.[1] This is a major advantage over static direct mechanisms, especially in combinatorial auctions in which an exponential number of bundles are sold.

Among many design issues concerning dynamic auctions, this paper addresses the following problem: to what extent should bid withdrawal be restricted? This problem has been discussed intensively in the context of the well-known *simultaneous ascending auction* (henceforth SAA; proposed by Paul Milgrom, Robert Wilson, and Preston McAfeesee; see Cramton (2006) for a survey). On the one hand, bid withdrawal brings the problem of undersell, i.e., "ending the auction at a point where demand is less than supply" (Ausubel et al. 2006, p.118). Bid withdrawal could also "facilitate undesirable gaming behavior" (Cramton 2006, p.100). For example, "a bidder may be tempted to first bid on several items to force the other bidders to bid (and thus reveal their valuations' estimates) and then withdraw her bid" (Haeringer 2018, p.106). On the other hand, in order to guarantee flexibility in choice, allowing bid withdrawal is essential, especially in the presence of the *exposure problem* (see, for example, Milgrom (2004, p.277)). Taking these negative/positive aspects into consideration, existing rules introduce a monetary punishment against bid withdrawal.[2]

Despite the importance of the issue of bid withdrawal in practice, in theory, not much attention has been paid to the problem. Instead, existing auctions tend to circumvent the problem by adopting an extreme idea, which creates practical problems. For example, auctions by Gul and Stacchetti (2000), Ausubel (2006) and Sun and Yang (2014) put essentially no restrictions on bid withdrawal; any bid in the previous round can be withdrawn in the current round. As a consequence, the problem of undersell could happen when bidders behave insincerely. In contrast, auctions by Mishra and Parkes (2007) and de Vries et al. (2007) do not allow any bid withdrawal. Similarly, in the ascending package auction by Ausubel and Milgrom (2002), it is specified that "[a] bidder can never reduce or withdraw a bid it has made on any package" (Ausubel and Milgrom 2002).[3] These rules make it difficult for bidders to respond to the updated information flexibly.

[1]This point has been emphasized in the literature; see Milgrom and Weber (1982), Section 5 of Milgrom (2004), Chapters 2, 4 and 10 of Cramton et al. (2006), and Chapter 6 of Haeringer (2018)).

[2]This penalty in turn introduces the incentive of strategic bidding delay, which is modulated by the so-called activity rule (see Cramton (2006) or Ausubel and Baranov (2020)).

[3]This rule is imposed when the auction is viewed as a dynamic auction game. When viewed as a direct revelation game (called the *proxy game*), the auction has the advantages of choosing a core outcome and supporting semi-sincere strategies as an equilibrium.

The purpose of the current paper is to explicitly formulate a withdrawal restriction rule and theoretically investigate a desirable one, with the aim to overcome the problems of extreme ideas. To pursue this investigation in a stylized setting, we focus on the unit-demand auction model due to Demange and Gale (1985). A *withdrawal restriction rule* specifies a set of items that were reported in the previous round and cannot be withdrawn in the current round. In view of the (dis)advantages of extreme rules mentioned above, the following question arises: what is the *mildest* restriction rule that induces a balance between demand and supply for any possible strategy profile? The answer to this question turns out to be an intuitive rule: it restricts withdrawing bids on items that have been affected most favorably by the price change. For example, if a bidder bids on items k and ℓ and only the price of ℓ increases, then k is affected more favorably than ℓ. The bidder must bid on k again but is allowed to withdraw her bid on ℓ.

Under this restriction rule, the resulting auction game adjusts prices monotonically and satisfies two desirable properties. First, on any termination, the auctioneer can find an equilibrium allocation with respect to the final prices and the reported demands. This is a stronger efficiency property than existing ones in which equilibrium allocation is guaranteed only for sincere bidding. Second, sincere bidding by all bidders constitutes an ex post Nash equilibrium. Furthermore, our rule satisfies least restrictiveness: any milder rule permits a strategy profile for which equilibrium allocation does not exist at the end of the auction. These results clarify the essential properties that should be satisfied by a withdrawal restriction rule.

In the latter part of the paper, we generalize the restriction rule in the unit-demand setting to the multi-demand setting due to Ausubel (2006). Bidders are assumed to have substitutable valuations and report a set of bundles at each round, as they are prohibited from withdrawing bids on the most favorably affected bundles.[4] We prove that the aforementioned equilibrium-realization/incentive properties are generalized to this setting. Along the way, we also show that Ausubel's (2006) original auction is not well-defined with respect to outcomes off the equilibrium path and propose a way to overcome this problem.

Related literature

Among the huge literature on auctions, we confine our attention to dynamic auctions that maintain anonymous prices and deal with heterogeneous commodities.

[4] It is noteworthy that Ausubel (2006) considers a set of bundles that have the minimal intersection with the items whose prices have been updated (see Definition 5 and Proposition 2 therein). These bundles are equivalent to those affected most favorably. While the author considers the bundles in the context of recovering indirect utility functions from demand reports, their connection to bid withdrawal is not discussed in the paper.

As mentioned earlier, one of the most successful auction formats is the SAA. We consider an auction format that shares the essential properties of the SAA, namely, prices are anonymous and rise monotonically, determining a winner for any item on any termination. Recently, a new auction design, called the clock-proxy auction (first proposed by Ausubel et al. (2006); see also Ausubel and Baranov (2014)) has been put into practice. The key idea of this auction is to divide the whole auction process into two parts, the dynamic clock rounds and the static supplementary round. While the clock-proxy auction avoids the problem of undersell by introducing the supplementary static round, we avoid it by restricting bid withdrawal. A distinguishing feature of our auction from the above ones is to guarantee incentive compatibility in a dynamic process.[5]

While the SAA asks bidders to report a price for items, we ask bidders to report a demand set. This idea can be seen in many existing auctions, such as those by Demange et al. (1986), Gul and Stacchetti (2000), Ausubel (2006), and Sun and Yang (2009). Similar to the market mechanism, these auctions proceed by iteratively adjusting the prices of the items in excess demand/supply.[6] However, different from markets, the number of participants is small and hence some individuals can undermine the balance between demand and supply. In Gul and Stacchetti's (2000) auction, demand and supply do not necessarily coincide off the equilibrium path. In particular, when excess demand is not well-defined at some point, the "no allocation" punishment is implemented, i.e., no one receives any item. In Ausubel (2006), if excess demand/supply occurs infinitely many times, then the bidders' payoffs are set to be $-\infty$. Sun and Yang's (2014) auction proceeds by accumulating demand reports and maintaining an anonymous price path.[7] Bidders can withdraw any bid in the previous round; more precisely, bidders can withdraw any past bid as long as the tentative prices have not been changed by more than one unit. If the bidders create excess demand/supply many times, then they suffer the "no allocation" punishment and a monetary penalty. Different from these auctions, our auction always reaches an equilibrium allocation.

A recent paper by Ausubel and Baranov (2020) investigates desirable restrictions on feasible bids in dynamic auctions. The authors' main focus is to find an activity rule (specifying which bids *can* be chosen) that enforces the law of demand. Different from this approach, our main focus is to find a withdrawal restriction rule (specifiyng which bids *must* be chosen) that induces a balance between demand and supply.

The proofs of our main theorems rely on the techniques in discrete convex analysis

[5]The SAA is known to be vulnerable to a strategic behavior called *demand reduction*. This fact has been widely documented in the literature; see, for example, Ausubel et al. (2014).

[6]Ausubel's (2006) auction is formulated in terms of the minimization algorithm of the Lyapuniv function. Yokote (2020b) proves that minimizing the Lyapunov function is equilvalent to adjusting the prices of items in excess demand/supply.

[7]Sun and Yang (2014) assume that the bidders view the items as complements, while we assume that they view the items as substitutes in the multi-demand setting. Neither assumption implies the other.

(Murota 2003). Recently, this theory has been proved to be particularly useful for dealing with two-sided markets under constraints; see Kojima et al. (2018) or Yokote (2020c). We translate the problem of restricting bid withdrawal into that of imposing constraints on a discrete choice set. To the best of our knowledge, this study is the first to combine discrete convex analysis with non-cooperative game theory. For a survey of the application of discrete convex analysis to economics, see Murota (2016).

Structure of this paper

Section 2 introduces preliminaries for formulating extensive-form games. Sections 3 and 4 analyze dynamic auctions for the unit-demand and multi-demand setting, respectively. Section 5 discusses directions for future research. All proofs are relegated to Section 6.

2 Extensive-form game: general description

Let $N = \{1, \ldots, n\}$ be a set of **bidders** and K be a set of indivisible **items**. Each item can be assigned to at most one bidder. A **price vector** is given by $p \in \mathbb{Z}^K_+$.[8]

Let Σ be an arbitrary finite set, representing a set of **demand reports** from which bidders can choose. We consider the following auction format: (i) in each discrete round t, the current prices $p(t)$ are displayed to bidders, (ii) bidders send a demand report $\sigma^i(t) \in \Sigma$ $(i \in N)$ simultaneously, and (iii) the auctioneer updates the current prices $p(t)$ to $p(t+1)$ according to some rule. Information is fully transparent; at each round t, every bidder is informed about the other bidders' bids up to round t and the prices up to and including round t. We describe an extensive-form auction game in terms of the set of possible sequences.[9]

The **set of price-report sequences** is given by

$$\mathcal{S} \equiv \bigcup_{t=1}^{\infty} \big((\mathbb{Z}^K_+ \times \Sigma^N)^{t-1} \times \mathbb{Z}^K_+, \tag{1}$$

where $(\mathbb{Z}^K_+ \times \Sigma^N)^0 \times \mathbb{Z}^K_+ \equiv \mathbb{Z}^K_+$. A generic sequence in the above set is denoted in three ways as

$$h(t) \equiv \langle \xi(t-1), p(t) \rangle \equiv \langle (p(1), \sigma^N(1)), \ldots, (p(t-1), \sigma^N(t-1)), p(t) \rangle,$$

which are interpreted as follows:

- $p(s)$ $(1 \leq s \leq t-1)$ represents the price vectors up to round t;

[8]Note that we consider linear and anonymous prices (see Definition 4 of Mishra and Parkes (2007)).

