History of Mathematics in Mathematics Education – An Overview

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***Zusammenfassung**: Dieser Beitrag gibt einen Überblick über das Forschungsfeld Geschichte der Mathematik und ihrer Didaktik (im Beitrag als „HPM-domain“[1] bezeichnet) mit Schwerpunkt auf einige der jüngsten Entwicklungen. In Abschnitt 1 wird das Ziel und die Struktur des Beitrags skizziert, während in Abschnitt 2 die dem Forschungsfeld zugrundeliegenden Begründungen erläutert werden (im Beitrag als „HPM perspective“ bezeichnet). Abschnitt 3, sowie der Anhang, geben einen kurzen historischen Überblick über die Entwicklung des Forschungsfeldes mit Bezug auf die wichtigsten internationalen Aktivitäten, deren Ergebnisse und einem besonderen Blick auf die Aktivitäten in Deutschland. Im Hauptteil des Beitrags (Abschnitt 4) sollen behandelte Schlüsselfragen des Forschungsfeldes formuliert werden, die die Aktivitäten strukturieren. In Abschnitt 5 werden schließlich zentrale Fragen kurz kommentiert, die in der Forschung weiterverfolgt werden können.*

***Abstract:** This paper aims to provide an overview of the research on the relations between History of Mathematics and Mathematics Education (what we call the “HPM domain”[1]) with emphasis on some of the more recent developments. In section 1 we outline the aim and structure of the paper, while in section 2 we explain the rationale underlying this domain (what we call the “HPM perspective”). Section 3 and the appendix provide a brief historical account of the development of the HPM domain with reference to the main international activities in its context and their outcomes and special reference to these activities in Germany. In the main part of the paper (section 4) we formulate the key issues addressed in this domain and attempt to present the work done structured by these issues. Finally, in section 5 we comment briefly on the central issues to be further pursued in research.*

1. Introduction

This paper is an elaborated up-to-date version of Clark, Kjeldsen, Schorcht, Tzanakis, and Wang (2016) and Clark, Kjeldsen, Schorcht, and Tzanakis (2018). The paper is an overview of the research field History in Mathematics Education with focus on the so-called *HPM* (History and Pedagogy of Mathematics) *perspective* and *domain* with emphasis on some of the more recent developments. It is an overview in the sense that it attempts to present the key issues dealt with in this field.

The aim of the paper is not to single out and present one coherent theoretical framework for history of mathematics in mathematics education; the aim is to present a (selection) of various ideas, issues and concerns which are present in the field, and results obtained.

Since it is impossible to present and give full justice to all the research done in this field, we have to a large extent drawn upon survey papers and collected works. This is also evident from the list of literature where some authors are listed extensively because they are (co-)editors of many of these collected works.

First we explain the rationale underlying the *HPM perspective* (section 2), followed by a brief historical account of the development of the *HPM domain*. Together with the appendix this is meant to give an outline of the main international activities and their outcomes concerning educational research and its implementation in educational practice, extended with the *HPM* activities in the context of the German GDM/DMV and “Arbeitskreis Mathematikgeschichte und Unterricht” (section 3). Section 4, which forms the main part of the paper, formulates the key questions addressed in this domain and attempts to present (parts of) the work done structured by these questions and concerns. Finally, in section 5 we comment briefly on the central issues to be further pursued in research.

2. Integrating the history of mathematics in mathematics education: The rationale

Presenting mathematics as a collection of definitions, axioms, theorems and proofs, has been a common way especially influenced by the axiomatic approach and formalism as a philosophical thesis.

However, the “polished” products of mathematical activity is just one aspect of what constitutes mathematical knowledge; specifically, the part of mathematics which is communicated, criticized (in order to be finally accepted or rejected) and serves as the basis for new work. Another aspect is the processes of its making. The fact that mathematics is a human intellectual enterprise with a long history and a vivid present, implies that mathematical knowledge is determined, not only by the circumstances in which it becomes a deductively structured corpus of knowledge, but also by the procedure that originally led or may lead to it (cf. Brousseau, 1983, p. 170; Hadamard, 1954, p. 104); in other words, "doing mathematics" is equally important. This includes several different processes: using heuristic arguments; making mistakes; having doubts or even misconceptions; retrogressing in the development and understanding of a subject etc. (Courant & Robbins, 1978, Introductory comments; Lakatos, 1976, Introduction and Appendix 2; Schoenfeld, 1992; Stewart, 1992, pp. 9–10; see also Barbin, 1997; Tzanakis et al., 2000, section 7.1).

In this perspective mathematics is conceived both as a logically structured collection of intellectual products and as processes of knowledge production. Therefore, learning mathematics is understood not only as the process by which one becomes acquainted with and competent in handling the symbols and the logical syntax of theories and in accumulating new results presented as finished products. It also includes the understanding of the implicit motivations, the sense-making actions and the reflective processes of mathematicians, which aim at the construction of meaning by linking old and new knowledge, and by extending and enhancing existing conceptual frameworks (Hiebert & Carpenter 1992, p. 67; Schoenfeld, Smith, & Arcavi, 1993). This active process carried out by mathematicians should also be integrated into mathematics teaching by giving learners the opportunity to "do mathematics". At the same time, this conception of mathematics should be central to the image of mathematics communicated to the outside world (Tzanakis et al., 2000, section 7.1; Wille, 2001; see also Clark et al., 2018, section 7.1).

Along these lines, putting emphasis on integrating historical and epistemological issues in mathematics teaching and learning is a way for exposing mathematics in the making that may help understanding specific parts of mathematics better and become more deeply aware of what mathematics as a discipline is and how it grows; more specifically, that mathematics:

- has undergone changes over time, underscored by shifting views of what mathematics is and how it should be taught and learnt;
- has been in constant dialogue with other scientific disciplines, technology, philosophy and the arts;
- has constituted a constant force for stimulating and supporting scientific, technical, artistic and social developments;
- is the result of contributions from many different cultures.

Conceiving mathematics as a human intellectual activity for the acquisition of knowledge, and the evolutionary character of this knowledge is important for mathematics education, both by supporting the doing, learning and teaching of specific pieces of mathematics and by helping to appreciate the relation of mathematics with other intellectual and cultural pursuits all along its historical development. This conception of mathematics as detailed in the preceding paragraphs is what we have called the *HPM* (*H*istory-*P*edagogy-*M*athematics) *perspective* (Clark et al., 2018, section 1.1), linking *History*, *Education* and *Mathematics* as three distinct but fruitfully interrelated dimensions for teaching and learning both mathematics and about mathematics that are complementary to each other in the sense that (Clark et al., 2018, p. 2; see also in this connection, Fried, 2007, p. 203):

- **History** points to the non-absolute nature of human knowledge: what is acceptable as valid knowledge is "time-dependent" and is potentially subject to changes; that is, *historicity* is one of its characteristics.
- **Education** stresses the fact that humans are different in several respects depending on age, social conditions, cultural tradition, individual characteristics, etc. In this way education helps to understand these differences and to become more tolerant towards views, preconceptions, misconceptions and possibly idiosyncratic ways of self-expression held by the learners and/or the teachers.
- **Mathematics** – more strongly than any other science – emphasizes the need for logical, rational and intellectual rigor and consistency in the human endeavor to understand both the mental and empirical aspects of the world.

Exploring the multifaceted interrelations of these three dimensions has formed the core and main concern of the approaches adopted towards integrating history and epistemology of mathematics in mathematics education; what we have called the *HPM domain* (Clark et al., 2018, p. 3). In the next section we give a brief historical account of its development with reference to the main activities in its context and their outcomes.

3. The historical development of the HPM domain

Integrating the history of mathematics in mathematics education has been advocated at least since the second half of the 19th century[2], when important mathematicians like De Morgan (1865), Zeuthen (1902), Poincaré (1908, ch. II.II), Klein (2016, pp. 256–257, viii–ix; Toeplitz, 1927) and others (Glaisher, 1890, p. 96; Barwell, 1913; Miller, 1916) pointed to this path and historians of mathematics like Cajori (1894, p. 3), Loria (1899) and Tannery (1907) showed an active interest on the role the history of mathematics can play in mathematics education (for details and further references on these earlier ideas and works see Furinghetti, to appear, section 2.2; Tzanakis et al., 2000, section 7.2 and references therein; see also Allmendinger, 2014; Bagni, Furinghetti & Spagnolo, 2004; Pineau, 2012; Schorcht, 2018). As a consequence of the debates on the foundations of mathematics in the 20th century, this interest was revived (British Ministry of Education, 1958; MAA, 1935; see also Kline, 1973, ch. 4; Leake, 1983) and history became a resource for various epistemological approaches, like Bachelard's (1938) historical epistemology, Piaget's genetic epistemology (Piaget & Garcia, 1989) and Freudenthal's (1983) phenomenological epistemology. This also stimulated the formulation of specific ideas and conclusions on the learning process (Brousseau, 1997; Ernest, 1994; Lakatos, 1976; see also Barbin & Tzanakis, 2014, p. 256; Furinghetti, to appear).

