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Research Article

SOME FIXED POINT RESULTS FOR CONTINUOUS FUNCTIONS ON AN ARBITRARY INTERVALS

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ABSTRACT

In this paper, we first give a necessary and sufficient condition for convergence of Picard-S iteration process to a fixed point of continuous functions on an arbitrary interval and prove equivalence of Picard-S and P iterative processes. We also compare the rate of convergence between Picard-S and some others iteration processes in the literature. Finally, some numerical examples for comparing the rate of convergence of those methods are also given.
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1. INTRODUCTION AND PRELIMINARIES

Iterative methods are popular tools to approximate fixed points of nonlinear mappings. They are designed to be applied in solving equations arising in physical formulation but there is no systematic study of the numerical aspects of these iterative schemes. In computational mathematics, it is of vital interest to know which of the given iterative procedures converge faster to a desired solution, commonly known as the rate of convergence. Thus, when studied an iterative procedure, it should be considered two criteria which are the faster and the simplify. In this direction, some of notable studies were conducted by Mann, Ishikawa, Noor, Suantai, Karakaya, Gursoy, Dogan, Yildirim, Karahan, Sainuan, Agarwal, Rhoades and Khan [1-17]. In addition, the fixed point mappings were studied as much as studies on the iterative methods. Different varieties of these mappings are available in the literature. The well known of them, are contraction mappings, nonexpansive mappings and Lipschitzian mappings, and these are the continuous ones. Therefore, in this study, we handle the general mapping which is a class of continuous mapping.

Let E be a closed interval on the real line and $\wp: E \to E$ be a continuous function. A point $p \in E$ is a fixed point of $\wp$ if $\wp(p) = p$. We denote by $F(\wp)$ the set of fixed points of $\wp$.

Now, we will consider some of these schemes related to this work. The sequence $\{x_n\}_{n=1}^{\infty}$ defined by

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$$\begin{cases} x_1 \in E, \\ x_{n+1} = \wp(x_n), (n \in \mathbb{N}), \end{cases} \tag{1.1}$$

is called to Picard iterat ive process.

Mann [3] introduced Mann iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \wp(x_n), (n \in \mathbb{N}), \end{cases} \tag{1.2}$$

where $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$. In 1953, Mann showed that if $\wp$ is a continuous real function on a unit interval of the real line with a unique fixed point, then the Mann iteration converges to a unique fixed point of $\wp$.

Ishikawa [4] introduced Ishikawa iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \wp(y_n) \\ y_n = (1-\beta_n)x_n + \beta_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. Clearly, the Mann iteration process are special case of the Ishikawa iteration process.

M.A. Noor [5] introduced Noor iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n \wp(y_n) \\ y_n = (1-\alpha_n)x_n + \alpha_n \wp(z_n) \\ z_n = (1-\beta_n)x_n + \beta_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.4}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty} \in [0,1]$. Clearly, the Mann and Ishikawa iteration processes are special cases of the Noor iteration process.

In 1974, Rhoades [6] obtained the convergence result to Mann iteration for a class of continuous and nondecreasing functions on a closed unit interval, and then he [7] extended the results for Ishikawa iterations. After that in 1991, Borwein and Borwein [8] obtained the convergence result to Mann iteration for continuous functions on a bounded closed interval. Qing and Qihou [9] extended results in [8] to an arbitrary interval and to Ishikawa iteration and presented a necessary and sufficient condition for the convergence of Ishikawa iteration of continuous functions on an arbitrary interval.

Phuengrattana and Suantai [5] introduced SP iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n \wp(y_n) \\ y_n = (1-\alpha_n)z_n + \alpha_n \wp(z_n) \\ z_n = (1-\beta_n)x_n + \beta_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty} \in [0,1]$.

They presented a necessary and sufficient condition for the convergence of SP-iteration [10] of continuous functions on an arbitrary interval. Also, they compared the rate of convergence of Mann, Ishikawa, Noor iterations and SP-iteration by numerical examples and concluded that SP-iteration converges faster than all of them.

In 2007, Agarwal et all [11] introduced S iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\alpha_n)\wp(x_n) + \alpha_n \wp(y_n) \\ y_n = (1-\beta_n)x_n + \beta_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.6}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. They proved some convergence theorems for uniformly continuous nearly asymptotically nonexpansive mappings. However, I. Karahan [18] proved some convergence theorems for S-iteration method for continuous functions defined on an arbitrary interval.

In 2013, Karakaya et al. [12] introduced a new three step iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\gamma_n - a_n)y_n + a_n \wp(y_n) + \gamma_n \wp(z_n) \\ y_n = (1-\alpha_n - b_n)z_n + b_n \wp(x_n) + \alpha_n \wp(z_n) \\ z_n = (1-\beta_n)x_n + \beta_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.7}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$and $\{\gamma_n\}_{n=1}^{\infty} \in [0,1]$.