[9]This description method is borrowed from Osborne and Rubinstein (1994) and Kaneko and Kline (2008).

- $\sigma^N(s) \equiv \big(\sigma^i(s)\big)_{i\in N}$ $(1 \leq s \leq t-1)$ represents the profiles of demand reports up to round t; and

- $p(t)$ is the price vector at round t.

For notational convenience, let $\langle \xi(0), p\rangle \equiv p$.

A non-empty finite subset $\mathcal{H} \subseteq \mathcal{S}$ is said to **induce an extensive-form game** (with finite horizon) if it satisfies the following conditions:

- If $p, p' \in \mathcal{H}$, then $p = p'$.

- If $\big\langle \xi(t-2), \big(p(t-1), \sigma(t-1)\big), p(t)\big\rangle \in \mathcal{H}$ with $t \geq 2$, then $\langle \xi(t-2), p(t-1)\rangle \in \mathcal{H}$.

- For any $\langle \xi(t-1), p(t)\rangle, \langle \xi'(t-1), p'(t)\rangle \in \mathcal{H}$ with $\xi(t-1) = \xi'(t-1)$, we have $p(t) = p'(t)$.

The second condition states that if a sequence is in $\mathcal{H}$, then its subsequence is also included in $\mathcal{H}$. The third condition states that if two sequences share the same path up to t, then the resulting prices must be the same. Under these conditions, the set $\mathcal{H}$ can be represented by a tree with nodes representing tentative prices and edges representing profiles of demand reports (we will specify active players and final payoffs in later analyses). In game-theoretic terminology, a sequence $h(t) \in \mathcal{H}$ is called a **history**. A history $\langle \xi(t-1), p(t)\rangle \in \mathcal{H}$ is called a **terminal history** if, for any $\sigma^N \in \Sigma^N$ and $p \in \mathbb{Z}^K_+$, it holds that $\langle \xi(t-1), (p(t), \sigma^N), p\rangle \notin \mathcal{H}$.

3 Unit-demand setting

In this section we consider the unit-demand setting and identify desirable restrictions on bid withdrawal. To this end we introduce two auction rules: the price update rule and the withdrawal restriction rule. The former specifies how the prices are updated and the latter specifies which bundles cannot be withdrawn. Before introducing these rules, we clarify the class of value functions we consider.

3.1 Value function

Let θ denote the **dummy item** that can be assigned to any number of bidders; in contrast to this appellation, we call an item in K a *tangible item*. Set $\bar{K} = K \cup \{\theta\}$.

A **value function** is a function $v : \bar{K} \to \mathbb{Z}$ satisfying $v(\theta) = 0$. Let $\bar{p} \in \mathbb{Z}^K_+$ denote a **ceiling price vector**, which represents a sufficiently high price vector at which no bidder demands any item. Let $\mathcal{V}$ denote the set of all value functions v such that $v(k) < \bar{p}_k$ for all $k \in K$. In our auction game, it is assumed that every bidder i first draws a value function v privately from $\mathcal{V}$. We impose the following assumption:

A1 (*Quasi-linear utility*): for any $v \in V$, i's utility under v from receiving $k \in K$ in return for the payment $t \geq 0$ is given by $v(k) - t$.

Let $\mathcal{K} \equiv 2^K$ and $\bar{\mathcal{K}} \equiv 2^{\bar{K}}$. Throughout this section, we set $\Sigma \equiv \bar{\mathcal{K}} \backslash \{\varnothing\}$.[10] If a bidder reports $A \in \bar{\mathcal{K}} \backslash \{\varnothing\}$, it is intended to mean that the bidder desires (any) one of the items in A.

3.2 Price update rule

A **price update rule** is a function $\pi : \mathbb{Z}_+^K \times \Sigma^N \to \mathbb{Z}_+^K$. Given a tentative price vector $p \in \mathbb{Z}_+^K$ and a profile of demand reports $\sigma^N \in \Sigma^N$, the rule specifies an element in $\mathbb{Z}_+^K$, the updated prices.

We adopt the same price update rule as that of Demange et al. (1986). For $\sigma \in \Sigma$, we define the **min-requirement function** $\hat{R}(\cdot|\sigma) : \mathcal{K} \to \{0, 1\}$ as follows:[11]

$$\hat{R}(A|\sigma) = \min\big\{|\{k\} \cap A| : k \in \sigma\big\} \text{ for all } A \in \mathcal{K}.$$

In words, $\hat{R}(A|\sigma)$ represents the minimum number of items in A necessary to satisfy the demand at σ. For $\sigma^N \in \Sigma^N$, with a slight abuse of notation, let $\hat{R}(A|\sigma^N) \equiv \sum_{i \in N} \hat{R}(A|\sigma^i)$. We define the **excess demand function** $E(\cdot|\sigma^N) : \mathcal{K} \to \mathbb{Z}$ as follows:

$$E(A|\sigma^N) = \hat{R}(A|\sigma^N) - |A| \text{ for all } A \in \mathcal{K}. \tag{2}$$

If $E(A|\sigma^N) > 0$ for some $A \in \mathcal{K}$, then the number of items in A is not sufficient to serve the bidders' demands at σ^N, representing excess demand.

One can verify that $E(\cdot|\sigma^N)$ is **supermodular**, i.e.,

$$E(A \cup B|\sigma^N) + E(A \cap B|\sigma^N) \geq E(A|\sigma^N) + E(B|\sigma^N) \text{ for all } A, B \in \mathcal{K}. \tag{3}$$

This inequality implies that there exists a unique **minimal maximizer** of $E(\cdot|\sigma^N)$, denoted as $\underline{E}(\sigma^N)$.

[10] We use $\emptyset$ to denote the empty set contained in K and $\varnothing$ to denote that contained in Σ.

[11] Equivalently,

$$\hat{R}(A|\sigma) = \begin{cases} 1 & \text{if } \sigma \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

We adopt the price update rule $\tilde{\pi}(\cdot,\cdot)$ defined as follows:

$$\tilde{\pi}(p,\sigma^N)_k = \begin{cases} \min\{p_k+1,\bar{p}\} & \text{if } k \in \underline{E}(\sigma^N), \\ p_k & \text{otherwise.} \end{cases} \tag{4}$$

3.3 Withdrawal restriction rule

This section addresses the problem of how to restrict bid withdrawal. A withdrawal restriction rule, simply a **restriction rule**, is a function $\rho : (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma \to \Sigma \cup \{\varnothing\}$ that satisfies

$$\rho(A,\sigma) \subseteq \sigma \text{ for all } (A,\sigma) \in (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma. \tag{5}$$

The first argument $A \in \mathcal{K}$ represents a set of items whose prices have been updated, while the second argument $\sigma \in \Sigma$ represents a bidder's demand report in the previous round. Given these inputs, $\rho(\cdot,\cdot)$ specifies the set of irrevocable items on which i must bid in the current round. By this interpretation, the output of $\rho(\cdot,\cdot)$ must be a part of the items reported in the previous round, which is stated in (5). When the output is $\varnothing$, the bidder is free from any restriction in that round. Here, we implicitly require a rule to depend only on two pieces of information: price changes and bids in the previous round. This formulation has the advantage of making the rule as simple and easy-to-understand as possible.

In restricting bid withdrawal, it is important to note that restrictions create the risk of forcing a bid on undesirable items for the bidder. To minimize the risk, one can consider forcing bids only on items that have been affected *most favorably* by the price change. Since price increases are undesirable for bidders, such items have the minimal intersection with A, i.e., they attain the minimum value of $\hat{R}(A|\sigma)$. Formally, we define $\tilde{\rho}(\cdot,\cdot)$ by

$$\tilde{\rho}(A,\sigma) = \big\{k \in \sigma : |\{k\} \cap A| = \hat{R}(A|\sigma)\big\}\backslash\{\theta\} \text{ for all } (A,\sigma) \in (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma. \tag{6}$$

We remark that the dummy item is eliminated from the set. This guarantees that a bidder is never forced to bid on the dummy item.

3.4 Extensive-form auction game

We define an extensive-form auction game $\widetilde{\mathcal{H}} \subseteq \mathcal{S}$ inductively as follows:

H1: $\mathbf{0} \in \widetilde{\mathcal{H}}$.

H2: If $\sigma^i \in \Sigma$ for all $i \in N$, then

$$\mathbf{0} \neq \tilde{\pi}\big(\mathbf{0}, \sigma^N\big) \Longrightarrow \big\langle \big(\mathbf{0}, \sigma^N\big), \tilde{\pi}\big(\mathbf{0}, \sigma^N\big)\big\rangle \in \widetilde{\mathcal{H}}.$$

H3: If $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}$, $\sigma^i \in \Sigma$ and $\sigma^i \supseteq \tilde{\rho}\big(\underline{E}(\sigma^N(t-1)), \sigma^i(t-1)\big)$ for all $i \in N$, then

$$p \neq \tilde{\pi}\big(p(t), \sigma^N\big) \Longrightarrow \big\langle \xi(t-1), \big(p(t), \sigma^N\big), \tilde{\pi}\big(p(t), \sigma^N\big)\big\rangle \in \widetilde{\mathcal{H}}.$$

The above game has the following structure. In each round $t = 1, 2, \ldots$, every bidder i observes $p(t)$ and simultaneously reports a set of items $\sigma^i \in \Sigma$ so that σ^i conforms to the restriction rule, after which the tentative prices $p(t)$ are updated to $\tilde{\pi}(p(t), \sigma^N)$. If the prices remain the same, then the game stops. Otherwise, the history is prolonged by incorporating the demand reports σ^N and the updated prices $\tilde{\pi}\big(p(t), \sigma^N\big)$.

A formal game procedure is given as follows:

Game procedure: [12] Prior to the start of the auction, nature according to a joint probability distribution function $F(\cdot)$ draws a profile $v^N \equiv (v^i)_{i \in N} \in \mathcal{V}^N$ and reveals to every bidder $i \in N$ only his own value function v^i. Then, the bidders play the game $\widetilde{\mathcal{H}}$.

We will specify payoffs at each terminal history in the next section.