The interest in introducing history in mathematics education became stronger in the period 1960–1980, after the *New Math* reform (Barbin & Tzanakis, 2014) since history helps to conceive mathematics as an evolving human activity. In 1969, the 31st Yearbook of the National Council of Teachers of Mathematics in USA was devoted to the history of mathematics as a teaching tool (NCTM, 1969). The same year the *First International Congress on Mathematical Education* (ICME-1) took place as a major event that helped to establish mathematics education as a standard subject in regularly organized international meetings. In 1972 at ICME-2, 38 different Working Groups (WG) were created on main themes of mathematics education; one of them concerning the "Relations between the History and Pedagogy of Mathematics" was organized by P. S. Jones and L. Rogers. This Working Group was continued at ICME-3 in 1976. Having acknowledged the importance and the widespread interest in historical-pedagogical studies in mathematics, a resolution was made to the International Commission on Mathematical Instruction (ICMI) proposing setting up a system to ensure regular sessions at future ICMEs on this theme. ICMI approved the affiliation of a new Study Group, which was originally called "*International Study Group on Relations between History and Pedagogy of Mathematics, cooperating with the International Commission on Mathematical Instruction*" (Fasanelli & Fauvel, 2006). The establishment of this group – now abbreviated as the *HPM Group* – greatly stimulated and supported the interest and educational research in this area at an international level. In fact, the main points which constituted the original focus and aim of this group as they were announced a little later, remain pertinent even today (HPM Group 1978; reprinted in Fasanelli & Fauvel, 2006, pp. xi).

Since then, integrating the history of mathematics in mathematics education has evolved into a worldwide intensively studied domain of new pedagogical practices and research activities inspired by the *HPM perspective*. More details and references can be found in Barbin (2013), Barbin and Tzanakis (2014), Barbin, Guillemette, and Tzanakis (to appear), Fasanelli and Fauvel (2006), Furinghetti (to appear).

The rising international interest in the *HPM perspective* and the activities related to the *HPM domain* led to the approval by ICMI in 1996 of a 4-year *ICMI Study* surveying the work done in this domain and reporting on the main issues for further research. After the Study co-chairs' *Discussion Document* (Fauvel & van Maanen, 1997) and a *Study Conference* in 1998, the Study culminated in the publication that was both comprehensive and the result of the collective work of many individuals (Fauvel & van Maanen, 2000). This volume became a landmark in establishing and making widely visible the *HPM perspective* as a promising line of research in the context of mathematics education, stimulating and enhancing the international interest of the educational community in the HPM domain, inspiring and motivating further research and actual implementations in education communicated in various ways.

In the appendix, we give an account of the main regular international activities and their outcomes concerning educational research and its implementation in educational practice relevant to the *HPM domain* and mainly (though not exclusively) realized in the context of the *HPM Group*, together with *HPM* activities in the context of the German GDM/DMV and "Arbeitskreis Mathematikgeschichte und Unterricht". In addition, several collective volumes and monographs have been published, special issues of journals have been devoted to this domain, numerous individual papers in scientific journals have appeared and doctoral theses have been written. An indicative sample is given below, and a sufficiently comprehensive and up-to-date bibliographical survey can be found in Clark et al. (2016, section 3).

Collective volumes: E.g. Barbin, 2010, 2012b, 2015, 2018; Barbin and Bénard, 2007; Bekken and Mosvold, 2003; Biegel, Reich, and Sonar, 2008; Boero, 2007; Calinger, 1996; Clark et al., 2018; Hanna, Jahnke, and Pulte, 2010; Katz, 2000; Katz and Tzanakis, 2011; Nickel, Helmerich, Krömer, Lengnink, and Rathgeb, 2018; Rathgeb, Helmerich, Krömer, Lengnink, and Nickel, 2013; Shell-Gellasch, 2008; Shell-Gellasch and Jardine, 2005, 2011; Sriraman, 2012; Swetz, Fauvel, Bekken, Johansson, and Katz,1995.

Special issues of journals: E.g. Clark and Thoo, 2014; Furinghetti, Radford, and Katz, 2007; Hupp and Siller, 2017; Jahnke et al., 1991, 1999, 2008; Karam, 2015; Katz, Jankvist, Fried, and Rowland, 2014; Siu and Tzanakis, 2004; Stedall, 2010; Winter et al., 1986.

Monographs and textbooks: E.g. Barbin and Moyon, 2013; Barbin, Moussard, and Bénard, 2018; Filloy, Rojano, and Puig, 2008; Hairer and Wanner, 1996; Knoebel, Laubenbacher, Lodder, and Pengeley, 2007; Ostermann and Wanner, 2012; Roth and Radford, 2011; Schubring, 2005; Shell-Gellasch and Thoo, 2015; Stein, 2010.

4. Addressing the key issues in the HPM domain

4.1 The basic questions

According to the rationale underlying the *HPM perspective*, as concisely presented in section 2, it is the fruitful and harmonious interplay among the three distinct but complementary dimensions – *History*, *Education* and *Mathematics* – that constitutes what could be in principle interesting, stimulating and beneficial for teaching and learning mathematics and about mathematics.

From section 3 and the appendix, it becomes evident that the last few decades have generated considerable research in the *HPM domain* covering a whole spectrum of different activities: the study and developments of theoretical and conceptual frameworks for integrating history in mathematics education, empirical research based on actual classroom implementations; the design of specific teaching units; the development of various kinds of teaching aids; the investigation and understanding of students' response to the introduction of the history of mathematics in teaching (teacher education included); designing, applying and evaluating interdisciplinary teaching; the exploration of eventual parallels between the historical development and learning in a modern classroom (i.e. if, to what extent, and in which way "ontogenesis recapitulates (aspects of) phylogenesis"; the old but still discussed issue of "historical parallelism", e.g. Furinghetti & Radford, 2008; Radford et al., 2000; Schubring, 2006, 2011; Thomaidis & Tzanakis, 2007); the study of theoretical constructs and conceptual frameworks developed in the context of other disciplines (in particular, philosophy, epistemology and cognitive science) and possible mutual benefits; and the evaluation of the effectiveness of all this in educational practice.

In the entire spectrum of these activities, the central issues permeating research as recurring themes that form its *leitmotif* are the following four questions (cf. Clark et al., 2018, p. 2):

Which history *is suitable, pertinent, and relevant to mathematics education?*

Which role *can the history of mathematics play in mathematics education and with* ***which objective****?*

In ***which way(s)*** *history can serve in educational practice by following* ***which approach(es)?***

How can ***all this be evaluated and assessed*** *and to what extent and in what sense does it contribute to the teaching and learning of mathematics?*

In fact, all attempts to explore the multifaceted relation between history of mathematics and mathematics education, explicitly or implicitly address, illuminate, and/or provide insights in one way or another on one or more of these questions (cf. Clark et al., 2016, section 2.3; Clark et al., 2018, section 1.3; Tzanakis, 2016, section 3).

They point to key issues to be addressed while integrating historical elements in mathematics education, and provide a spectrum of possible relevant aspects to be considered when designing and implementing teacher interventions, learning procedures etc. in practice. However, there are also other factors, not directly related to either history or mathematics, which can influence or even be decisive in implementing the *HPM perspective* in practice: approaches may vary in size and scope, according to the specific didactical aim, the subject matter, the level and orientation of the learners, the available didactical time, and important external constraints like curriculum regulations, the number of learners in a classroom etc. As will be seen below, some of these factors are at the heart of various counterarguments and objections that have been raised against the *HPM perspective.*

As outlined in section 3, the *HPM perspective* is an integrated research area in the field of mathematics education. However, the question whether the history of mathematics is appropriate or even relevant at all to the teaching and/or learning of mathematics, is an issue that is debated and several counterarguments and objections have been raised in this connection (Siu, 2006, pp. 268–269; Tzanakis et al., 2000, p. 203; see also Furinghetti, to appear, section 7). A selection of these is listed in the ICMI study (Tzanakis et al., 2000, p. 203) and presented below. They have been classified into two main categories (A and B) as given below (Clark et al., 2018, pp. 6–7; Tzanakis & Thomaidis, 2012, section 3.4; cf. Nickel, 2013; Panasuk & Horton, 2012, p. 12).

A. Counterarguments and objections of an epistemological and methodological nature

(a) *On the nature of mathematics*

1. This is not mathematics! The subject should be taught first and then its history.

2. Progress in mathematics is to make difficult problems a routine, so why bother to look back?

3. What happened historically can be very tortuous. Telling its actual development can produce confusion rather than enlightenment!

(b) *On the difficulties inherent to the HPM perspective*

1. Does it really help to read historically important original texts, which is a very difficult and time-consuming task?

2. Is it liable to breed cultural chauvinism and parochial nationalism?

3. Students (especially the younger ones) have an erratic sense of the past that makes historical contextualization of mathematics difficult or even impossible.