In 2013, Kadioglu and Yildirim, [13] introduced KY iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\gamma_n - a_n)x_n + a_n \wp(y_n) + \gamma_n \wp(z_n) \\ y_n = (1-\alpha_n - b_n)x_n + b_n \wp(x_n) + \alpha_n \wp(z_n) \\ z_n = (1-\beta_n)x_n + \beta_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.8}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$and $\{\gamma_n\}_{n=1}^{\infty} \in [0,1]$.

In 2013, Khan [14] introduced Picard-Mann Hybrid iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = \wp(y_n) \\ y_n = (1-\alpha_n)x_n + \alpha_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.9}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. In [15], Karahan and Ozdemir presented a necessary and sufficient condition for the convergence of the PMH-iterative process of continuous nondecreasing functions on an arbitrary interval. They also gave the numerical examples for the PMH-iterative process to compare with the Mann, Ishikawa, Noor and SP iterative processes and concluded that PMH-iterative pracess converges faster than the others.

Gürsoy and Karakaya [16] introduced Picard-S iterative process as follows:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = \wp(y_n) \\ y_n = (1-\alpha_n)\wp(x_n) + \alpha_n \wp(z_n) \\ z_n = (1-\beta_n)x_n + \beta_n \wp(x_n) (n \in \mathbb{N}), \end{cases} \tag{1.10}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$.

Recently, P. Sainuan [17] introduced P- iterative process as follows:

$$\begin{cases} x_1 \in E, \\ s_{n+1} = (1-\gamma_n)\wp(p_n) + \gamma_n \wp(r_n) \\ p_n = (1-\alpha_n)r_n + \alpha_n \wp(r_n) \\ r_n = (1-\beta_n)s_n + \beta_n \wp(s_n) (n \in \mathbb{N}), \end{cases} \tag{1.11}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty} \in [0,1]$.

In this paper, we give a necessary and sufficient condition for the convergence of the Picard-S iterative process of continuous functions on an arbitrary interval. We also prove that the Mann, Ishikawa, Noor, SP, S, PMH, KDGE (1.7), KY, P and SP- iterative processes are equivalent and the Picard–S iterative process converges faster than the others for the class of continuous and nondecreasing functions. We also compare the rate of convergence of them by numerical examples.

Now, we will give some useful Lemmas, Definitions and Theorems for proofs of our main results.

Lemma 1 [17] Let E be a closed interval on the real line and $\wp\colon \mathrm{E} \to \mathrm{E}$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be defined by (1.11). Then the following hold:

i) If $\wp(x_1) < x_1$, then $\wp(x_n) \leq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-increasing.
ii) If $\wp(x_1) > x_1$, then $\wp(x_n) \geq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-decreasing.

Definition 1 [10] Let E be a closed interval on the real line and $\wp: E \to E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ two iterative processes which converge to the fixed point q of $\wp$. Then, $\{x_n\}_{n=1}^{\infty}$ is said to converge faster than $\{s_n\}_{n=1}^{\infty}$ if

$$|x_n - q| \leq |s_n - q| \quad \text{for all } n \in \mathbb{N}.$$

Theorem 1 [17] Let E be a closed interval on the real line and $\wp: E \to E$ be a continuous, non-decreasing function such that $F(\wp)$ is nonempty and bounded and $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. Also for the initial values $x_1 = s_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ be defined by (1.11) and (1.6), respectively. If $\{s_n\}_{n=1}^{\infty}$ iterative process converges to a fixed point $q \in F(\wp)$, then $\{x_n\}_{n=1}^{\infty}$ itertive process converges to same fixed point $q \in F(\wp)$. Moreover, the P- iterative process converges faster than the S- iterative process.

2. MAIN RESULTS

Lemma 2 Let E be a closed interval on the real line and $\wp: E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be defined by (1.10). Then the followings hold:

i) If $\wp(x_1) < x_1$, then $\wp(x_n) \leq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-increasing.
ii) If $\wp(x_1) > x_1$, then $\wp(x_n) \geq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-decreasing.