3.5 Realization of equilibrium

We turn our attention to the problem of how to allocate the items at the end of the game. For $p \in \mathbb{Z}_+^K$ and $\sigma^N \in \Sigma^N$, an **equilibrium allocation** with respect to p and σ^N is a tuple $(k^i)_{i \in N} \in \bar{K}^N$ such that[13]

$$\begin{aligned} &k^i \neq k^j \text{ for all } i, j \in N \text{ with } i \neq j, \\ &k^i \in \sigma^i \text{ for all } i \in N, \\ &k \in K \setminus \cup_{i \in N} \{k^i\} \Longrightarrow p_k = 0. \end{aligned} \tag{7}$$

This is a standard equilibrium notion and also embodies our purpose to avoid undersell, i.e., to allocate all the items (with positive prices) to some bidder.

For $p \in \mathbb{Z}_+^K$, we write $p < \bar{p}$ to denote $p_k < \bar{p}_k$ for all $k \in K$. We are ready to state the first main result:

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[12] The first sentence is cited from Sun and Yang (2014) with a slight change in notation.

[13] Here, we implicitly assume that the seller values all the items at zero.

Theorem 1 (Realization of equilibrium allocation: unit-demand case). *Let $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1), p(t)\rangle \in \widetilde{\mathcal{H}}$ be a terminal history such that $p(t) < \bar{p}$. Then, there exists an equilibrium allocation with respect to $p(t)$ and $\sigma^N(t-1)$.*

Proof. See Section 6.2. □

In real-world auctions, we can expect that the final prices do not reach the sufficiently high ceiling prices and that the price of every item is updated at least once. With this qualification, our auction always ends up allocating all the items to the satisfaction of bidders. This is in sharp contrast to existing auctions (Gul and Stacchetti (2000), Ausubel (2006), Sun and Yang (2014)) where undersell could occur and a certain form of punishment is implemented even when the going prices do not reach the ceiling prices.

Using Theorem 1, we specify the allocation and payment rules.

Allocation rule: For any terminal history $h(t) \equiv \langle \xi(t-1), p(t)\rangle \in \widetilde{\mathcal{H}}$ with $p(t) < \bar{p}$, the auctioneer chooses an (arbitrary) equilibrium allocation $(k^i)_{i \in N}$ and implements it; for other terminal histories, no one receives anything.

Payment rule: For any terminal history $h(t) \equiv \langle \xi(t-1), p(t)\rangle \in \widetilde{\mathcal{H}}$ with $p(t) < \bar{p}$, bidder $i \in N$ pays the price of k^i at $p(t)$ if k^i is a tangible item and pays nothing if $k^i = \theta$; for other terminal histories, everyone pays nothing.

3.6 Restrictiveness of rules

While Theorem 1 guarantees the realization of equilibrium allocation on termination, this is not the only criterion for desirable restriction rules; in view of bidders' freedom of choice, less restrictive rules are better. Our next theorem states that $\tilde{\rho}$ is least restrictive in the sense specified below.

Let ρ, ρ' be two restriction rules. We say that ρ is **less restrictive** than ρ' if

$$\rho(A,\sigma) \subseteq \rho'(A,\sigma) \text{ for all } (A,\sigma) \in (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma, \text{ and}$$
$$\rho(A,\sigma) \subsetneq \rho'(A,\sigma) \text{ for some } (A,\sigma) \in (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma.$$

Recall that the output of ρ specifies the items on which a bidder must bid. If this set becomes strictly smaller in the above sense, then the bidder can enjoy a larger choice set.

Theorem 2 (Least restrictiveness). *Suppose $n \geq |K| + 2$ and $\bar{p}_k \geq 3$ for all $k \in K$. Let ρ be a restriction rule that is less restrictive than $\tilde{\rho}$. Consider the dynamic auction game defined as in Section 3.4, with the only difference that $\tilde{\rho}$ is replaced with ρ. Then, there exists a* *terminal history* $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}$ *with* $p(t) < \bar{p}$ *at which there exists no equilibrium allocation with respect to* $p(t)$ *and* $\sigma^N(t-1)$.

Proof. See Section 6.3. □

By the definition of the price adjustment rule, excess demand never occurs at terminal histories. Hence, the above non-existence result implies that some terminal histories permit excess supply, i.e., there are some items priced positively but not demanded by any bidder. This in turn suggests the need to decrease prices, creating a cyclic and never-ending price adjustment process. Hence, if an auctioneer prioritizes balancing demand and supply as well as respecting freedom, our rule offers a desirable candidate.

3.7 Incentive compatibility

Building on existing results, we establish (dynamic) incentive compatibility, a fundamental desideratum in market design. A bidder i's **strategy** $s^i(\cdot,\cdot)$ maps a value function $v \in \mathcal{V}$ and a nonterminal history $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}$ to an element in $\tilde{\rho}\big(\underline{E}(\sigma^N(t-1)), \sigma^i(t-1)\big)$. We often write $s^i_v(\cdot)$ to denote i's strategy under value function v. A strategy is a contingency plan specifying which items to report at each nonterminal history subject to the restriction rule. It is assumed that this plan is made in the ex ante stage, i.e., before the realization of one's value function.

We say that a bidder i **bids sincerely** if i chooses a strategy $s^i(\cdot,\cdot)$ such that, for any $v \in \mathcal{V}$ and any nonterminal history $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}$,

$$s^i(v, h(t)) = \arg\max_{k \in \bar{K}} \{v(k) - p_k\} \cup \tilde{\rho}\big(\underline{E}(\sigma^N(t-1)), \sigma^i(t-1)\big), \tag{8}$$

where p_θ is defined to be 0. In words, except for the irrevocable bids, the bidder bids on the utility-maximizing items. We say that a sincere bidding defined by (8) is **feasible** at $h(t)$ if sincere bidding does not contradict the restriction rule, i.e.,

$$\arg\max_{k \in \bar{K}} \{v(k) - p_k\} \supseteq \tilde{\rho}\big(\underline{E}(\sigma^N(t-1)), \sigma^i(t-1)\big)$$

One can verify that, if sincere bidding is feasible at $h(t-1)$ and the bidder indeed bids only on the utility-maximizing items, then it is also feasible at $h(t)$. Namely, the restriction rule enables sincere bidding.

Let S^i denote the set of all i's strategies and let $S^N \equiv \times_{i \in N} S^i$, with generic element $s^N \in S^N$. For any $i \in N$, $v^N \in \mathcal{V}^N$ and $s^N \in S^N$, let $u^i((s^j_{v^j})_{j \in N} | v^i)$ denote i's utility in $\widetilde{\mathcal{H}}$ when the bidders have value functions v^N and follow s^N. We say that a strategy profile

$s^N \in S^N$ is an **ex post Nash equilibrium** if, for any $i \in N$ and $v^N \in \mathcal{V}^N$, it holds that

$$u^i((s^j_{v^j})_{j\in N}|v^i) \geq u^i(\tilde{s}^i_{v^i}, (s^j_{v^j})_{j\in N\setminus\{i\}}|v^i) \text{ for all } \tilde{s}^i \in S^i.$$

Milgrom (2004 p.189) explains the attractive features of ex post equilibria in ascending auctions.

Since the strategy space in our auction is narrower than that in Gul and Stacchetti's (2000) auction,[14] the following proposition holds:

Proposition 1. *Sincere bidding by all bidders is an ex post Nash equilibrium.*

Together with Theorem 1, sincere bidding results in an efficient resource allocation with respect to the true value functions.

We remark that Ausubel (2006) and Sun and Yang (2014) consider ex post *perfect* equilibrium; in any subgame, sincere bidding is the best strategy if all the others bid sincerely. However, in our auction game, there does not necessarily exist a subgame perfect equilibrium in pure strategies[15] because sincerely bidding is not always feasible in a subgame. Meanwhile, we can establish the following weakened form of subgame perfection: for any subgame in which sincere bidding is feasible, sincere bidding by all bidders is a Nash equilibrium.

4 Multi-demand setting

In this section, we prove that the theoretical results in Section 3 carry forward to the multi-demand setting due to Ausubel (2006). We introduce notations in a parallel manner to Section 3.

4.1 Value function

A **value function** is a function $v : \mathcal{K} \to \mathbb{Z}$ that satisfies $v(\emptyset) = 0$. For $p \in \mathbb{Z}^K_+$ and $A \in \mathcal{K}$, let $p(A) \equiv \sum_{k\in K} p_k$ and $v[p](A) \equiv v(A) - p(A)$. In addition to A1 (quasi-linearity), we impose the following two assumptions:

A2 (*Monotonicity*): for any $A, A' \in \mathcal{K}$ with $A \subseteq A'$, it holds that $v(A) \leq v(A')$.

[14] Gul and Stacchetti's (2000) theorem (Theorem 5) refers to perfect Bayesian equilibrium, but their proof can be directly applied to prove ex post Nash equilibrium in our auction because the auction realizes the minimum equilibrium price vector that induces the VCG payments; see Demange et al. (1986) and Leonard (1983).

[15] We will prove this claim in an updated version of this paper.

A3 (*Substitutability*): for any $p, p' \in \mathbb{Z}_+^K$ with $p \le p'$ and any $A \in \underset{A\in\mathcal{K}}{\operatorname{argmax}}\, v[p](A)$, there exists $A' \in \underset{A\in\mathcal{K}}{\operatorname{argmax}}\, v[p'](A)$ such that

$$[p_k = p'_k, k \in A] \Longrightarrow k \in A'.$$

The domain of value functions $\mathcal{V}$ is given by

$$\mathcal{V} = \{v : v \text{ satisfies A1-A3 and } v[\bar{p}](A) < 0 \text{ for all } A \in \mathcal{K}\backslash\{\emptyset\}\},$$

where $\bar{p}$ is the ceiling price vector (see Section 3.1).

Let $\mathcal{B} \equiv 2^{\mathcal{K}}$, representing the **family of sets of bundles**. Throughout this section, we set $\Sigma \equiv \mathcal{B}\backslash\{\varnothing\}$. Namely, each bidder reports a set of bundles that she desires at the tentative prices.