B. Counterarguments and objections of a didactical and more practical nature

(a) *The background and attitude of the teachers*

1. There is lack of didactical time in class.

2. Teachers should be well educated in history: Since they are not professional historians of mathematics, how can they be sure of their exposition's accuracy?

3. Teachers are not sufficiently trained to implement a historical perspective in their teaching.

4. There is lack of appropriate didactical and resource material.

(b) *The background and attitude of the students*

1. They regard it as history and they dislike history class!

2. They regard it non-interesting just like mathematics itself.

3. They do not have enough knowledge of general history and culture to appreciate it.

(c) *Assessment issues*

1. How can one set questions on it in a test or examination?

2. Is there any empirical evidence that students learn better when the history of mathematics is integrated into mathematics teaching?

We should remark that in view of the rationale advocated in section 2, some of the opinions underlying (A) above reflect an understanding of the nature of mathematics and/or history rather different from that underlying the *HPM perspective*. Some of the counterarguments and objections under (B) point to obstacles faced in any attempt to integrate the history of mathematics in mathematics education (Tzanakis et al., 2000, p. 212). However, either implicitly or explicitly, both concern particular points inherent in possible answers to the general questions raised above, which will be explored in more detail in the subsections that follow.

4.2 Which history?

Implicit to some of the objections under (A) above might be the idea that the term "history" is used in the same sense by historians, mathematicians, mathematics educators, or teachers of mathematics. That this is not so was stressed quite early by Grattan-Guinness (1973) and by many researchers later on. In the 1980s, d'Ambrosio emphasized the need to develop three separate conceptions of the history of mathematics: history as taught in schools, history as understood through the development of mathematics, and the history of that mathematics which is practiced among identifiable cultural groups (Fasanelli & Fauvel, 2006, p. xv); thus introducing the concept of ethnomathematics[3] in contrast to academic (or learned) mathematics, i.e. the mathematics which is taught and learned in schools and universities (d'Ambrosio, 1986, particularly p. 5; Meserve & Booker, 1986, p. 257; Gerdes, n.d.). The question "***which history** is suitable, pertinent, and relevant to mathematics education?*" has been a recurrently debated issue among historians and educators with an interest in the HPM perspective. In recent years this issue has been considered from various perspectives yielding important insights into this fundamental question (see particularly Fried, 2001, 2011; Kjeldsen, 2011a, 2011b, 2012, 2019; Kjeldsen & Blomhøj, 2012, and references therein). Other objections under (A) above address the issue that historical developments often took place along complicated paths, led to dead ends and included notions, methods and problems no longer used in mathematics nowadays. Therefore, its integration in education is nontrivial, posing the question why it must be done at all, since in this way history may be forced "to serve aims not only foreign to its own but even antithetical to them" (Fried, 2011, p. 13). There is the danger of either simplifying or/and distorting history to serve education as still another of its tools by adopting what the British historian Herbert Butterfield (1931) called a "Whig" (approach to) history, where history is written from the view of the present (see also Kragh, 1989, p. 89). It is a present-centeredness approach to history the results of which, as phrased by Wilson and Ashplant (1988, p. 11), is a history that is

> constrained by the perceptual and conceptual categories of the present, bound within the framework of the present, deploying a perceptual 'set' derived from the present.

In other words, creating

> a distortion of the past not only by reading modern intentions and conceptions into the doings and writings of thinkers in the past, [...] but also by forcing the past through a sieve keeping out ideas foreign to a modern way of looking at things and letting through those that can be related to modern interests (Fried, 2011, p. 16).

As discussed in a recent paper by Fried (2018) there are many ways of relating to mathematics of the past. He analyzes various relations on a scale from "non-historical" to "historical". Such analyses are helpful in order to clarify existing conflicts and tensions between various approaches to mathematical knowledge. Here we will restrict ourselves, with due attention to the relevance of history to mathematics education, to the distinction between *History* and *Heritage* made by Grattan-Guinness (2004a, 2004b). With the "history" of a particular mathematical notion[4] he refers to

> the development of [this notion] during a particular period: its launch and early forms, its impact [in the following years and decades], and applications in and/or outside mathematics. It addresses the question 'What happened in the past?' by offering descriptions. Maybe some kinds of explanation will also be attempted to answer the companion question 'Why did it happen?' [...] false starts, missed opportunities [...], sleepers, and repeats are noted and maybe explained [...] *differences* between [this notion] and seemingly similar more modern notions are likely to be emphasized. (Grattan-Guinness, 2004b, p. 1; 2004a, p. 164; italics in the original)

With the "heritage" of a particular mathematical notion, he refers

> to the impact of [this notion] upon later work, both at the time and afterward, especially the forms which it may take, or be embodied, in later contexts. Some modern form of [this notion] is usually the main focus, with attention paid to the course of its development. [...] the mathematical relationships will be noted, but historical ones [...] will hold much less interest. [It] addresses the question 'how did we get here?' [...] The modern notions are inserted into [the notion] when appropriate, and thereby [the notion] is unveiled [...] *similarities* between [this notion] and its more modern notions are likely to be emphasized; the present is *photocopied* onto the past. (Grattan-Guinness, 2004a, p. 165; italics in the original)

While "*both kinds of activity* [i.e. a history-based and a heritage-based approach] *are quite legitimate*, and [...] important in their own right" (Grattan-Guinness, 2004a, p. 165; italics in the original), mathematicians, mathematics teachers and educators should be aware of such distinctions when dealing with historical considerations.[5]

This distinction is close to similar ones between pairs of methodological approaches like explicit and implicit use of history, direct and indirect genetic approach, forward and backward heuristics (Toeplitz, 1927; Tzanakis et al., 2000, pp. 209–210). Hence, it is of potential relevance to education as a possible (but non-unique) conceptual tool while considering specific historical cases in an educational context (Tzanakis & Thomaidis, 2012).

Such distinctions may contribute towards an operational answer to the recurrent question: Why history and which history is appropriate to be used for educational purposes (Barbin, 1997)?

4.3 With which *role* and *objective*?

It is a question that has been extensively discussed from several points of view quite early (see e.g. Grattan-Guinness, 1978), especially in relation to the appropriateness and pertinence of original historical sources in mathematics education. It has been analyzed mostly on the basis of both a priori theoretical and epistemological arguments and of empirical research.

According to the literature, there is a more or less general consensus that the history of mathematics can have three distinct **roles** or **functions**, mutually complementing and supplementing each other (Barbin, 1997, 2006; Barbin & Tzanakis, 2014; Barbin et al., to appear; Furinghetti, to appear, section 5; Furinghetti, Jahnke, & van Maanen, 2006a, pp. 1286–1287; Jahnke et al., 2000, section 9.1; Jankvist, 2013, section7):

A ***replacement*** role (*fonction vicariante* in French): This is the possibility offered by history to approach mathematics differently from the way it is often presented (i.e. a corpus of knowledge consisting of final, polished intellectual products; an externally given set of techniques for solving problems given from outside; school units useful for examinations etc.); that is, not only as final results, but also as mental processes that may lead to them; hence to perceive mathematics both as a collection of well-defined and deductively organized results, and as a vivid intellectual activity. In this connection, history allows for a deeper analysis of mathematical activities, motivating and stimulating research in relation to “activity-based teaching”, via which it can be realized that in different historical periods there were different conceptions of mathematical notions, including such fundamental ones like number, function, existence (of a mathematical object), rigor, evidence, proof etc. That is, (meta)ideas and (meta)concepts that today are taken for granted in their present form, are the result of a historical development; in other words, that *historicity* is inherent to them (Barbin & Bénard, 2007).

A ***reorientation*** role (*fonction dépaysante* in French): Considering a mathematical subject in historical perspective, that is, not in relation to our present knowledge and understanding, but in the context it was originally conceived, formulated and applied, may be a source of “epistemological astonishment” because by deciphering and/or doing mathematics that are very different from those of knowledge and procedures that “have been taken for granted” so far, this knowledge and procedures may become questioned (Barbin & Tzanakis, 2014, p. 259). This might especially be established through contact with (carefully chosen) original texts (and other historical sources). This change of what is (supposed to be) familiar, into something unfamiliar, may challenge the learners’ and teachers’ conventional perception of mathematical knowledge as something that has always been existing in its currently established form. This in turn may lead to a deeper awareness that mathematical knowledge is an evolving human intellectual activity, based on a dialectical interplay between the human mind’s creativity and careful intelligent (mental and/or real) experimentation. And in this way, on the one hand teachers can be encouraged to investigate why contemporaries do not understand such a novelty and why students do not understand, either (Barbin, 2012a). On the other hand, students may learn something about their own mathematics by experiencing and “reflecting on the contrast between modern concepts and their historical counterparts” (Fried, Guillemette, & Jahnke, 2016, p. 218), and think about the mathematicians in their context. As a result, the teachers are helped to become more attentive to their students and the students enhance their ability to do mathematics and to think about mathematics.