Proof. (i)Let $\wp(x_1) < x_1$. Assume that $\wp(x_k) \leq x_k$ for $k > 1$. Then by (1.10) we gate $\wp(x_k) \leq z_k \leq x_k$. Since $\wp$ non-decreasing, we have $\wp(z_k) \leq \wp(x_k) \leq z_k \leq x_k$. Again using the same arguments, we obtain

$$\wp(z_k) \leq y_k \leq \wp(x_k) \leq z_k \leq x_k$$

$$\wp(y_k) \leq \wp(z_k) \leq y_k \leq \wp(x_k) \leq z_k \leq x_k$$

and since

$$x_{k+1} = \wp(y_k),$$

we have

$$x_{k+1} = \wp(y_k) \leq \wp(z_k) \leq y_k \leq \wp(x_k) \leq z_k \leq x_k.$$

Using the non-decreasing property of $\wp$, we obtain

$$\wp(x_{k+1}) \leq x_{k+1} = \wp(y_k) \leq \wp(z_k) \leq y_k \leq \wp(x_k) \leq z_k \leq x_k. \tag{2.1}$$

By the induction, we have

$$\wp(x_n) \leq x_n.$$

Hence

$$\wp(y_n) \leq \wp(z_n) \leq \wp(x_n).$$

Considering (2.1), we can conclude that

$$x_{n+1} = \wp(y_n) \leq \wp(z_n) \leq \wp(x_n) \leq x_n, \text{for all } n \in \mathbb{N}.$$

Therefore, the sequence $(x_n)_{n=1}^{\infty}$ is non-increasing.

(ii) By using the same argument as in (i), We obtain the desired result.

Theorem 2 Let E be a closed interval on the real line and $\wp: E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be defined by (1.10). Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of $\wp$.

Proof. Assume that $(x_n)_{n=1}^{\infty}$ is bounded. If $\wp(x_1) = x_1$, we have

$$x_2 = \wp(y_1) = x_1$$
$$y_1 = (1-\alpha_1)\wp(x_1) + \alpha_1\wp(z_1) = x_1$$
$$z_1 = (1-\beta_1)x_1 + \beta_1\wp(x_1) = x_1.$$

It is clear that $x_n = x_1$ and $\lim_{n\to\infty} x_n = x_1$, for all $n \geq 1$.

If $\wp(x_1) < x_1$ or $\wp(x_1) > x_1$, then, by Lemma (2), we obtain that $(x_n)_{n=1}^{\infty}$ is non-increasing or non-decreasing. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, it implies that $\{x_n\}_{n=1}^{\infty}$ is convergent.

Since $(x_n)_{n=1}^{\infty}$ is convergent, there is a $\lim_{n\to\infty} x_n = q \in E$. Using the continuity of $\wp$ and boundedness of $(x_n)_{n=1}^{\infty}$, we obtain $\{\wp(x_n)\}_{n=1}^{\infty}$ is bounded. In addition, Picard-S iteration method can be edited as follows:

$$x_{n+1} = \wp(y_n)$$
$$y_n - \wp(x_n) = \alpha_n[\wp(z_n) - \wp(x_n)]$$
$$z_n - x_n = \beta_n[\wp(x_n) - x_n].$$

We show in two steps that q is a fixed point of $\wp$.

Step 1 If $\wp(x_1) < x_1$, then $\wp(x_n) \leq x_n$ for all $n \geq 1$ and since $\lim_{n\to\infty} x_n = q \in E$, it is clear that $\lim_{n\to\infty}\wp(x_n) = \wp(q) \leq \lim_{n\to\infty} x_n = q \in E$. Also, the following inequality was obtained by Lemma (2)

$$x_{n+1} = \wp(y_n) \leq \wp(z_n) \leq \wp(x_n), \qquad \text{for all } n \in \mathbb{N}.$$

Hence

$$q = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\wp(y_n) \leq \lim_{n\to\infty}\wp(z_n) \leq \lim_{n\to\infty}\wp(x_n) = \wp(q).$$

It contradicts our assumption. Therefore, $\wp(q) = q$.

Step 2 If $\wp(x_1) > x_1$, then $\wp(x_n) \geq x_n$ for all $n \geq 1$ and since $\lim_{n\to\infty} x_n = q \in E$, it is clear that $\lim_{n\to\infty} x_n = q \leq \lim_{n\to\infty}\wp(x_n) = \wp(q) \in E$. Also, the following inequalty was obtained by Lemma (2)

$$x_{n+1} = \wp(y_n) \geq \wp(z_n) \geq \wp(x_n), \text{forall} n \in \mathbb{N}.$$

Hence

$$q = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\wp(y_n) \geq \lim_{n\to\infty}\wp(z_n) \geq \lim_{n\to\infty}\wp(x_n) = \wp(q).$$

It contradicts our assumption. Therefore, $\wp(q) = q$.

Hence q is a fixed point of $\wp$ and $\{x_n\}_{n=1}^{\infty}$ converge to q.

Lemma 3 Let E be a closed interval on the real line and $\wp\colon \mathrm{E} \to \mathrm{E}$ be a continuous, non-decreasing function and $(\alpha_n)_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. Also for the initial value $x_1 \in \mathrm{E}$, let $\{x_n\}_{n=1}^{\infty}$ be defined by (1.10). Then, the following assertions are true.

- **i)** If $q \in F(\wp)$ with $x_1 > q$, then $x_n \geq q$ for all $n \geq 1$.
- **ii)** If $q \in F(\wp)$ with $x_1 < q$, then $x_n \leq q$ for all $n \geq 1$.