4.2 Price-update and withdrawal restriction rules

In parallel to Section 3.2, a **price update rule** is a function $\pi : \mathbb{Z}_+^K \times \Sigma^N \to \mathbb{Z}_+^K$. For $\sigma \in \Sigma$, we define the **min-requirement function** $\hat{R}(\cdot|\sigma) : \mathcal{K} \to \mathbb{Z}_+$ as follows:

$$\hat{R}(A|\sigma) = \min\big\{|B \cap A| : B \in \sigma\big\} \text{ for all } A \in \mathcal{K}. \tag{9}$$

For $\sigma^N \in \Sigma^N$, we define the **excess demand function** $E(\cdot|\sigma^N)$ as in (2). As will be proven in Section 6.1.1, under the bidding rule we impose (see (12) in Section 4.3), $E(\cdot|\sigma^N)$ is supermodular (see (3)). This implies that there exists a unique **minimal maximizer** of $E(\cdot|\sigma^N)$, denoted as $\underline{E}(\sigma^N)$. We adopt the same price update rule $\tilde{\pi}(\cdot,\cdot)$ as (4). Yokote (2020b) proves that the price update process induced by this rule coincides with the minimization algorithm of the Lyapunov function due to Ausubel (2006).

In parallel to Section 3.3, a **restriction rule** is a function $\rho : (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma \to \Sigma \cup \{\varnothing\}$. We define $\tilde{\rho}$ by

$$\tilde{\rho}(A,\sigma) = \big\{B \in \sigma : |B \cap A| = \hat{R}(A|\sigma),\ |B| \ge |B'| \text{ for all } B' \in \sigma \text{ with } |B \cap A| = \hat{R}(A|\sigma)\big\}\backslash\{\emptyset\}$$
$$\text{for all } (A,\sigma) \in (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma. \tag{10}$$

This is a straightforward generalization of (6) to the multi-demand setting, with one nontrivial twist that only maximum-size bundles are chosen. We will later introduce two propositions (Proposition A in Section 4.3 and Proposition 3 in Section 6.1.1) which guarantee that this restriction rule enables sincere bidding.

4.3 Remark on the well-definedness of Ausubel's (2006) auction

Before proceeding to the formal description of our auction game, we demonstrate that Ausubel's (2006) original auction is not well-defined with respect to outcomes off the equilibrium path and that the problem can be circumvented by imposing an additional rule on bidding behavior.

In this subsection, we set the domain of price vectors to be $\mathbb{Z}^K$ rather than $\mathbb{Z}^K_+$. The following assumption on a value function v is the same as A3 except for the domain of price vectors:

A3* (*Substitutability*): for any $p, p' \in \mathbb{Z}^K$ with $p \leq p'$ and any $A \in \operatorname*{argmax}_{A \in \mathcal{K}} v[p](A)$, there exists $A' \in \operatorname*{argmax}_{A \in \mathcal{K}} v[p'](A)$ such that

$$[p_k = p'_k, k \in A] \Longrightarrow k \in A'.$$

We introduce additional notations. For $A \in \mathcal{K}$, let $\mathbb{1}^A \in \{0,1\}^K$ denote the **characteristic vector** for A, i.e., $\mathbb{1}^A_k = 1$ if $k \in K$ and $\mathbb{1}^A_k = 0$ otherwise. When a value function v for $i \in N$ is specified, it induces a **demand correspondence** $D^i : \mathbb{Z}^K \to \Sigma$ and an **indirect utility function** $V^i : \mathbb{Z}^K \to \mathbb{Z}$ defined as follows:

$$D^i(p) = \arg\max_{A \in \mathcal{K}} v[p] \text{ for all } p \in \mathbb{Z}^K,$$
$$V^i(p) = \max_{A \in \mathcal{K}} v[p](A) \text{ for all } p \in \mathbb{Z}^K.$$

For notational simplicity, we suppress the dependence of $D^i(\cdot)$ and $V^i(\cdot)$ on v. We define $\tilde{\rho}' : (\mathcal{K} \backslash \{\emptyset\}) \times \Sigma \to \Sigma$ by

$$\tilde{\rho}'(A, \sigma) = \big\{B \in \sigma : |B \cap A| = \hat{R}(A|\sigma)\big\} \text{ for all } (A, \sigma) \in (\mathcal{K} \backslash \{\emptyset\}) \times \Sigma.$$

In words, $\tilde{\rho}'(\cdot, \cdot)$ collects the set of bundles that attain the minimum value of $\hat{R}(\cdot|\cdot)$. The output of $\tilde{\rho}'$ is always larger than that of $\tilde{\rho}$ (see (10)).

We now revisit Ausubel's (2006) price adjustment process. Fix $p \in \mathbb{Z}^K$ and $i \in N$. Suppose that i has a value function v satisfying A3* and reports sincerely the bundles in $D^i(p)$. Then, the auctioneer can identify i's optimal bundles in a "local" space around p:

> *[u]sing equation (19) and Proposition 2, the auctioneer can extend the report to identify an optimal bundle at every point in the unit K-dimentional cube $\{p + \Delta : 0 \leq \Delta \leq \mathbb{1}^K\}$ and $\{p - \Delta : \mathbf{0} \leq \Delta \leq \mathbb{1}^K\}$.* (p.626, line 5)

Here, Proposition 2 states the following:

Proposition A (Ausubel (2006), Proposition 2). *Suppose that i's value function v satisfies A3*. Then, for any* $p \in \mathbb{Z}^K$, $A \in \mathcal{K}$ *and* $B \in \tilde{\rho}'(A, D^i(p))$, *it holds that* $B \in D^i(p + \mathbb{1}^A)$.

Namely, among the utility-maximizing bundles at p, those contained in $\tilde{\rho}'(A, D^i(p))$ remain optimal at $p + \mathbb{1}^A$. Using this property,

> *the indirect utility function* $V^i(\cdot)$ *of each bidder* i, *once specified at* $p(t)$, *has a unique extension to the unit* K*-dimensional cubes.* (p.626, line 9)

As an example, fix $A \in \mathcal{K}$ and consider the price change from p to $p + \mathbb{1}^A$. By Proposition A, any bundle $B \in \tilde{\rho}'(A, D^i(p))$ is optimal both at p and at $p + \mathbb{1}^A$, which means that i's indirect utility decreases by $\hat{R}(A|D^i(p))$ because

$$V^i(p) - V^i(p + \mathbb{1}^A) = v[p](B) - v[p + \mathbb{1}^A](B) = |B \cap A| = \hat{R}(A|D^i(p)).$$

Ausubel (2006) continues as:

> *[t]he auctioneer then determines the price vector on the lattice* $\{p + \Delta : 0 \leq \Delta \leq 1\}$ *that minimizes the Lyapunov function* $L(\cdot)$ *and uses this as the next price vector,* $p(t+1)$. (p.626, line 12)

Here, the Lyapunov function $L : \mathbb{Z}^K \to \mathbb{Z}$ is defined by

$$L(p) = \sum_{k \in K} p_k + \sum_{i \in N} V^i(p) \text{ for all } p \in \mathbb{Z}^K.$$

For example, if the auctioneer changes p to $p + \mathbb{1}^A$, the former term on the right-hand side increases by $|A|$, while the latter term decreases by $\hat{R}(A|D^i(p))$ for each $i \in N$. In this way, the minimizers of $L(p + \mathbb{1}^A)$, where the minimum is taken over $A \in \mathcal{K}$, can be identified in the local space around p. The auctioneer chooses a minimal minimizer of the Lyapunov function, which becomes the next price vector.

The argument so far assumes that the bidders have substitutable valuations and behave sincerely. However, the point here is that the process of identifying minimizers (namely, calculating the value of $\hat{R}(A|D^i(p))$) can be done only by using the information of the demand report $D^i(p)$. Hence, the process can be extended to any demand report $\sigma^i \in \Sigma$ by calculating $\hat{R}(A|\sigma^i)$ instead of $\hat{R}(A|D^i(p))$. On this procedure, Ausubel (2006) writes

> *in the event of untruthful reporting, any minimal minimizer can be selected.* (∗)
>
> (p.619, left column, (c), line 5)

If the price update process stops, then

> *By Proposition 1, there exists an allocation* $(x_i^*, \ldots, x_n^*)$ *such that* $x_i^* \in Q_i(p)$, *for every* $i = 1, \ldots, n$. (p.619, left column, (d), line 5)

Here, Proposition 1 states the equivalence between minimizers of $L(\cdot)$ and equilibrium allocations. This proposition is supposed to guarantee the existence of a *feasible allocation*, an allocation at which every bidder receives what she demands.

We show that the last claim is incorrect via a counterexample. Let $N = \{1, 2, 3\}$ and $K = \{k_1, k_2, k_3, k_4\}$. Let an initial price vector $p \in \mathbb{Z}^K$ be arbitrarily given. Suppose that the bidders report the following bundles at round 1:

$$\sigma^1 = \big\{\{k_1, k_2\}, \{k_3, k_4\}\big\},\ \sigma^2 = \big\{\{k_1\}, \{k_2\}\big\},\ \sigma^3 = \big\{\{k_3\}, \{k_4\}\big\}. \tag{11}$$

Following the price update rule mentioned above, we calculate $-\hat{R}(A|\sigma^i)$ (decrease in the indirect utilities for all A) and $|A| - \hat{R}(\mathbb{1}^A|\sigma^N)$ (decrease in the Lyapunov function) for all $A \in \mathcal{K}$.

A	$-\hat{R}\,(A\|\sigma^1)$	$-\hat{R}\,(A\|\sigma^2)$	$-\hat{R}\,(A\|\sigma^3)$	$\|A\| - \hat{R}(A\|\sigma^N)$
$\emptyset$	0	0	0	0
$\{k_1\}$	0	0	0	1
$\{k_2\}$	0	0	0	1
$\{k_3\}$	0	0	0	1
$\{k_4\}$	0	0	0	1
$\{k_1, k_2\}$	0	-1	0	1
$\{k_1, k_3\}$	-1	0	0	1
$\{k_1, k_4\}$	-1	0	0	1
$\{k_2, k_3\}$	-1	0	0	1
$\{k_2, k_4\}$	-1	0	0	1
$\{k_3, k_4\}$	0	0	-1	1
$\{k_1, k_2, k_3\}$	-1	-1	0	1
$\{k_1, k_2, k_4\}$	-1	-1	0	1
$\{k_1, k_3, k_4\}$	-1	0	-1	1
$\{k_2, k_3, k_4\}$	-1	0	-1	1
K	-2	-1	-1	0

The minimizers of the Lyapunov function are

$$\Big\{p, p + \mathbb{1}^K\Big\},$$

where the minimal minimizer is p. Namely, the prices are not updated and the auction stops. However, there does not exist an allocation that satisfies the bidders' demands: if bidder 1 receives $\{k_1, k_2\}$, then bidder 2 cannot obtain what she demands; if bidder 1 receives $\{k_3, k_4\}$, then bidder 3 cannot obtain what she demands.