A ***cultural*** role (*fonction culturelle* in French): History makes it possible to appreciate that the development of mathematics always takes place in a specific scientific, technological and societal context at a given time and place (e.g. Epple, 2000). This means that learning mathematics not only concerns learning to solve problems, or mastering formal language, but also getting aware of the historical, cultural, social or ethical dimensions of the mathematical activity; thus appreciating mathematical knowledge as an integral part of human intellectual history in the development of society. In this way, mathematics is perceived from perspectives that lie beyond its currently established boundaries as a universal, objective and practically a-historical discipline (cf. e.g. d’Ambrosio, 2018; Gerdes, n.d.). Seen in this perspective, the implementation of different approaches in teaching and learning mathematics depending on the societal and cultural context of the learners and their social environment, becomes important, reflections on the universality of mathematics can be revisited and questions on the internationalization of mathematical curricula can be raised.

In relation to the possible **objective** of integrating the history of mathematics in mathematics education, an analytical survey was done for the ICMI study. They identified five main areas, paraphrased and quoted below in A–E, in which the *HPM perspective* could be beneficial (Tzanakis et al., 2000, section 7.2; see also Tzanakis & Thomaidis, 2012, section 3). Their analysis offers a more detailed task-directed description of the history’s role(s) in the educational process (see also the critical discussion in Jankvist, 2009b, sections 3–5).

A. The learning of mathematics

1. *Historical development vs. polished mathematics*: Integrating (some of) the key steps in the historical development into teaching can help to uncover the meaning and significance of pieces of mathematics new to students; hence, to present these pieces in a way that illustrates how they were developed, minimizing logical gaps and ad hoc introduction of concepts, methods or proofs.

2. *History as a resource*: History constitutes a rich, almost unlimited reservoir of meaningful questions, and problems, which in principle can motivate, raise the interest and engage the learner by linking present knowledge and learning processes to knowledge and problems in the past.

3. *History as a bridge between mathematics and other disciplines*: History points to interrelations and interdependence among different mathematical domains, or, between mathematics and other disciplines. This may help to connect domains at first glance appearing unrelated, and to appreciate that fruitful research in a scientific domain is often motivated by questions and problems coming from apparently unrelated disciplines.

4. *The more general educational value of history*: Engaging students (teacher students included) in historically-oriented study projects supports and reinforces personal growth and skills not necessarily related to mathematics: reading, writing, looking for resources, documenting, discussing, analyzing and “talking about” (as complementary to “doing”) mathematics.[6]

B. Understanding the nature of mathematics and mathematical activity

1. *Content*: History offers different perspectives of mathematics: illuminating concepts, conjectures, techniques and proofs via mathematically relevant and historically important questions and problems; stressing the evolutionary nature of mathematical knowledge and meta-concepts such as proof, rigor, evidence, error etc.; developing the awareness that mistakes, heuristics arguments, uncertainties, doubts, intuitive arguments, controversies and idiosyncratic approaches are an integral part of mathematics in the making.

2. *Form*: History makes clear the evolutionary character of mathematical notations, terminology, favorite computational methods, modes of expression and representation. This helps to become aware of the verbal and/or symbolic mathematical language of a given period; to re-evaluate the role of visual, intuitive and non-formal approaches of the past; to understand better the advantages and/or disadvantages of current formulations in mathematics; and in this way, it might help to motivate learning by stressing clarity, conciseness and logical completeness.

C. The didactical background of teachers and their pedagogical repertoire

Teachers can enlarge their pedagogical repertoire and their didactical background by studying the history of mathematics to strengthen their abilities in the following areas:

1. *Identifying motivations*: Questions and problems that historically motivated the introduction of new ideas, concepts and methods may help to uncover the rationale underlying the introduction of new knowledge and the substratum that sustained further progress.

2. *Getting aware of difficulties and obstacles*: Historically important cases make possible: To identify difficulties and obstacles, that appeared in history and bear analogies with students’ difficulties caused by (a combination of) epistemological and didactical factors; and to realize that often new knowledge resulted gradually and slowly, on the basis of questions and problems presupposing a mathematical maturity on the part of the students that may not exist yet. Either case may lead to a more adequate teaching design and implementation.

3. *Getting involved and/or becoming aware of “doing mathematics” as a creative process*: Considering questions and problems in historical context engages in doing (usually known) mathematics. Nevertheless focusing on solving historical problems that are interesting per se, or relating them with others and benefiting from this interrelation may enrich mathematical literacy. Specifically, it may provide a wider and deeper understanding of particular mathematical issues and their evolution, and/or point to meta-mathematical issues inherent in the questions and problems considered.

4. *Enriching the didactical repertoire*: Since history constitutes a stimulating and insightful resource of authentic questions and problems, it is potentially a natural path to enrich teachers’ didactical repertoire and increase their ability to explain, approach and understand particular issues in mathematics and about mathematics.

5. *Deciphering and understanding idiosyncratic and/or non-conventional approaches to mathematics*: Studying a known piece of “correct mathematics” as it was originally presented and treated may help to learn how to work on known mathematics in a different (old-fashioned, or abandoned) way and therefore to become more sensitive and tolerant towards non-conventional, idiosyncratic, or incorrect mathematics adopted, or developed by students.

D. The affective predisposition towards mathematics

1. *Understanding mathematics as a human endeavour*: By its very nature, history is a privileged domain to appreciate the evolutionary nature of mathematics, and consequently, that mathematics constitutes a corpus of human knowledge subject to change, and not a finished system of eternal rigid truths.

2. *Persisting with ideas, attempting lines of inquiry and posing questions*: As a human intellectual endeavor, mathematics involves posing questions, arguing heuristically and following unconventional approaches. Considering such cases in history may indirectly help to persist with one’s own ideas, to undertake lines of inquiry, to pose questions and to feel free and legitimized to develop creative or idiosyncratic ways of thought.

3. *Not getting discouraged by failures, mistakes, uncertainties and misunderstandings*: It is a historical fact that uncertainties, doubts, controversies, mistakes and blind alleys have always formed an integral part of mathematical development. Looking

in detail at such historical examples may help the learner to not get discouraged by failure, mistakes, uncertainties or misunderstandings, and the teacher to appreciate the fact that often they form a fruitful part of learning and doing mathematics.

E. The appreciation of mathematics as a cultural-human endeavour

1. *Mathematics evolves under the influence of factors intrinsic to it*: History offers examples to get aware of the diversity of factors internal to it that influenced its development – like aesthetic criteria, intellectual curiosity, challenge and pleasure, recreational purposes etc. – thus appreciating that mathematics is not driven by practical factors only.

2. *Mathematics evolves under the influence of factors extrinsic to it*: History also offers examples of how the internal development of mathematics, whether driven by external or internal factors, has been influenced, or even determined by social and cultural factors.

3. *Mathematics form part of local cultures*: History shows that mathematics is the result of several different cultures, each one developing its own perception of mathematics and influenced by it. Studying specific examples from this perspective helps to become aware of the multicultural nature of mathematics. And in this way to re-evaluate local cultural heritage as a means of appropriately designing teaching and developing tolerance and respect among fellow students.

In these five areas (A–E) in which the *HPM perspective* could be beneficial in the sense described above, the past is used in different ways and the purposes for integrating history varies. Jankvist (2009a, 2009b, 2009c, 2009d) suggested to distinguish between the use of history in mathematics education as

(i) “a *tool* in the sense of assisting the actual learning of mathematics (mathematical concepts, theories, and so forth)” be it “a motivational or affective tool, and […] as a cognitive tool” (Jankvist 2009d, p. 8, emphasis added);

(ii) “a *goal* in itself for example, by bringing about meta-aspects concerning the history of mathematics in mathematics education” (Jankvist, 2009d, p. 8, emphasis added). Hereby history serves the primary purpose of

> posing and suggesting answers to questions about the evolution and development of mathematics, about the inner and outer driving forces of this evolution, or the cultural and societal aspects of mathematics and its history (Jankvist, 2009c, p. 69, quoting Niss (2001, p. 10)).

These should be considered as complementary purposes for integrating history in mathematics education, in the sense that although they may co-exist in a specific case, usually they will not have equal weights, depending on the other factors analyzed in this section. They could help to analyze further the possible roles and objectives of the history of mathematics in mathematics education, in relation to the variety of their possible implementations in practice.

Another related distinction was the result of a literature review by Furinghetti (2004), where she searched for “the pedagogical meaning of linking history and mathematics teaching” (p. 11). She classified existing research work and actual implementations into two main streams (Furinghetti, 2004, pp. 2–3; see also Furinghetti, to appear, section 5):

(i) “History for constructing mathematical objects”, concerning the core of the problems related to the teaching and learning of mathematics.

(ii) “History for reflecting on the nature of mathematics as a socio-cultural process”, which among other things considers the history as a means to promote mathematics in the classroom in order to humanize mathematics, though this latter concept, as she pointed out, has no clear-cut universally accepted meaning.

4.4 In which *way* – following *which approaches*?

As already mentioned, methodological issues concerning implementation of the integration of the history of mathematics in mathematics education are highly nontrivial.