Proof. (i) By using our claim and $\wp$'s non-decreasing, we obtain $\wp(x_1) \geq \wp(q)$. By Picard-S iterative process, we have

$$z_1 = (1-\beta_1)x_1 + \beta_1\wp(x_1) \geq (1-\beta_1)q + \beta_1\wp(q) = q$$
$$y_1 = (1-\alpha_1)\wp(x_1) + \alpha_1\wp(z_1) \geq (1-\alpha_1)\wp(q) + \alpha_1\wp(q) = q.$$

They imply that $\wp(z_1) \geq \wp(q)$ and $\wp(y_1) \geq \wp(q)$. Again we rehandle the Picard-S iterative process, we obtain

$$x_2 = \wp(y_1) \geq \wp(q) = q.$$

Suppose that $x_k \geq q$ for $k > 2$. Then $\wp(x_k) \geq \wp(q) = q$.

By Picard-S iterative process, we have

$$z_k = (1-\beta_k)x_k + \beta_k\wp(x_k) \geq (1-\beta_k)q + \beta_k\wp(q) = q$$
$$y_k = (1-\alpha_k)\wp(x_k) + \alpha_k\wp(z_k) \geq (1-\alpha_k)\wp(q) + \alpha_k\wp(q) = q.$$

Thus $\wp(z_k) \geq \wp(q) = q$ and $\wp(y_k) \geq \wp(q) = q$. Also, we obtain

$$x_{k+1} = \wp(y_k) \geq \wp(q) = q.$$

By induction, we have

$$x_n \geq p, \qquad \text{for all } n \geq 1.$$

(ii) Using the same arguments in (i), it can easily be shown that this assertion is correct. For this reason, the proof will not be given.

Lemma 4 Let E be a closed interval on the real line and $\wp\colon \mathrm{E} \to \mathrm{E}$ be a continuous, non-decreasing function and $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$. Also for the initial values $x_1 = s_1 \in \mathrm{E}$, let $\{x_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ be defined by (1.10) and (1.11), respectively. Then, the following assertions are true.

i) If $\wp(s_1) < s_1$, then $x_n \leq s_n$ for all $n \geq 1$.
ii) If $\wp(s_1) > s_1$, then $x_n \geq s_n$ for all $n \geq 1$.

Proof. (i) Since $x_1 = s_1$ we have $\wp(x_1) < x_1$. By (1.10) and $\wp$ is non-decreasing, we obtain

$$\wp(z_1) \leq \wp(x_1) \leq z_1 \leq x_1$$

and

$$\wp(y_1) \leq \wp(z_1) \leq \wp(x_1) \leq z_1 \leq x_1.$$

From Lemma 3.1 in the [17], we have

$$\wp(p_n) \leq \wp(r_n) \leq \wp(s_n) \leq r_n \leq s_n.$$

Thus, by (1.10) and (1.11), we get

$$z_1 - s_1 = (1-\beta_1)x_1 + \beta_1\wp(x_1) - s_1 \leq 0$$

which implies that $z_1 \leq s_1$. That is, $\wp(y_1) \leq \wp(z_1) \leq \wp(s_1)$,

$$z_1 - r_1 = (1-\beta_1)x_1 + \beta_1\wp(x_1) - (1-\beta_1)s_1 - \beta_1\wp(s_1) = 0$$

which implies that $z_1 = r_1$. That is, $\wp(y_1) \leq \wp(z_1) = \wp(r_1)$,

$$\begin{aligned} y_1 - p_1 &= (1-\alpha_1)\wp(x_1) + \alpha_1\wp(z_1) - (1-\alpha_1)r_1 - \alpha_1\wp(r_1) \\ &= (1-\alpha_1)(\wp(x_1) - r_1) \leq 0 \end{aligned}$$

which implies that $y_1 \leq p_1$. That is, $\wp(y_1) \leq \wp(p_1) \leq \wp(r_1)$ and

$$\begin{aligned} x_2 - s_2 &= \wp(y_1) - (1-\lambda_1)\wp(p_1) - \lambda_1\wp(r_1) \\ &\leq \wp(y_1) - (1-\lambda_1)\wp(p_1) - \lambda_1\wp(p_1) \\ &\leq \wp(y_1) - \wp(p_1) \\ &\leq 0 \end{aligned}$$

which implies that $x_2 \leq s_2$. That is, $\wp(x_2) \leq \wp(s_2)$.