In addition to the ascending auction, Ausubel (2006) considers the descending auction and the hybrid version of them, called the *global Walrasian tâtonnement algorithm.* In the appendix, we show that the descending phase applied to the above example also concludes that the prices are not updated. Hence, regardless of which auction format (ascending, descending, or global) we choose, the prices remain the same, and the auction terminates with no feasible allocation.

This negative result comes from the usage of Proposition 1. As Ausubel (2006) writes, "making Assumptions (A1')-(A4') assures that the hypothesis of Proposition 1 is satisfied", where (A4') corresponds to A3* (substitutability) in our paper. However, it is allowed that a bidder submits an untruthful demand report that cannot be supported as the maximizers of a substitutable valuation, for which the validity of Proposition 1 cannot be guaranteed (this point is overlooked in the explanation $(*)$). Indeed, in the counterexample, bidder 1's demand report can never be supported as the maximizers of a substitutable valuation (see Section 6.1.1).

The above discussion suggests the need to introduce an additional rule on bidding behavior. Throughout this section, for any $\mathcal{H} \subseteq \mathcal{S}$ that induces an extensive-form game, it is assumed that the following rule is satisfied:

Bidding rule: For any $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t) \rangle \in \mathcal{H}$ and $i \in N$, there exists v such that v satisfies A3* and

$$\sigma^i(t-1) = \arg\max_{A \in \mathcal{K}} v[p(t-1)](A). \qquad (12)$$

It is worthwhile to emphasize that the underlying value function v does not need to be identical across rounds. Under this rule, the assumptions behind Proposition 1 hold, recovering the results in Ausubel (2006). In Section 6.1.1, we show that this rule is equivalent to requiring bidders to report a set of bundles satisfying a certain convexity assumption.

4.4 Extensive-form auction game and its properties

Our auction game is the same as that of Ausubel's (2006) except for the restriction rule part. A characteristic of this auction is to maintain $n+1$ price paths.

For $j = 1, \ldots, n$, the j-th price path is obtained as a result of playing a game $\widetilde{\mathcal{H}}_{-j} \subseteq \mathcal{S}$. This game, defined inductively below, requires player j to be *inactive* in the sense that j always bids on $\{\emptyset\}$:

H1: $\mathbf{0} \in \widetilde{\mathcal{H}}_{-j}$.

H2: If $\sigma^i \in \Sigma$ for all $i \in N\backslash\{j\}$ and $\sigma^j = \{\emptyset\}$, then

$$\mathbf{0} \neq \tilde{\pi}\big(\mathbf{0}, \sigma^N\big) \Longrightarrow \big\langle \big(\mathbf{0}, \sigma^N\big), \tilde{\pi}\big(\mathbf{0}, \sigma^N\big)\big\rangle \in \widetilde{\mathcal{H}}_{-j}.$$

H3: If $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}_{-j}$, $\sigma^i \in \Sigma$ and $\sigma^i \supseteq \tilde{\rho}\big(\underline{E}(\sigma^N(t-1)), \sigma^i(t-1)\big)$ for all $i \in N\backslash\{j\}$, and $\sigma^j = \{\emptyset\}$, then

$$p \neq \tilde{\pi}\big(p(t), \sigma^N\big) \Longrightarrow \big\langle \xi(t-1), \big(p(t), \sigma^N\big), \tilde{\pi}\big(p(t), \sigma^N\big)\big\rangle \in \widetilde{\mathcal{H}}_{-j}.$$

The final $(n+1)$-th price path is obtained as a result of playing a game $\widetilde{\mathcal{H}} \subseteq \mathcal{S}$, which follows the n-th game $\widetilde{\mathcal{H}}_{-n}$[16] and allows all the n bidders to be active. The game is defined inductively as follows:

H1: $\widetilde{\mathcal{H}}_{-n} \subseteq \widetilde{\mathcal{H}}$.

H2: If $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle$ is a terminal history in $\widetilde{\mathcal{H}}_{-n}$, $\sigma^i = \sigma^i(t-1)$ for all $i \in N\backslash\{n\}$ and $\sigma^n \in \Sigma$, then

$$p(t) \neq \tilde{\pi}\big(p(t), \sigma^N\big) \Longrightarrow \big\langle \xi(t-1), \big(p(t), \sigma^N\big), \tilde{\pi}\big(p(t), \sigma^N\big)\big\rangle \in \widetilde{\mathcal{H}}.$$

H3: If $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}_{-j}$, $\sigma^i \in \Sigma$ and $\sigma^i \supseteq \tilde{\rho}\big(\underline{E}(\sigma^N(t-1)), \sigma^i(t-1)\big)$ for all $i \in N$, then

$$p \neq \tilde{\pi}\big(p(t), \sigma^N\big) \Longrightarrow \big\langle \xi(t-1), \big(p(t), \sigma^N\big), \tilde{\pi}\big(p(t), \sigma^N\big)\big\rangle \in \widetilde{\mathcal{H}}.$$

We are in a position to fully describe the game procedure.

Game procedure:[17] Prior to the start of the auction, nature according to a joint probability distribution function $F(\cdot)$ draws a profile $v^N \in \mathcal{V}^N$ and reveals to every bidder $i \in N$ only his own value function v^i. Then, the bidders play $n+1$ games in two stages.
Stage 1: The bidders simultaneously play the n games $\widetilde{\mathcal{H}}_{-j}$ for $j = 1, \ldots, n$, each of which realizes one terminal history $h_{-j}(t_{-j})$.
Stage 2: Starting from history $h_{-n}(t_{-n})$, the bidders play $\widetilde{\mathcal{H}}$, which realizes one terminal history $h(t)$ with $t \geq t_{-n} + 1$.

[16]In view of recovering VCG payments, we can choose any one of the n games $\widetilde{\mathcal{H}}_{-j}$ for $j = 1, \ldots, n$. Here, for presentational simplicity and in line with Ausubel's (2006) description, we choose the n-th game.

[17]The first sentence is cited from Sun and Yang (2014) with a slight change in notation.

We define equilibrium allocation as in (7). The following theorem is a multi-demand counterpart of Theorem 1:

Theorem 3 (Realization of equilibrium allocation: multi-demand case)**.** *Let $j \in N$ and $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1), p(t)\rangle \in \widetilde{\mathcal{H}}_{-j}$ be a terminal history such that $p(t) < \bar{p}$. Then, there exists an equilibrium allocation $(A^i_{-j})_{i \in N\setminus\{j\}}$ with respect to $p(t)$ and $\sigma^N(t-1)$.*

Proof. See Section 6.2. □

Corollary 1. *Let $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1), p(t)\rangle \in \widetilde{\mathcal{H}}$ be a terminal history such that $p(t) < \bar{p}$. Then, there exists an equilibrium allocation $(A^i)_{i \in N}$ with respect to $p(t)$ and $\sigma^N(t-1)$.*

Proof. See Section 6.2. □

Using the equilibrium allocations in this theorem and corollary, we specify the allocation and payment rules.

Allocation rule: For any terminal history $h(t) \equiv \langle \xi(t-1), p(t)\rangle \in \widetilde{\mathcal{H}}$ with $p(t) < \bar{p}$, the auctioneer chooses an (arbitrary) equilibrium allocation $(A^i)_{i \in N}$ and implements it; for other terminal histories, no one receives anything.

Payment rule: For any terminal history $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}$ with $p(t) < \bar{p}$, bidder i's ($i = 1, \ldots, n-1$) payment is determined together with the information stored in $h_{-i}(t_{-i}) \equiv \langle \xi_{-i}(t_{-i}-2), (p_{-i}(t_{-i}-1), \sigma^N_{-i}(t_{-i})), p_{-i}(t_{-i})\rangle$ and $h_{-n}(t_{-n})$ as

$$-\sum_{j \in N\setminus\{i\}} p(t)(A^j) + \sum_{j \in N\setminus\{i\}} p_{-i}(t_{-i})(A^j_{-i}) - \sum_{j \in N\setminus\{i\}} \sum_{s=1}^{t_{-i}-1} \hat{R}\Big(\underline{E}\big(\sigma^N_{-i}(s)\big)|\sigma^j_{-i}(s)\Big)$$
$$+ \sum_{j \in N\setminus\{i,n\}} \sum_{s=1}^{t_{-n}-1} \hat{R}\Big(\underline{E}\big(\sigma^N_{-n}(s)\big)|\sigma^j_{-n}(s)\Big) + \sum_{j \in N\setminus\{i\}} \sum_{s=t_{-i}}^{t-1} \hat{R}\Big(\underline{E}\big(\sigma^N(s)\big)|\sigma^j(s)\Big),$$

and bidder n's payment is

$$-\sum_{j \in N\setminus\{n\}} p(t)(A^j) + \sum_{j \in N\setminus\{n\}} p_{-n}(t_{-n})(A^j_{-n}) + \sum_{j \in N\setminus\{n\}} \sum_{s=t_{-n}}^{t-1} \hat{R}\Big(\underline{E}\big(\sigma^N(s)\big)|\sigma^j(s)\Big).$$

For other terminal histories, we assume that everyone pays nothing.

Importantly, the bidders' payments coincide with the VCG payments (see Ausubel (2006)). To see the underlying idea, recall from Section 4.3 that $\hat{R}(\cdot,\cdot)$ represents a decrease in utilities caused by the price change. Using this information, we can calculate how the bidders' total utilities change from a market with $n-1$ bidders to the market with n bidders. As is well known in the mechanism design literature, this utility gap corresponds to VCG payments. Defining strategies and ex post Nash equilibria as in Section 3.7, we obtain:

Proposition 2. *Sincere bidding by all bidders is an ex post Nash equilibrium.*

This proposition follows from the fact that the strategy space in our auction is narrower than that in Ausubel's (2006) auction.