In a literature survey of the manners employed for integrating history in mathematics education, Jankvist (2009b, sections 6, 9) classifies them into three broad categories (cf. Clark et al., 2018, section 1.3.3):

Illumination approaches: Here the teaching and learning of mathematics, in the classroom or textbooks, is supplemented by historical information of varying size and emphasis. Both issues related to mathematical concepts, theories, disciplines, methods etc. and meta-aspects in mathematics (i.e. history as a tool and as a goal) can be considered in this context, where history enlightens the learner by illustrative examples and insightful comments pertaining to the subject under consideration.

Module approaches: Here history forms an instructional unit often based on the detailed study of specific cases, tied to or adjoining the mathematics curriculum. History appears more or less directly, possibly based on original documents. The content of such modules varies greatly in size, from sharply focused on a well-defined “local” issue, to a full course or textbook having the wider scope to present conceptual mathematical developments, historical facts, or both.

History-based approaches: Here history permeates teaching, shaping the sequence and the way of presentation. In other words, the historical development is not necessarily discussed in the open, but often sets the agenda for the order and the way in which mathematical topics are presented. Therefore, even though teaching may lean heavily upon history, the influence of history is implicit rather than explicit.

On the other hand, an analytical survey of ways to accomplish the integration of history in mathematics education done for the ICMI Study (Tzanakis et al., 2000, section 7.3), pointed out three different types of approaches (A, B, and C, below) distinguished by their emphasis on a different principal aim, which nevertheless may be combined and thus complement each other (see also Jankvist, 2009b, sections 7, 8; Tzanakis & Thomaidis, 2012, section 3.2).

A. *To provide direct historical information*, aiming *to learn history*

This involves a whole spectrum of different types of information and (didactical) material associated to it: (i) the inclusion of isolated "factual information" (names, dates, events, biographies, famous problems and questions, facsimiles etc.) in the form of historical snippets, annotated and commented images, paintings, photographs etc.; (ii) courses, books, or individual book chapters on (particular issues from) the history of mathematics; (iii) historical notes, annotated bibliographical surveys; guides for further reading etc.

Here the emphasis is more on becoming aware of history than on learning mathematics.

B. *To implement a teaching approach* (explicitly or implicitly) *inspired by history*, aiming *to learn mathematics*

This is an approach, in which history inspires teaching and learning as a natural, integral part of teaching and/or the didactical material associated with it. In this context history may appear either explicitly, or implicitly (see Tzanakis et al., 2000, section 7.3.2; cf. Toeplitz, 1927).

Here, the emphasis is more on learning mathematics, than on learning history. There are many variations in form, content and specific methodology, e.g.:

(a) Teaching modules and/or mathematical textbooks explicitly based on or implicitly permeated by historical elements; (b) Student research projects; (c) Worksheets often based on original documents either as a set of questions to introduce a topic, a set of problems etc.; or a collection of exercises, recreational problems and games to elaborate on a method, consolidate a topic etc.; (d) Excerpts from original texts to introduce a topic with annotations to help better grasp its content; (e) Self-contained collections of materials, as "historical packages" ready for use in the classroom; (f) History-based teaching capsules and/or didactical material as a means to support the teaching and learning of specific pieces of mathematics, by taking advantage of errors, alternative conceptions, change of perspective, revision of implicit assumptions, intuitive arguments etc. that appeared in history; (g) Insights into past and/or recent developments, in relation to outstanding old issues and problems of diverse character (problems with clever, alternative or exemplary solutions, having provoked and/or anticipated important developments, still unsolved, unsolvable, or simply recreational), constituting good examples to fascinate students and reveal the evolutionary nature of mathematics.

C. *To focus on mathematics as a discipline and the cultural and social context* in which it developed, aiming *to create or enhance mathematical awareness*

Here *mathematical awareness* means the ability to recognize and understand the general characteristics of mathematics both as a corpus of knowledge and as a human activity, appreciating their significance for mathematics itself and in its relation to other scientific disciplines, culture and society, as well as the influence on its development of factors both intrinsic and extrinsic to it.

This kind of approach may focus on issues intrinsic to the mathematical activity: (i) the role played by specific conceptual frameworks in the development of, and in, particular mathematical domains; (ii) the evolutionary nature of mathematics both in content and form: notation, terminology, computational methods, meta-mathematical concepts (e.g. axiom, proof, rigor, evidence), modes of expression and representation; (iii) the role played by specific characteristics of the mathematical activity itself, like doubts, paradoxes, contradictions, heuristics, intuitions, impossibility arguments etc. in fundamental mathematical processes like generalization, abstraction, or formalization.

It may focus on extrinsic characteristics of the mathematical activity, as well, by unfolding, exploring and emphasizing (i) the interrelations between (specific pieces of) mathematics and the sciences, philosophy, the arts and social sciences; (ii) the influence of the social and cultural contexts in the development of particular mathematical subjects; (iii) specific characteristics of mathematics as an integral part of different civilizations and cultural traditions; (iv) issues from the history of mathematics education and the influence they exerted on mathematics education itself.

In a more didactically-oriented perspective, Nickel (2013, 2016) describes four usages of the history of mathematics in the classroom and their eventual caricatures, each one intended to emphasize the "vital function" ("lebensdienliche Funktion"; Nickel, 2013, p. 254) of the history of mathematics, while its caricature is tantamount to a misuse:

The ***anecdotal*** **or** ***comforting usage*** (*anekdotischer bzw. tröstender Gebrauch* in German) occurs when history is integrated from an affective point of view. This is usually done by presenting short stories – such as those of Little Gauss – and depictions of great personalities. Especially in comforting use, pupils are told that even great mathematicians occasionally faced problems in mathematics (cf. (A) above). The anecdotal or comforting use can also fall into the opposite if it drifts into a **monumental or jovial caricature** (*monumentalische bzw. jovial Karikatur* in German). The monumental caricature describes (the great) mathematicians as heroes whose intellect is hardly attainable, while by considering the historical development as a continuous progress forward, the jovial one describes them as humans, who add knowledge to what already exists along a cumulative path towards perfection (this is close to a "Whiggish" view of history; cf. section 4.2).

The ***genetic usage*** (*genetischer Gebrauch* in German) is used in contrast to the representation of strictly deductive and formal mathematics. Developments, alternative ways or intended applications can be shown. Mathematics is unveiled to the learner in terms of its historical genesis either explicitly or implicitly (cf. (B) above). However, on the other hand, strictly following the complexity of the historical development of mathematics can be quite confusing, since this development includes not only steps forward, but also mistakes, misconceptions, retrogressions, dead ends and zigzag routes. Especially for students, who do not have a sufficiently wide overview of mathematics, this is a caricature of the genetic usage that Nickel calls **history as an obstacle to understanding** (*Geschichte als Verstehenshindernis* in German).

The ***alienating usage*** (*verfremdende Gebrauch* in German) takes place when already known mathematical facts are presented in a new way. This alienation opens the view to historically realized alternatives and makes clear that the way one usually thinks, proves, and calculates in today's mathematics is by no means so simple and self-evident. This appreciation of one's own present, however, can take place only when experiences with alternatives have been made possible (cf. the *reorientation* role mentioned in section 4.3).

Finally, the ***authentic exemplary usage*** (*authentisch exemplarischer Gebrauch* in German) gives students insights into mathematicians' present and past actions, so that they may experience a genuine approach to doing mathematics. This approach is an authentic one to the history of mathematics, possibly exhibiting interdisciplinary traits as well. However, a purely positivist usage of history in this context constitutes an **antiquarian caricature** (*antiquarische Karikatur* in German), in which the unreflecting teacher collects curiosities, excluding inquiry and contemporary needs. This means that working on a specific mathematical content is replaced by the purely referential presentation of the work of the historical personalities related to it.

As already highlighted in section 4.2, the above approaches may vary in size and scope, according to the specific didactical aim (A(b1)), the mathematical subject (A(a1)), the level of instruction and the age and orientation of the learners (A(b1), B(b3)), the available didactical time (B(a1)), and other important external constraints, like the official curriculum to be followed and the constraints it imposes (e.g. specific types of assessment; (B(c1)), number of students in a classroom etc.).

4.5 How can the HPM perspective be *evaluated and assessed*?

There is a demand for sufficient empirical evidence about the effectiveness of the *HPM perspective* in improving mathematics education from the point of view of both teaching and learning mathematics. This is a key issue that has already been addressed (e.g. Jankvist, 2007; Siu & Tzanakis, 2004, p. 3), realizing that any such evaluation goes side by side with actual classroom implementations, in-school teaching and teacher pre- and in-service education. Therefore, most works referring to such implementations necessarily address evaluation issues about their effectiveness (e.g. those listed in section 5). On the other hand, quite early it became clear that evaluating the effectiveness of the *HPM perspective* is a complex process based more on qualitative than quantitative methodologies, which also include considering changes induced in teachers' own perception of mathematics, examining how this may influence the way they teach mathematics and exploring if and in which ways this affects students' perception and understanding of mathematics (Barbin et al., 2000, particularly sections 3.1, 3.2).