We suppose that $x_k \leq s_k$, for $k \in \mathbb{N}$. Then $\wp(x_k) \leq \wp(s_k)$. From Lemma 3.1 in [17], we have $\wp(s_k) \leq s_k$ and from Lemma 2, we have $\wp(x_k) \leq x_k$. This follows that

$$\wp(x_k) \leq z_k \leq x_k \leq s_k.$$

From properties of $\wp$, we get

$$\wp(y_k) \leq \wp(z_k) \leq \wp(s_k).$$

Also, by (1.10) and (1.11), we get

$$\begin{aligned} y_k - p_k &= (1-\alpha_k)\wp(x_k) + \alpha_k\wp(z_k) - (1-\alpha_k)r_k - \alpha_k\wp(r_k) \\ &\leq (1-\alpha_k)(\wp(x_k) - r_k) \end{aligned}$$

$$\leq 0$$

which implies that $y_k \leq p_k$. That is, $\wp(y_k) \leq \wp(p_k) \leq \wp(r_k)$ and

$$\begin{aligned} x_{k+1} - s_{k+1} &= \wp(y_k) - (1-\lambda_k)\wp(p_k) - \lambda_k \wp(r_k) \\ &\leq \wp(y_k) - (1-\lambda_k)\wp(p_k) - \lambda_k \wp(p_k) \\ &\leq \wp(y_k) - \wp(p_k) \\ &\leq 0 \end{aligned}$$

we conclude that $x_{k+1} \leq s_{k+1}$. That is, $\wp(x_{k+1}) \leq \wp(s_{k+1})$.

By induction, we obtain the desired result $x_n \leq s_n$, for all $n \geq 1$.

(ii) Using the same arguments in (i), we can easily show that this assertion. For this reason, the proof of (ii) will not be given.

The next proposition shows that the convergence of Picard–S iterative process depends on how far the initial point from the fixed point set.

Proposition 1 Let E be a closed interval on the real line and $\wp\colon \mathrm{E} \to \mathrm{E}$ be a continuous, non-decreasing function and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \in [0,1]$. Also for the initial values $x_1 \in \mathrm{E}$, let $\{x_n\}_{n=1}^{\infty}$ be defined by (1.10). Then, the following assertions are true.

i) $F(\wp)$ is nonempty and bounded with $x_1 < \inf\{q \in E\colon q = \wp(q)\}$. If $\wp(x_1) < x_1$, then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by Picard–S iterative process does not converge to a fixed point of $\wp$.

ii) $F(\wp)$ is nonempty and bounded with $x_1 > \sup\{q \in E\colon q = \wp(q)\}$. If $\wp(x_1) > x_1$, then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by Picard–S iterative process does not converge to a fixed point of $\wp$.

Proof. If $\wp(a) = a$ or $\wp(b) = b$, we are done. Otherwise, $\wp(a) > a$ and $\wp(b) < b$. Consider the function $g(x) = \wp(x) - x$. Then $g(a) > 0$ while $g(b) < 0$. By the Intermediate Value Theorem, since g is also continuous, there exists $x_0 \in [a,b]$ such that $g(x_0) = 0$, or $\wp(x_0) = x_0$. Thus, we have $F(\wp) \neq \emptyset$. Since $\wp(x) \in [a,b]$ for all $n \in N$ and $[a,b]$ is bounded, then $F(\wp)$ is bounded too.)

Now we will show that the (i) and (ii) claims are provided.

(i) By Lemma 2 and by assertion of (i), $\{x_n\}_{n=1}^{\infty}$ is non-increasing and $x_1 < \inf\{q \in E\colon q = \wp(q)\}$, respectively. Therefore, q cannot be one of the term of the sequence $\{x_n\}_{n=1}^{\infty}$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by Picard–S iterative process does not converge to a fixed point of $\wp$.

(ii) By Lemma 2 and by assertion of (i), $\{x_n\}_{n=1}^{\infty}$ is non-decreasing and $x_1 > \sup\{q \in E\colon q = \wp(q)\}$, respectively. Therefore, q cannot be one of the term of the sequence $\{x_n\}_{n=1}^{\infty}$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by Picard–S iterative process does not converge to a fixed point of $\wp$.

Theorem 3 *Let E be a closed interval on the real line and $\wp\colon E \to E$ be a continuous, non-decreasing function such that $F(\wp)$ is nonempty and bounded and $\{\lambda_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \in [0,1]$. Also for the initial values $x_1 = s_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ be defined by (1.10) and (1.11) respectively. If $\{x_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ converge to same fixed point $q \in F(\wp)$, then, the Picard–S iterative process converges faster than the P- iterative process.*

Proof. In [17], it was shown that P iterative process converges to fixed point of $\wp$. Let $k = \inf\{q \in E\colon q = \wp(q)\}$ and $t = \sup\{q \in E\colon q = \wp(q)\}$. Our proof will be analyzed in three cases.