5 Discussion

We discuss three related topics, which will be explored in more detail in future research.

5.1 Laboratory experiments

To understand the effects of introducing a new auction rule, it is helpful to conduct a laboratory experiment. We can compare various aspects of the auction outcomes in settings with and without restriction rules. For example, it might be interesting to compare the rate of sincere bidding.[18] A notable characteristic of our restriction rule is to display (part of) utility-maximizing items to bidders as long as the bidder behaves sincerely up until the going round. In this environment, bidders can identify utility-maximizing items easily, which seems to contribute to increasing the rate of sincere bidding.

5.2 Restrictions from the middle of the auction

In our analysis, restrictions are imposed on bidders from the beginning of the auction. To further exploit the advantage of price discovery, it would also be possible to impose restrictions from the middle of the auction.[19] More specifically, one can implement the following auction: bidders can withdraw any bid for a certain period, but after a predetermined round, prices are decreased until no excess supply occurs. Then, the bidders move on to the ascending phase subject to restrictions. It can be shown that this auction also reaches an equilibrium allocation on any termination.

[18] There are many studies on this issue; we refer the reader to a recent paper by Masuda et al. (2019) and the references therein.

[19] This idea is borrowed from the clock-proxy auction due to Ausubel et al. (2006); bidders can withdraw bids in the first half of the auction, but they are required to "finalize" their bids in later rounds.

5.3 Relationship to the DA algorithm

The celebrated *deferred acceptance algorithm* (Gale and Shapley 1962) forms the basis of matching theory. Putting the algorithm into the context of job-matching markets (see Kelso and Crawford (1982), Hatfield and Milgrom (2005), Echenique (2012)), its key feature is that a firm must repeat its offer to workers if their wages remain the same. Here, invariance of wages means that the workers have been affected more favorably than others whose wages have been increased. In other words, firms are prohibited from withdrawing offers that have been affected most favorably. From this perspective, one can view our auction as a variation of the DA algorithm that accommodates indifferences and anonymous prices.[20] More generally, it might be possible to deal with existing algorithms in a unified manner using the language of withdrawal restriction.

6 Proofs

6.1 Preliminaries

In this section, we consider the multi-demand setting in Section 4 (recall that $\Sigma \equiv \mathcal{B}\backslash\{\varnothing\}$).

6.1.1 Discrete convex analysis

For $A \subseteq K$, $k \in K\backslash A$ and $\ell \in A$, let $A + k \equiv A \cup \{k\}$, $A - \ell \equiv A\backslash\{\ell\}$, and $A + k - \ell \equiv (A + k) - \ell$. For an auxiliary simbol ϕ, let $A - \phi \equiv A$.

We say that $\sigma \in \mathcal{B}\backslash\{\varnothing\}$ is an **M$^\natural$-convex set** (Murota 2003) if, for any $A, A' \in \sigma$ and $k \in A\backslash A'$, there exists $\ell \in (A'\backslash A) \cup \{\phi\}$ such that

$$A - k + \ell \in \sigma,\ A' + k - \ell \in \sigma.$$

Proposition 3. *Suppose that v satisfies A2 and A3. Then, v satisfies A3*.*

Proof. This proposition follows from Fujishige and Yang (2003) (see also Remark 21 of Murota and Tamura (2003)).[21] □

[20] Although not as closely related to our study, the "clinching" rule in Ausubel's (2004) auction for homogeneous commodities can be interpreted as a kind of withdrawal restriction rule. To see this point, we draw attention to Bikhchandani and Ostroy's (2006) finding: Ausubel's (2004) auction with $|K|$ commodities can be regarded as an auction where each bidder i has $|K|$ copies, each of whom has a unit-demand preference with a monetary evaluation being equal to i's marginal utility. If a bidder i "clinches" an item, it is interpreted to mean that one of i's copies receives the item and exits the auction. We can reinterpret this situation in the context of withdrawal restriction: the copy is forced to bid on the item (whose price is kept unchanged) until the end of the auction.

[21] While these authors assume that the domain of price vectors is $\mathbb{R}^K$, in our current context, the domain

Proposition 4. *Let $\mathcal{H} \subseteq \mathcal{S}$ be a set of sequences that induces an extensive-form game. Then, the bidding rule (12) is satisfied in this game if and only if, for any $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \mathcal{H}$ and $i \in N$, i's demand report $\sigma^i(t-1)$ forms an M$^\natural$-convex set.*

Proof. The *only if* part follows from Fujishige and Yang (2003) and Theorem 6.30 of Murota (2003). To prove the *if* part, it suffices to prove that, for an arbitrarily chosen $p \in \mathbb{Z}^K$ and an M$^\natural$-convex set $\sigma \in \mathcal{B}\backslash\{\varnothing\}$, there exists a value function v such that v satisfies A3* and $\arg\max_{A\in\mathcal{K}} v[p](A) = \sigma$. We define $f_1, f_2 : \mathbb{Z}^K \to \mathbb{Z} \cup \{+\infty\}$ by

$$f_1(x) = \begin{cases} 0 & \text{if } x = \mathbb{1}^A \text{ for some } A \in \sigma, \\ +\infty & \text{otherwise,} \end{cases} \qquad f_2(x) = x^2 \text{ for all } x \in \mathbb{Z}^K.$$

These functions satisfy a notion of discrete convexity called M$^\natural$-convexity (see Section 6.3 of Murota (2003)). By Theorem 6.15 of Murota (2003), the infimal convlution g of f_1 and f_2 is an M$^\natural$-convex function and satisfies $g(x) = 0$ if $x = \mathbb{1}^A$ for some $A \in \sigma$ and $g(x) > 0$ otherwise. Define $v : \mathcal{K} \to \mathbb{Z}$ by $v(A) = -g(\mathbb{1}^A) + p(A)$ for all $A \in \mathcal{K}$. Again by Theorem 6.15 of Murota (2003) and Fujishige and Yang (2003), the desired condition holds for v. □

Two remarks are in order. First, in the counterexample given by (11), σ^1 is not M$^\natural$-convex, which, together with Proposition 4, means that the bidding rule is violated. Second, by Proposition 4 and the discrete conjugacy theorem (see Theorem 8.4 of Murota 2003), $E(\cdot|\sigma^N)$ is supermodular.

6.1.2 Characterization of equilibrium price vectors

For $A \in \mathcal{K}$ and $p \in \mathbb{Z}^K_+$, we define

$$I^+(A, p) = \{k \in A : p_k > 0\}.$$

Paralelling the min-requirement function (see (9)), for $\sigma \in \Sigma$, we define the **max-requirement function** as follows: $\check{R}(\cdot|\sigma) : \mathcal{K} \to \mathbb{Z}_+$ by

$$\check{R}(A|\sigma) = \max\big\{|B \cap A| : B \in \sigma\big\} \text{ for all } A \in \mathcal{K}. \tag{13}$$

For $A \in \mathcal{K}$ and $\sigma^N \in \Sigma^N$, let $\check{R}(A|\sigma^N) \equiv \sum_{i\in N} \check{R}(A|\sigma^i)$.

can be restricted to $\mathbb{Z}^K$. We will prove this point in a separate paper.

Proposition 5. *Let $p \in \mathbb{Z}_+^K$ and $\sigma^N \in \Sigma^N$. Then, there exists an equilibrium allocation with respect to p and σ^N if and only if*

$$|I^+(A,p)| \leq \check{R}(A|\sigma^N) \text{ and } \hat{R}(A|\sigma^N) \leq |A| \text{ for all } A \in \mathcal{K}.$$

Proof. The proof is by the discrete separation theorem; see Yokote (2020b). □

6.2 Proof of Theorem 1, Theorem 3 and Corollary 1

We prove Theorem 3 while omit the proofs of Theorem 1 and Corollary 1 which can be dealt with analogously. We consider the multi-demand setting in Section 4 (recall that $\Sigma \equiv \mathcal{B}\backslash\{\varnothing\}$). By Propositions 3 and 4, a bidder's demand set always forms an M$^\natural$-convex set. In the remaining part, fix $j \in N$.

Claim 1. *Let $h(t) \equiv \langle (p(1), \sigma^N(1)), \ldots, (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}_{-j}$ with $t \geq 3$, $i \in N\backslash\{j\}$ and $B \in \mathcal{K}$. Then,*

$$\check{R}(B|\sigma^i(t-1)) \geq \hat{R}(\underline{A}|\sigma^i(t-2)) - \hat{R}(\underline{A}\backslash B|\sigma^i(t-2)) + \check{R}(B\backslash\underline{A}|\sigma^i(t-2)), \tag{14}$$

where $\underline{A} \equiv \underline{E}(\sigma^N(t-2))$.

Proof.[22] If $\underline{A} = \emptyset$, then $p(t-2) = p(t-1)$, a contradiction to $h(t) \in \widetilde{\mathcal{H}}_{-j}$. Hence, $\underline{A} \neq \emptyset$. If $\tilde{\rho}(\underline{A}, \sigma^i(t-2)) = \varnothing$, by the definition of $\tilde{\rho}$ (see (10)),

$$\{A \in \sigma^i(t-2) : |A \cap \underline{A}| = \hat{R}(\underline{A}|\sigma(t-2))\} = \{\emptyset\}.$$

This means that

$$\emptyset \in \sigma^i(t-2) \text{ and } |A \cap \underline{A}| \geq 1 \text{ for all } A \in \sigma^i(t-2)\backslash\{\emptyset\}. \tag{15}$$

If $\check{R}(B\backslash\underline{A}|\sigma^i(t-2)) \geq 1$, then there exists $A \in \sigma^i(t-2)$ such that $A \cap (B\backslash\underline{A}) \neq \emptyset$. By M$^\natural$-convexity applied to $\emptyset, A \in \sigma^i(t-2)$ and $k \in (A\backslash\emptyset) \cap (B\backslash\underline{A})$, we have $\{k\} \in \sigma^i(t-2)$, a contradiction to (15). Hence, we must have $\check{R}(B\backslash\underline{A}|\sigma^i(t-2)) = 0$. In this case, together with $\emptyset \in \sigma^i(t-2)$, the right-hand side of (14) is equal to 0 and hence (14) immediately follows.