This is an area of currently active research (see e.g. Bütüner, 2015; Leng, 2006) with no general consensus on its results yet, because of a complex network of factors interfering with each other that do not allow an easy and direct comparison of empirical findings of different research works. These include (Clark et al., 2018, section 1.3.4):

(a) Different and sometimes strong preconceptions, misconceptions, predispositions either of the teachers or the students which are not easily and/or quickly modified.

(b) The instructional level (primary, secondary, tertiary) and the orientation of the students, teacher students included (science or humanities; elementary or secondary school teachers etc.), as well as students' previous educational path. These have determined their knowledge of, attitude towards, and preconceptions about mathematics, which are decisive factors that strongly differentiate otherwise similar didactical implementations of the *HPM perspective*.

(c) Additional factors that are largely independent of the perspective adopted in teaching and learning mathematics, but nevertheless may favor, enhance, impede, or prevent the implementation of an approach based on the *HPM perspective*. Typical examples are the content of the curriculum and the specific constraints it imposes (e.g. the type of tests to be used in examinations; teaching based on the same textbook to be used by all teachers; teaching time devoted to mathematics per week etc.); the number of students in the class; the structure of the educational system (e.g. in a centralized system teachers have less freedom, and therefore fewer possibilities to apply an innovative teaching approach that may fall outside the official curriculum regulations).

(d) Not all mathematical subjects are equally accessible or appropriate to be taught and/or learned in a historically motivated or driven context.

Therefore, despite many thoughtfully designed and carefully applied empirical investigations, much work is still needed to evaluate the effectiveness of implementing the *HPM perspective*.

5. Concluding remarks

The *HPM perspective* presented in section 2 emerged gradually over the last decades as a perception of mathematics worth exploring, thanks to research and teaching work done worldwide. This has established the *HPM domain* as a valuable research area in the context of mathematics education. At the same time this led the field to realize that in conducting research within this domain and implementing its results in educational practice, the following issues are central:

1) The crucial role *pre- and in-service teachers' education* has as a necessary prerequisite for the *HPM perspective* to be realized in practice at all. The need to put emphasis on this has been stressed repeatedly (e.g. Alpaslan, Işıksal, & Haser, 2014, pp. 160–162; Barbin et al., 2000, p. 70; Barbin, Furinghetti, Lawrence, & Smestad, 2011b; Furinghetti, 2004, p. 4; Gazit, 2013, section 4; Horton, 2011; Huntley & Flores, 2010, section 1; Schorcht, 2015). In particular, it has been advocated that beliefs and views about mathematics and its teaching may be positively affected by history (Buchholtz & Schorcht, 2014, 2016, to appear; Charalambous, Panaoura, & Philippou, 2009; Furinghetti, 1997; Jankvist, 2009b; Schorcht & Buchholtz, 2015; Spies & Witzke, 2018), though scepticism has been also expressed in this connection (see Furinghetti, 2007, and references therein; Philippou & Christou, 1998).

Though accommodating the *HPM perspective* in an essential way into the official national curricula does not seem to have attained wide applicability[7], intensive efforts have been made to train teachers and explore changes in their attitude and/or teaching (some indicative examples are Arcavi & Isoda, 2007; Bruckheimer & Arcavi, 2000; Burns, 2010; Clark, 2011; Clark et al., 2018, chs. 4, 11, 14, 18; Kjeldsen & Carter, 2014; Liu, 2003; Mosvold, Jacobsen, & Jankvist, 2014; Povey, 2014; Smestad, 2011; Waldegg, 2004).

2) The equally crucial significance of designing, producing, making available and disseminating diverse *didactical source material* in the form of anthologies of original sources, annotated bibliography, description of teaching sequences or modules that could serve as a source of inspiration and/or as generic examples for classroom implementation, educational aids of various types, appropriate websites etc. (see e.g. Panasuk & Horton, 2012, p. 16; Pengelley, 2011, pp. 3–4; Percival, 2004, p. iii; Tzanakis et al., 2000, pp. 212–213). This need has been satisfied to a considerable extent in the last 15 years, so that such material is available nowadays in a variety of forms, aiming also to increase teachers' interest and participation in national and international events related to the *HPM perspective*. Indicative examples are:

- The wide spectrum of resource material in *Convergence*[8]; e.g. see the review of some examples in Barnett et al. (2017), Beery (2015) or Clark (2009) for the detailed description of a teaching module.

- Katz and Michalowicz (2005): didactical source material in 11 modules.
- Siu (2007): a useful survey of the literature and available resources.
- Biegel, Reich, and Sonar (2008): A collective volume in German with many examples for integrating history of mathematics.
- Pengelley et al. (2009): Didactical material for discrete mathematics based on original sources.
- Pengelley and Laubenbacher (2014): A website with many references to published work and material available online[9].
- Barnett, Lodder, and Pengelley (2014): Extensive information on teaching with historical sources and bibliography on its theoretical framework and available resource material.
- Books with material that can be used directly and/or inspire teaching (e.g. Barbin, 2015; Clark et al., 2018, part III, chs. 5, 16, 17; Demattè, 2006; Moyon & Tournès, 2018; Shell-Gellasch & Thoo, 2015; Stein, 2010).

3) The need of systematically and carefully designed and applied empirical research in order to examine in detail and evaluate convincingly the effectiveness of the *HPM perspective*, as well as students' and teachers' awareness of mathematics as a discipline and their disposition towards it. An indicative, but far from exhaustive selection from existing publications appears below.

- Fauvel and van Maanen (2000): Chs. 7, 8 provide a variety of examples of possible classroom implementations, for several mathematical subjects; Ch. 9 gives examples of using original sources in the classroom and specific didactical strategies to do so.
- Clark et al. (2018, Parts III and IV); Katz and Tzanakis (2011, chs. 9, 10, 13, 14, 16, 19) and Sriraman (2012, chs. 2, 7, 14) provide particular examples, most of them emphasizing empirical results of actual implementations.
- Katz et al. (2014): This volume is rich on recent work in the HPM domain, including a sufficiently comprehensive old and recent bibliography in the editors' introduction and in its 12 papers. They concern theoretical issues on the history, philosophy and epistemology of mathematics, and on empirical investigations both in school and teacher education.
- Doctoral dissertations have been written with considerable work on both the theoretical issues of the *HPM perspective* and on empirical investigation and evaluation of actual implementations (e.g. Clark, 2006; Glaubitz, 2010; Guevara Casanova, 2015; Jankvist, 2009a; Schorcht, 2018; Su, 2005; van Amerom, 2002).

4) The importance of acquiring a deeper understanding of theoretical ideas put forward in the *HPM domain* and carefully developing them into coherent theoretical frameworks and methodological schemes that will serve as a foundation for further research and applications in this area (indicative examples are Barbin, 2018; Barbin & Bénard, 2007; Clark et al., 2018, part I, ch. 8; Fried, 2001, 2011; Hanna et al., 2010; Jahnke, 1995a, 1995b, 1996; Jankvist, 2009b; Jankvist & Kjeldsen, 2011; most of the papers in Karam, 2015; Kjeldsen, 2012, 2018; Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014; Schubring, 2011; Thomaidis & Tzanakis, 2007).

We believe that elaborating on these four central issues will improve our understanding of the benefits that possibly underlie the *HPM perspective* as a main epistemological and educational thesis on the multifarious interrelations among history, education and mathematics.

Remarks

[1] **H**istory and **P**edagogy of **M**athematics: An acronym abbreviating the name of the *International Study Group on the relations between the History and Pedagogy of Mathematics*, one of the oldest study groups affiliated to ICMI (*International Commission on Mathematical Instruction*) and established within the educational community as the *HPM Group*. Here this acronym is used as a terminus technicus (see section 3).

[2]An earlier advocate in Germany was the schoolteacher F. W. Lindner (1808).

[3] "…we will call ethnomathematics the mathematics which is practiced among identifiable cultural groups, such as national-tribal societies, labor groups, children of a certain age bracket, professional classes, and so on" (d'Ambrosio, 1985, p. 45); for a recent, broader conception of ethnomathematics, see d'Ambrosio, 2018, pp. 230ff.

[4] Here "notion" serves "as the umbrella term to cover a theory (or definition, proof-method, technique, algorithm, notation(s), whole branch of mathematics, …)" (Grattan-Guinness, 2004a, p. 164).

[5] Further details on this distinction are concisely tabulated in Grattan-Guinness (2004b, p. 4).

[6] This is close to some mathematical "competencies" as described by Niss (2003, pp. 119–121; 2004, pp. 184–186). It is also in tune with Fried's (2007, p. 203) argument that the history of mathematics may "contribute to students' growing into whole human beings".

[7] One exception is Denmark (see Jankvist, 2013, section 3; Kjeldsen, 2011b, section 15.2; Kjeldsen & Carter, 2014; Niss & Højgaard, 2011, ch. 4). For a recent discussion and survey see Boyé, Demattè, Lakoma, and Tzanakis (2011).