Case 1: Let $t < x_1 = s_1$. From Proposition 1, we get $\wp(x_1) < x_1$ and $\wp(s_1) < s_1$. From Lemma 4 (i), we have $x_n \leq s_n$ for all $n \geq 1$. By Using Picard-S iterative process and mathematical induction, we can show that $t \leq x_n$. Thus, we obtain

$$q \leq x_n \leq s_n,$$

so

$$|x_n - q| \leq |s_n - q|$$

for all $n \geq 1$. That is, $\{x_n\}_{n=1}^{\infty}$ itartive process converges to $q \in F(\wp)$ faster than $\{s_n\}_{n=1}^{\infty}$ iterative process.

Case 2: Let $k > x_1 = s_1$. From Proposition 1, we get $\wp(x_1) > x_1$ and $\wp(s_1) > s_1$. From Lemma 4 (ii), we have $x_n \geq s_n$ for all $n \geq 1$. By Using Picard-S iterative process and mathematical induction, we can show that $x_n \leq k$. Thus, we obtain

$$s_n \leq x_n \leq q,$$

so

$$0 \leq |x_n - q| \leq |s_n - q|$$

for all $n \geq 1$. It follows that $\{x_n\}_{n=1}^{\infty}$ itartive process converges to $q \in F(\wp)$ faster than $\{s_n\}_{n=1}^{\infty}$ iterative process.

Case 3: Let $t < x_1 = s_1 < k$. Suppose that $\wp(x_1) \neq x_1$. If $\wp(x_1) < x_1$, then by Lemma 3.4 in [17] we get that $\{s_n\}_{n=1}^{\infty}$ iterative process is non-increasing with limit q. This implies that $q \leq s_n$ for all $n \geq 1$. From Lemma 4 and Lemma 3, we obtain $q \leq x_n \leq s_n$. That is,

$$0 \leq |x_n - q| \leq |s_n - q|$$

it follows that $\{x_n\}_{n=1}^{\infty}$ iterative process converges to $q \in F(\wp)$ faster than $\{s_n\}_{n=1}^{\infty}$ iterative process.

Assume that $\wp(x_1) > x_1$, then by Lemma 3.4 in [17] we get that $\{s_n\}_{n=1}^{\infty}$ iterative process is non-decraisng with limit q. This implies that $s_n \leq q$ for all $n \geq 1$. From Lemma 4 and Lemma 3, we obtain $s_n \leq x_n \leq q$. That is,

$$0 \leq |x_n - q| \leq |s_n - q|$$

it follows that $\{x_n\}_{n=1}^{\infty}$ itartive process converges to $q \in F(\wp)$ faster than $\{s_n\}_{n=1}^{\infty}$ iterative process.

Example 1 [15] Let $\wp: [0,4] \to [0,4]$ defined by $\wp(\mathrm{x}) = \frac{\mathrm{x}^2 + 2\sqrt{\mathrm{x}} + 5}{8}$. Then, it is clear that the function $\wp$ is continuous and nondecreasing with the fixed point $\mathrm{q} = 1$. In the following tables, the comparison of the convergences for the Picard-S, P, Noor, SP, Mann, Ishikawa, S, KY, KDGE and Picard-Mann iterative processes are given with the initial value $\mathrm{x}_1 = \mathrm{y}_1 = \mathrm{z}_1 = \mathrm{w}_1 = \mathrm{v}_1 = \mathrm{s}_1 = \mathrm{k}_1 = \mathrm{m}_1 = \mathrm{t}_1 = \mathrm{l}_1 = 3$ and the sequences $\alpha_n = \beta_n = \gamma_n = \frac{1}{n^2+1}$ and $a_n = b_n = \frac{1}{n+1}$. we see that Picard-S iteration process converges to $\mathrm{q} = 1$ faster than the others.

Table 1. Comparison rate of convergence among some iteration methods

x_n	Picard	Picard-S	$x_n - \wp(x_n)$	KY
x_1	3	3	3	3
x_2	1,743193394231300	1,493320921813210	0,816987298107781	1,680131487141580
x_3	1,284893693492310	1,080486065149830	0,284066223114702	1,436450705945190
…	…	…	…	…
x_{20}	1,000000039283770	1,000000000000000	0,000000000000009	1,108877680153870
x_{21}	1,000000014705910	1,000000000000000	0,000000000000001	1,105276143812450
x_{22}	1,000000005506100	1,000000000000000	0,000000000000000	1,101969996245930
	…	…	…	…
x_{37}	1,000000000000000	1,000000000000000	0,000000000000000	1,072000399357590
…	…	…	…	…
x_{2000}	1,000000000000000	1,000000000000000	0,000000000000000	1,005785688451760

Table 1 shows that Picard-S iteration reaches the fixed point at the 20 th step while Picard iterative method reach 37 thstep. Also, KY iterative method can not reach to fixed point when we took it to 2000th step.