In the remaining part, suppose that $\tilde{\rho}(\underline{A}, \sigma^i(t-2)) \neq \varnothing$. Let $A_\circ \in \tilde{\rho}(\underline{A}, \sigma^i(t-2))$ be such that

$$|A_\circ \cap B| \geq |A \cap B| \text{ for all } A \in \tilde{\rho}(\underline{A}, \sigma^i(t-2)). \tag{16}$$

[22]We partly mimic the proof of Claim 3 of Yokote (2020a).

Since $A_\circ \in \tilde{\rho}(\underline{A}, \sigma^i(t-2)) \subseteq \sigma^i(t-1)$ and $\check{R}(B|\sigma^i(t-1))$ takes the maximum value among the bundles in $\sigma^i(t-1)$, we obtain $\check{R}(B|\sigma^i(t-1)) \geq |A_\circ \cap B|$. Hence, to prove the desired inequality, it suffices to prove that

$$|A_\circ \cap B| \geq \hat{R}(\underline{A}|\sigma^i(t-2)) - \hat{R}(\underline{A}\backslash B|\sigma^i(t-2)) + \check{R}(B\backslash \underline{A}|\sigma^i(t-2)).$$

Since $|A_\circ \cap B| = |A_\circ \cap \underline{A}| - |A_\circ \cap (\underline{A}\backslash B)| + |A_\circ \cap (B\backslash \underline{A})|$, the above inequality is rephrased as

$$\begin{aligned}&|A_\circ \cap \underline{A}| - |A_\circ \cap (\underline{A}\backslash B)| + |A_\circ \cap (B\backslash \underline{A})| \\ &\geq \hat{R}(\underline{A}|\sigma^i(t-2)) - \hat{R}(\underline{A}\backslash B|\sigma^i(t-2)) + \check{R}(B\backslash \underline{A}|\sigma^i(t-2)).\end{aligned}$$

Since $A_\circ \in \tilde{\rho}(\underline{A}, \sigma^i(t-2)) \subseteq \sigma^i(t-2)$ and $\hat{R}(\underline{A}|\sigma^i(t-2))$ takes the minimum value among the bundles in $\sigma^i(t-2)$, we obtain $|A_\circ \cap \underline{A}| \geq \hat{R}(\underline{A}|\sigma^i(t-2))$. Hence, to prove the above inequality, it suffices to prove the following:

$$|A_\circ \cap (\underline{A}\backslash B)| \leq \hat{R}(\underline{A}\backslash B|\sigma^i(t-2)), \tag{17}$$

$$|A_\circ \cap (B\backslash \underline{A})| \geq \check{R}(B\backslash \underline{A}|\sigma^i(t-2)). \tag{18}$$

Proof of (17): Suppose to the contrary that $|A_\circ \cap (\underline{A}\backslash B)| > \hat{R}(\underline{A}\backslash B|\sigma^i(t-2))$. Let $A_\bullet \in \sigma^i(t-2)$ be such that

$$\begin{aligned}&|A_\bullet \cap (\underline{A}\backslash B)| = \hat{R}(\underline{A}\backslash B|\sigma^i(t-2)), \\ &|A_\circ \backslash A_\bullet| \leq |A_\circ \backslash A| \text{ for all } A \in \sigma^i(t-2) \text{ with } |A \cap (\underline{A}\backslash B)| = \hat{R}(\underline{A}\backslash B|\sigma^i(t-2)).\end{aligned} \tag{19}$$

By the supposition, there exists $k \in (A_\circ \backslash A_\bullet) \cap (\underline{A}\backslash B)$. By M$^\natural$-convexity, there exists $\ell \in (A_\bullet \backslash A_\circ) \cup \{\phi\}$ such that

$$A_\circ - k + \ell \in \sigma^i(t-2),\ A_\bullet + k - \ell \in \sigma^i(t-2).$$

We consider two cases.

<u>Case 1:</u> Suppose $\ell \in \underline{A}\backslash B$. Since $k \in (A_\circ \backslash A_\bullet) \cap (\underline{A}\backslash B)$, we have

$$|(A_\bullet + k - \ell) \cap (\underline{A}\backslash B)| = |A_\bullet \cap (\underline{A}\backslash B)| \text{ and } |(A_\circ \backslash (A_\bullet + k - \ell)| < |A_\circ \backslash A_\bullet|,$$

a contradiction to (19).

<u>Case 2:</u> Suppose $\ell \notin \underline{A}\backslash B$.

<u>Subcase 2-1:</u> Suppose $\ell \notin \underline{A} \cap B$. Together with $k \in \underline{A}\backslash B$, we have $|(A_\circ - k + \ell) \cap \underline{A}| < |A_\circ \cap \underline{A}|$, a contradiction to $A_\circ \in \tilde{\rho}(\underline{A}, \sigma^i(t-2))$.

Subcase 2-2: Suppose $\ell \in \underline{A} \cap B$. In this case, $|(A_\circ - k + \ell) \cap \underline{A}| = |A_\circ \cap \underline{A}|$. Together with $|A_\circ - k + \ell| = |A_\circ|$, we have $A_\circ - k + \ell \in \tilde{\rho}(\underline{A}, \sigma^i(t-2))$. Moreover, by the choice of k and ℓ,

$$|(A_\circ - k + \ell) \cap B| > |A_\circ \cap B|,$$

a contradiction to (16).

Proof of (18): Suppose to the contrary that

$$|A_\circ \cap (B \backslash \underline{A})| < \check{R}(B \backslash \underline{A} | \sigma^i(t-2)).$$

Let $A_\bullet \in \sigma^i(t-2)$ be such that

$$\begin{aligned} &|A_\bullet \cap (B \backslash \underline{A})| = \check{R}(B \backslash \underline{A} | \sigma^i(t-2)), \\ &|A_\circ \backslash A_\bullet| \le |A_\circ \backslash A| \text{ for all } A \in \sigma^i(t-2) \text{ with } |A \cap (B \backslash \underline{A})| = \check{R}(B \backslash \underline{A} | \sigma^i(t-2)). \end{aligned} \tag{20}$$

By the supposition, there exists $k \in (A_\bullet \backslash A_\circ) \cap (B \backslash \underline{A})$. By M$^\natural$-convexity, there exists $\ell \in (A_\circ \backslash A_\bullet) \cup \{\phi\}$ such that

$$A_\bullet - k + \ell \in \sigma^i(t-2), \ A_\circ + k - \ell \in \sigma^i(t-2).$$

We consider two cases.

Case 1: Suppose $\ell \in B \backslash \underline{A}$. Since $k \in (A_\bullet \backslash A_\circ) \cap (B \backslash \underline{A})$, we have

$$|(A_\bullet - k + \ell) \cap (B \backslash \underline{A})| = |A_\bullet \cap (B \backslash \underline{A})| \text{ and } |(A_\circ \backslash (A_\bullet - k + \ell)| < |A_\circ \backslash A_\bullet|,$$

a contradiction to (20).

Case 2: Suppose $\ell \notin B \backslash \underline{A}$.

Subcase 2-1: Suppose $\ell \in \underline{A}$. Together with $k \in B \backslash \underline{A}$, we have $|(A_\circ + k - \ell) \cap \underline{A}| < |A_\circ \cap \underline{A}|$, a contradiction to $A_\circ \in \tilde{\rho}(\underline{A}, \sigma^i(t-2))$.

Subcase 2-2: The remaining possibility is that $\ell \notin B \cup \underline{A}$. In this case, $|(A_\circ + k - \ell) \cap \underline{A}| = |A_\circ \cap \underline{A}|$. If $\ell = \phi$, then $|A_\circ + k - \ell| > |A_\circ|$, a contradiction to $A_\circ \in \tilde{\rho}(\underline{A}, \sigma^i(t-2))$. Hence, $\ell \ne \phi$. We obtain $|A_\circ + k - \ell| = |A_\circ|$, which implies $A_\circ + k - \ell \in \tilde{\rho}(\underline{A}, \sigma^i(t-2))$. Moreover, by the choice of k and ℓ,

$$|(A_\circ + k - \ell) \cap B| > |A_\circ \cap B|,$$

a contradiction to (16). □

Claim 2. *Let $B \in \mathcal{K}$. Then, for any $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1)), p(t) \rangle \in \widetilde{\mathcal{H}}_{-j}$ with*

$t \geq 2$, *it holds that*

$$\check{R}(B|\sigma^N(t-1)) \geq |I^+(B, p(t-1))|.$$

Proof. We proceed by induction on t.

Induction base: If $t = 2$, then $p(t-1) = \mathbf{0}$. This means that $|I^+(B, p(t-1))| = 0$ and hence the claim holds.

Induction step: Suppose that the claim holds for all $h(t-1) \in \widetilde{\mathcal{H}}_{-j}$ and we prove the claim for $h(t) \equiv \langle \xi(t-3), (p(t-2), \sigma^N(t-2)), (p(t-1), \sigma^N(t-1)), p(t)\rangle \in \widetilde{\mathcal{H}}_{-j}$, where $t \geq 3$. Let $\underline{A} \equiv \underline{E}(\sigma^N(t-2))$.