[8] www.maa.org/publications/periodicals/convergence

[9] This website will be disabled soon; all projects will be moved to: https://blogs.ursinus.edu/triumphs/, where many other primary source projects can be located.

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Appendix

Below we give a brief account of the main regular international activities related to the HPM domain and their outcomes (section A.1), and a short presentation of journals and newsletters (section A.2).

A.1 Meetings and related collective volumes

A.1.1 *ICME Satellite Meetings of the HPM Group* (HPM meetings)

These quadrennial conferences are a major activity that bring together individuals with a keen interest in the relationship between the history of mathematics and mathematics education: researchers in mathematics education interested in the history of mathematics in relation to mathematical thinking, mathematics teachers at all levels eager to gain insights into the *HPM perspective*, historians of mathematics wishing to talk about their research, mathematicians wanting to learn about new possibilities to teach their discipline, and all those with an interest in the *HPM domain*.

They are organized just after, or before the ICME:

1984 Adelaide; *ICME 5* in Adelaide

1988 Florence; *ICME 6* in Budapest

1992 Toronto; *ICME 7* in Quebec

1996 Braga, *HEM Braga 96* conjointly with the *2nd ESU*; *ICME 8* in Seville

2000 Taipei, *HPM 2000*; *ICME 9* in Tokyo-Makuhari

2004 Uppsala, *HPM 2004*, conjointly with the *4th ESU*; *ICME 10* in Copenhagen

2008 Mexico City, *HPM 2008*; *ICME 11* in Monterrey

2012 Daejeon, *HPM 2012*; *ICME 12* in Seoul

2016 Montpellier, *HPM 2016*; *ICME 13* in Hamburg

2020 Macao, *HPM 2020*; *ICME 14* in Shanghai

The books published as a result of these *HPM* meetings are listed in chronological order:

Swetz et al. (1995) after ICME-6; Calinger (1996) after HPM 1992; Lagarto et al. (1996) during HPM 1996; Katz (2000) after HPM 1996; Horng and Lin, (2000) at HPM 2000; Bekken and Mosvold (2003) before ICME 10 and HPM 2004; Horng, Lin, Ning, and Tso (2004) before HPM 2004; Furinghetti, Kaisjer, and Tzanakis (2006b) after HPM 2004; Cantoral, Fasanelli, Garciadiego, Stein and Tzanakis at HPM 2008; Barbin, Hwang, and Tzanakis (2012) at HPM 2012 (a revised edition is in progress); and Radford, Furinghetti, and Hausberger (2016) during HPM 2016.

A.1.2 The *European Summer University on the History and Epistemology in Mathematics Education* (ESU)

The initiative of organizing a *Summer University* (SU) on the *History and Epistemology in Mathematics Education* belongs to the French mathematics education community in the early 1980s. The French IREMs (Instituts de Recherche *sur l'Enseignement des Mathématiques)* organized the first interdisciplinary meeting in 1984, in Le Mans, France, followed by another three in France. The next one was organized in 1993 on a European scale; the 1st *European Summer University on the History and Epistemology in Mathematics Education*, (a name coined since then, abbreviated as ESU since 2004), though many participants come from outside Europe. Since 2010, ESU is organized every four years to avoid coincidence with the HPM meetings.

Since its original conception, ESU has been developed and established into one of the major activities in the HPM domain. It mainly aims to: provide a school for working on a historical, epistemological, and cultural approach to mathematics and its teaching, with emphasis on actual implementation; give the opportunity to mathematics teachers, educators, and researchers to share their teaching ideas and classroom experience related to a historical perspective in teaching; and motivate further collaboration along these lines among teachers of mathematics and researchers on the History of Mathematics and Mathematics Education in Europe and beyond, attempting to reveal and strengthen the *HPM perspective*. Below is a list of the ESUs:

1993, *ESU* 1 Montpellier

1996, *ESU* 2 Braga (conjointly with *HEM Braga 96*)

1999, *ESU* 3 Leuven & Louvain-la-Neuve

2004, *ESU* 4 Uppsala (conjointly with *HPM 2004*)

2007, *ESU* 5 Prague

2010, *ESU* 6 Vienna

2014, *ESU* 7 Copenhagen

2018, *ESU* 8 Oslo

The following works were published after the *ESU*s: Lalande, Jaboeuf, and Nouazé (1995); Lagarto, Vieira, and Veloso (1996); Radelet-de-Grave and Brichard (2001); Furinghetti et al. (2006b); Barbin, Stehlikova, and Tzanakis (2008); Barbin, Kronfellner, and Tzanakis (2011a); Barbin, Jankvist, and Kjeldsen (2015); Tzanakis, Barbin, Jankvist, Kjeldsen, and Smestad (to appear).

A.1.3 The *HPM domain* at ICMEs

Activities related to the *HPM perspective* have always been present in the ICMEs (Fasanelli & Fauvel, 2006, for ICMEs before 2000). Since 2000, such activities have formed an established part of the ICMEs' scientific program:

(a) ICME 9, Tokyo, Japan, 2000

WG for Action 13: History and Culture in Mathematics Education coordinated by J. van Maanen and W.-S. Horng.

Summary in H. Fujita, Y. Hashimoto, B. R. Hodgson, P. Y. Lee, S. Lerman and T. Sawada (Eds.) (2004). *Proc. of the 9th ICME* (pp. 287–291). Dordrecht: Kluwer Academic Publishers, in CD (Available online at http://www.icmihistory.unito.it/icme9.php)

(b) ICME-10, Copenhagen, Denmark, 2004

TSG 17 (Abbreviation for *Topic Study Group*): The role of the history of mathematics in mathematics education, organized by M. K. Siu, C. Tzanakis, A. El Idrissi, S. Kaisjer, and L. Radford.

Summary in M. Niss (Ed.) (2008). *Proc. of the 10th ICME* (pp. 363–367). IMFUFA, Roskilde University, in CD (Available online at http://www.icmihistory.unito.it/icme10.php)

This TSG led to a post-conference publication (Siu & Tzanakis, 2004)

(c) ICME 11 Monterrey, Mexico, 2008

TSG 23: The role of the history of mathematics in mathematics education, organized by A. El Idrissi, A. Miguel, F. Furinghetti, A. Garciadiego, and É. Barbin (summary available online at https://www.mathunion.org/fileadmin/ICMI/files/Digital_Library/ICMEs/TSG_23_Report_BB_FF.pdf)

(d) ICME 12 Seoul, Korea, 2012

TSG 20: The role of history of mathematics in mathematics education, organized by R. Chorlay, W.-S. Horng, M. Kronfellner, K. Clark, A. El Idrissi, and H. Chang.

Summary in S. J. Cho (Ed.) (2015). *Proc. of the 12th ICME: Intellectual and attitudinal challenges*. New York: Springer, pp. 485–487.

(e) ICME 13 Hamburg, Germany 2016

TSG 25: The role of history of mathematics in mathematics education, organized by C. Tzanakis, X. Wang, K. Clark, T. H. Kjeldsen, and S. Schorcht (Available online at http://www.icme13.org/files/tsg/TSG_25.pdf). Summary in Clark, Kjeldsen, Schorcht, Tzanakis, and Wang (2017).

A.1.4 The *HPM domain* at CERME

CERME (Congress of European Research in Mathematics Education) is a regular activity of the *European Society for Research in Mathematics Education* (ERME), organized every two years in the form of presentations, discussions, and debates within thematic working group (WG). Though relatively new, the *HPM perspective* has exhibited great potential at CERME and is expected to play a central role in the future:

(a) *CERME 6*, *Lyon*, France, 2009

WG 15: *Theory and research on the role of history in mathematics education*, organized by F. Furinghetti, J-L. Dorier, U. T. Jankvist, J. van Maanen, and C. Tzanakis. A new WG structured along 7 themes; 13 papers and 1 poster accepted and included in the proceedings (Furinghetti, Dorier, Jankvist, van Maanen, & Tzanakis, 2010).

(b) *CERME 7*, Rzeszów, Poland, 2011

WG 12: *History in Mathematics Education*, organized by U. T. Jankvist, S. Lawrence, C. Tzanakis, and J. van Maanen. Structured along 9 themes; 13 papers and 1 poster accepted and included in the proceedings (Jankvist, Lawrence, Tzanakis, & van Maanen, 2011).

(c) *CERME 8*, Antalya, Turkey, 2013

WG 12: *History in Mathematics Education*, organized by U. T. Jankvist, K. Clark, S. Lawrence, and J. van Maanen. Structured along 9 themes (the same as CERME 7); 12 papers and 3 posters accepted and included in the proceedings (Jankvist, Clark, Lawrence, & van Maanen, 2013).