Table 2. Comparison rate of convergence among some iteration methods

x_n	KDGE	SP	*Noor*	S
x_1	3	3	3	3
x_2	1,680131487141580	1,924334468850340	2,377606159933700	2,024977377181080
…	…	…	…	…
x_{38}	1,007950056477930	1,356067810399450	2,018874358352420	1,000000000000000
…	…	…	…	…
x_{2000}	1,000051985649190	1,339494412884360	2,004720715565150	1,000000000000000

Table 2 shows that S iteration reaches the fixed point at the 38 th step while KDGE, SP and Noor iterative methods can not reach to fixed point when we took to the values in 2000th step.

Table 3. Comparison rate of convergence among some iteration methods

x_n	Picard-Mann	Ishikawa	Mann
x_1	3	3	3
x_2	1,866942052469940	2,433471026234970	2,433471026234970
…	…	…	…
x_{37}	1,000000000000000	2,066367297691530	2,220142312185830
…	…	…	…
x_{2000}	1,000000000000000	2,051282598845450	2,203449635776570

Table 3 shows that Picard-Mann iteration reaches fixed point at the 37 th step while Ishikawa and Mann iterative methods can not reach to fixed point when we took to the values in 2000th step.

The following figures are graphical presentations of the above results.

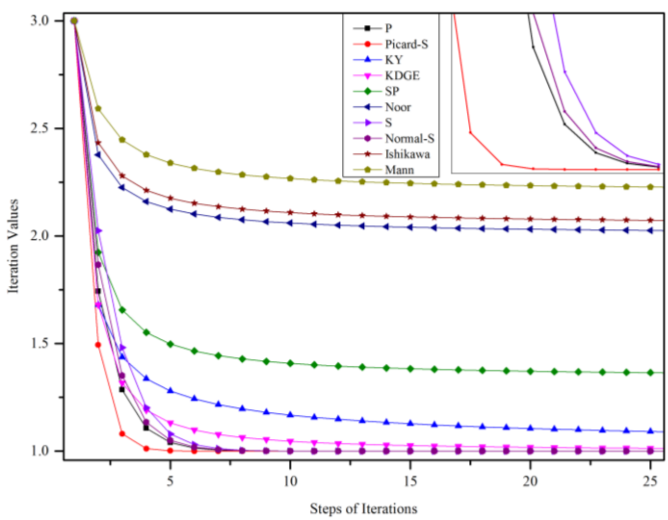

Figure 1. Comparison the rate of convergence of the iterative schemes

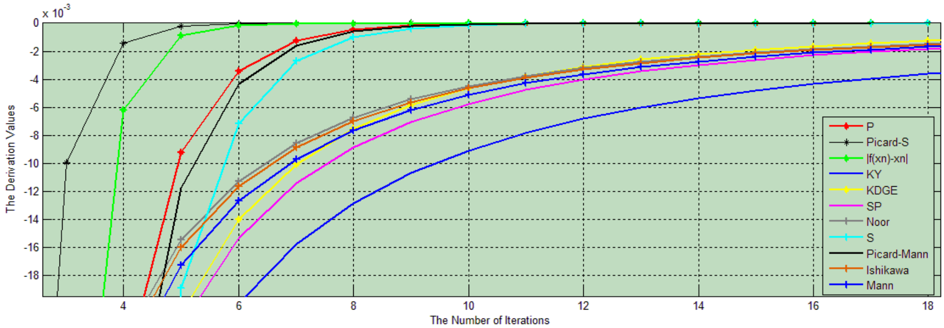

Figure 2. Comparison of the variation between the successive steps of the iteration methods

Example 2 Let $\wp: [0,4] \rightarrow [0,4]$ defined by $\wp(x) = \frac{x^2 - \sqrt{x} + 5}{5}$. Then, it is clear that the function $\wp$ is continuous and nondecreasing with the fixed point $q = 1$. In Table 1, the comparison of the convergences for the Picard-S, P, Noor, SP, Mann, Ishikawa, S, KY, KDGE and Picard-Mann iterative processes are given with the initial value $x_1 = y_1 = z_1 = w_1 = v_1 = s_1 = k_1 = m_1 = t_1 = l_1 = 3$ and the sequences $\alpha_n = \beta_n = \gamma_n = \frac{1}{n^2+1}$ and $a_n = b_n = \frac{1}{n+1}$. we see that Picard-S iteration process converges to $q = 1$ faster than the others.