Suppose to the contrary that $\check{R}(B|\sigma^N(t-1)) < |I^+(B, p(t-1))|$. By Claim 1 and the fact that j always bids on $\{\emptyset\}$,

$$|I^+(B, p(t-1))| > \hat{R}(\underline{A}|\sigma^N(t-2)) - \hat{R}(\underline{A}\backslash B|\sigma^N(t-2)) + \check{R}(B\backslash \underline{A}|\sigma^N(t-2)). \tag{21}$$

Since only the prices of items in $\underline{A}$ increase from $p(t-2)$ to $p(t-1)$, we have

$$\begin{aligned} |I^+(B, p(t-1))| &= |I^+(\underline{A}, p(t-1))| - |I^+(\underline{A}\backslash B, p(t-1))| + |I^+(B\backslash \underline{A}, p(t-1))| \\ &= |\underline{A}| - |\underline{A}\backslash B| + |I^+(B\backslash \underline{A}, p(t-2))|. \end{aligned} \tag{22}$$

By (21) and (22),

$$\begin{aligned} &|\underline{A}| - |\underline{A}\backslash B| + |I^+(B\backslash \underline{A}, p(t-2))| \\ &> \hat{R}(\underline{A}|\sigma^N(t-2)) - \hat{R}(\underline{A}\backslash B|\sigma^N(t-2)) + \check{R}(B\backslash \underline{A}|\sigma^N(t-2)), \\ 0 &> \Big[\hat{R}(\underline{A}|\sigma^N(t-2)) - |\underline{A}|\Big] - \Big[\hat{R}(\underline{A}\backslash B|\sigma^N(t-2)) - |\underline{A}\backslash B|\Big] \\ &+ \Big[\check{R}(B\backslash \underline{A}|\sigma^N(t-2)) - |I^+(B\backslash \underline{A}, p(t-2))|\Big]. \end{aligned} \tag{23}$$

By the induction hypothesis,

$$\check{R}(B\backslash \underline{A}|\sigma^N(t-2)) - |I^+(B\backslash \underline{A}, p(t-2))| \geq 0. \tag{24}$$

By (23) and (24),

$$0 > \Big[\hat{R}(\underline{A}|\sigma^N(t-2)) - |\underline{A}|\Big] - \Big[\hat{R}(\underline{A}\backslash B|\sigma^N(t-2)) - |\underline{A}\backslash B|\Big]. \tag{25}$$

We consider two cases.

Case 1: Suppose $\underline{A} \cap B = \emptyset$, which is equivalent to $\underline{A}\backslash B = \underline{A}$. Then, the right-hand side of (25) is equal to 0, which is impossible.

Case 2: Suppose $\underline{A} \cap B \neq \emptyset$. Then, (25) exhibits a contradiction to the fact that $\underline{A}$ maximizes $E(\cdot|\sigma^N(t-2))$. □

We resume the proof of Theorem 3. Let $h(t) \equiv \langle \xi(t-2), (p(t-1), \sigma^N(t-1), p(t) \rangle \in \widetilde{\mathcal{H}}_{-j}$ be a terminal history such that $p(t) < \bar{p}$. By the definition of the price-update rule, $\hat{R}(A|\sigma^N(t-1)) \leq |A|$ for all $A \in \mathcal{K}$. By Claim 2, $|I^+(A, p(t))| = |I^+(A, p(t-1))| \leq \check{R}(A \mid \sigma^N(t-1))$ for all $A \in \mathcal{K}$. By Proposition 5, we obtain the desired claim. □

6.3 Proof of Theorem 2

We consider the model setup in Section 3 (recall that $\Sigma \equiv \bar{\mathcal{K}} \backslash \{\varnothing\}$). Choose an arbitrary restriction rule ρ that is less restrictive than $\tilde{\rho}$. Then, there exists $(A_*, \sigma_*) \in (\mathcal{K}\backslash\{\emptyset\}) \times \Sigma$ such that

$$\rho(A_*, \sigma_*) \subsetneq \tilde{\rho}(A_*, \sigma_*). \tag{26}$$

Moreover, by the definition of restrictiveness, $\rho(A, \{\theta\}) \subseteq \tilde{\rho}(A, \{\theta\}) = \varnothing$ for all $A \in \mathcal{K}\backslash\{\emptyset\}$. Hence, if a bidder bids on $\{\theta\}$ at some round, then she can bid on any items in the next round. With this in mind, we consider two cases.

<u>**Case 1:**</u> Suppose $(\sigma_* \backslash \{\theta\}) \subseteq A_*$. If $\theta \in \sigma_*$, then $\tilde{\rho}(A_*, \sigma_*) = \varnothing$, a contradiction to (26). The remaining possibility is that $\theta \notin \sigma_*$.

Subcase 1-1: Suppose $\sigma_* = A_*$. Consider the bidding behavior given by the table below, where the rows represent rounds and the columns represent (i) tentative prices, (ii) bidders' demand reports σ^N (note: bidders not listed in the table are assumed to bid on $\{\theta\}$ in every round), and (iii) the items whose prices are updated (i.e., $\underline{E}(\sigma^N)$):

	$p(t)$	$1, \ldots, \lvert\sigma_*\rvert + 1$	$\underline{E}(\sigma^N)$
Round 1	$\mathbf{0}$	σ_*	σ_*
Round 2	$\mathbb{1}^{A_*} (= \mathbb{1}^{\sigma_*})$	$\rho(A_*, \sigma_*) \cup \{\theta\}$	$\emptyset$

By definition, $\tilde{\rho}(A_*, \sigma_*) = \sigma^*$. By (26), there exists $k \in \sigma_* \backslash \rho(A_*, \sigma_*)$. At the end of the game, the price of k is positive, but no one demands it. Hence, there exists no equilibrium allocation.

Subcase 1-2: Suppose $\sigma_* \subsetneq A_*$. Consider the following bidding behavior:

	$p(t)$	$1, \ldots, \lvert\sigma_*\rvert + 1$	$\lvert\sigma_*\rvert + 2, \ldots, \lvert A_*\rvert + 2$	$\underline{E}(\sigma_*^N)$
Round 1	$\mathbf{0}$	σ_*	$\{\theta\}$	σ_*
Round 2	$\mathbb{1}^{\sigma_*}$	σ_*	$A_*\backslash\sigma_*$	A_*
Round 3	$\mathbb{1}^{\sigma_*} + \mathbb{1}^{A_*}$	$\rho(A_*, \sigma_*) \cup \{\theta\}$	$(A_*\backslash\sigma_*) \cup \{\theta\}$	$\emptyset$

Then, at the end of the game, the price of an item $k \in \sigma_*\backslash\rho(A_*, \sigma_*)$ is positive, but no one demands it. Hence, there exists no equilibrium allocation.

Case 2: Suppose $(\sigma_*\backslash\{\theta\}) \nsubseteq A_*$.

Subcase 2-1: Suppose $\theta \notin \sigma_*$. Then, there exist $A' \subseteq A_*$ and $A'' \subseteq K\backslash A_*$ with $A'' \neq \emptyset$ such that $\sigma_* = A' \cup A''$. By definition, $\tilde{\rho}(A_*, \sigma_*) = A''$. By (26), there exists $k \in A''\backslash\rho(A_*, \sigma_*)$. Consider the following bidding behavior:

	$p(t)$	1	2	$3, \ldots, \lvert A_*\rvert + 3$	$\underline{E}(\sigma_*^N)$
Round 1	$\mathbf{0}$	$\{k\}$	$\{k\}$	$\{\theta\}$	$\{k\}$
Round 2	$\mathbb{1}^{\{k\}}$	σ_*	$\sigma_* \cup \{\theta\}$	A_*	A_*
Round 3	$\mathbb{1}^{\{k\}} + \mathbb{1}^{A_*}$	$\rho(A_*, \sigma_*) \cup \{\theta\}$	σ_*	A_*	A_*
Round 4	$\mathbb{1}^{\{k\}} + 2 \cdot \mathbb{1}^{A_*}$	$\rho(A_*, \sigma_*) \cup \{\theta\}$	$\rho(A_*, \sigma_*) \cup \{\theta\}$	$A_* \cup \{\theta\}$	$\emptyset$

Then, at the end of the game, the price of k is positive but no one demands it. Note that, by $\lvert A_*\rvert \leq \lvert K\rvert - 1$, it is sufficient to have $\lvert K\rvert + 2$ in order to realize the above bidding behavior.

Subcase 2-2: Suppose $\theta \in \sigma_*$. Then, there exist $A' \subseteq A_*$ and $A'' \subseteq K\backslash A_*$ with $A'' \neq \emptyset$ such that $\sigma_* = A' \cup A'' \cup \{\theta\}$. By definition, $\tilde{\rho}(A_*, \sigma_*) = A''$. By (26), there exists $k \in A''\backslash\rho(A_*, \sigma_*)$. Consider the following bidding behavior:

	$p(t)$	$1, \ldots, \lvert A''\rvert + 1$	$\lvert A''\rvert + 2, \ldots, \lvert A_*\rvert + \lvert A''\rvert + 2$	$\underline{E}(\sigma_*^N)$
Round 1	$\mathbf{0}$	A''	$\{\theta\}$	A''
Round 2	$\mathbb{1}^{A''}$	σ_*	A_*	A_*
Round 3	$\mathbb{1}^{A''} + \mathbb{1}^{A_*}$	$\rho(A_*, \sigma_*) \cup \{\theta\}$	$A_* \cup \{\theta\}$	$\emptyset$

Then, at the end of the game, the price of $k \in A''$ is positive but no one demands it.

Appendix

We complete the argument on the well-definedness of Ausubel's (2006) auction in Section 4.3. Consider the demand reports given by (11). In the decending phase, we calculate

$\check{R}(A|\sigma^i)$ (decrease in the indirect utilities for all A; recall the definition (13)) and $-|A| + \check{R}(A|\sigma^N)$ (decrease in the Lyapunov function) for all $A \in \mathcal{K}$.

A	$\check{R}(A\|\sigma^1)$	$\check{R}(A\|\sigma^2)$	$\check{R}(A\|\sigma^3)$	$-\|A\| + \check{R}(A\|\sigma^N)$
$\emptyset$	0	0	0	0
$\{k_1\}$	1	1	0	1
$\{k_2\}$	1	1	0	1
$\{k_3\}$	1	0	1	1
$\{k_4\}$	1	0	1	1
$\{k_1, k_2\}$	2	1	0	1
$\{k_1, k_3\}$	1	1	1	1
$\{k_1, k_4\}$	1	1	1	1
$\{k_2, k_3\}$	1	1	1	1
$\{k_2, k_4\}$	1	1	1	1
$\{k_3, k_4\}$	2	0	1	1
$\{k_1, k_2, k_3\}$	2	1	1	1
$\{k_1, k_2, k_4\}$	2	1	1	1
$\{k_1, k_3, k_4\}$	2	1	1	1
$\{k_2, k_3, k_4\}$	2	1	1	1
K	2	1	1	0

The minimizers of the Lyapunov function are

$$\left\{p, p - \mathbb{1}^K\right\},$$

where the maximal minimizer is p. Namely, the prices are not updated and the auction stops.

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