(d) *CERME 9*, *Prague*, Czech Republic, 2015

Thematic WG 12: *History in Mathematics Education*, organized by R. Chorlay, U. T. Jankvist, K. Clark, and J. van Maanen. The WG themes were modified considerably, becoming more specific. This reflects further deepening of research in this area, with emphasis both on empirical work and its assessment on sharpening theoretical ideas, and developing conceptual frameworks adequate for describing and understanding phenomena relevant to the *HPM perspective*; 14 papers and 2 posters accepted and included in the proceedings (Chorlay, Jankvist, Clark, & van Maanen, 2015).

(e) *CERME 10*, *Dublin*, Ireland, 2017

Thematic WG 12: *History in Mathematics Education*, organized by R. Chorlay, K. Clark, K. Gosztonyi, and S. Lawrence. The program was structured along 4 main themes, concerning: the design and/or assessment of teaching/learning materials using the history of mathematics; surveys on the existing uses of history or epistemology in mathematics education; theoretical and methodological issues also in relation to other areas of mathematics education; and the history of mathematics education; 16 papers and 2 posters accepted (Chorlay, Clark, Gosztonyi, & Lawrence, 2017)

(f) *CERME 11*, *Utrecht*, Netherlands, 2019

Thematic WG 12: *History in Mathematics Education*, organized by R. Chorlay, A. Bernardes, T. Hamann, and A. M. Oller-Marcén (https://cerme11.org/wp-content/uploads/2018/03/TWG_12_cfp.pdf). Structured along 4 themes (the same as CERME 10).

A.1.5 The *HPM domain* at GDM/ DMV and „Arbeitskreis Mathematikgeschichte und Unterricht“

The “Arbeitskreis Mathematikgeschichte und Unterricht” (Working Group Mathematics History and Education) of the German society, „GDM – Gesellschaft für Didaktik der Mathematik“ (Society for the Didactics of Mathematics) has existed since 1995. The working group brings together historians, mathematicians and educators interested in the history of mathematics. GDM’s annual conference is an appropriate forum for working groups, including the working group on the HPM domain. The “Österreichische Symposien zur Geschichte der Mathematik” (Austrian Symposia on the History of Mathematics) also take place every two years in Miesenbach. Moreover, approximately every two years is a joint conference of the “Arbeitskreis Mathematikgeschichte und Unterricht” of GDM and the “Fachsektion Mathematikgeschichte” (Section History of Mathematics) of the “DMV – Deutsche Mathematiker-Vereinigung” (German Mathematical Society) takes place:

1991, Gosen/ Berlin, Organization: H. Bernhardt and P. Schreiber

1993, Wuppertal, Organization: E. Scholz

1995, Rummelsberg, Organization: G. Löffladt and M. Toepell (Proceedings: Toepell, M. (Ed.) (1998). *Mathematik im Wandel 1*. Franzbecker: Hildesheim & Berlin.)

1997, Calw, Organization: M. von Renteln

1999, Bautzen-Schmochtitz, Organization: S. Deschauer and W. Voß

2001, Zingst, Organization: P. Schreiber (Proceedings: Toepell, M. (Ed.) (2009). *Mathematik im Wandel 4*. Franzbecker: Hildesheim.)

2002, Erfurt, Organization: H. Roloff and M. Weidauer (Proceedings: Weidauer, M. & Roloff, H. (Eds.) (2004). *Wege zu Adam Ries: Tagung zur Geschichte der Mathematik*. Rauner: Augsburg.)

2003, Attendorn/Neu-Listernohl, Organization: W. Hein and P. Ullrich (Proceedings: Hein, W. & Ullrich, P. (Eds.) (2004). *Mathematik im Fluss der Zeit: Tagung zur Geschichte der Mathematik*. Rauner: Augsburg.)

2005, Rummelsberg, Organization: G. Löffladt (Proceedings: Reich, U. & Hyksova, M. (Eds.) (2006). *Wanderschaft in der Mathematik: Tagung zur Geschichte der Mathematik in Rummelsberg*. Rauner: Augsburg.)

2007, Lambrecht/ Pfalz, Organization: P. Ullrich and I. Hupp (Proceedings: Hupp, I. & Ullrich, P. (Eds.) (2017). *Mathematische Streiflichter: Tagung zur Geschichte der Mathematikin Lambrecht (Pfalz) vom 16. bis 20. Mai 2007*. Rauner: Augsburg.)

2009, Pfalzgrafenweiler, Organization: M. von Renteln and U. Reich (Proceedings: Hyksova, M. & Reich, U. (Eds.) (2011). *Eintauchen in die mathematische Vergangenheit: Tagung zur Geschichte der Mathematik in Pfalzgrafenweiler im Schwarzwald*. Rauner: Augsburg.)

2011, Freisingen, Organization: H. Fischer (Proceedings: Fischer, H. & Deschauer, S. (Eds.) (2013). *Zeitläufte der Mathematik: Tagung zur Geschichte der Mathematik in Freising 2011*. Rauner: Augsburg.)

2013, Jena, Organization: M. Fothe, M. Schmitz, B. Skorsetz, and R. Tobies (Proceedings: Fothe, M., Schmitz, M., Skorsetz, B., & Tobies, R. (Eds.) (2014). *Mathematik und Anwendungen*. Thüringer Institut für Lehrerfortbildung: Bad Berka.)

2015, Hamburg/Seevetal, Organization: G. Wolfschmidt and H. Fischer (Proceedings: Wolfschmidt, G. (Ed.) (2017). *Proceedings of the Christoph J. Scriba Memorial Meeting: History of Mathematics*. Nuncius Hamburgensis, Band 36. Tredition: Hamburg.)

2017, Wittenberg, Organization: K. Richter

2019, Erbacher Hof/ Mainz, Organization: Y. Weiss

A.2 Journals and Newsletters

A.2.1 *Convergence: Where Mathematics, History, and Teaching Interact*

Since 2004, the MAA has published *Convergence*: *Where Mathematics, History and Teaching Interact* (Available online at http://www.maa.org/press/periodicals/convergence/about-convergence), a free online journal in HM and its use in teaching.

Aimed at teachers of mathematics at both the secondary and collegiate levels, *Convergence* includes topics from grades 8–16 mathematics, with special emphasis on grades 8–14. Its resources for using the HM in mathematics teaching include informative articles about the HM, translations of original sources, classroom activities, projects and modules, teaching tools such as its *Mathematical Treasures* (Available online at http://www.maa.org/press/periodicals/convergence/mathematical-treasures-from-the-smith-and-plimpton-collections-at-columbia-university), reviews of new and old books, websites, *Problems from Another Time* (Available online at *http://www.maa.org/press/periodicals/convergence/mathematical-treasures-from-the-smith-and-plimpton-collections-at-columbia-university*), and other teaching aids that focus on utility in the classroom.

A.2.2 The *British Journal for the History of Mathematics*

This is the former *Bulletin of the British Society for the History of Mathematics (BSHM Bulletin)*; (Available online at https://www.tandfonline.com/toc/tbsh21/current). It aims to promote research into the HM and to encourage its use at all levels of ME. Articles on local HM and the use of HM in ME are particularly encouraged. It was originally published as a Newsletter, until 2004 when its 50th issue became Bulletin 1. Under the influence of the late J. Fauvel, president of BSHM (1992–1994), editor of its Newsletter (1995–2001) and chair of the HPM Group (1992–1996) and his successor, the late J. Stedall, the Newsletter changed from providing information to members into a scientific journal with a regular Education Section directly related to issues relevant to the HPM perspective since 2002 (issue No 46).

A.2.3 SieB – *Siegener Beiträge zur Geschichte und Philosophie der Mathematik*

„Siegener Beiträge zur Geschichte und Philosophie der Mathematik – SieB" provides a forum for a discourse in the domain of the history and philosophy of mathematics in Germany. The presented essays are published once a year as a collective volume edited by Ralf Krömer (Bergische Universität Wuppertal) and Gregor Nickel (Universität Siegen) including several articles from an HPM perspective, though some of these volumes are monographs; e.g.:

Spies, S. (2013). *Ästhetische Erfahrung Mathematik: Über das Phänomen schöner Beweise und den Mathematiker als Künstler*;

Allmendinger, H. (2014). *Felix Kleins „Elementarmathematik vom höheren Standpunkte aus": Eine Analyse aus historischer und mathematikdidaktischer Sicht*;

Rathgeb, M. (2016). *George Spencer Browns „Laws of Form" zwischen Mathematik und Philosophie: Gehalt - Genese – Geltung* and

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A.2.4 The *HPM Newsletter*

The Newsletter appears three times per year since 1980. Originally it was available by contacting the regional distributors; however, for the last 13 years it is also available online from the HPM Group website (Available online at http://www.clab.edc.uoc.gr/hpm/) and its *Newsletter* web page (Available online at http://www.clab.edc.uoc.gr/hpm/NewsLetters.htm).

It includes a calendar of upcoming events, a guest editorial, a '*Have You Read These?*' section, short reviews and announcements of meetings and activities. Furthermore, for the last 13 years it has also included short articles, reports on research projects and PhD theses, book reviews, lists of relevant websites, and particular themes that are suggested for further research.