Table 4. Comparison rate of convergence among some iteration methods

x_n	Picard	Picard-S	$x_n - \wp(x_n)$	KY
x_1	3	3	3	3
x_2	2,018390920190360	1,759136735931020	0,546410161513776	1,957979323347730
x_3	1,448279406433830	1,129285550067650	0,405489229060364	1,635704266021610
…	…	…	…	…
x_{17}	1,000000088521140	1,000000000000000	0,000000000000029	1,170683966268190
x_{18}	1,000000026483730	1,000000000000000	0,000000000000003	1,163443203639200
x_{19}	1,000000007925880	1,000000000000000	0,000000000000000	1,156912923643230
…	…	…	…	…
x_{31}	1,000000000000000	1,000000000000000	0,000000000000000	1,109326153000230
…	…	…	…	…
x_{2000}	1,000000000000000	1,000000000000000	0,000000000000000	1,005819415489230

Table 4 shows that Picard-S iteration reaches the fixed point at the 17 th step while Picard iterative method reach 31 thstep. Also, KY iterative method can not reach to fixed point when we took it to 2000th step.

Table 5. Comparison rate of convergence among some iteration methods

x_n	KDGE	SP	$Noor$	S
x_1	3	3	3	3
x_2	1,957979323347730	2,160884172310030	2,504245993858570	2,305205874625330
…	…	…	…	…
x_{32}	1,011245858126640	1,483936011677190	2,171848713776710	1,000000000000000
…	…	…	…	…
x_{2000}	1,000031094935700	1,456784715385330	2,155013135026180	1,000000000000000

Table 5 shows that S iteration reaches the fixed point at the 32 th step while KDGE, SP and Noor iterative methods can not reach to fixed point when we took to the values in 2000th step.

Table 6. Comparison rate of convergence among some iteration methods

x_n	Picard-Mann	Ishikawa	Mann
x_1	3	3	3
x_2	2,156821910764440	2,578410955382220	2,726794919243110
…	…	…	…
x_{32}	1,000000000000000	2,244732325255310	2,417811816780250
…	…	…	…
x_{2000}	1,000000000000000	2,227456768974440	2,399780047518960

Table 3 shows that Picard-Mann iteration reaches fixed point at the 32 th step while Ishikawa and Mann iterative methods can not reach to fixed point when we took to the values in 2000th step..

The following figures are graphical presentations of the above results:

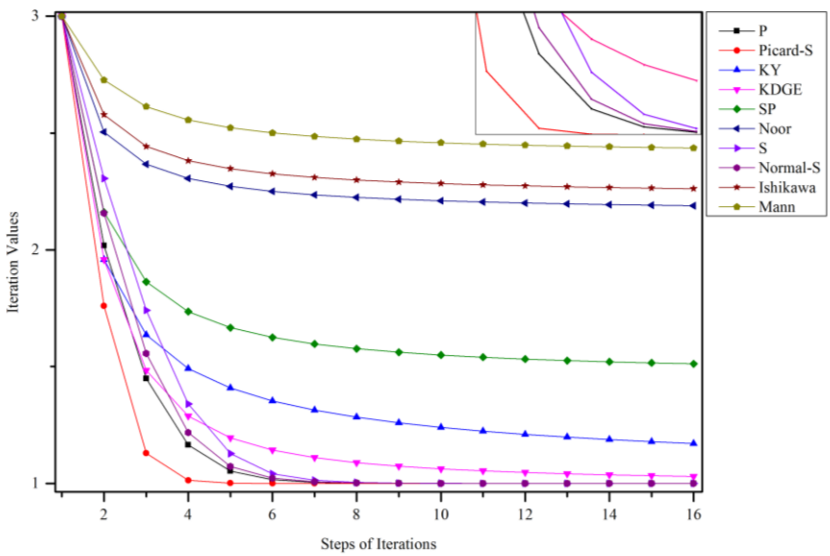

Figure 3. Comparison the rate of convergence of the iterative schemes

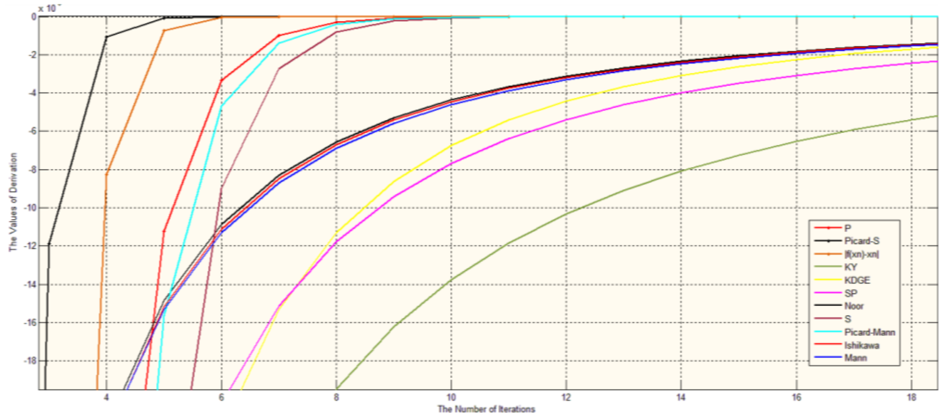

Figure 4. Comparison of the variation between the successive steps of the iteration methods

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